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Cyclic morphisms in the category of pairs and generalized G-sequences

By

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1. Introduction

In [1,2,3], D.H. Gottlieb introduced and studied the evaluation subgroups (or Gottlieb groups) $G_n(X)$ of $\pi_n(X)$. He used the concept of cyclic homotopies in the definition of Gottlieb groups. Varadarajian [12] transfered the epithet "cyclic" to the maps rather than homotopies and used the concept of cyclic maps to define a subset $G(A,X)$ of $\Pi(A,X)$ the set of homotopy class of maps from A to X. Furthermore, he used the subset $G(A,X)$ to study the role of cyclic map and cocyclic map in the set-up of Eckmann-Hilton duality.

Since then, many authors have studied and generalized $G_n(X)$, for instance, G.E. Lang [6], K.L. Lim [8], N. Oda [9], J. Siegel [11], J. Kim and the authors [5, 7, 10, 13]. In [5], the second author and J. Kim have generalized $G_n(X)$ to $G_n(X,A)$ for a CW-pair (X,A). In [7], the authors introduced the subgroups $G_n^{Rel}(X,A)$ of the relative homotopy groups $\pi_n(X,A)$ and showed that for a CW-pair (X,A), $G_n(A)$, $G_n(X,A)$ and $G_n^{Rel}(X,A)$ make a sequence

$$\cdots \to G_n(A) \xrightarrow{i_*} G_n(X,A) \xrightarrow{j_*} G_n^{Rel}(X,A) \xrightarrow{\partial} \cdots \to G_1^{Rel}(X,A) \to G_0(A) \to G_0(X,A),$$

where i_*, j_* and ∂ are restrictions of the usual homomorphisms of the homotopy sequence

$$\cdots \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \to \cdots \to \pi_0(A) \to \pi_0(X).$$

We call this sequence the G-sequence of a pair (X,A). We showed that if the inclusion $i: A \to X$ is homotopic to a constant map or has a left homotopy inverse then the G-sequence of the CW-pair (X,A) is exact. Recently, Oda [9] introduced the set of the homotopy classes of the axes of pairings as a generalization of the Varadarajin set $G(A,X)$ and the generalized evaluation subgroup $G_n^h(X,A)$ (in [5]).

In this paper, we introduce the concept of "cyclic morphism" as a generalization of cyclic map and we use this concept to define a set in the category of pairs. We show that this set is a generalization of all subgroups

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mentioned above, that is, Gottlieb groups $G_n(X)$, Varadarajian's set $G(A, X)$, Oda's set $G^h(A, X)$, generalized evaluation groups $G_n(X, A)$ and relative evaluation groups $G_n^{Rel}(X, A)$. Furthermore, we study the conditions for the sets to be homotopy invariant or groups. We also use the sets to study the role of cyclic morphisms in the category of pairs. We generalize the concept of G-sequence of a CW-pair to that of the category of pairs and study the conditions for this new sequence to be exact. By exactness, we obtain a nice form of computations for the generalized Gottlieb subsets.

Throughout this paper, all spaces will be connected and of the homotopy type of CW-complexes. Hence the exponential law of function spaces holds and all base points denoted by $*$ are nondegenerate.

2. Definitions and Notation

For $n \geqq 0$, let $\Sigma^n A$ be the n-th suspension of A, CA the cone of A and $i(A): A \to CA$ the natural inclusion given by $i(A)(x) = (x, 0)$. Then we are able to identify $C\Sigma^n A$ with $\Sigma^n CA$ and $i(\Sigma^n A)$ with $\Sigma^n(i(A))$ by bringing the last coordinate forward. So $\Sigma^n A$ and $C\Sigma^n A$ have the co-Hopf structure. Let $i(\Sigma^n A)$ be denoted by i_{n+1}. We denote by $\Pi(A, X)$ the set of homotopy classes of maps from A to X preserving base point. It is well-known that $\Pi_n(A, X) = \Pi(\Sigma^n A, X)$ is a group if $n \geqq 1$ and is abelian for $n \geqq 2$.

The category of pairs is the category in which the "objects" are maps $(A, *) \to (B, *)$ and a "map" from α to β is a pair of maps (f_1, f_2) such that the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow[\beta]{} & B_2 \end{array}$$

commutes [4].

We shall call the maps in this category just "morphisms" to distinguish from maps between spaces. Two morphisms $(f_1, f_2), (g_1, g_2): \alpha \to \beta$ are called *homotopic* if there is a morphism $(H_1, H_2): \alpha \times 1_I \to \beta$ such that H_1 is a homotopy between f_1 and g_1 and H_2 is a homotopy between f_2 and g_2, where 1_I is the indentity map of the unit interval I into iteself.

The set $\Pi(\alpha, \beta)$ is the set of homotopy classes of morphisms from α to β in the category of pairs. In particular, $\Pi_n(\alpha, \beta) = \Pi(\Sigma^n \alpha, \beta)$ is a group if $n \geqq 1$ and is abelian for $n \geqq 2$. If $\alpha = i_n: \Sigma^{n-1}A \to C\Sigma^{n-1}A$ is the natural inclusion, $\Pi(\alpha, \beta)$ is denoted by $\Pi_n(A, \beta)$ and is called the *n-th homotopy group of β rel.* A. If β is an inclusion and $A = S^0$, then we get the ordinary relative homotopy groups. Furthermore, if $\beta: * \to B$, then $\Pi_n(A, \beta) = \Pi_n(A, B)$ and if $\beta: B \to *$, then $\Pi_n(A, \beta) = \Pi_{n-1}(A, B)$.

Let Y^X be the function space of maps from X to Y with compact open topology,

$(Y^X; f)$ the path component of f in Y^X and $\omega\colon Y^X \to Y$ be the evaluation map given by $\omega(f)=f(*)$. Then ω is always continuous map for CW-complexes.

Here we recall several generalizations of Gottlieb groups and cyclic homotopies.

Definition 2.1 ([12]). A map $f\colon A \to X$ is said to be *cyclic* if there eixist a map $H\colon A \times X \to X$ such that the diagram

$$\begin{array}{ccc} A \times X & \xrightarrow{H} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ A \vee X & \xrightarrow[f \vee 1]{} & X \vee X \end{array}$$

is homotopy commutative, where j is the inclusion map and ∇ is the folding map.

Definition 2.2 ([12]). $G(A,X)=\{[f]\in \Pi(A,X) \mid f \text{ is a cyclic map}\}$, equivalently, $G(A,X)=\omega_*\Pi(A,X^X)$, where $\omega\colon X^X \to X$ is the evaluation map. In particular, $G(\Sigma^n A,X)$ is denoted by $\mathcal{G}_n(A,X)$. Equivalently, $\omega_*\Pi_n(A,X^X)=\mathcal{G}_n(A,X)$.

The subgroup $\mathcal{G}_n(A,X)$ is a generalization of $G(A,X)$ and the Gottlieb group $G_n(X)$. In fact, $\mathcal{G}_0(A,X)=G(A,X)$ and $\mathcal{G}_n(S^0,X)=G_n(X)$.

Definition 2.3 ([7]). A pair map $f\colon (B^n, S^{n-1}) \to (X,A)$ is *relative cyclic* if there exists a map $H\colon (B^n \times X, S^{n-1} \times A) \to (X,A)$ such that $H|_{B^n \times *}=f$ and $H|_{* \times X}=1_{(X,A)}$

In fact, for the CW-Pair (X,A), $\exists H\colon (B^n \times X, S^{n-1} \times A) \to (X,A)$ such that $H|_{B^n \times *}=f$ and $H|_{* \times X}=1_{(X,A)}$ if and only if $\exists H'\colon (B^n \times A, S^{n-1} \times A) \to (X,A)$ such that $H'|_{B^n \times *}=f$ and $H'|_{* \times A}=i_A$.

Definition 2.4 ([7]). $G_n^{Rel}(X,A)=\{[f]\in \pi_n(X,A) \mid f \text{ is relative cyclic.}\}$

Definition 2.5. *Let* $h\colon B \to X$ *be a map. A map* $f\colon A \to X$ *is called a cyclic map with respect to* h *if there exists a map* $H\colon A \times B \to X$ *such that the diagram*

$$\begin{array}{ccc} A \times B & \xrightarrow{H} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ A \vee B & \xrightarrow[f \vee h]{} & X \vee X \end{array}$$

is homotopy commutative.

In [9], Oda introduced the following set to generalize some of the results on the Varadarajin [12].

Definition 2.6 [9]. $\mathscr{G}^h(A,X)=\{[f]\in\Pi(A,X)\mid f$ is a cyclic map with respect to $h\}$, equivalently, $\mathscr{G}^h(A,X)=\omega_*\Pi(A,X^B;h)$, where $(X^B;h)$ means the component of h in the function space from B to X. In fact, Oda denoted this set h (A,X).

If h is the identity map of X, then the Oda's set $\mathscr{G}^h(A,X)$ is just the Varadarajin set $G(A,X)$. In particular, we denote $\mathscr{G}^h(\Sigma^n A,X)$ by $\mathscr{G}^h_n(A,X)$ and is equivalent to the image of $\omega_*;\Pi_n(A,X^B;h)\to\Pi_n(A,X)$. The subgroup $\mathscr{G}^h_n(A,X)$ is a generalization of the several subgroups mentioned above. In fact, we have $\mathscr{G}^{1_X}_0(A,X)=G(A,X)$, $\mathscr{G}^h_n(S^0,X)=G^h_n(X,B)$ (in [5]) and $\mathscr{G}^h_0(A,X)=\mathscr{G}^h(A,X)$.

3. Cyclic morphisms and their homotopy classes in the category of pairs

In this section, we introduce the notion of cyclic morphisms and study the set of their homotopy classes in the category of pairs.

Definition 3.1. Let $h:X\to B_1$ be a map. A map $(f_1,f_2):\alpha\to\beta$ is called a *cyclic morphism* with respect to h if there exists a map $(H_1,H_2):\alpha\times 1_X\to\beta$ such that $(H_1,H_2)|_\alpha=(f_1,f_2)$ and $(H_1,H_2)|_{1_X}=(h,\beta h)$, that is, the following diagram commutes

$$\begin{tikzcd}
A_1\vee X \arrow[r, "f_1\vee h"] \arrow[d, "j"'] & B1_1\vee B_1 \arrow[d, "\nabla"] \\
A_1\times X \arrow[r, "H_1"] \arrow[d, "\alpha\times 1"'] & B_1 \arrow[d, "\beta"] \\
A_2\times X \arrow[r, "H_2"'] & B_2 \\
A_2\vee X \arrow[r, "f_2\vee\beta h"'] \arrow[u, "j"] & B_2\vee B_2 \arrow[u, "\nabla"']
\end{tikzcd}$$

(H_1,H_2) is called an *affiliated morphism* of (f_1,f_2) with respect to h. If $h:B_1\to B_1$ is the identity, then (f_1,f_2) is called just a *cyclic morphism*.

Remark. If $\beta:B_1\to *$ is the trivial map, then it is easy to show that $(f_1,*):\alpha\to\beta$ is a cyclic morphism with respect to h if and only if $f_1:A_1\to B_1$ is a cyclic map with respect to h.

Let $i_n:S^{n-1}\to B^n$ and $i_A:A\to X$ be the inclusions. Then a pair map $f:(B^n,S^{n-1})\to(X,A)$ is relative cyclic if and only if $(f|_{S^{n-1}},f):i_n\to i_A$ is a cyclic morphism. So the concept of cyclic morphism is a generalization of relative cyclic map.

Definition 3.2. We define the subset $\mathscr{G}^h(\alpha,\beta)$ of $\Pi(\alpha,\beta)$ as the set of homotopy classes of cyclic morphisms with respect to h. That is,

$$\mathscr{G}^h(\alpha,\beta)=\{[f_1,f_2]\in\Pi(\alpha,\beta)\,|\,(f_1,f_2) \text{ is a cyclic morphism with respect to } h\}.$$

We denote $\mathscr{G}^h(\Sigma^n\alpha,\beta)$ by $\mathscr{G}^h_n(\alpha,\beta)$, where $\Sigma^n\alpha:\Sigma^n A_2\to\Sigma^n A_2$ is the map between two suspensions induced by α which is called a suspension map. In particular, if $i_n:\Sigma^{n-1}A\to C\Sigma^{n-1}A$ is the natural inclusion, then we denote $\mathscr{G}^h(i_n,\beta)$ by $\mathscr{G}^h_n(A,\beta)$. Moreover, we denote $\mathscr{G}^h_n(A,\beta)$ by $\mathscr{G}_n(A,\beta)$ if $h:B_1\to B_1$ is the identity map. $\mathscr{G}_n(A,\beta)$ is a generalization of $G_n^{Rel}(B_2,B_1)$ because $\mathscr{G}_n(S^0,i)=G_n^{Rel}(B_2,B_1)$, where $i:B_1\to B_1$ are the inclusion.

Define $\bar{\beta}:(B_1^X,h)\to(B_2^X,\beta h)$ by $\bar{\beta}(g)=\beta g$, where $\beta:B_1\to B_2$ is a map and let $\omega_1:B_1^X\to B_1$ and $\omega_2:B_2^X\to B_2$ be evaluation maps. Then $(\omega_1,\omega_2):\bar{\beta}\to\beta$ is a morphism and it induces a map $(\omega_1,\omega_2)_*:\Pi(\alpha,\bar{\beta})\to\Pi(\alpha,\beta)$.

Theorem 3.3. *Let $\beta:B_1\to B_2$ be a map and $\bar{\beta}:(B_1^X,h)\to(B_2^X,\beta h)$ be a map. Then $(\omega_1,\omega_2)_*\Pi(\alpha,\bar{\beta})=\mathscr{G}^h(\alpha,\beta)$.*

Proof. Let $[f_1,f_2]\in\mathscr{G}^h(\alpha,\beta)$. Then there exists an affiliated morphism $(H_1,H_2):\alpha\times 1_X\to\beta$ of (f_1,f_2) w.r.t. h. Let $\bar{f}_i$ be the adjoint of H_i given by $\bar{f}_i(a_i)(x)=H_i(a_i,x)$, where $a_i\in A_i$ and $x\in X$, for $i=1,2$. Since $\bar{\beta}\bar{f}_1=\bar{f}_2\alpha$, $(\bar{f}_1,\bar{f}_2)$ is a morphism from α to $\bar{\beta}$. So $[\bar{f}_1,\bar{f}_2]\in\Pi(\alpha,\bar{\beta})$. Since $[f_1,f_2]=[\omega_1\bar{f}_1,\omega_2\bar{f}_2]=(\omega_1,\omega_2)_*[\bar{f}_1,\bar{f}_2]$, we have $[f_1,f_2]\in(\omega_1,\omega_2)_*\Pi(\alpha,\bar{\beta})$. Therefore we obtain $\mathscr{G}^h(\alpha,\beta)\subset(\omega_1,\omega_2)_*\Pi(\alpha,\bar{\beta})$.

Similarly, we have $(\omega_1,\omega_2)_*\Pi(\alpha,\bar{\beta})\subset\mathscr{G}^h(\alpha,\beta)$.

By Theorem 3.3, if α is a suspension map, then $\mathscr{G}^h(\alpha,\beta)$ is a group. In particular, $\mathscr{G}^h_n(\alpha,\beta)$ is a group, for $n\geqq 1$. It is easy to prove the following theorem by using lemma.

Lemma 3.4. *Let $(g_1,g_2):\gamma\to\alpha$ be a morphism. If $(f_1,f_2):\alpha\to\beta$ is a cyclic morphism with respect to h, then the composition $(f_1,f_2)\circ(g_1,g_2):\gamma\to\beta$ is a cyclic morphism with respect to h.*

Theorem 3.5. *If $(g_1,g_2):\gamma\to\alpha$ is a morphism, then the induced map $(g_1,g_2)^*:\Pi(\alpha,\beta)\to\Pi(\gamma,\beta)$ carries $\mathscr{G}^h(\alpha,\beta)$ into $\mathscr{G}^h(\gamma,\beta)$.*

Lemma 3.6. *If $(g_1,g_2):\beta\to\gamma$ is a morphism and $(f_1,f_2):\alpha\to\beta$ is a cyclic morphism with respect to h, then the composition $(g_1,g_2)\circ(f_1,f_2)$ is a cyclic morphism with respect to g_1h.*

Proof. Let $(H_1,H_2):\alpha\times 1_X\to\beta$ be an affiliated morphism of (f_1,f_2). Then $(g_1H_1,g_2H_2):\alpha\times 1_X\to\gamma$ is an affiliated morphism of $(g_1,g_2)\circ(f_1,f_2)$. It is explained by the following diagram and the fact that $g_2\beta=\gamma g_1$;

$$\begin{tikzcd}
A_1 \vee X \arrow[r, "f_1 \vee h"] \arrow[d, "j"'] & B_1 \vee B_1 \arrow[r, "g_1 \vee g_1"] \arrow[d, "\nabla"'] & C_1 \vee C_1 \arrow[d, "\nabla"] \\
A_1 \times X \arrow[r, "H_1"] \arrow[d, "\alpha \times 1_X"'] & B_1 \arrow[r, "g_1"] \arrow[d, "\beta"'] & C_1 \arrow[d, "\gamma"] \\
A_2 \times X \arrow[r, "H_2"] & B_2 \arrow[r, "g_2"] & C_2 \\
A_2 \vee X \arrow[r, "f_2 \vee \beta h"'] \arrow[u, "j"] & B_2 \vee B_2 \arrow[r, "g_2 \vee g_2"'] \arrow[u, "\nabla"'] & C_2 \vee C_2 \arrow[u, "\nabla"']
\end{tikzcd}$$

Theorem 3.7. *If* $(g_1, g_2): \beta \to \gamma$ *is a morphism, then the induced map* $(g_1, g_2)_*: \Pi(\alpha, \beta) \to \Pi(\alpha, \gamma)$ *carries* $\mathscr{G}^h(\alpha, \beta)$ *into* $\mathscr{G}^{g_1 h}(\alpha, \gamma)$.

If $\alpha \times 1_X: A_1 \times X \to A_2 \times X$ is a cofibration, then $\mathscr{G}^h(\alpha, \beta)$ is determined by the homotopy class of h.

Lemma 3.8. *Let* $\alpha \times 1_X: A_1 \times X \to A_2 \times X$ *be a cofibration and* $h, h': X \to B_1$ *be maps. If h is homotopic to h', then* $\mathscr{G}^h(\alpha, \beta) = \mathscr{G}^{h'}(\alpha, \beta)$.

Proof. It is sufficient to show that one of them contains the other. Let $(f_1, f_2): \alpha \to \beta$ be a cyclic morphism with respect to h. Then there is an affiliated morphism $(H_1, H_2): \alpha \times 1_X \to \beta$ with respect to h such that the following diagram are commutative

$$\begin{tikzcd}
A_1 \vee X \arrow[r, "f_1 \vee h"] \arrow[d, "j"'] & B_1 \vee B_1 \arrow[d, "\nabla"] \\
A_1 \times X \arrow[r, "H_1"] \arrow[d, "\alpha \times 1"'] & B_1 \arrow[d, "\beta"] \\
A_2 \times X \arrow[r, "H_2"'] & B_2 \\
A_2 \vee X \arrow[r, "f_2 \vee \beta h"'] \arrow[u, "j"] & B_2 \vee B_2 \arrow[u, "\nabla"']
\end{tikzcd}$$

Moreover, since h is homotopic to h', there exists a homotopy F from h to h'. Define

$$\hat{H}_1 : (A_1 \vee X) \times I \quad A_1 \times X \times 0 \to B_1$$

by

$$\hat{H}_1|_{A_1\times *\times I}=f_1,\ \hat{H}_1|_{*\times X\times I}=F \text{ and } \hat{H}_1|_{A_1\times X\times 0}=H_1.$$

Then since the inclusion $A_1 \vee X \subsetneq A_1\times X$ is a cofibration, there is an extention $\bar{H}_1: A_1\times X\times I\to B_1$ of $\hat{H}_1$. Consider the map $\beta\bar{H}_1: A_1\times X\times I\to B_2$ and the following diagram

$$\begin{tikzcd}
A_1\times X\times 0 \arrow[rr, hook] \arrow[dd, "\alpha\times 1_X\times 1_0"'] & & A_1\times X\times I \arrow[dl, "\beta\bar{H}_1"'] \arrow[dd, "\alpha\times 1_X\times 1_I"] \\
& B_2 \arrow[dr, "?"] & \\
A_2\times X\times 0 \arrow[ur, "H_2"] \arrow[rr, hook] & & A_2\times X\times I
\end{tikzcd}$$

Since $\alpha\times 1_X$ is a cofibration, there exists a map $\bar{H}_2: A_2\times X\times I\to B_2$ such that $\beta\bar{H}_1=\bar{H}_2(\alpha\times 1_X\times 1_I)$ and $H_2=\bar{H}_2|_{A_2\times X\times 0}$. Let $H_1'=\bar{H}_1|_{A_1\times X\times 1}$ and $H_2'=\bar{H}_2|_{A_2\times X\times 1}$. Then $H_1'|_{A_1\times *}=f_1$, $H_1'|_{*\times X}=h'$ and $H_2'|_{A_2\times *}=\bar{H}_2|_{A_2\times *\times 1}\sim \bar{H}_2|_{A_2\times *\times 0}=f_2$. Furthermore, $H_2'(*,x)=\bar{H}_2(*,x,1)=\bar{H}_2(\alpha(*),x,1)=\beta\bar{H}_1(*,x,1)=\beta h'(x)$. Thus $[f_1,f_2]=[f_1,H_2'|_{A_2\times *}]\in\mathscr{G}^{h'}(\alpha,\beta)$.

In Lemma 3.8, (H_1', H_2') may not be an affiliated morphism of (f_1,f_2) with respect to h' because we can prove only $H_2'|_{A_2\times *}$ is homotopic to f_2 rather than equal. But if $\alpha=i: A_1\to A_2$ is the inclusion, then we can obtain an affilated morphism of (f_1,f_2) with respect to h'. So we can get a stronger theorem than Lemma 3.8. In the proof of Lemma 3.8, define

$$\hat{H}_2: A_1\times X\times I \quad A_2\times *\times I\to B_2$$

by $\hat{H}_2=\beta\hat{H}_1 \coprod f_2$. Then it is well-defined since $\beta\hat{H}_1(a,*,t)=\beta f_1(a)=f_2(a)$. If we substitute $\hat{H}_2$ for $\beta\hat{H}_1$ and apply the property of cofibration, then we obtain a map $H_2': A_2\times X\to B_2$ such that $H_2'|_{A_2\times *}=f_2$. So (H_1', H_2') is an affiliated morphism of (f_1,f_2) with respect to h'. Thus we have the following theorem.

Theorem 3.9. *Let $\alpha=i: A_1\to A_2$ be the inclusion and let $h,h': X\to B_1$ be homotopic. Then $(f_1,f_2):\alpha\to\beta$ is a cyclic morphism with respect to h if and only if it is a cyclic morphism with respect to h'.*

Corollary 3.10. *If $h,h': X\to B_1$ are homotopic, then $\mathscr{G}^h_n(A,\beta)=\mathscr{G}^{h'}_n(A,\beta)$.*

Let $(f_1,f_2):\alpha\to\beta$ be a morphism. A morphism $(g_1, g_2):\beta\to\alpha$ is called a left homotopy inverse of (f_1,f_2) if $(g_1,g_2)\circ(f_1,f_2)$ is homotopic to 1_α, that is, there exists a morphism $(H_1, H_2):\alpha\times 1_I\to\alpha$ such that $(H_1, H_2)|_{\alpha\times 0}=(g_1,g_2)\circ(f_1,f_2)$ and $(H_1, H_2)|_{\alpha\times 1}=1_\alpha$. Similarly, $(g_1,g_2):\beta\to\alpha$ is called a right homotopy inverse of $(f_1,f_2):\alpha\to\beta$ if $(f_1,f_2)\circ(g_1,g_2)$ is homotopic to 1_β. In particular, $(f_1,f_2):\alpha\to\beta$ is called a homotopy equivalence if it has a right and left homotopy inverse.

Corollary 3.11. *If the morphism* $(g_1,g_2):\gamma\to\alpha$ *is a homotopy equivalence, then the induced map* $(g_1,g_2)^*:\mathscr{G}^h(\alpha,\beta)\to\mathscr{G}^h(\gamma,\beta)$ *is an isomorphism of sets.*

Proof. It follows from Corollary 3.5 and Corollary 3.10.

Lemma 3.12. *Let* $\alpha\times 1_X:A_1\times X\to A_2\times X$ *be a cofibration. If the morphism* $(g_1,g_2):\beta\to\gamma$ *is a homotopy equivalence, then the induced map* $(g_1,g_2)_*:\mathscr{G}^h(\alpha,\beta)\to\mathscr{G}^{g_1h}(\alpha,\gamma)$ *is an isomorphism of sets. If* α *is a suspension map, then it is a group isomorphism.*

Proof. Let $(r_1,r_2):\gamma\to\beta$ is a homotopy inverse of (g_1,g_2). Then $(r_1,r_2)_*$ carries $\mathscr{G}^{g_1h}(\alpha,\gamma)$ into $\mathscr{G}^{r_1g_1h}(\alpha,\beta)$ by Theorem 3.7. By Lemma 3.8, we have $\mathscr{G}^{r_1g_1h}(\alpha,\beta)=\mathscr{G}^h(\alpha,\beta)$ and hence this completes the proof.

Theorem 3.13. *The subgroup* $\mathscr{G}_n(A,\beta)$ *of* $\Pi_n(A,\beta)$ *is homotopy invariant with respect to two variables.*

Proof. For the first variable, the theorem is true by Corollary 3.11. We show that it is true for second variable.

Let $(g_1,g_2):\beta\to\gamma$ is homotopy equivalent with homotopy inverse (r_1,r_2) where $\beta:B_1\to B_2$ and $\gamma:C_1\to C_2$. By Lemma 3.12, $(g_1,g_2)_*:\mathscr{G}_n(A,\beta)\to\mathscr{G}_n^{g_1}(A,\gamma)$ is an isomorphism. So it is sufficient to show $\mathscr{G}_n^{g_1}(A,\gamma)=\mathscr{G}_n(A,\gamma)$. Let $[f_1,f_2]\in\mathscr{G}_n^{g_1}(A,\gamma)$. Then there is an affiliated morphism $(H_1,H_2):i_n\times 1_{B_1}\to\gamma$ such that $(H_1,H_2)|_{i_n}=(f_1,f_2)$ and $(H_1,H_2)|_{1_{B_1}}=(g_1,\gamma g_1)$. Let $H_1'=H_1(1_{\Sigma^{n-1}A}\times r_1)$ and $H_2'=H_2(1_{C\Sigma^{n-1}A}\times r_1)$. Then $(H_1',H_2')|_{i_n}=(f_1,f_2)$ and $(H_1',H_2')|_{1_{C_1}}=(g_1r_1,\gamma g_1r_1)$. So $[f_1,f_2]\in\mathscr{G}_n^{g_1r_1}(A,\gamma)$. Since $\mathscr{G}_n^{g_1\gamma_1}(A,\gamma)=\mathscr{G}_n(A,\gamma)$ by Lemma 3.8, we have $\mathscr{G}_n^{g_1}(A,\gamma)\subset\mathscr{G}_n(A,\gamma)$. Similarly, $\mathscr{G}_n^{g_1}(A,\gamma)\supset\mathscr{G}_n(A,\gamma)$.

4. A generalization of G-sequence to the category of pairs

Let $\bar\beta:(B_1{}^{B_1},1_{B_1})\to(B_2{}^{B_1},\beta)$ be a map given by $\bar\beta(g)=\beta g$ and let $\omega_1:B_1{}^{B_1}\to B_1$ and $\omega_2:B_2{}^{B_1}\to B_2$ be evaluation maps given by $\omega_1(g)=g(*)$ and $\omega_2(g')=g'(*)$ respectively, where $*$ is a base point of B_1. Then $(\omega_1,\omega_2):\bar\beta\to\beta$ is a map and it induces a homomorphism $(\omega_1,\omega_2)_*:\Pi_n(A,\bar\beta)\to\Pi_n(A,\beta)$. By Theorem 3.3, we have $(\omega_1,\omega_2)_*\Pi_n(A,\bar\beta)=\mathscr{G}_n(A,\beta)$. Therefore, if $\beta:*\to B_2$, then $\mathscr{G}_n(A,\beta)=\Pi_n(A,B_2)$ and if $\beta:B_1\to *$, $\mathscr{G}_n(A,\beta)=\omega_{1_*}\Pi_{n-1}(A,B_1{}^{B_1})=\mathscr{G}_{n-1}(A,B_1)$.

Let $\beta:B_1\to B_2$ be a map. Then there exists an exact sequence

$$\cdots\to\Pi_n(A,B_1)\xrightarrow{\beta_*}\Pi_n(A,B_2)\xrightarrow{J}\Pi_n(A,\beta)\xrightarrow{\partial}\Pi_{n-1}(A,B_1)\to\cdots,$$

where β_* is the induced map, J is explained by the diagram

$$\begin{array}{ccccc}
\Sigma^{n-1}A & \xrightarrow{f_1} & * & \longrightarrow & B_1 \\
{\scriptstyle i_n}\downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} \\
C\Sigma^{n-1}A & \xrightarrow{f_2} & B_2 & \xrightarrow{1_{B_2}} & B_2
\end{array}$$

and ∂ by

$$\begin{array}{ccccc}
\Sigma^{n-1}A & \xrightarrow{f_1} & B_1 & \xrightarrow{1_{B_1}} & B_1 \\
{\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow \\
C\Sigma^{n-1}A & \xrightarrow{f_2} & B_2 & \longrightarrow & *
\end{array}$$

If $\beta: B_1 \to B_2$ is a map and $\bar{\beta}: B_1{}^{B_1} \to B_2{}^{B_1}$ is the map defined by $\bar{\beta}(g) = \beta g$, then we have an exact commutative ladder by the naturality

$$\begin{array}{ccccccccc}
\cdots \to & \Pi_n(A, B_1{}^{B_1}; 1_{B_1}) & \xrightarrow{\bar{\beta}_*} & \Pi_n(A, B_2{}^{B_1}; \beta) & \xrightarrow{J} & \Pi_n(A, \bar{\beta}) & \xrightarrow{\partial} & \Pi_{n-1}(A, B_1{}^{B_1}; 1_{B_1}) & \to \cdots \\
& \downarrow{\scriptstyle \omega_{1*}} & & \downarrow{\scriptstyle \omega_{2*}} & & \downarrow{\scriptstyle (\omega_1, \omega_2)_*} & & \downarrow{\scriptstyle \omega_{1*}} & \\
\cdots \to & \Pi_n(A, B_1) & \xrightarrow{\beta_*} & \Pi_n(A, B_2) & \xrightarrow{J} & \Pi_n(A, \beta) & \xrightarrow{\partial} & \Pi_{n-1}(A, B_1) & \to \cdots
\end{array}$$

By Definition 2.2 and Definition 2.7, we can make a subsequence

$$\cdots \to \mathcal{G}_n(A, B_1) \xrightarrow{\beta_*} \mathcal{G}_n^{\beta}(A, B_2) \xrightarrow{J} \mathcal{G}_n(A, \beta) \xrightarrow{\partial} \mathcal{G}_{n-1}(A, B_1) \to \cdots$$

from the above commutative ladder. We call this sequence *G-sequence of β rel. A* in the category of pairs.

Theorem 4.1. *If $\beta: B_1 \to B_2$ is null homotopic, then the G-sequence of β rel. A is exact.*

Before we prove the Theorem 4.1, we shall show the following lemma.

Lemma 4.2. *If $\beta: B_1 \to B_2$ is null homotopic, then $\mathcal{G}_n^{\beta}(A, B_2) = \Pi_n(A, B_2)$.*

Proof. Since β is homotopic to a constant map $c: B_1 \to B_2$ such that $c(b) = *$, there is a path $l: I \to B_2{}^{B_1}$ from β to c. So we have a natural isomorphism $l_*: \Pi_n(A, B_2{}^{B_1}; \beta) \to \Pi_n(A, B_2{}^{B_1}; c)$. Moreover, $\omega_* l_* = \omega_*$ in the diagram

$$\begin{tikzcd}
\Pi_n(A, B_2{}^{B_1}; \beta) \arrow[r, "i_*"] \arrow[d, "\omega_*"'] & \Pi_n(A, B_2{}^{B_1}; c) \arrow[d, "\omega_*"] \\
\Pi_n(A, B_2; *) \arrow[r, "1"] & \Pi_n(A, B_2; *)
\end{tikzcd}$$

where $\omega: B_2{}^{B_1} \to B_2$ is the evaluation given by $\omega(g) = g(*)$. Let $f: (\Sigma^n A, *) \to (B_2, *)$ be a map. If we define $\bar{f}: (\Sigma^n A, *) \to (B_2{}^{B_1}, c)$ by $\bar{f}(a)(b) = f(a)$, then $[\bar{f}] \in \Pi_n(A, B_2{}^{B_1}; c)$ and $\omega_*[\bar{f}] = [f] \in \Pi_n(A, B_2; *)$. So we have $\omega_*(\Pi_n(A, B_2{}^{B_1}; c)) = \Pi_n(A, B_2; *)$. Thus we have $\mathcal{G}_n^\beta(A, B_2) = \Pi_n(A, B_2; *)$.

Corollary 4.3. *If $\beta: B_1 \to B_2$ is null homotopic, then $J(\Pi_n(A, B_2)) \subset \mathcal{G}_n(A, \beta)$.*

Proof of Theorem 4.1. Consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Pi_n(A, B_1{}^{B_1}; 1_{B_1}) \arrow[r, "\bar{\beta}_*"] \arrow[d, "\omega_{1*}"] & \Pi_n(A, B_2{}^{B_1}; \beta) \arrow[r, "\bar{J}"] \arrow[d, "\omega_{2*}"] & \Pi_n(A, \bar{\beta}) \arrow[r, "\bar{\partial}"] \arrow[d, "(\omega_1, \omega_2)_*"] & \Pi_{n-1}(A, B_1{}^{B_1}; 1_{B_1}) \arrow[r] \arrow[d, "\omega_{1*}"] & \cdots \\
\cdots \arrow[r] & \mathcal{G}_n(A, B_1) \arrow[r, "\beta'"] \arrow[d, "I"'] & \mathcal{G}_n^\beta(A, B_2) \arrow[r, "J'"] \arrow[d, "I"'] & \mathcal{G}_n(A, \beta) \arrow[r, "\partial'"] \arrow[d, "I"'] & \mathcal{G}_{n-1}(A, B_1) \arrow[r] \arrow[d, "I"'] & \cdots \\
\cdots \arrow[r] & \Pi_n(A, B_1) \arrow[r, "\beta_*"] & \Pi_n(A, B_2) \arrow[r, "J"] & \Pi_n(A, \beta) \arrow[r, "\partial"] & \Pi_{n-1}(A, B_1) \arrow[r] & \cdots
\end{tikzcd}$$

where $\beta_*' = \beta_*|_{\mathcal{G}_n(A, B_1)}$, $J' = J|_{\mathcal{G}_n^\beta(A, B_2)}$, $\partial' = \partial|_{\mathcal{G}_n(A, B)}$ and I's are inclusions.

Since β is null homotopic, $\bar{\beta}: B_1{}^{B_1} \to B_2{}^{B_1}$ is null homotopic. Thus β_* and $\bar{\beta}_*$ are 0-homomorphisms and J, $\bar{J}$ are monomorphisms. From this fact, the G-sequence of β rel. A is exact at $\mathcal{G}_n^\beta(A, B_2)$. Furthermore, the sequence is exact at $\mathcal{G}_n(A, \beta)$ by Corollary 4.3. We must show that the sequence is exact at $\mathcal{G}_{n-1}(A, B_1)$ but it is sufficient to show that $\partial'(\mathcal{G}_n(A, \beta)) = \mathcal{G}_{n-1}(A, B_1)$. Since $\bar{\partial}$ is an epimorphism, $\partial'(\mathcal{G}_n(A, \beta)) = \partial'(\omega_1, \omega_2)_*(\Pi_n(A, \bar{\beta})) = \omega_{1_*}\bar{\partial}(\Pi_n(A, \bar{\beta})) = \omega_{1_*}(\Pi_{n-1}(A, B_1{}^{B_1})) = \mathcal{G}_{n-1}(A, B_1)$.

Theorem 4.4. *If $\beta: B_1 \to B_2$ has a left homotopy inverse, then the G-sequence of β rel. A in the category of pairs is exact.*

Before we prove Theorem 4.4, we need to show the following lemma

Lemma 4.5. *If $\beta: B_1 \to B_2$ has a left homotopy inverse, then we have*

$$\beta_*(\mathcal{G}_n(A, B_1)) = \beta_*(\Pi_n(A, B_1)) \cap \mathcal{G}_n^\beta(A, B_2).$$

Proof. Since $\beta_*(\mathcal{G}_n(A, B_1)) \subset \beta_*(\Pi_n(A, B_1)) \cap \mathcal{G}_n^\beta(A, B_2)$, it is sufficient to show that $\beta_*(\mathcal{G}_n(A, B_1)) \supset \beta_*(\Pi_n(A, B_1)) \cap \mathcal{G}_n^\beta(A, B_2)$. Let $\gamma: B_2 \to B_1$ be a left homotopy inverse of β and $[f] \in \beta_*(\Pi_n(A, B_1)) \cap \mathcal{G}_n^\beta(A, B_2)$. Then there is an element

$[g]\in\Pi_n(A,B_1)$ and a map $H:\Sigma^nA\times B_1\to B_2$ such that $\beta_*[g]=[f]$ and the following diagram commutes homotopically

$$\begin{tikzcd}
\Sigma^nA\times B_1 \arrow[r, "H"] & B_2 \\
\Sigma^nA\vee B_1 \arrow[u, "j"] \arrow[r, "f\vee\beta"'] & B_2\vee B_2 \arrow[u, "\nabla"']
\end{tikzcd}$$

Define $H'=\gamma H$. Then $\gamma H|_{\Sigma^nA\times *}\sim\gamma f$ and $\gamma H|_{*\times B_1}\sim\gamma\beta\sim 1_{B_2}$. Thus H' is an affileated map of γf. So $\gamma_*[f]=[\gamma f]\in\mathscr{G}_n(A,B_1)$. Furthermore, $[g]=\gamma_*\beta_*[g]=\gamma_*[f]\in\mathscr{G}_n(A,B_1)$. So $\beta_*[g]=[f]\in\beta_*\mathscr{G}_n(A,B_1)$.

Corollary 4.6. *If $\beta:B_1\to B_2$ has a left homomtopy inverse γ, then $\gamma_*(\mathscr{G}_n^\beta(A,B_2))\subset\mathscr{G}_n(A,B_1)$.*

Proof of Theorem 4.4. Let $\gamma:B_2\to B_1$ be a left homotopy inverse of β. Then we have commutative ladder

$$\begin{tikzcd}
{} \arrow[r] & \Pi_n(A,{B_1}^{B_1};1_{B_1}) \arrow[r, shift left, "\bar\beta_*"] \arrow[d, "\omega_{1*}"] & \Pi_n(A,{B_2}^{B_1};\beta) \arrow[l, shift left, "\bar\gamma_*"] \arrow[r, "\bar J"] \arrow[d, "\omega_{2*}"] & \Pi_n(A,\bar\beta) \arrow[r, "\bar\partial"] \arrow[d, "(\omega_1{,}\omega_2)_*"] & \Pi_{n-1}(A,{B_1}^{B_1};1_{B_1}) \arrow[r] \arrow[d, "\omega_{1*}"] & {} \\
{} \arrow[r] & \mathscr{G}_n(A,B_1) \arrow[r, "\beta'_*"] \arrow[d, "I"'] & \mathscr{G}_n^\beta(A,B_2) \arrow[r, "J'"] \arrow[d, "I"'] & \mathscr{G}_n(A,\beta) \arrow[r, "\partial'"] \arrow[d, "I"'] & \mathscr{G}_{n-1}(A,B_1) \arrow[r] \arrow[d, "I"'] & {} \\
{} \arrow[r] & \Pi_n(A,B_1) \arrow[r, shift left, "\beta_*"] & \Pi_n(A,B_2) \arrow[l, shift left, "\gamma_*"] \arrow[r, "J"] & \Pi_n(A,\beta) \arrow[r, "\partial"] & \Pi_{n-1}(A,B_1) \arrow[r] & {}
\end{tikzcd}$$

where $\beta_*'=\beta_*|_{\mathscr{G}_n(A,B_1)}$, $J'=J|_{\mathscr{G}_n^\beta(A,B_2)}$, $\partial'=\partial|_{\mathscr{G}_n(A,\beta)}$, I's are inclusions, $\bar\beta:{B_1}^{B_1}\to{B_2}^{B_1}$ is given by $\bar\beta(f)=\beta f$ and $\bar\gamma:{B_2}^{B_1}\to{B_1}^{B_1}$ is given by $\bar\gamma(g)=\gamma g$.

Since the lower sequence is exact and $\gamma_*\beta_*=1$, β_* is a monomorphism and so ∂ is a 0-homomorphism. Therefore, the G-sequence is exact at $\mathscr{G}_n(A,\beta)$. By Lemma 4.5, we have

$$\beta'_*(\mathscr{G}_n(A,B_1))=\beta_*(\Pi_n(A,B_1))\cap\mathscr{G}_n^\beta(A,B_2)=\operatorname{Ker}J\cap\mathscr{G}_n^\beta(A,B_2)=\operatorname{Ker}J'.$$

So the G-sequence is exact at $\mathscr{G}_n^\beta(A,B_2)$.

Finally, we show the G-sequence is exact at $\mathscr{G}_n(A,\beta)$. Since $\gamma\beta\sim 1_{B_1}$, $\bar\gamma\bar\beta\sim 1_{{B_1}^{B_1}}$. Let $F:B_1\times I\to B_1$ is a homotopy from $\gamma\beta$ to 1_{B_1}. Then $\bar F:{B_1}^{B_1}\times I\to{B_1}^{B_1}$ given by $\bar F(f,t)(b)=F(f(b),t)$ is a homotopy from $\bar\gamma\bar\beta$ to $1_{{B_1}^{B_1}}$. Thus $\bar\beta_*$ is a monomorphism and $\bar J$ is an epimorphism. Therefore, we have $J'(\mathscr{G}_n^\beta(A,B_2))=\mathscr{G}_n(A,\beta)=\operatorname{Ker}\partial\cap\mathscr{G}_n(A,\beta)=\operatorname{Ker}\partial'$.

Corollary 4.7. *If $\beta: B_1 \to B_2$ has a left homotopy inverse, then we have*

$$\mathscr{G}_n^{\beta}(A, B_2) \cong \mathscr{G}_n(A, B_1) \oplus \mathscr{G}_n(A, \beta).$$

Given a differential triple $B_0 \xrightarrow{v} B_1 \xrightarrow{\beta} B_2$ with $\beta v = *$, there exists a homomorphism $\varepsilon: \Pi_n(A, v) \to \Pi_n(A, B_2)$ given by $\varepsilon[f_1, f_2] = (*, \beta)_*[f_1, f_2]$ in the following commutative diagram

$$\begin{array}{ccccc}
\Sigma^{n-1}A & \xrightarrow{f_1} & B_0 & \xrightarrow{*} & * \\
{\scriptstyle i_n}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle *} \\
C\Sigma^{n-1}A & \xrightarrow[f_2]{} & B_1 & \xrightarrow[\beta]{} & B_2
\end{array}$$

In particular, if $B_0 \xrightarrow{v} B_1 \xrightarrow{\beta} B_2$ is a fibration, then $\varepsilon: \Pi_n(A, v) \to \Pi_n(A, B_2)$ is an isomorphism.

Lemma 4.8. *If the following diagram of two differential triples*

$$\begin{array}{ccccc}
B_0 & \xrightarrow{v} & B_1 & \xrightarrow{\beta} & B_2 \\
{\scriptstyle \alpha_0}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
B'_0 & \xrightarrow[v']{} & B'_1 & \xrightarrow[\beta']{} & B'_1
\end{array}$$

is commutative, then we have the following commutative diagram

$$\begin{array}{ccc}
\Pi_n(A, v) & \xrightarrow{\varepsilon} & \Pi_n(A, B_2) \\
{\scriptstyle (\alpha_0, \alpha_1)_*}\downarrow & & \downarrow{\scriptstyle \alpha_{2*}} \\
\Pi_n(A, v') & \xrightarrow[\varepsilon']{} & \Pi_n(A, B'_2)
\end{array}$$

Proof. We can prove the lemma by the following two diagrams

$$\begin{array}{ccccccc}
\Sigma^{n-1}A & \xrightarrow{f_1} & B_0 & \xrightarrow{\alpha_0} & B_1 & \to & * \\
{\scriptstyle i_n}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} & & \downarrow \\
C\Sigma^{n-1}A & \xrightarrow[f_2]{} & B_1 & \xrightarrow[\alpha_1]{} & B'_1 & \xrightarrow{\beta'} & B'_2
\end{array}$$

and

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \xrightarrow{f_1} & B_0 & \xrightarrow{*} & * & \xrightarrow{*} & * \\ {\scriptstyle i_n}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow & & \downarrow \\ C\Sigma^{n-1}A & \xrightarrow[f_2]{} & B_1 & \xrightarrow[\beta]{} & B_2 & \xrightarrow[\alpha_2]{} & B_2' \end{array}$$

Since $\alpha_{2*}\varepsilon[f_1, f_2] = [*, \alpha_2\beta f_2]$, $\varepsilon'(\alpha_0, \alpha_1)_*[f_1, f_2] = [*, \beta'\alpha_1 f_2]$ and $\alpha_2\beta = \beta'\alpha_1$, the lemma was proved.

Theorem 4.9. *If $B_0 \xrightarrow{v} B_1 \xrightarrow{\beta} B_2$ is a fibration, then we have $\mathscr{G}_n(A, v) = \Pi_n(A, B_2)$.*

Proof. It is sufficient to show that $\varepsilon|_{\mathscr{G}_n(A,v)}$ is an epimorphism. If we define

$$\bar{v}: (B_0{}^{B_0}, 1_{B_0{}^{B}{}_0}) \to (B_1{}^{B_0}, v) \text{ and } \bar{\beta}: (B_1{}^{B_0}, v) \to (B_2{}^{B_0}, c),$$

by $\bar{v}(f) = vf$ and $\bar{\beta}(g) = \beta g$, then the triple $(B_0{}^{B_0}, 1_{B_0{}^{B}{}_0}) \xrightarrow{\bar{v}} (B_1{}^{B_0}, v) \xrightarrow{\bar{\beta}} (B_2{}^{B_0}, c)$ is a fibration, where c is the constant map. So there exists an isomorphism $\bar{\varepsilon}: \Pi_n(A, \bar{v}) \to \Pi_n(A, B_2{}^{B_0})$. Since the diagram

$$\begin{array}{ccccc} (B_0{}^{B_0}, 1_{B_0{}^{B}{}_0}) & \xrightarrow{\bar{v}} & (B_1{}^{B_0}, v) & \xrightarrow{\bar{\beta}} & (B_2{}^{B_0}, c) \\ {\scriptstyle \omega_0}\downarrow & & {\scriptstyle \omega_1}\downarrow & & {\scriptstyle \omega_2}\downarrow \\ (B_0, *) & \xrightarrow{v} & (B_1, *) & \xrightarrow{\beta} & (B_2, *) \end{array}$$

is commutative, we have the following commutative diagram

$$\begin{array}{ccc} \Pi_n(A, \bar{v}) & \xrightarrow{\bar{\varepsilon}} & \Pi_n(A, (B_2{}^{B_0}, c)) \\ {\scriptstyle (\omega_0, \omega_1)_*}\downarrow & & \downarrow{\scriptstyle \omega_{2*}} \\ \Pi_n(A, v) & \xrightarrow[\varepsilon]{} & \Pi_n(A, B_2). \end{array}$$

By the fact ω_{2*} is an epimorphism, we can prove $\mathscr{G}_n(A, v) = \Pi_n(A, B_2)$.

By Theorem 4.9, we have the following corollaries.

Corollary 4.10. *If $B_0 \xrightarrow{v} B_1 \xrightarrow{\beta} B_2$ is a fibration, then we have the following sequence*

$$\cdots \to \mathscr{G}_n(A, B_0) \xrightarrow{v_*} \mathscr{G}_n^v(A, B_1) \xrightarrow{\beta_*} \Pi_n(A, B_n) \xrightarrow{\partial} \mathscr{G}_{n-1}(A, B_0) \to \cdots$$

This sequence is called *the G-sequence of the fibration rel. A* in the category of pairs. If $B_0 \xrightarrow{\nu} B_1 \xrightarrow{\beta} B_2$ is a fibration, we can easily check that the G-sequence of ν is exact if and only if the G-sequence of the fibration in the category of pairs is exact. Thus if ν is null homotopic or has a left homotopy inverse, then the G-sequence of the fibration in the category of pairs is exact. Especially, if ν has a left homotopy inverse, then we have the following corollary.

Corollary 4.11. *If $B_0 \xrightarrow{\nu} B_1 \xrightarrow{\beta} B_2$ is a fibration and ν has a left homotopy inverse, then we have*

$$\mathcal{G}_n^{\nu}(A, B_1) \cong \mathcal{G}_n(A, B_0) \oplus \Pi_n(A, B_2).$$

Consider the following commutative ladder which consists of G-sequence of the fibration and the homotopy sequence of the fibration

$$\begin{array}{ccccccc}
\to \mathcal{G}_n(A, B_0) & \xrightarrow{\nu_*} & \mathcal{G}_n^{\nu}(A, B_1) & \xrightarrow{\beta_*} & \Pi_n(A, B_2) & \xrightarrow{\partial} \\
\downarrow I_n^0 & & \downarrow I_{n-1}^1 & & \downarrow 1 & \\
\to \Pi_n(A, B_0) & \xrightarrow{\nu_*} & \Pi_n(A, B_1) & \xrightarrow{\beta_*} & \Pi_n(A, B_2) & \xrightarrow{\partial}
\end{array}$$

$$\begin{array}{ccccccc}
\mathcal{G}_{n-1}(A, B_0) & \xrightarrow{\nu_*} & \mathcal{G}_{n-1}^{\nu}(A, B_1) & \xrightarrow{\beta_*} & \Pi_{n-1}(A, B_2) & \to \\
\downarrow I_{n-1}^0 & & \downarrow I_{n-1}^1 & & \downarrow 1 & \\
\Pi_{n-1}(A, B_0) & \xrightarrow{\nu_*} & \Pi_{n-1}(A, B_1) & \xrightarrow{\beta_*} & \Pi_{n-1}(A, B_2) & \to
\end{array}$$

where the I's are inclusions and 1 is the identity. If the upper sequence (G-sequence of the fibration rel. A) is exact, then by the theorem of Barratt and Whitehead, we have the following theorem.

Theorem 4.12. *Let $B_0 \xrightarrow{\nu} B_1 \xrightarrow{\beta} B_2$ is a fibration. If ν is null homotopic or has a left homotopy inverse, then we have following long exact sequence*

$$\cdots \to \mathcal{G}_n(A, B_0) \xrightarrow{(I_n^0,\ \nu_*)} \Pi_n(A, B_0) \oplus \mathcal{G}_n^{\nu}(A, B_1) \xrightarrow{\nu_* - I_n^1} \Pi_n(A, B_1) \xrightarrow{\partial\beta_*} \mathcal{G}_{n-1}(A, B_0) \xrightarrow{(I_{n-1}^0,\ \nu_*)}$$
$$\Pi_{n-1}(A, B_0) \oplus \mathcal{G}_{n-1}^{\nu}(A, B_1) \xrightarrow{\nu_* - I_{n-1}^1} \cdots.$$

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