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Global existence and energy decay of small solutions to the Kirchhoff equation with linear dissipation localized near infinity

In memory of Nobuhisa Iwasaki, a dearest friend and research colleague

By

Kiyoshi MOCHIZUKI

1. Introduction

We consider the initial value problem

$$(1.1) \qquad \begin{cases} w_{tt} - \sigma(\|\nabla w(t)\|^2)\Delta w + b(x,t)w_t = 0, & (x,t) \in \mathbf{R}^N \times (0,\infty) \\ w(x,0) = f_1(x), w_t(x,0) = f_2(x), & x \in \mathbf{R}^N, \end{cases}$$

where $w_t = \partial w/\partial t$, $w_{tt} = \partial^2 w/\partial t^2$, $\nabla w = (\partial w/\partial x_1, \dots, \partial w/\partial x_N)$, $\Delta = \sum_{j=1}^N \partial^2/\partial x_j^2$ and $\|\cdot\|$ is the norm of $L^2(\mathbf{R}^N)$. Here $\sigma(s)$, $s \ge 0$, is a uniformly positive C^1-function and $b(x,t)$ is a nonnegative C^1-function.

Equation (1.1) was introduced by Kirchhoff [7] in case of $N = 1$ to describe a nonlinear vibrations of elastic string. After the pioneering work [2] of Bernstein, many authors ([1], [3], [4], [5], [6], [9], [10], [13], [14], [15], [16]) have investigated this equation. Among them the global existence results with non-analytic data were obtained by Greenberg-Hu [6], Yamada [15], Nishihara-Yamada [14], D'Ancona-Spagnolo [3] (see also [4]), Yamazaki [16] and Mizumachi [10] under some smallness assumptions on the initial data $\{f_1(x), f_2(x)\}$. [3], [6] and [16] studied the conservative case $b(x,t) \equiv 0$, and [10], [14] and [15] studied the dissipative case $b(x,t) \equiv b_0 > 0$. Note here that in [3], [10] and [14] is treated a more general degenerate problem with $\sigma(s) \ge 0$.

In this paper, we shall restrict ourselves to the nondegenerate problem (1.1) and extend results of [15] to the case where $b(x,t)$ is effective only near infinity. Note that the presence of the dissipative term $b_0 w_t$ with $b_0 > 0$, which is equally effective in the whole $\mathbf{R}^N$, is crucial in [15] to show not only the global existence but also the energy decay of solutions. We shall loosen the role of the dissipative term by employing additional estimates which control the local energy of solutions.

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In the following we require

(A1) There exists $a_0 > 0$ such that

$$\sigma(s) \geq a_0 > 0 \qquad \text{for } s \geq 0.$$

(A2) There exists $R_0 > 0$ and $0 < b_0 \leq b_1$ such that

$$\begin{cases} b(x,t) \geq b_0 & \text{in } A(R_0) \times [0,\infty), \\ b(x,t) \leq b_1 & \text{in } \mathbf{R}^N \times [0,\infty), \end{cases}$$

where $A(R_0) = \{x \in \mathbf{R}^N; |x| > R_0\}$.

(A3) There exists $b_2 > 0$ and nonnegative function $\beta(t) \in L^1((0,\infty))$ such that

$$|b_t(x,t)| + |\nabla b(x,t)| \leq b_2 \quad \text{in } \mathbf{R}^N \times (0,\infty),$$

$$b_t(x,t) \leq \beta(t) \quad \text{in } \mathbf{R}^N \times (0,\infty).$$

We use the following notation: H^k $(k = 0, 1, 2)$ is the usual Sobolev space with norm

$$\|f\|_{H^k} = \left\{ \sum_{|\alpha| \leq k} \int_{\mathbf{R}^N} |\nabla^\alpha f(x)|^2 dx \right\}^{1/2}$$

(α being multi-indices); $H^0 = L^2$ and we write $\|f\|_{L^2} = \|f\|$; E is the space of all pairs $f = \{f_1, f_2\}$ of functions such that

$$\|f\|_E^2 = \|\{f_1, f_2\}\|_E^2 = \tfrac{1}{2}\{\|f_2\|^2 + \sigma_1(\|\nabla f_1\|^2)\} < \infty,$$

where $\sigma_1(s) = \int_0^s \sigma(\tau) d\tau$; For solution $w(t)$ of (1.1), we simply write

$$\|w(t)\|_E^2 = \|\{w(t), w_t(t)\}\|_E^2$$

and call it the energy of $w(t)$ at time t.

Now our results are summarized in the following two theorems.

Theorem 1. *Assume* (A1)$\sim$(A3) *and let* $\{f_1, f_2\} \in H^2 \times H^1$.

(i) *There exists* $\delta_0 > 0$ *such that if* $\|f_1\|_{H^2} < \delta_0$ *and* $\|f_2\|_{H^1} < \delta_0$, *then problem* (1.1) *has a unique global solution*

$$w(\cdot, t) \in Q \equiv C^0([0,\infty); H^2) \cap C^1([0,\infty); H^1) \cap C^2([0,\infty); L^2).$$

(ii) *For this solution we have*

$$\|w(t)\|^2 = O(1) \qquad \text{as } t \to \infty, \tag{1.2}$$

$$\|w(t)\|_E^2 = O(t^{-1}) \qquad \text{as } t \to \infty, \tag{1.3}$$

$$\|w_{tt}\|^2 + \|\nabla w_t(t)\|^2 + \|\nabla^2 w(t)\|^2 = O(1) \qquad \text{as } t \to \infty. \tag{1.4}$$

Theorem 2. *Assume further the following*

(A4) *There exist* $C > 0$ *and* $\mu \geq 0$ *such that*

$$|b_t(x,t)| \leq C(1+t)^{-\mu} b(x,t) \quad in \ (x,t) \in \mathbf{R}^N \times (0,\infty).$$

Then we have

$$\|w_{tt}\|^2 + \|\nabla w_t\|^2 = O(t^{-1-\min\{1,2\mu\}}) \qquad as \ t \to \infty, \tag{1.5}$$

$$\|\nabla^2 w\|^2 = O(t^{-1-\min\{1/2,\mu\}}) \qquad as \ t \to \infty. \tag{1.6}$$

Remark. If $b(x,t)$ is independent of t, then (A4) is always satisfied with $\mu \geq 1/2$. In this case, the decay order $O(t^{-2})$ of (1.5) is the same with that of Yamada [15]. However, the decay order $O(t^{-3/2})$ of (1.6) is weaker than his.

Our argument is based on weighted energy inequalities (other than [15], cf., Matsumura [8] and Mochizuki [11]). To show the integrability of $\|w(t)\|_E^2$ in $t \in (0,\infty)$, we use two inequalities obtained from equations (1.1) multiplied by $\varphi_0 w_t + w$ and by $\psi(r)(w_r + (\alpha/2r)w)$, where $\varphi_0 > 0, \alpha \geq 0$ and $\psi(r)$ is a bounded, nondecreasing, positive function of $r = |x| > 0$. (We also use inequalities which are similarly obtained after differentiating equation (1.1).) If $b(x,t)$ is uniformly positive in the whole space $\mathbf{R}^N$, the first inequality is enough to obtain the integrability of $\|w(t)\|_E^2$ (cf., [15]). The second inequality is used to estimate the local energy which is not controled by the dissipative term.

Note that for the classical wave equation

$$w_{tt} - \Delta w + b(x,t)w_t = 0 \quad \text{in } \mathbf{R}^N \times (0,\infty),$$

our method can be applied to a more general $b(x,t)$ which may also decay as $|x| \to \infty$ (Mochizuki-Nakazawa [12]). See also Zuazua [17] where is treated the energy decay for the Klein-Gordon equation with locally distributed dissipation.

The rest of the paper is organized as follows: In §2 we give apriori inequalities for up to the second derivatives of solutions to (1.1). In §3, after discussing the local solvability of (1.1), we apply the results of §2 to prove Theorem 1. Finally in §4 we prove Theorem 2.

2. Weighted energy estimates

In this section we shall give apriori estimates for solutions $w(t)$ to (1.1) requiring $w(t) \in Q = \bigcap_{j=0}^{2} C^j([0,\infty); H^{2-j})$.

For the sake of simplicity, we put $a(t) = \sigma(\|\nabla w(t)\|^2)$ in (1.1).

We multiply (1.1) by w_t and integrate by parts over $\mathbf{R}^N$. Then

$$\frac{d}{dt}\|w(t)\|_E^2 + \int_{\mathbf{R}^N} b(x,t) w_t^2 dx = 0, \tag{2.1}$$

which implies the energy identity

$$\|w(t)\|_E^2 + \int_0^t \int_{\mathbf{R}^N} b(x,t) w_t^2 dx d\tau = \|w(0)\|_E^2. \tag{2.2}$$

Next we multiply (1.1) by $2(\varphi_0 w_t + w)$, where $\varphi_0 > 0$. Integrating by parts then gives

$$\frac{d}{dt}\left\{2\varphi_0\|w(t)\|_E^2 + \int_{\mathbf{R}^N}(2w_t w + bw^2)dx\right\} + \int_{\mathbf{R}^N}\{2(\varphi_0 b - 1)w_t^2 + 2a|\nabla w|^2 - b_t w^2\}dxd\tau = 0. \tag{2.3}$$

We shall use this to establish the boundedness of $\|w(t)\|^2$ and the integrability in $t \in (0, \infty)$ of $\|w(t)\|_E^2$. For these purposes we have to make up for the defect of $b(x, t)$ in $|x| < R_0$.

Let $\psi' = \psi'(r), r > 0$, be defined by

$$\psi' = \psi'(r) = \begin{cases} \dfrac{2\psi_0}{3R_0}, & 0 < r < R_0, \\ -\dfrac{2\psi_0}{3R_0^2}(r - R_0) + \dfrac{2\psi_0}{3R_0}, & R_0 \le r < 2R_0, \\ 0, & r \ge 2R_0. \end{cases} \tag{2.4}$$

where $\psi_0 > 0$. Then its indefinite integral $\psi = \psi(r)$ is given by

$$\psi(r) = \int_0^r \psi'(\rho)d\rho = \begin{cases} \dfrac{2\psi_0}{3R_0}r, & 0 < r < R_0, \\ -\dfrac{\psi_0}{3R_0^2}(r - R_0)^2 + \dfrac{2\psi_0}{3R_0}r, & R_0 \le r < 2R_0, \\ \psi_0, & r \ge 2R_0. \end{cases} \tag{2.5}$$

As is easily seen, $\psi(r)$ is a piecewise C^2-function and

$$\psi'(r) \ge 0, \qquad \psi''(r) \le 0, \qquad r^{-1}\psi(r) - \psi'(r) \ge 0 \quad \text{in } r > 0. \tag{2.6}$$

We multiply (1.1) by $2\psi(w_r + (\alpha/2r)w)$, where $\alpha = 0$ if $N = 1, 2$ and $\alpha = N - 1$ if $N \ge 3$, and integrate by parts over $\mathbf{R}^N$. Then since

$$\begin{aligned} w_{tt}2\psi\left(w_r + \frac{\alpha}{2r}w\right) &= \left\{w_t 2\psi\left(w_r + \frac{\alpha}{2r}w\right)\right\}_t - \nabla\cdot\left(\frac{x}{r}\psi w_t^2\right) \\ &\quad + \left(\frac{N - 1 - \alpha}{r}\psi + \psi'\right)w_t^2, \\ -\Delta w 2\psi\left(w_r + \frac{\alpha}{2r}w\right) &= -\nabla\cdot\left(2\psi\nabla w w_r - \frac{x}{r}\psi|\nabla w|^2\right) + 2\psi' w_r^2 \\ &\quad + 2r^{-1}\psi(|\nabla w|^2 - w_r^2) - \left(\frac{N - 1}{r}\psi + \psi'\right)|\nabla w|^2 \\ &\quad - \nabla\cdot\left\{\frac{\alpha}{r}\psi\nabla w w + \frac{x}{r}\left(\frac{\alpha}{2r^2}\psi - \frac{\alpha}{2r}\psi'\right)w^2\right\} \\ &\quad + \frac{\alpha}{r}\psi|\nabla w|^2 + (r^{-1}\psi - \psi')\frac{\alpha(N - 3)}{2r^2}w^2 - \psi''\frac{\alpha}{2r}w^2, \\ bw_t 2\psi\left(w_r + \frac{\alpha}{2r}w\right) &= 2\psi b w_t w_r + \left(\psi b\frac{\alpha}{2r}w^2\right)_t - \psi b_t\frac{\alpha}{2r}w^2, \end{aligned}$$

it follows that

$$
\tag{2.7}
\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^N} \psi\Big(2w_t w_r + \frac{\alpha}{r} w_t w + b\frac{\alpha}{2r} w^2\Big) dx + \int_{\mathbf{R}^N} \Big[2\psi b w_t w_r - \psi b_t \frac{\alpha}{2r} w^2 \\
&\quad + \Big(\frac{N-1-\alpha}{r}\psi + \psi'\Big) w_t^2 + \Big(\frac{\alpha - N + 1}{r}\psi + \psi'\Big) a|\nabla w|^2 - \psi'' \frac{\alpha}{2r} a w^2 \\
&\quad + 2(r^{-1}\psi - \psi') a\Big\{|\nabla w|^2 - w_r^2 + \frac{\alpha(N-3)}{4r^2} w^2\Big\}\Big] dx = 0.
\end{aligned}
$$

Now, we put together (2.3) and (2.7). Then noting the inequalities of (2.6), we obtain

$$
\tag{2.8}
\frac{d}{dt} X(t) + Z(t) - \int_{\mathbf{R}^N} \Big(1 + \frac{\alpha\psi_0}{2r}\Big) b_t w^2 dx \le 0,
$$

where

$$
X(t) = 2\varphi_0 \|w(t)\|_E^2 + \int_{\mathbf{R}^N} \Big\{ \Big(1 + \frac{\alpha\psi}{2r}\Big)(2w_t w + b w^2) + 2\psi w_t w_r \Big\} dx,
$$

$$
\begin{aligned}
Z(t) = \int_{\mathbf{R}^N} \Big\{ &2(\varphi_0 b - 1) w_t^2 + 2a|\nabla w|^2 + 2\psi b w_t w_r \\
&+ \Big(\frac{N-1-\alpha}{r}\psi + \psi'\Big) w_t^2 + \Big(\frac{\alpha - N + 1}{r}\psi + \psi'\Big) a|\nabla w|^2 \Big\} dx.
\end{aligned}
$$

Lemma 2.1. *For each $\varphi_0, \psi_0 > 0$ there exist constants $C_j = C_j(\varphi_0, \psi_0) > 0$ $(j = 1, 2, 3)$ such that for any $t \ge 0$,*

$$
X(t) \ge C_1 \|w(t)\|^2 - C_2 \|w(t)\|_E^2,
$$

$$
X(t) \le C_3 \{\|w(t)\|_E^2 + \|w(t)\|^2\}.
$$

Proof. Note that $0 < \alpha\psi/2r \le \alpha\psi_0/3R_0$ and

$$
\tag{2.9}
\tfrac{1}{2}\{\|w_t\|^2 + a_0\|\nabla w\|^2\} \le \|w(t)\|_E^2.
$$

Then by the Schwarz inequality we have for any $\varepsilon > 0$,

$$
\tag{2.10}
\begin{aligned}
X(t) \ge 2\Big(\varphi_0 - \frac{\psi_0}{\sqrt{a_0}}\Big)\|w(t)\|_E^2 + \int_{\mathbf{R}^N} b w^2 dx \\
- \Big(1 + \frac{\alpha\psi_0}{3R_0}\Big)\Big\{\varepsilon\|w\|^2 + \frac{1}{\varepsilon}\|w_t\|^2\Big\}.
\end{aligned}
$$

Let $\chi = \chi(r), r = |x|$, be a monotone C^1-function such that $\chi(r) = 1$ for $r \le R_0$ and $= 0$ for $r \ge R > R_0$. Then since

$$[\chi(r)w(r\cdot)]^2 = \left[\int_r^R \{\chi'(\rho)w(\rho\cdot) + \chi(\rho)w_r(\rho\cdot)\}d\rho\right]^2$$

$$\leq 2\int_r^R \chi'^2\rho^{-N+1}d\rho \int_{R_0}^R w(\rho\cdot)^2\rho^{N-1}d\rho$$

$$+ 2\int_r^R \chi^2\rho^{-N+1}d\rho \int_0^R w_r(\rho\cdot)^2\rho^{N-1}d\rho,$$

integrating both sides over $B(R) = \{x; |x| < R\}$ gives

$$\int_{\mathbf{R}^N} (\chi w)^2 dx \leq \frac{L^2R^2}{N}\int_{R_0<|x|<R} w^2 dx + \frac{R^2}{N}\int_{B(R)} w_r^2 dx,$$

where $L = \max_{R_0 \leq r \leq R} |\chi'(r)|$. So,

$$\varepsilon\|w\|^2 \leq \varepsilon\int_{A(R_0)} w^2 dx - \varepsilon\int_{B(R_0)} w^2 dx$$

$$+ 2\varepsilon\left\{\frac{L^2R^2}{N}\int_{R_0<|x|<R} w^2 dx + \frac{R^2}{N}\int_{B(R)} w_r^2 dx\right\}.$$

Substituting this in (2.10), we have

$$X(t) \geq 2\left(\varphi_0 - \frac{\psi_0}{\sqrt{a_0}}\right)\|w(t)\|_E^2$$

$$+ \left\{b_0 - \varepsilon\left(1 + \frac{\alpha\psi_0}{3R_0}\right)\left(1 + \frac{R^2L^2}{N}\right)\right\}\int_{A(R_0)} w^2 dx$$

$$+ \left(1 + \frac{\alpha\psi_0}{3R_0}\right)\left\{\varepsilon\int_{B(R_0)} w^2 dx - \frac{2R^2}{N}\|w_r\|^2 - \frac{1}{\varepsilon}\|w_t\|^2\right\}.$$

In this inequality, let ε be chosen to satisfy

$$b_0 = \varepsilon\left(1 + \frac{\alpha\psi_0}{3R_0}\right)\left(2 + \frac{R^2L^2}{N}\right)$$

and put

$$C_1 = \varepsilon\left(1 + \frac{\alpha\psi_0}{3R_0}\right),$$

$$C_2 > -2\varphi_0 + \frac{2\psi_0}{\sqrt{a_0}} + 2\left(1 + \frac{\alpha\psi_0}{3R_0}\right)\max\left\{\frac{1}{\varepsilon}, \frac{2\varepsilon R^2}{Na_0}\right\}.$$

Then the first inequality of the lemma follows.

On the other hand, since we have

$$X(t) \le 2\varphi_0\|w(t)\|_E^2 + \frac{\psi_0}{\sqrt{a_0}}\{\|w_t\|^2 + \|\nabla w\|^2\}$$

$$+ \left(1 + \frac{\alpha\psi_0}{3R_0}\right)\{\|w_t\|^2 + (1 + b_1)\|w\|^2\},$$

the second inequality also holds for suitablly chosen $C_3(\varphi_0, \psi_0) > 0$.

Lemma 2.2. *Let φ_0, ψ_0 be chosen to satisfy*

$$\text{(2.11)} \qquad \begin{cases} \psi_0 > 3R_0, \quad 4a_0(\varphi_0 - b_0^{-1}) > \psi_0^2 b_1 & \text{if } N \ne 2 \\ \dfrac{3}{2}R_0 < \psi_0 < 4R_0, \quad \left(4 - \dfrac{\psi_0}{R_0}\right)a_0(\varphi_0 - b_0^{-1}) > \psi_0^2 b_1 & \text{if } N = 2. \end{cases}$$

Then there exists $C_4 = C_4(\varphi_0, \psi_0) > 0$ such that

$$Z(t) \ge C_4\{\|w_t\|^2 + a\|\nabla w\|^2\}.$$

Proof. Let $N \ne 2$. Then

$$Z(t) = \int_{\mathbf{R}^N}\{(2\varphi_0 b - 2 + \psi')w_t^2 + (2 + \psi')a|\nabla w|^2 + 2\psi b w_t w_r\}dx.$$

Note that $\psi'(r) \ge (2\psi_0/3R_0)\chi_{R_0}(r)$, where χ_{R_0} is the characteristic function on $r \in (0, R_0)$. Then by (2.11) it follows that

$$2\varphi_0 b - 2 + \psi' \ge 2(\varphi_0 - b_0^{-1})b + 2\left(\frac{\psi_0}{3R_0} - 1\right)\chi_{R_0}(r) > 0,$$

$$(2\varphi_0 b - 2 + \psi')(2 + \psi)a - \psi^2 b^2$$

$$\ge [4a_0(\varphi_0 - b_0^{-1}) - \psi_0^2 b_1]b + 4a_0\left(\frac{2\psi_0}{3R_0} - 2\right)\chi_{R_0}(r)$$

$$\ge \min\left\{[4a_0(\varphi_0 - b_0^{-1}) - \psi_0^2 b_1]b_0, 4a_0\left(\frac{\psi_0}{3R_0} - 1\right)\right\} > 0.$$

Thus, we have the assertion of the lemma.

Next let $N = 2$. Then

$$Z(t) = \int_{\mathbf{R}^2}\{(2\varphi_0 b - 2 + r^{-1}\psi + \psi')w_t^2$$

$$+ (2 - r^{-1}\psi + \psi')a|\nabla w|^2 + 2\psi b w_t w_r\}dx.$$

By (2.11) it follows that

$$2\varphi_0 b - 2 + r^{-1}\psi + \psi' \ge 2(\varphi_0 - b_0^{-1})b + 2\left(\frac{2\psi_0}{3R_0} - 1\right)\chi_{R_0}(r) > 0,$$

$$(2 - r^{-1}\psi + \psi')a \geq \left(2 - \frac{\psi_0}{2R_0}\right)a_0 > 0,$$

$$(2\varphi_0 b - 2 + r^{-1}\psi + \psi')(2 - r^{-1}\psi + \psi')a - \psi^2 b^2$$
$$\geq \min\left\{\left[\left(4 - \frac{\psi_0}{R_0}\right)a_0(\varphi_0 - b_0^{-1}) - \psi_0^2 b_1\right]b_0,\ \left(4 - \frac{\psi_0}{R_0}\right)a_0\left(\frac{2\psi_0}{3R_0} - 1\right)\right\} > 0.$$

Thus, the assertion also holds in this case.

Proposition 2.3. *Let $w(t)$ be the solution to* (1.1). *Then there exist $C_5 > 0$ such that for any $t \geq 0$,*

$$\|w(t)\|^2 + \int_0^t \{\|w_t\|^2 + a\|\nabla w\|^2\}d\tau \leq C_5\{\|w(0)\|_E^2 + \|f_1\|^2\}.$$

Proof. Integrate (2.8) over $(0, t)$, where φ_0, ψ_0 are chosen to satisfy (2.11). Then applying Lemmas 2.1 and 2.2, we obtain

$$C_1\|w(t)\|^2 + C_4\int_0^t \{\|w_t\|^2 + a\|\nabla w\|^2\}d\tau$$
$$\leq (C_2 + C_3)\{\|w(0)\|_E^2 + \|f_1\|^2\} + \left(1 + \frac{\alpha\psi_0}{3R_0}\right)\int_0^t \beta(\tau)\|w(\tau)\|^2 d\tau.$$

Since $\beta(t) \in L^1$, we can apply the Gronwall inequality to obtain the desired inequality.

Next, we shall give apriori estimates for second order derivatives of solutions. For this aim we differentiate both sides of (1.1) by x_j $(j = 1, \ldots, N)$. Let u stand for each component of ∇w. Then u satisfies the equation

$$u_{tt} - a(t)\Delta u + b(x,t)u_t + c(x,t)w_t = 0, \qquad (x,t) \in \mathbf{R}^N \times (0, \infty), \tag{2.12}$$

where c is the corresponding component of ∇b.

In the following, for the sake of simplicity, u is required to be in Q. However, this requirement is not necessary for our final results. In fact, if we put

$$u_\varepsilon = \rho_\varepsilon * u = \int_{\mathbf{R}^N} \rho_\varepsilon(x-y)u(y,t)dy,$$

where $\rho_\varepsilon *$ is the Friedrichs mollifier, then $u_\varepsilon \in Q$ and satisfies

$$u_{\varepsilon tt} - a(t)\Delta u_\varepsilon + \rho_\varepsilon * (bu_t + cw_t) = 0, \qquad (x,t) \in \mathbf{R}^N \times [0, \infty). \tag{2.12$_\varepsilon$}$$

Starting from this equation, by the limit procedure, we can obtain the same conclusion without assuming $u \in Q$.

We multiply (2.12) by $2(\varphi_0 u_t + u)$ and integrate by parts over $\mathbf{R}^N$. Then

$$\frac{d}{dt}\left[\varphi_0\{\|u_t\|^2 + a\|\nabla u\|^2\} + \int_{\mathbf{R}^N}(2u_t u + bu^2)dx\right] + \int_{\mathbf{R}^N}\{2(\varphi_0 b - 1)u_t^2 \tag{2.13}$$
$$+ (2a - \varphi_0 a')|\nabla u|^2 - b_t u^2 + 2cw_t(\varphi_0 u_t + u)\}dx = 0.$$

Next we multiply (2.12) by $2\psi(u_r + (\alpha/2r)u)$ and integrate over $\mathbf{R}^N$. Then as in the case of w (cf., identity (2.7)), it follows that

$$\begin{aligned}(2.14)\qquad & \frac{d}{dt}\int_{\mathbf{R}^N} \psi\Big(2u_t u_r + \frac{\alpha}{r}u_t u + b\frac{\alpha}{2r}u^2\Big)dx \\ & + \int_{\mathbf{R}^N}\Big\{2\psi b_t u_r - \psi b_t\frac{\alpha}{2r}u^2 + \Big(\frac{N-1+\alpha}{r}\psi + \psi'\Big)u_t^2 \\ & + \Big(\frac{\alpha-N+1}{r}\psi + \psi'\Big)a|\nabla u|^2 + 2\psi c w_t\Big(u_r + \frac{\alpha}{2r}u\Big)\Big\}dx d\tau \le 0.\end{aligned}$$

We put together (2.13) and (2.14). Then

$$\begin{aligned}(2.15)\qquad & \frac{d}{dt}X_1(t) + Z_1(t) - \varphi_0 a'\|\nabla u\|^2 \\ & - \int_{\mathbf{R}^N}\Big[\Big(1+\frac{\alpha\psi}{2r}\Big)b_t u^2 - 2cw_t\Big\{\varphi_0 u_t + \psi u_r + \Big(1+\frac{\alpha\psi}{2r}\Big)u\Big\}\Big]dx \le 0,\end{aligned}$$

where

$$X_1(t) = \varphi_0\{\|u_t\|^2 + a\|\nabla u\|^2\} + \int_{\mathbf{R}^N}\Big\{\Big(1+\frac{\alpha\psi}{2r}\Big)(2u_t u + bu^2) + 2\psi u_t u_r\Big\}dx,$$

$$\begin{aligned}Z_1(t) = \int_{\mathbf{R}^N}\Big[&2(\varphi_0 b - 1)u_t^2 + 2a|\nabla u|^2 + 2\psi b u_t u_r \\ & + \Big(\frac{N-1+\alpha}{r}\tilde\psi + \tilde\psi'\Big)u_t^2 + \Big(\frac{\alpha-N+1}{r}\tilde\psi + \tilde\psi'\Big)a|\nabla u|^2\Big]dx.\end{aligned}$$

Lemma 2.4. *Let φ_0, ψ_0 be chosen to satisfy*

$$(2.16)\qquad \varphi_0 > \frac{\psi_0}{\sqrt{a_0}}.$$

Then there exist constants $C_j = C_j(\varphi_0, \psi_0) > 0$ $(j = 6, 7, 8)$ such that for any $t \ge 0$,

$$X_1(t) \ge C_6\{\|u_t\|^2 + a\|\nabla u\|^2\} - C_7\|w(t)\|_E^2,$$

$$X_1(t) \le C_0\{\|u_t\|^2 + a\|\nabla u\|^2 + \|w(t)\|_E^2\}.$$

Proof. By the Schwarz inequality we have for any $\varepsilon > 0$,

$$X_1(t) \ge \Big(\varphi_0 - \frac{\psi_0}{\sqrt{a_0}}\Big)\{\|u_t\|^2 + a\|\nabla u\|^2\} - \Big(1 + \frac{\alpha\psi_0}{3R_0}\Big)\Big\{\varepsilon\|u_t\|^2 + \frac{1}{\varepsilon}\|u\|^2\Big\}.$$

This and (2.16) show the first inequality. The second inequality is similarly proved.

The same argument as in Lemma 2.2 can be applied to show the following

Lemma 2.5. Let φ_0, ψ_0 *be chosen to satisfy* (2.11). *Then there exists* $C_9 = C_9(\varphi_0, \psi_0) > 0$ *such that*

$$Z_1(t) \geq C_9\{\|u_t\|^2 + a\|\nabla u\|^2\}.$$

Proposition 2.6. *Let* w *be the solution to* (1.1), *and let* u *stand for each component of* ∇w. *Then there exist* $0 < C_{10} < 1$ *and* $C_{11} > 0$ *such that for any* $t > 0$,

$$\|u_t\|^2 + a_0\|\nabla u\|^2 + \int_0^t \{\|u_t\|^2 + (C_{10}a - \tilde{\varphi}_0 a')\|\nabla u\|^2\}d\tau$$
$$\leq C_{11}\{\|u_t(0)\|^2 + a(0)\|\nabla u(0)\|^2 + \|w(0)\|_E^2\}.$$

Proof. Since we have $1 + \alpha\psi/2r \leq 1 + \alpha\psi_0/3R_0$ and $|b_t(x,t)| + |c(x,t)| \leq b_2$, by the Schwarz inequality, it follows that

$$\int_{\mathbf{R}^N}\left[\left(1 + \frac{\alpha\psi}{2r}\right)b_t u^2 - 2cw_t\left\{\varphi_0 u_t + \psi u_r + \left(1 + \frac{\alpha\psi}{2r}\right)u\right\}\right]dx \tag{2.17}$$
$$\leq \varepsilon\{\|u_t\|^2 + a_0\|\nabla u\|^2\} + C(\varepsilon)\{\|w_t\|^2 + a_0\|u\|^2\}$$

for any $\varepsilon > 0$.

Now integrate (2.15) over $(0, t)$, where φ_0, ψ_0 are chosen to satisfy (2.11) and (2.16), and apply Lemmas 2.4, 2.5 and (2.17) with $\varepsilon < C_9$ to the left side. Then the desired inequality follows since we have

$$\int_0^t \{\|w_t\|^2 + a_0\|u\|^2\}dt \leq C_5\{\|w(0)\|_E^2 + \|f_1\|^2\}$$

by Proposition 2.3.

3. Proof of Theorem 1

First we shall discuss the local existence of solutions to (1.1). Let θ be a positive number satisfying $\theta \geq b_2/2$. We choose the triplet T, M and δ of positive numbers as follows:

$$M = \{1 + \sigma(\delta^2)\}\delta^2 e^{2\alpha T}, \tag{3.1}$$

$$\frac{mM}{a_0} \leq \theta, \tag{3.2}$$

$$2mMa_0Te^{2mMT/a_0} < 1. \tag{3.3}$$

where $m = \sup_{s \leq M}|\sigma'(s)|$.

Theorem 3.1. *Let M, T and δ be as above, and let*

$$\|f_1\|_{H^2} \le \delta \quad and \quad \|f_2\|_{H^1} \le \delta. \tag{3.4}$$

Then (1.1) *has a unique solution* $w(t) \in Q(T) = \bigcap_{j=0}^{2} C^j([0,T]; H^{2-j})$ *which also satisfies*

$$\sup_{0\le t\le T} \|w_t(t)\|_{H^1}^2 \le M \quad and \quad \sup_{0\le t\le T} \|\nabla w(t)\|_{H^1}^2 \le M. \tag{3.5}$$

Sketch of Proof (cf., [15; §3]). Let K be the set of all functions $v \in Q(T)$ such that

$$v(0) = f_1 \quad \text{and} \quad v_t(0) = f_2,$$

$$\sup_{0\le t\le T} \|v_t(t)\|_{H^1}^2 \le M \quad \text{and} \quad \sup_{0\le t\le T} \|\nabla v(t)\|_{H^1}^2 \le M.$$

For each $v \in K$, we consider the initial value problem

$$\begin{cases} w_{tt} - \sigma(\|\nabla v(t)\|^2)\Delta w + b(x,t)w_t = 0, & (x,t) \in \mathbf{R}^N \times (0,\infty) \\ w(x,0) = f_1(x), \quad w_t(x,0) = f_2(x), & x \in \mathbf{R}^N. \end{cases} \tag{3.6}$$

Since $\sigma(\|\nabla v(t)\|^2) \in C^1([0,T])$ and is uniformly positive, this linear problem has a unique solution in $Q(T)$. As in §2 (cf., (2.1) and (2.13)), we have

$$\frac{1}{2}\frac{d}{dt}\{\|w_t\|_{H^1}^2 + \sigma(\|\nabla v\|^2)\|\nabla w\|_{H^1}^2\} - \sigma'(\|\nabla v\|^2)(\nabla v, \nabla v_t)\|\nabla w\|_{H^1}^2$$
$$+ \int_{\mathbf{R}^N} \{b(w_t^2 + |\nabla w_t|^2) + w_t \nabla b \cdot \nabla w_t\} dx = 0.$$

Here

$$\int_{\mathbf{R}^N} |w_t \nabla b \cdot \nabla w_t| dx \le \frac{b_2}{2}\|w_t\|_{H^1}^2 \le \theta \|w_t\|_{H^1}^2.$$

Moreover, since $v \in K$,

$$\sigma'(\|\nabla v\|^2)|(\nabla v, \nabla v_t)| \le mM \le \frac{mM}{a_0}\sigma(\|\nabla v\|^2).$$

Thus, noting (3.2), we have

$$\frac{1}{2}\frac{d}{dt}\{\|w_t\|_{H^1}^2 + \sigma(\|\nabla v\|^2)\|\nabla w\|_{H^1}^2\} \le \theta\{\|w_t\|_{H^1}^2 + \sigma(\|\nabla v\|^2)\|\nabla w\|_{H^1}^2\},$$

from which it follows that

$$\|w_t\|_{H^1}^2 + a_0\|\nabla w\|_{H^1}^2 \le \{1 + \sigma(\delta^2)\}\delta^2 e^{2\theta t}.$$

Hence, $w(t) \in K$ by (3.1).

We define the map S by $w = Sv$. As is proved, S maps K into itself. Let w^0 be any element in K, and define $\{w^n\} \subset K$ successively by

$$w^{n+1} = Sw^n, \qquad n = 0, 1, \dots.$$

Put $u^n = w^n - w^{n-1}$. Then u^n satisfies the equation

$$u_{tt}^n - \sigma(\|\nabla w^{n-1}\|^2)\Delta u^n + bu_t^n = \{\sigma(\|\nabla w^{n-1}\|^2) - \sigma(\|\nabla w^{n-2}\|)\}\Delta w^{n-1},$$

from which it follows that

$$\frac{1}{2}\{\|u_t^n(t)\|^2 + a_0\|\nabla u^n\|^2\} \leq mM\int_0^t \{\|\nabla u^n\|^2 + 2\|\nabla u^{n-1}\|\,\|u_t^n\|\}d\tau$$

for $0 \leq t \leq T$. Thus, applying the Gronwall inequality, we obtain

$$\|u_t^n\|^2 + a_0\|\nabla u^n\|^2 \leq 2mMa_0e^{2mMt/a_0}\int_0^T \|\nabla u^{n-1}\|^2 d\tau. \tag{3.7}$$

for $0 \leq t \leq T$. This and (3.3) imply that $\{w^n\}$ is a Cauchy sequence in $C^0([0, T]; H^1) \cap C^1([0, T]; L^2)$.

Let w be the limit function. Then we see $w \in C([0, T]; H^2)$ since $\{w^n\}$ is weakly compact in H^2 uniformly in $t \in [0, T]$. This and equation (3.6) show $w \in C^2([0, t]; L^2)$, and finally we conclude that w solves (1.1).

To see the uniqueness of solutions, let $u = w - v$ for two solutions $w, v \in Q(T)$. Then u satisfies an estimate similar to (3.7) showing $u \equiv 0$.

Since the unique existence of local solution $w \in Q(T)$ is guaranteed by Theorem 3.1, we can now enter into the proof of Theorem 1.

Proof of Theorem 1. (i) It suffices to obtain apriori bounds for $\|w_t(t)\|_{H^1}$ and $\|w(t)\|_{H^2}$.

Let $\{f_1, f_2\}$ satisfy (3.4) for some $\delta > 0$. Then by (2.2), (2.9) and Proposition 2.3 we obtain

$$\|w_t\|^2 + \|w\|_{H^1}^2 \leq C_{12}\{\delta^2 + \sigma_1(\delta^2)\} \equiv M_1(\delta^2). \tag{3.8}$$

Next we shall use Proposition 2.6. Note that

$$C_{10}a - \tilde{\varphi}_0 a' \geq C_{10}a_0 - \varphi_0\sigma'(\|\nabla w\|^2)\|\nabla w\|\,\|\nabla w_t\|.$$

Let

$$m_0(s) = \sup_{\tau \leq s}\sigma(\tau) \qquad \text{and} \qquad m(s) = \sup_{\tau \leq s}\sigma'(\tau).$$

Then by means of (3.8) we have

$$C_{10}a - \tilde{\varphi}_0 a' \geq C_{10}a_0 - \tilde{\varphi}_0 m(M_1(\delta^2))\sqrt{M_1(\delta^2)}\|\nabla w_t\|.$$

Assume

$$C_{10}a_0 - \tilde{\varphi}_0 m(M_1(\delta^2))\sqrt{M_1(\delta^2)}\|\nabla w_t\| \geq 0$$

in some $0 \le t \le t_1$. Then Proposition 2.6 shows that

$$\|\nabla w_t\|^2 + \|\Delta w\|^2 \le C_{13}\{\delta^2 + m_0(\delta^2)\delta^2 + \sigma_1(\delta^2)\} \equiv M_2(\delta^2) \tag{3.9}$$

in the same interval.

Note that $M_j(\delta^2)$ $(j = 1, 2)$ is monotone increasing in δ^2 and $M_j(0) = 0$. Then there exists a unique $\delta_0 > 0$ satisfying

$$C_{10}a_0 = \tilde{\varphi}_0 m(M_1(\delta_0^2))\sqrt{M_1(\delta_0^2)M_2(\delta_0^2)}.$$

So, if δ in (3.4) is chosen less than δ_0, (3.9) holds as long as the solution exists.

(3.8) and (3.9) are thus the desired apriori estimates which guarantee the global existence of solutions to (1.1).

Proof of Theorem 1. (ii) (1.2) is proved in Proposition 2.3 and is already used in the above proof.

To show (1.3) we multiply (2.1) by t and integrate over $(0, t)$. Then

$$t\|w(t)\|_E^2 + \int_0^t \int_{\mathbf{R}^N} \tau b w_t^2 dx d\tau = \int_0^t \|w(\tau)\|_E^2 d\tau.$$

Since

$$\|w(t)\|_E^2 \le \tfrac{1}{2}\{\|w_t\|^2 + m_0(M_1(\delta^2))\|\nabla w\|^2\},$$

it follows from Proposition 2.3 that

$$t\|w(t)\|_E^2 + \int_0^t \tau \int_{\mathbf{R}^N} b w_t^2 dx d\tau \le C_{14}(\delta^2)\{\|w(0)\|_E^2 + \|f_1\|^2\} < \infty, \tag{3.10}$$

which proves (1.3).

Since we have (3.9), assertion (1.4) is concluded if $\|w_{tt}\|$ is shown to be bounded. By equation (1.1)

$$\|w_{tt}\| \le m(M_1(\delta^2))\|\Delta w\| + b_1\|w_t\|.$$

Then the boundedness is obvious from (3.8) and (3.9).

4. Proof of Theorem 2

We begin with a lemma.

Lemma 4.1. *We have*

$$\int_0^t \{\|\nabla^2 w\|^2 + \|\nabla w_t\|^2 + \|w_{tt}\|^2\} d\tau = O(1) \qquad \textit{as } t \to \infty.$$

Proof. By use of (1.3) and (1.4) we have

$$|a'(t)| \le \sigma'(\|\nabla w\|^2)\|\nabla w\|\,\|\nabla w_t\| = O(t^{-1/2}). \tag{4.1}$$

Then it follows from Proposition 2.6 that

$$\int_0^t \{\|\nabla w_t\|^2 + \|\nabla^2 w\|^2\} d\tau = O(1).$$

Moreover, by use of this estimate and (2.2), we have from equation (1.1)

$$\int_0^t \|w_{tt}\|^2 d\tau \leq 2\int_0^t \left\{a^2\|\Delta w\|^2 + b_1 \int_{\mathbf{R}^N} bw_t^2 dx\right\} d\tau = O(1)$$

Thus, the lemma is proved.

To proceed into the proof of Theorem 2, we need extra "energy" estimates based on the equation

$$w_{ttt} - a\Delta w_t + bw_{tt} = a'\Delta w - b_t w_t. \tag{4.2}$$

We obtain this differentiating (1.1) by t. As in the case of (2.12) we require here $w_t \in Q$. However, this requirement is not necessary for our final results.

Lemma 4.2. *We have for any* $t > 0$

$$\begin{aligned} &\frac{d}{dt}\{\|w_{tt}\|^2 + a\|\nabla w_t\|^2\} + \int_{\mathbf{R}^N} bw_{tt}^2 dx \\ &\qquad \leq |a'(t)|\{\|\Delta w\|^2 + \|w_{tt}\|^2 + \|\nabla w_t\|^2\} + C^2 b_1 (1+t)^{-2\mu} \int_{\mathbf{R}^N} bw_t^2 dx, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\frac{d}{dt} X_2(t) + C_{12}\{\|w_{tt}\|^2 + \|\nabla w_t\|^2\} \leq C_{13}(1+t)^{-\mu} \int_{\mathbf{R}^N} bw_t^2 dx \\ &\qquad + C_{14}|a'|\{\|\Delta w\|^2 + \|w_{tt}\|^2 + \|\nabla w_t\|^2 + \|w_t\|^2\}, \end{aligned} \tag{4.4}$$

where $X_2(t)$ *is the function* $X_1(t)$ *in* (2.15) *with* u *replaced by* w_t *and* C_j $(j = 12, 13, 14)$ *are some positive constants.*

Proof. We multiply (4.2) by $2w_{tt}$ and integrate by parts over $\mathbf{R}^N$. Then

$$\begin{aligned} &\frac{d}{dt}\{\|w_{tt}\|^2 + a\|\nabla w_t\|^2\} + 2\int_{\mathbf{R}^N} bw_{tt}^2 dx \\ &\qquad = \int_{\mathbf{R}^N} \{2a'\Delta w w_{tt} + a'|\nabla w_t|^2 - 2b_t w_t w_{tt}\} dx, \end{aligned}$$

and the Schwarz inequality and (A4) show (4.3).

Next, we multiply (4.2) by $2(\varphi_0 w_{tt} + w_t) + 2\psi(w_{tr} + (\alpha/2r)w_t)$ and integrate by parts over $\mathbf{R}^N$. Then similarly to (2.15) we have

$$\begin{aligned} &\frac{d}{dx} X_2(t) + Z_2(t) - \varphi_0 a'\|\nabla w_t\|^2 \\ &\quad - \int_{\mathbf{R}^N} \left[\left(1 + \frac{\alpha\psi}{2r}\right) b_t w_t^2 - 2(a'\Delta w - b_t w_t)\left\{\varphi_0 w_{tt} + \psi w_{tr} + \left(1 + \frac{\alpha\psi}{2r}\right) w_t\right\}\right] dx \leq 0. \end{aligned}$$

By the Schwarz inequality we have for any $\varepsilon > 0$,

$$\frac{d}{dt}X_2(t) + Z_2(t) \le \varepsilon\{\|w_{tt}\|^2 + \|\nabla w_t\|^2\} + \left(3 + \frac{\alpha\psi_0}{R_0} + \frac{1}{\varepsilon}\right)\int_{\mathbf{R}^N} |b_t| w_t^2 dx$$
$$+ C_{14}|a'|\{\|\Delta w\|^2 + \|w_{tt}\|^2 + \|\nabla w_t\|^2 + \|w_t\|^2\},$$

where $C_{14} = \max\{\varphi_0 + \psi_0, 1 + \alpha\psi_0/3R_0\}$. Thus, (4.4) follows if we use Lemma 2.5 with $u = w_t$ and (A4).

Proof of Theorem 2. For the sake of simplicity, in the following, we assume $0 \le \mu \le 1/2$.

First, we multiply (4.4) by $t^{1/2}$ and integrate over $(0, t)$. Then since $a'(t) = O(t^{-1/2})$ by (4.1),

$$t^{1/2}X_2(t) - \frac{1}{2}\int_0^t \tau^{-1/2}X_2(\tau)d\tau + C_{12}\int_0^t \tau^{1/2}\{\|w_{tt}\|^2 + \|\nabla w_t\|^2\}d\tau$$
$$\le C_{13}\int_0^t \tau^{1/2-\mu}\int_{\mathbf{R}^N} bw_t^2 dxd\tau$$
$$+ C_{15}\int_0^t \{\|\Delta w\|^2 + \|w_{tt}\|^2 + \|\nabla w_t\|^2 + \|w_t\|^2\}d\tau = O(1),$$

where in the last estimate we have used (3.10) and Lemma 4.1. Moreover, by Lemma 2.4 with u replaced by w_t,

$$t^{1/2}X_2(t) - \frac{1}{2}\int_0^t \tau^{-1/2}X_2(\tau)d\tau \ge -C_7 t^{1/2}\|w(t)\|_E^2$$
$$-\frac{1}{2}C_8\int_0^t \{\|w_{tt}\|^2 + \|\nabla w_t\|^2 + \|w(\tau)\|_E^2\}d\tau = O(1),$$

where in the last estimate we have used (1.3), Proposition 2.6 and Lemma 4.1.

Summarizing these inequalities, we obtain

$$\int_0^t t^{1/2}\{\|w_{tt}\|^2 + \|\nabla w_t\|^2\}d\tau = O(1). \tag{4.5}$$

This and (3.10) also show

$$\int_0^t t^{1/2}\|\Delta w\|^2 d\tau = O(1). \tag{4.6}$$

Next we multiply (4.3) by t and integrate over $(0, t)$. Then since $ta'(t) = O(t^{1/2})$, it follows that

$$t\{\|w_{tt}\|^2 + a\|\nabla w_t\|^2\} - \int_0^t \{\|w_{tt}\|^2 + \|\nabla w_t\|^2\}d\tau$$
$$\le C_{16}\int_0^t \tau^{1/2}\{\|\Delta w\|^2 + \|w_{tt}\|^2 + \|\nabla w_t\|^2\}d\tau + C_{17}\int_0^t \tau^{1-2\mu}\int_{\mathbf{R}^N} bw_t^2 dxd\tau.$$

By means of Lemma 4.1, (4.5), (4.6) and (3.10), this yields

$$t\{\|w_{tt}\|^2 + \|\nabla w_t\|^2\} = O(1). \tag{4.7}$$

(4.7) implies

$$a'(t) = O(t^{-1}). \tag{4.8}$$

Taking account of this, we next multiply (4.4) by t. Then we can follow the argument of the 1-st step of this proof to obtain

$$\int_0^t \tau\{\|w_{tt}\|^2 + \|\nabla w_t\|^2 + \|\Delta w\|^2\}d\tau = O(1). \tag{4.9}$$

Finally, we multiply (4.3) by $t^{1+2\mu}$. Then applying (4.8) and (4.9), we obtain

$$\begin{aligned} t^{1+2\mu}\{\|w_{tt}\|^2 + a\|\nabla w_t\|^2\} &+ \int_0^t \tau^{1+2\mu}\int_{\mathbf{R}^N} bw_{tt}^2 dx d\tau \\ &\leq C_{18}\int_0^t \tau^{2\mu}\{\|\Delta w\|^2 + \|w_{tt}\|^2 + \|\nabla w_t\|^2\}d\tau \\ &\quad + C_{19}\int_0^t \tau\int_{\mathbf{R}^N} bw_t^2 dx d\tau = O(1), \end{aligned} \tag{4.10}$$

which proves (1.5).

Note that (3.10) and (4.10) imply

$$\int_0^t \tau^{1+\mu}\int_{\mathbf{R}^N} b|w_t w_{tt}| dx d\tau = O(1).$$

Then since

$$\frac{d}{dt}\left[t^{1+\mu}\int_{\mathbf{R}^N} bw_t^2 dx\right] \leq \{(1+\mu)t^\mu + Ct\}\int_{\mathbf{R}^N} bw_t^2 dx + 2t^{1+\mu}\int_{\mathbf{R}^N} b|w_t w_{tt}| dx$$

by (A4), integration gives us

$$\int_{\mathbf{R}^N} bw_t^2 dx = O(t^{-1-\mu}). \tag{4.11}$$

(4.10) and (4.11) show (1.6) if we use equation (1.1).

The proof of Theorem 2 is thus complete.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY

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