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Geometric Bogomolov's conjecture for curves of genus 3 over function fields

By

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Introduction

Throughout this paper, we always use a fixed algebraically closed field k.

Let X be a smooth projective surface over k, Y a smooth projective curve over k, and let $f : X \to Y$ be a generically smooth semistable curve of genus $g \geq 2$ over Y. Let K be the function field of Y, $\overline{K}$ the algebraic closure of K, and let C be the generic fiber of f. Let $j : C(\overline{K}) \to \mathrm{Pic}^0(C)(\overline{K})$ be a morphism defined by $j(x) = (2g-2)x - \omega_C$, where ω_C is the dualizing sheaf of C, and let $\|\cdot\|_{NT}$ be the semi-norm arising from the canonical Néron-Tate pairing on $\mathrm{Pic}^0(C)(\overline{K})$. We set

$$B_C(P; r) = \left\{ x \in C(\overline{K}) \;\middle|\; \|j(x) - P\|_{NT} \leq r \right\}$$

for $P \in \mathrm{Pic}^0(C)(\overline{K})$ and $r \geq 0$, and set

$$r_C(P) = \begin{cases} -\infty & \text{if} \quad \#\big(B_C(P;0)\big) = \infty, \\ \sup\left\{ r \geq 0 \;\middle|\; \#\big(B_C(P;r)\big) < \infty \right\} & \text{otherwise.} \end{cases}$$

Then, we have the following conjectures due to Bogomolov.

Conjecture (Geometric Bogomolov's conjecture)**.** If f is non-isotrivial, then $r_C(P) > 0$ for all P.

We have also an effective version of Bogomolov's conjecture.

Conjecture (Effective version of geometric Bogomolov's conjecture)**.** If f is non-isotrivial, then there exists an effectively calculated positive number r_0 such that

$$\inf_{P \in \mathrm{Pic}^0(C)(\overline{K})} r_C(P) \geq r_0.$$

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The meaning of "effectively calculated" is to give a concrete algorithm or formula to find r_0.

An arithmetic version of Bogomolov's conjecture has been completely solved by Ullmo [8], Zhang [11] and Moriwaki [6]. As remarked in [6, Remark 8.3], due to the poorness of geometric heights, geometric Bogomolov's conjecture is a rather subtle problem. In this paper, we give an answer to the effective version of geometric Bogomolov's conjecture for non-hyperelliptic curves of genus 3.

In order to describe r_0 above, we introduce the types of nodes of a semistable curve. Let P be a node of a semistable curve Z of genus g over k. We can assign a number i to the node P, called the *type* of P, in the following way. Let $\nu : Z_P \to Z$ be the partial normalization at P. If Z_P is connected, then $i = 0$. Otherwise, i is the minimum of the arithmetic genera of the two connected components of Z_P. We denote by $\delta_i(Z)$ the number of nodes of type i, and by $\delta_i(X/Y)$ the number of nodes of type i in all the fibers of $f : X \to Y$, i.e., $\delta_i(X/Y) = \sum_{y \in Y} \delta_i(X_y)$.

Our main theorem of this paper is the following.

Main Theorem. *Let X be a smooth projective surface over k, Y a smooth projective curve over k, and let $f : X \to Y$ be a generically smooth semistable curve of genus 3 over Y. If f is not smooth and the generic fiber is not hyperelliptic, then we have*

$$\inf_{P \in \mathrm{Pic}^0(C)(\overline{K})} r_C(P) \geq \sqrt{\frac{2}{99}\delta_0(X/Y) + \frac{8}{3}\delta_1(X/Y)}.$$

Some partial answers to the effective version of geometric Bogomolov's conjecture have already been given by Moriwaki and the author. Moriwaki gave an answer in [5] for the case where every singular fiber of f is a tree of stable components under the assumption of $\mathrm{char}(k) = 0$. He also gave answers for the case that f has only irreducible fibers in [2] and for the case of $g = 2$ in [3] without assumptions on the characteristic. Recently, the author gave an answer in [9] for the case where the generic fiber is hyperelliptic under the assumption of $\mathrm{char}(k) = 0$.

We will explain how to prove it. First of all, note the following essential fact (cf. [10, Theorem 5.6], [2, Corollary 2.3] and [3, Theorem 2.1]).

Key Fact. If $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a > 0$, then we have

$$\inf_{P \in \mathrm{Pic}^0(C)(\overline{K})} r_C(P) \geq \sqrt{(g-1)(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a},$$

where $(\cdot)_a$ is the admissible pairing.

By virtue of this fact, our main purpose is to find an effectively calculated positive number which bounds $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a$ below. By the definition, we have

$$(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a = (\omega_{X/Y} \cdot \omega_{X/Y}) - \sum_{y \in Y} \epsilon(\bar{G}_y, \omega_y),$$

where $\epsilon(\bar{G}_y, \omega_y)$ is a real number arising from the polarized metrized dual graph $(\bar{G}_y, \omega_y)$ of X_y introduced by S. Zhang in [10]. If we find numbers r_1 and r_2 such that $(\omega_{X/Y} \cdot \omega_{X/Y}) \geq r_1$, $\epsilon(G_y, \omega_y) \leq r_2$ and $r_1 - r_2 > 0$, and if $r_1 - r_2$ can be effectively calculated, then $r_1 - r_2$ gives us an answer. Every partial answer mentioned above was obtained in this way, and we will also prove Main Theorem by finding such r_1 and r_2.

1. Review and some remarks on the admissible constants

In this section, we recall several facts on Green's functions on metrized graphs and give some remarks on the admissible constants. See [10] for details on metrized graphs and Green's function, and see [9] for details on rigidified graphs, contraction and irreducible decomposition.

Definition 1.1 (cf. [9]). A *rigidification* on a graph G is a finite subset V of G such that $G \setminus V$ is a disjoint union of open line segments. A *rigidified graph* $[G]$ is a pair of a graph G and a rigidification on G.

For a rigidified graph $[G]$, the above V is denoted by $\mathrm{Vert}[G]$ and its member is called a vertex. We can also define the notion of edges. We denote by $\mathrm{Ed}[G]$ the set of edges. An $\mathbb{R}$-divisor supported in $\mathrm{Vert}[G]$ is called a *polarization* on $[G]$.

Let $[G]$ be a rigidified graph and let S be a subset of $\mathrm{Ed}[G]$. We have defined in [9] a contraction of S, that is, a graph obtained by contracting all the edges in S, which is usually denoted by $p_S : [G] \to [G_S]$. When G has a Lebesgue measure, that on G_S is naturally induced. To simplify the notation, we write $p^S : [G] \to [G^S]$ for $p_{(\mathrm{Ed}[G] \setminus S)} : [G] \to [G_{(\mathrm{Ed}[G] \setminus S)}]$. Note that we have the canonical identification $\mathrm{Ed}[G^S] \cong S$ and the induced map $\mathrm{Vert}[G] \to \mathrm{Vert}[G^S]$. For a polarization D, we have also the polarization D_S and D^S on $[G_S]$ and $[G^S]$ respectively. We say a polarized rigidified graph $([H], E)$ is *dominated* by $([G], D)$ if $([H], E) \cong ([G_S], D_S)$ for some subset S of $\mathrm{Ed}[G]$.

A rigidified graph $[G]$ is said to be *reducible* if there is a vertex P of $[G]$ such that $G \setminus \{P\}$ is not connected. $[G]$ is said to be *irreducible* if it is not reducible. For any graph $[G]$, we have the *irreducible decomposition* of $[G]$. (See [9] for details.) Each irreducible component of $[G]$ can be metrized as a subspace of G when G is metrized. Note that every irreducible component $[H]$ can be canonically regarded as $[G^{\mathrm{Ed}[H]}]$. If D is a polarization and $[H]$ is an irreducible component of $[G]$, then we also call $\left([H], D^{\mathrm{Ed}[H]}\right)$ an irreducible component of a polarized rigidified graph $([G], D)$.

Next, let us recall Green's function. Let $\bar{G}$ be a metrized graph and D an $\mathbb{R}$-divisor on G. If $\deg(D) \neq -2$, then there are a unique measure $\mu_{(\bar{G},D)}$ on G and a unique function $g_{(\bar{G},D)}$ on $G \times G$ with the following properties (cf. [10]).

(a) $\displaystyle\int_G \mu_{(\bar{G},D)} = 1$.

(b) $g_{(\bar{G},D)}(x, y)$ is symmetric and continuous on $G \times G$.

(c) For a fixed $x \in G$, $\Delta_y(g_{(\bar{G},D)}(x, y)) = \delta_x - \mu_{(\bar{G},D)}$.

(d) For a fixed $x \in G$, $\displaystyle\int_G g_{(\bar{G},D)}(x,y)\mu_{(\bar{G},D)}(y) = 0$.

(e) $g_{(\bar{G},D)}(D,y) + g_{(\bar{G},D)}(y,y)$ is a constant function on $y \in G$.

The constant $g_{(\bar{G},D)}(D,y) + g_{(\bar{G},D)}(y,y)$ is denoted by $c(\bar{G},D)$. Further, we set

$$\epsilon(\bar{G},D) = 2\deg(D)c(\bar{G},D) - g_{(\bar{G},D)}(D,D),$$

which we call the *admissible constant* of $(\bar{G},D)$.

For a polarized rigidified metrized graph $([\bar{G}],D)$, Green's function does not depend on its rigidification $\mathrm{Vert}[G]$, neither does the admissible constant of course. Moreover, if we give another Lebesgue measure to G with respect to which the length of no edges change, then the admissible constant does not change. Hence, as far as we talk on the admissible constants of pairs $([\bar{G}],D)$'s, we may consider a Lebesgue measure on $[G]$ as a family of positive real numbers $\{l_e\}_{e\in\mathrm{Ed}[G]}$, each of which should be called the length of the edge.

The next proposition implies the properness of the definition of irreducible components of polarized rigidified graphs and gives us a fundamental tool for calculating the admissible constants.

Proposition 1.2 ([9]). *Let $([\bar{G}],D)$ be a polarized metrized rigidified graph with $\deg D \neq -2$ and let $\{([\bar{G}_i],D_i)\}_{i=1,\dots,r}$ be the set of irreducible components of $([\bar{G}],D)$. Then, we have*

$$\epsilon(\bar{G},D) = \epsilon(\bar{G}_1,D_1) + \cdots + \epsilon(\bar{G}_r,D_r).$$

We have another convenient formula.

Lemma 1.3. *Let $([\bar{G}],D)$ be a polarized rigidified metrized graph with $\deg D \neq -2$. Let e_1 be an edge of $[G]$ of length l_1. Then, we have*

$$\lim_{l_1\to 0}\epsilon(\bar{G},D) = \epsilon\big(\bar{G}_{\{e_1\}},D_{\{e_1\}}\big).$$

The idea of proof Lemma 1.3 is simple, but concrete description needs some pages. We will, hence, give proof in the last section.

Before proceeding the argument, we make sure the definition of *polarized metrized rigidified dual graph* of a semistable curve C. It is the choice of rigidifications of the dual graph is that we would like to emphasize. The base metrized space $\bar{G}$ and its canonical divisor are the ordinary ones which are dealt with in [2] or [10]. We define V as the set of points of G corresponding to irreducible components of C which are not (-2)-smooth rational curves. Then, we can easily see that $G \setminus V$ is a disjoint union of open segments, hence V is a rigidification. Moreover, it is immediate that the canonical divisor is supported in V. We call this $(\bar{G},V)$ with the canonical polarization the polarized metrized rigidified dual graph of C in this paper.

Let $([G],D)$ be a polarized rigidified graph. We can assign numbers to $P \in \mathrm{Vert}[G]$ in the following way. Let $m(G,D)_P$ be the coefficient of P in D and let $b(G)_P$ be the number of branches at P, that is, that of directions going away from P.

Definition 1.4. Let $([G], D)$ and $([\tilde{G}], \tilde{D})$ be polarized rigidified graphs.

(1) $([\tilde{G}], \tilde{D})$ is said to be *maximal* if $m(\tilde{G}, \tilde{D})_P = 1$ and $b(\tilde{G})_P = 3$ for all $P \in \mathrm{Vert}[\tilde{G}]$.

(2) $([\tilde{G}], \tilde{D})$ is called a *maximal model* of $([G], D)$ if $([\tilde{G}], \tilde{D})$ is maximal and dominates $([G], D)$.

Definition 1.4 may seem artificial, but it is quite natural for the polarized rigidified dual graphs of semistable curves.

Proposition 1.5. *Let $([G], D)$ be a polarized rigidified graph such that for any $P \in \mathrm{Vert}[G]$,*

(a) *$b(G)_P \geq 2$,*

(b) *$m(G, D)_P > 0$,*

(c) *$m(G, D)_P - b(G)_P = 2k$, where $k = -1, 0, 1, 2, \ldots$.*

Then, there exists a maximal model of $([G], D)$. In particular, if $([G], D)$ is an irreducible component of the polarized rigidified dual graph of a semistable curve and G is not a closed segment, then $([G], D)$ has a maximal model.

Proof. First we set

$$n([G], D)_P = \#\{Q \in \mathrm{Vert}[G] \mid (m(G, D)_Q, b(G)_Q) = (m(G, D)_P, b(G)_P)\}$$

for each $P \in \mathrm{Vert}[G]$. We put the lexicographic order in $\mathbb{N}^3$, and we will prove our assertion by induction on

$$\big(m(G, D), b(G), n([G], D)\big) := \max_{P \in \mathrm{Vert}[G]} \big\{\big(m(G, D)_P, b(G)_P, n([G], D)_P\big) \in \mathbb{N}^3\big\}.$$

If $(m(G, D), b(G), n([G], D)) = (1, 3, *)$, then it is already maximal.

Let P_0 be a vertex which gives $(m(G, D), b(G), n([G], D))$. We consider the case of $m(G, D) - b(G) \geq 0$ first. Then, $m(G, D) \geq 2$ by the assumptions (a) and (c), and hence we can take a polarized rigidified graph $([G'], D')$ like the following: there exist two edges e_1 and e_2 which are the closed intervals, and exist two vertices $P_{0,1}$ and $P_{0,2}$ which are the terminal points of both e_1 and e_2, such that

$$\begin{aligned} \big(m(G, D)_{P_{0,1}}, b(G)_{P_{0,1}}\big) &= (1, 3), \\ \big(m(G, D)_{P_{0,2}}, b(G)_{P_{0,2}}\big) &= \big(m(G, D) - 1, b(G) + 1\big), \\ \big(\big[G'_{\{e_1, e_2\}}\big], D'_{\{e_1, e_2\}}\big) &= ([G], D) \end{aligned}$$

(cf. Figure 1). We can easily see that $([G'], D')$ satisfies the conditions (a), (b) and (c), and that $\big(m(G, D), b(G), n([G], D)\big) > \big(m(G', D'), b(G'), n([G'], D')\big)$.

Next let us suppose that $m(G, D) - b(G) = -2$. If $m(G, D) = 1$, then $b(G) = 3$ and it is already maximal. If $m(G, D) \geq 2$, then we can take a polarized rigidified graph $([G'], D')$ like the following: there exists an edge e homeomorphic to the closed interval such that

$$\begin{aligned} \big(m(G, D)_{P_{0,1}}, b(G)_{P_{0,1}}\big) &= (1, 3), \\ \big(m(G, D)_{P_{0,2}}, b(G)_{P_{0,2}}\big) &= \big(m(G, D) - 1, b(G) - 1\big), \\ \big(\big[G'_{\{e\}}\big], D'_{\{e\}}\big) &= ([G], D) \end{aligned}$$

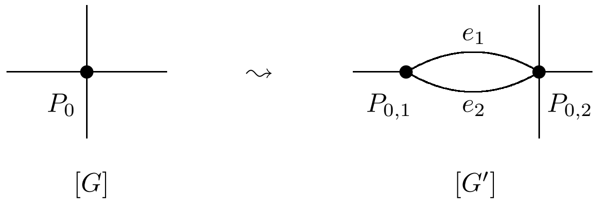

Figure 1: $b(G)_{P_0} = 4$.

where $P_{0,1}$ and $P_{0,2}$ are the terminal points of e (cf. Figure 2). Then we can again easily see that $([G'], D')$ satisfies the conditions (a), (b) and (c), and that $\big(m(G,D), b(G), n([G],D)\big) > \big(m(G',D'), b(G'), n([G'],D')\big)$.

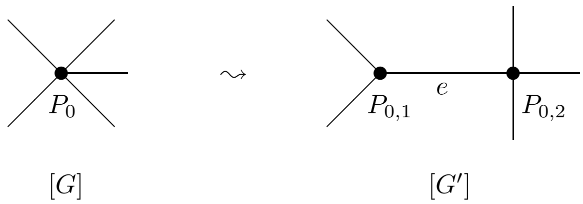

Figure 2: $(m(G,D)_{P_0}, b(G)_{P_0}) = (3,5)$.

In each of the above cases, $([G'], D')$ has a maximal model by the induction hypothesis, and it is a maximal model of $([G], D)$.

The latter part is immediate since its irreducible component satisfies the conditions (a), (b) and (c) unless it is a closed interval. Thus, we complete the proof. $\square$

By virtue of Proposition 1.2, Lemma 1.3 and the above proposition, we see it is essential to calculate the admissible constants of irreducible maximal metrized graphs.

2. On a basis of the direct image of the relative dualizing sheaf

In this section, we find a certain local basis of $f_*\omega_{X/Y}$, which we will use later to estimate $(\omega_{X/Y} \cdot \omega_{X/Y})$.

First, we introduce some notions and words. We mean a *nodal* curve by a reduced projective curve over k which has at most nodes as singularities.

Definition 2.1. We say a connected nodal curve C is *quasi-irreducible* if its every node is of type 0.

For a connected nodal curve C, let us consider the set of nodes of positive type. If we normalize C at such nodes all at once, we have the *quasi-irreducible* *decomposition* of C. Each of the components is called a *quasi-irreducible component* of C. We fix the following notations:

$$\begin{aligned}\operatorname{Comp}(C) &:= \text{the set of connected closed subspaces consist of irreducible}\\ &\qquad \text{components of } C,\\ \operatorname{QIrr}(C) &:= \{D \in \operatorname{Comp}(C) \mid D \text{ is a quasi-irreducible component of } C\},\\ \operatorname{Irr}(C) &:= \{D \in \operatorname{Comp}(C) \mid D \text{ is irreducible}\}.\end{aligned}$$

We sometimes regard a member of $\operatorname{Comp}(C)$ as a reduced scheme.

Let D_1 and D_2 be elements of $\operatorname{Comp}(C)$. We say that $D_3 \in \operatorname{Comp}(C)$ *connects* D_2 *with* D_1 if D_3 has common irreducible components with D_1 and D_2.

Definition 2.2. The *distance from* D_1 *to* D_2, denoted by $d_{D_1}(D_2)$, is a non-negative integer defined by

$$d_{D_1}(D_2) := \min\{\# \operatorname{Irr}(D) \mid D \in \operatorname{Comp}(C) \text{ connects } D_2 \text{ with } D_1\} - 1.$$

We call $D \in \operatorname{Comp}(C)$ which attains the distance from D_1 to D_2 a *path from* D_1 *to* D_2.

Note that $d_{D_1}(D_2) = d_{D_2}(D_1)$ and that a path is not uniquely determined in general.

Remark 2.3. The distance between two components is equal to that of the two corresponding points in its metrized dual graph.

Let R be a discrete valuation ring with residue field k and put $S := \operatorname{Spec} R$. Let $\mathcal{C}$ be a regular S-scheme whose structure morphism f is a semistable curve of genus g. Let s be the closed point of S. For any reduced connected divisor C_0, we have a natural injective map $\omega_{\mathcal{C}/S}(C_0 - \mathcal{C}_s)|_{C_0} \hookrightarrow \omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$. By the adjunction formula, there is an isomorphism $t : \omega_{C_0} \to \omega_{\mathcal{C}/S}(C_0 - \mathcal{C}_s)|_{C_0}$. We call here the composite injective map $\omega_{C_0} \hookrightarrow \omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$ the canonical injection.

Lemma 2.4. *Let C_0 be a reduced connected vertical divisor of $\mathcal{C}$ and let η_0 be any non-zero global section of ω_{C_0}.*

(1) *There exists a global section $\tilde{\eta}_0$ of $\omega_{\mathcal{C}/S}$ with the following properties.*

(a) *If η_0' is the image of η_0 by the canonical injection $H^0(\omega_{C_0}) \to H^0(\omega_{\mathcal{C}_s})$, then $\eta_0' = \tilde{\eta}_0|_{\mathcal{C}_s}$.*

(b) $\operatorname{div}(\tilde{\eta}_0) - \sum_{C \in \operatorname{Irr}(\mathcal{C}_s)} d_{C_0}(C)C$ *is an effective divisor.*

We call such $\tilde{\eta}_0$ a stepwise extension of η_0.

(2) *Suppose that a basis $\{\eta_{C,i}\}_{i=1,\dots,g(C)}$ of $H^0(C, \omega_C)$ is given for any quasi-irreducible component C of $\mathcal{C}_s$ not isomorphic to $\mathbb{P}^1$. Let $\tilde{\eta}_{C,i}$ be a stepwise extension of $\eta_{C,i}$. Then, the set $\bigcup_C \{\tilde{\eta}_{C,i}\}_i$ of the stepwise extensions is a basis of $f_*\omega_{\mathcal{C}/S}$.*

Proof. (1) We simply write $d(C)$ for $d_{C_0}(C)$. Put $m := \sup_{C\in \mathrm{Irr}(\mathcal{C}_s)}\{d(C)\}$, and put

$$D_j := \sum_{d(C)\le j} (j-d(C))C, \qquad\qquad D'_j := \sum_{d(C)\le j} C$$

for $j = 1, 2, \dots, m$. Furthermore, we put $F_j = j\mathcal{C}_s$ and $E'_j = \mathcal{C}_s - D'_j$ to simplify the notation. Note that D'_j and E'_j are reduced curves and that $D_{j+1} = D_j + D'_j$. Let $\Sigma^{(j)}$ be a reduced closed subscheme of dimension 0 defined as $D'_j \cap E'_j$. Note that $\mathrm{Supp}(\Sigma^{(j)}) \neq \emptyset$ for $j \neq m$ and that $\Sigma^{(j)}$ can be regarded as a Cartier divisor on D_{j+1} and D'_j.

Let j be a positive integer less than m. Since we have an exact sequence

$$0 \to \mathcal{O}_{D'_j} \to \mathcal{O}_{D_{j+1}}(D_j) \to \mathcal{O}_{D_j}(D_j) \to 0,$$

we obtain, by tensoring $\omega_{\mathcal{C}/S}(-F_j)$,

$$0 \to \omega_{\mathcal{C}/S}(-F_j)|_{D'_j} \to \omega_{\mathcal{C}/S}(D_j - F_j)|_{D_{j+1}} \to \omega_{\mathcal{C}/S}(D_j - F_j)|_{D_j} \to 0.$$

By the adjunction formula, we have

$$\omega_{\mathcal{C}/S}(-F_j)|_{D'_j} \cong \omega_{D'_j}(\Sigma^{(j)}),$$

hence we obtain $h^1(\omega_{\mathcal{C}/S}(-F_j)|_{D'_j}) = 0$ by Serre duality. Therefore, we have the following exact sequence:

$$\begin{aligned} 0 &\to H^0(\omega_{\mathcal{C}/S}(-F_j)|_{D'_j}) \\ &\to H^0(\omega_{\mathcal{C}/S}(D_j - F_j)|_{D_{j+1}}) \\ &\to H^0(\omega_{\mathcal{C}/S}(D_j - F_j)|_{D_j}) \\ &\to 0. \end{aligned}$$

If a section $\eta_j \in H^0(\omega_{\mathcal{C}/S}(D_j - F_j)|_{D_j})$ is given, then by the above exact sequence, we can take a section $\eta'_{j+1} \in H^0(\omega_{\mathcal{C}/S}(D_j - F_j)|_{D_{j+1}})$ lying over η_j.

On the other hand, taking account that

$$\begin{aligned} (D'_j - F_{j+1}) + E'_j &= -F_j, \\ (D_{j+1} - F_{j+1}) + E'_j &= D_j - F_j, \\ (D'_j - F_{j+1}) + D_j &= D_{j+1} - F_{j+1}, \end{aligned}$$

we have the following natural diagram in which both the horizontal lines are exact.

$$\begin{tikzcd} 0 \arrow[r] & \omega_{\mathcal{C}/S}(D'_j - F_{j+1}) \arrow[r] \arrow[d] & \omega_{\mathcal{C}/S}(-F_j) \arrow[r] \arrow[d] & \omega_{\mathcal{C}/S}(-F_j)|_{E'_j} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1}) \arrow[r] & \omega_{\mathcal{C}/S}(D_j - F_j) \arrow[r] & \omega_{\mathcal{C}/S}(D_j - F_j)|_{E'_j} \arrow[r] & 0 \end{tikzcd}$$

Since any non-zero local section of the ideal sheaf of E'_j does not vanish at any associated point of $\mathcal{O}_{D_{j+1}}$ or $\mathcal{O}_{D'_j}$, we restrict the first horizontal line to D'_j and the second one to D_{j+1}, to obtain the following diagram in which both the horizontal lines are exact, where $k(\Sigma^{(j)})$ is the skyscraper sheaf of k with support $\Sigma^{(j)}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_{\mathcal{C}/S}(D'_j - F_{j+1})|_{D'_j} & \longrightarrow & \omega_{\mathcal{C}/S}(-F_j)|_{D'_j} & \longrightarrow & k\big(\Sigma^{(j)}\big) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & \omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1})|_{D_{j+1}} & \longrightarrow & \omega_{\mathcal{C}/S}(D_j - F_j)|_{D_{j+1}} & \longrightarrow & k\big(\Sigma^{(j)}\big) & \longrightarrow & 0 \end{array}$$

Taking the cohomology of the diagram, we have the following diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & H^0(\omega_{\mathcal{C}/S}(D'_j - F_{j+1})|_{D'_j}) & \longrightarrow & H^0(\omega_{\mathcal{C}/S}(-F_j)|_{D'_j}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(\omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1})|_{D_{j+1}}) & \longrightarrow & H^0(\omega_{\mathcal{C}/S}(D_j - F_j)|_{D_{j+1}}) \end{array}$$

$$\begin{array}{cccc} \longrightarrow & k(\Sigma^{(j)}) & \longrightarrow & H^1(\omega_{\mathcal{C}/S}(D'_j - F_{j+1})|_{D'_j}) \\ & \downarrow\cong & & \downarrow\alpha \\ \longrightarrow & k(\Sigma^{(j)}) & \longrightarrow & H^1(\omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1})|_{D_{j+1}}) \end{array}$$

We claim that α is injective. If it is not injective, then the map

$$k(\Sigma^{(j)}) \to H^1(\omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1})|_{D_{j+1}})$$

is the zero-map since $h^1(\omega_{\mathcal{C}/S}(D'_j - F_{j+1})|_{D'_j}) = 1$ by Serre duality, hence we see

$$H^1(\omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1})|_{D_{j+1}}) \cong H^1(\omega_{\mathcal{C}/S}(D_j - F_j)|_{D_{j+1}})$$

by the long exact sequence of the cohomology of the second horizontal line. Therefore, noting

$$\omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1})|_{D_{j+1}} \cong \omega_{D_{j+1}}$$

and

$$\begin{aligned} \omega_{\mathcal{C}/S}(D_j - F_j)|_{D_{j+1}} &\cong \omega_{\mathcal{C}/S}(D_{j+1} - F_{j+1})|_{D_{j+1}} \otimes_{\mathcal{O}_{D_{j+1}}} \mathcal{O}_{D_{j+1}}(\Sigma^{(j)}) \\ &\cong \omega_{D_{j+1}} \otimes_{\mathcal{O}_{D_{j+1}}} \mathcal{O}_{D_{j+1}}(\Sigma^{(j)}) \end{aligned}$$

and using Serre duality, we obtain

$$h^0(\mathcal{O}_{D_{j+1}}) = h^0(\mathcal{O}_{D_{j+1}}(-\Sigma^{(j)})).$$

That is a contradiction, and we see that α is injective.

Now chasing the diagram, we can easily see that there exist a global section η_{j+1} of $\omega_{\mathcal{C}/S}(D_{j+1}-F_{j+1})|_{D_{j+1}}$ and a global section η_j'' of $\omega_{\mathcal{C}/S}(-F_j)|_{D_j'}$ such that $\eta_j''+\eta_{j+1}=\eta_{j+1}'$ in $H^0(\omega_{\mathcal{C}/S}(D_{j+1}-F_{j+1})|_{D_{j+1}})$. In other words, we have a section $\eta_{j+1}\in H^0(\omega_{\mathcal{C}/S}(D_{j+1}-F_{j+1})|_{D_{j+1}})$ such that its image by the composite map

$$\phi_j:\omega_{\mathcal{C}/S}(D_{j+1}-F_{j+1})|_{D_{j+1}}\hookrightarrow\omega_{\mathcal{C}/S}(D_j-F_j)|_{D_{j+1}}\twoheadrightarrow\omega_{\mathcal{C}/S}(D_j-F_j)|_{D_j}$$

is η_j. Let us put

$$\Phi:=(\phi_1\circ\phi_2\circ\cdots\circ\phi_{m-1}):\omega_{\mathcal{C}/S}(D_m-F_m)|_{D_m}\to\omega_{\mathcal{C}/S}(D_1-F_1)|_{D_1}.$$

Let $\iota:\omega_{\mathcal{C}/S}(D_m-F_m)|_{D_m}\hookrightarrow\omega_{\mathcal{C}/S}|_{F_m}$ be a homomorphism obtained by tensoring $\omega_{\mathcal{C}/S}$ to the canonical injection $\mathcal{O}_{D_m}(D_m-F_m)\hookrightarrow\mathcal{O}_{F_m}$ and let $r_m:\omega_{\mathcal{C}/S}\twoheadrightarrow\omega_{\mathcal{C}/S}|_{F_m}$ be the restriction homomorphism. Then, we have a commutative diagram as follows, where $r:\omega_{\mathcal{C}/S}\twoheadrightarrow\omega_{\mathcal{C}_s}$ is the restriction map and $t:\omega_{C_0}\to\omega_{\mathcal{C}/S}(D_1-F_1)|_{D_1}$ is an isomorphism.

$$\begin{tikzcd}
\omega_{C_0} \arrow[r] \arrow[d, "t"', "\cong"] & \omega_{\mathcal{C}_s} & \omega_{\mathcal{C}/S} \arrow[l, "r"'] \arrow[d, "r_m"] \\
\omega_{\mathcal{C}/S}(D_1-F_1)|_{D_1} & \omega_{\mathcal{C}/S}(D_m-F_m)|_{D_m} \arrow[l, "\Phi"'] \arrow[r, "\iota"] & \omega_{\mathcal{C}/S}|_{F_m}
\end{tikzcd}$$

In the situation of our lemma, a section η_0 of ω_{C_0} is already given. By the above discussion, there is a section η_m of $\omega_{\mathcal{C}/S}(D_m-F_m)|_{D_m}$ with $t(\eta_0)=\Phi(\eta_m)$. Let us put $\tilde{\eta}_0$ to be a global section of $\omega_{\mathcal{C}/S}$ such that $r_m(\tilde{\eta}_0)=\iota(\eta_m)$. Then, it is immediate that $\tilde{\eta}_0$ has the property (a) and we can also see that $\tilde{\eta}_0$ has the property (b) because its image by r_m comes from $H^0(\omega_{\mathcal{C}/S}(D_m-F_m)|_{D_m})$. Thus, we complete the proof of (1).

(2) The set of such stepwise extensions is a basis at the closed point, hence the members generate $f_*\omega_{\mathcal{C}/S}$. Since the set of such stepwise extensions consists of g members and $f_*\omega_{\mathcal{C}/S}$ is free module of rank g, it is a basis. $\square$

3. Proof of the main theorem

Let us start the proof of Main Theorem. The proof consists of two parts as we announced in the introduction. One is the estimation of the admissible constants of the metrized dual graph of singular fibers and the other is that of the self-intersection of the relative dualizing sheaf.

3.1. Calculations of the admissible constants

Let $f:X\to Y$ be a generically smooth semistable curve of genus 3 as in Main Theorem and let $([G_y],\omega_y)$ be the polarized rigidified dual graph of a singular fiber X_y. Since $\deg(\omega_y)=4$, the maximal models of any irreducible component of $([G_y],\omega_y)$ not isomorphic to the closed interval have exactly four vertices. We can easily see that there exist only two maximal models as in Figure 3.

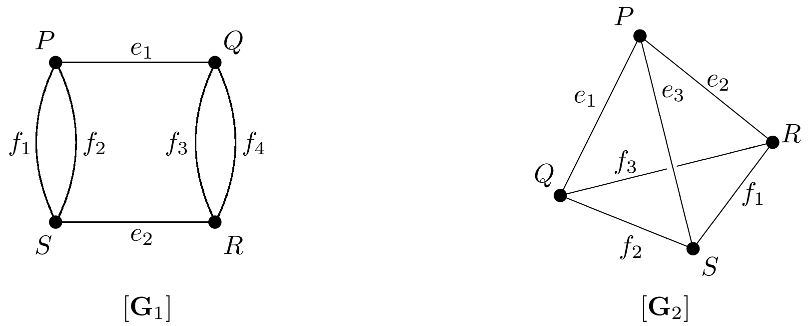

Figure 3: The maximal models.

Let $\sigma_i = \sigma_i(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ denote the i-th elementary symmetric polynomial on $\{X_1, X_2, X_3, Y_1, Y_2, Y_3\}$ and put

$$\begin{aligned} L(X_1, X_2, X_3, Y_1, Y_2, Y_3) &:= \sigma_3 - (X_1X_2X_3 + X_1Y_2Y_3 + X_2Y_3Y_1 + X_3Y_1Y_2), \\ F(X_1, X_2, X_3, Y_1, Y_2, Y_3) &:= 5\sigma_4 - (X_1X_2Y_1Y_2 + X_2X_3Y_2Y_3 + X_3X_1Y_3Y_1). \end{aligned}$$

We have the following results on the admissible constants of the maximal models.

Proposition 3.1. *Suppose that a Lebesgue measure is given to* $\mathbf{G}_1$ *and* $\mathbf{G}_2$.

(1) *If the length of* e_i *is* l_i *for* $i = 1, 2$ *and that of* f_j *is* m_j *for* $j = 1, 2, 3, 4$, *then we have*

$$\begin{aligned} \epsilon(\bar{\mathbf{G}}_1, D_1) = &\frac{2}{9}(l_1 + l_2 + m_1 + m_2 + m_3 + m_4) + \frac{F(l_1 + l_2, m_1, m_2, 0, m_3, m_4)}{9L(l_1 + l_2, m_1, m_2, 0, m_3, m_4)} \\ &+ \frac{4l_1l_2(m_1 + m_2)(m_3 + m_4)}{3L(l_1 + l_2, m_1, m_2, 0, m_3, m_4)}. \end{aligned}$$

(2) *If, for* $i = 1, 2, 3$, *the length of* e_i *is* l_i *and that of* f_i *is* m_i, *then we have*

$$\epsilon(\bar{\mathbf{G}}_2, D_2) = \frac{2}{9}(l_1 + l_2 + l_3 + m_1 + m_2 + m_3) + \frac{F(l_1, l_2, l_3, m_1, m_2, m_3)}{9L(l_1, l_2, l_3, m_1, m_2, m_3)}.$$

Proof. (1) Let us put $L_1 := L(l_1 + l_2, m_1, m_2, 0, m_3, m_4)$. Let $s_i : e_i \to [0, l_i]$ and $t_i : f_i \to [0, m_i]$ be the arc-length parameters such that

$$\begin{aligned} &s_1(P) = s_2(S) = t_1(P) = t_2(P) = t_3(Q) = t_4(Q) = 0, \\ &s_1(Q) = l_1, \quad s_2(R) = l_2, \\ &t_1(S) = m_1, \quad t_2(S) = m_2, \quad t_3(R) = m_3, \quad t_4(R) = m_4. \end{aligned}$$

Put

$$\alpha_1 = \alpha_2 = (m_1 + m_2)(m_3 + m_4)$$

and

$$\begin{aligned} A_1 &= (l_1 + l_2 + m_2)(m_3 + m_4) + m_3 m_4, \\ A_2 &= (l_1 + l_2 + m_1)(m_3 + m_4) + m_3 m_4, \\ A_3 &= (l_1 + l_2 + m_4)(m_1 + m_2) + m_1 m_2, \\ A_4 &= (l_1 + l_2 + m_3)(m_1 + m_2) + m_1 m_2. \end{aligned}$$

By [10, Lemma 3.7], we have

$$\mu_{(\mathbf{G}_1, D)} = \frac{\alpha_1}{3L_1} ds_1 + \frac{\alpha_2}{3L_1} ds_2 + \frac{A_1}{3L_1} dt_1 + \frac{A_2}{3L_1} dt_2 + \frac{A_3}{3L_1} dt_3 + \frac{A_4}{3L_1} dt_4.$$

Further, we put

$$\begin{aligned} \beta_1 &= \big((3l_1 + l_2)(m_3 + m_4) + 2m_3 m_4\big)(m_1 + m_2) - 4L_1, \\ \beta_2 &= -\big((5l_1 + 3l_2)(m_3 + m_4) + 4m_3 m_4\big)(m_1 + m_2) + 2L_1, \\ B_1 &= -\big((3l_1 + l_2)(m_3 + m_4) + 2m_3 m_4\big)m_2 - L_1, \\ B_2 &= -\big((3l_1 + l_2)(m_3 + m_4) + 2m_3 m_4\big)m_1 - L_1, \\ B_3 &= -\big((-3l_1 + l_2)(m_1 + m_2) + 2m_1 m_2\big)m_4 - L_1, \\ B_4 &= -\big((-3l_1 + l_2)(m_1 + m_2) + 2m_1 m_2\big)m_3 - L_1, \end{aligned}$$

$$\begin{aligned} \gamma_1 = {} & (l_1 + l_2 + m_1 + m_2 + m_3 + m_4)L_1 \\ & + 7(l_1 + l_2)(m_1 m_2 m_3 + m_2 m_3 m_4 + m_3 m_4 m_1 + m_4 m_1 m_2) \\ & + 24 l_1 l_2 (m_1 + m_2)(m_3 + m_4) + 20 m_1 m_2 m_3 m_4 + 36 l_1 m_1 m_2 (m_3 + m_4), \end{aligned}$$

$$C_1 = C_2 = \gamma_1, \qquad C_3 = C_4 = 18\alpha_1 {l_1}^2 + 18\beta_1 l_1 + \gamma_1,$$

and

$$\gamma_2 = 18 A_1 {m_1}^2 + 18 B_1 m_1 + C_1.$$

Now, consider the following function on $\mathbf{G}_1$:

$$g(x) = \begin{cases} \dfrac{\alpha_i}{6L_1} s_i(x)^2 + \dfrac{\beta_i}{6L_1} s_i(x) + \dfrac{\gamma_i}{108L_1} & \text{on} \quad e_i, \\[2ex] \dfrac{A_i}{6L_1} t_i(x)^2 + \dfrac{B_i}{6L_1} t_i(x) + \dfrac{C_i}{108L_1} & \text{on} \quad f_i. \end{cases}$$

Then, we can check by direct calculations that g is continuous, $\Delta(g) = \delta_P - \mu_{(\mathbf{G}_1,D)}$, and $\int_{\mathbf{G}_1} g\mu_{(\mathbf{G}_1,D)} = 0$. Thus, we have $g_{(\mathbf{G}_1,D)}(P,x) = g(x)$, and by [4, Lemma 4.1], we obtain the formula.

(2) Let us put $\bar{L} := L(l_1, l_2, l_3, m_1, m_2, m_3)$. Let $s_i : e_i \to [0, l_i]$ and $t_i : f_i \to [0, m_i]$ be the arc-length parameters such that

$$\begin{aligned}
&s_1(P) = s_2(P) = s_2(P) = t_1(R) = t_2(S) = t_3(Q) = 0,\\
&s_1(Q) = l_1, \quad s_2(R) = l_2, \quad s_2(S) = l_3,\\
&t_1(R) = m_1, \quad t_2(Q) = m_2, \quad t_3(R) = m_3.
\end{aligned}$$

Put

$$E(X_1, X_2, X_3, Y_1, Y_2, Y_3) := Y_1(X_2 + X_3 + Y_2 + Y_3) + (X_2 + X_3)(Y_2 + Y_3),$$

and put

$$\begin{aligned}
&E_1 := E(l_1, l_2, l_3, m_1, m_2, m_3), &&E_2 := E(l_2, l_3, l_1, m_2, m_3, m_1),\\
&E_3 := E(l_3, l_1, l_2, m_3, m_1, m_2), &&\\
&F_1 := E(m_1, l_2, m_3, l_1, m_2, l_3), &&F_2 := E(m_2, l_3, m_1, l_2, m_3, l_1),\\
&F_3 := E(m_3, l_1, m_2, l_3, m_1, l_2). &&
\end{aligned}$$

By [10, Lemma 3.7], we have

$$\mu_{(\mathbf{G}_2,D)} = \frac{E_1}{3\bar{L}}ds_1 + \frac{E_2}{3\bar{L}}ds_2 + \frac{E_3}{3\bar{L}}ds_3 + \frac{F_1}{3\bar{L}}dt_1 + \frac{F_2}{3\bar{L}}dt_2 + \frac{F_3}{3\bar{L}}dt_3.$$

Further we put

$$\begin{aligned}
\beta_1 = &- l_1E_1 - 4l_2l_3(m_1 + m_2 + m_3)\\
&- 2m_2m_3(m_1 + l_2 + l_3) - 3m_1(l_2m_2 + l_3m_3),\\
\beta_2 = &- l_2E_2 - 4l_3l_1(m_1 + m_2 + m_3)\\
&- 2m_3m_1(m_2 + l_3 + l_1) - 3m_2(l_3m_3 + l_1m_1),\\
\beta_3 = &- l_3E_3 - 4l_1l_2(m_1 + m_2 + m_3)\\
&- 2m_1m_2(m_3 + l_1 + l_2) - 3m_3(l_1m_1 + l_2m_2),
\end{aligned}$$

$$\begin{aligned}
B_1 &= \frac{1}{m_1}(E_3{l_3}^2 + \beta_3l_3 - E_2{l_2}^2 - \beta_2l_2 - F_1{m_1}^2),\\
B_2 &= \frac{1}{m_2}(E_1{l_1}^2 + \beta_1l_1 - E_3{l_3}^3 - \beta_3l_3 - F_2{m_2}^2),\\
B_3 &= \frac{1}{m_3}(E_2{l_2}^2 + \beta_2l_2 - E_1{l_1}^2 - \beta_1l_1 - F_3{m_3}^2),
\end{aligned}$$

$$\begin{aligned}
\gamma = &(l_1 + l_2 + l_3 + m_1 + m_2 + m_3)\bar{L} + 43l_1l_2l_3(m_1 + m_2 + m_3)\\
&+ 7(l_1 + l_2 + l_3)m_1m_2m_3) + 7(l_1l_2 + l_2l_3 + l_3l_1)(m_1m_2 + m_2m_3 + m_3m_1)\\
&+ 13(l_1l_2m_1m_2 + l_2l_3m_2m_3 + l_3l_1m_3m_1),
\end{aligned}$$

and,

$$C_1 = 18E_2 + 18\beta_2 + \gamma, \quad C_2 = 18E_3 + 18\beta_3 + \gamma, \quad C_3 = 18E_1 + 18\beta_1 + \gamma.$$

Now, consider the following function on $\mathbf{G}_2$:

$$g(x) = \begin{cases} \dfrac{E_i}{6\bar{L}} s_i(x)^2 + \dfrac{\beta_i}{6\bar{L}} s_i(x) + \dfrac{\gamma}{108\bar{L}} & \text{on} \quad e_i, \\[2ex] \dfrac{F_i}{6\bar{L}} t_i(x)^2 + \dfrac{B_i}{6\bar{L}} t_i(x) + \dfrac{C_i}{108\bar{L}} & \text{on} \quad f_i. \end{cases}$$

Then, we can check by direct calculations that g is continuous, $\Delta(g) = \delta_P - \mu_{(\mathbf{G}_2,D)}$, and $\int_{\mathbf{G}_2} g\mu_{(\mathbf{G}_2,D)} = 0$. Thus, we have $g_{(\mathbf{G}_2,D)}(P,x) = g(x)$, and by [4, Lemma 4.1], we obtain the formula. □

Thus, we know their admissible constants. Here are elementary inequalities, which we will use to estimate the admissible constants.

Lemma 3.2. *For any non-negative real numbers $l_1, \dots, m_3$, we have*

$$F(l_1,l_2,l_3,m_1,m_2,m_3) \leq \frac{10}{11}\bar{L}(l_1+l_2+l_3+m_1+m_2+m_3),$$
$$l_1 l_2 (m_1+m_2)(m_3+m_4) \leq L(l_1+l_2, m_1, m_2, 0, m_3, m_4)\min\{l_1, l_2\},$$

where $\bar{L} = L(l_1,l_2,l_3,m_1,m_2,m_3)$.

Proof. First of all, we note

$$\begin{aligned} \bar{L} = {} & (l_1l_2 + l_2l_3 + l_3l_1)(m_1+m_2+m_3) \\ & + l_1m_1(m_2+m_3) + l_2m_2(m_3+m_1) + l_3m_3(m_1+m_2) + m_1m_2m_3, \end{aligned}$$

which is immediate from the definition. Then, taking account of elementary inequalities

$$\begin{aligned} (l_1+l_2+l_3)(l_1l_2+l_2l_3+l_3l_1) &\geq 9l_1l_2l_3, \\ (m_1+m_2+m_3)^2 &\geq 3(m_1m_2+m_2m_3+m_3m_1), \\ (m_i+m_j)^2 &\geq 4m_im_j, \end{aligned}$$

we obtain

$$\begin{aligned} \bar{L}(l_1+l_2+l_3) \geq {} & 9l_1l_2l_3(m_1+m_2+m_3) \\ & + l_1(l_2+l_3)m_1(m_2+m_3) + l_2(l_3+l_1)m_2(m_3+m_1) \\ & + l_3(l_1+l_2)m_3(m_1+m_2) + (l_1+l_2+l_3)m_1m_2m_3 \end{aligned}$$

and

$$\begin{aligned} \bar{L}(m_1+m_2+m_3) \geq {} & 3(l_1l_2+l_2l_3+l_3l_1)(m_1m_2+m_2m_3+m_3m_1) \\ & + 4(l_1+l_2+l_3)m_1m_2m_3. \end{aligned}$$

Since $\bar{L}$ is stable under the permutations

$$(l_1, l_2, l_3, m_1, m_2, m_3) \mapsto \begin{cases} (l_1, m_2, m_3, m_1, l_2, l_3), \\ (l_2, m_3, m_1, m_2, l_3, l_1), \\ (l_3, m_1, m_2, m_3, l_1, l_2), \end{cases}$$

we have also the following inequalities:

$$\begin{aligned} \bar{L}(l_1 + m_2 + m_3) \geq\ & 9l_1m_2m_3(m_1 + l_2 + l_3) \\ & + l_1(m_2 + m_3)m_1(l_2 + l_3) + l_2(m_3 + l_1)m_2(l_3 + m_1) \\ & + l_3(l_1 + m_2)m_3(m_1 + l_2) + (l_1 + m_2 + m_3)m_1l_2l_3, \end{aligned}$$

$$\begin{aligned} \bar{L}(m_1 + l_2 + l_3) \geq\ & 3(l_1m_2 + m_2m_3 + m_3l_1)(m_1l_2 + l_2l_3 + l_3m_1) \\ & + 4(l_1 + m_2 + m_3)m_1l_2l_3, \end{aligned}$$

$$\begin{aligned} \bar{L}(l_2 + m_3 + m_1) \geq\ & 9l_2m_3m_1(m_2 + l_3 + l_1) \\ & + l_2(m_3 + m_1)m_2(l_3 + l_1) + l_3(m_1 + l_2)m_3(l_1 + m_2) \\ & + l_1(l_2 + m_3)m_1(m_2 + l_3) + (l_2 + m_3 + m_1)m_2l_3l_1, \end{aligned}$$

$$\begin{aligned} \bar{L}(m_2 + l_3 + l_1) \geq\ & 3(l_2m_3 + m_3m_1 + m_1l_2)(m_2l_3 + l_3l_1 + l_1m_2) \\ & + 4(l_2 + m_3 + m_1)m_2l_3l_1, \end{aligned}$$

$$\begin{aligned} \bar{L}(l_3 + m_1 + m_2) \geq\ & 9l_3m_1m_2(m_3 + l_1 + l_2) \\ & + l_3(m_1 + m_2)m_3(l_1 + l_2) + l_1(m_2 + l_3)m_1(l_2 + m_3) \\ & + l_2(l_3 + m_1)m_2(m_3 + l_1) + (l_3 + m_1 + m_2)m_3l_1l_2, \end{aligned}$$

and

$$\begin{aligned} \bar{L}(m_3 + l_1 + l_2) \geq\ & 3(l_3m_1 + m_1m_2 + m_2l_3)(m_3l_1 + l_1l_2 + l_2m_3) \\ & + 4(l_3 + m_1 + m_2)m_3l_1l_2. \end{aligned}$$

Therefore, we have straightforwardly

$$\begin{aligned} & \bar{L}(l_1 + l_2 + l_3 + m_1 + m_2 + m_3) \\ & \quad = \frac{1}{4}\bar{L}\big((l_1 + l_2 + l_3) + (m_1 + m_2 + m_3) + (l_1 + m_2 + m_3) + (m_1 + l_2 + l_3) \\ & \qquad + (l_2 + m_3 + m_1) + (m_2 + l_3 + l_1) + (l_3 + m_1 + m_2) + (m_3 + l_1 + l_2)\big) \\ & \quad \geq \frac{1}{4}\big(22l_1l_2l_3(m_1 + m_2 + m_3) + 22l_1m_2m_3(m_1 + l_2 + l_3) \\ & \qquad + 22l_2m_3m_1(m_2 + l_3 + l_1) + 22l_3m_1m_2(m_3 + l_1 + l_2) \\ & \qquad + 20l_1l_2m_1m_2 + 20l_2l_3m_2m_3 + 20l_3l_1m_3m_1\big) \\ & \quad \geq \frac{11}{10}\big(5\sigma_4 - (l_1l_2m_1m_2 + l_2l_3m_2m_3 + l_3l_1m_3m_1)\big). \end{aligned}$$

Thus, we obtain the first inequality.

For the second inequality, we may assume $l_1 \leq l_2$. Then, we have

$$\begin{aligned} L(l_1+l_2, m_1, m_2, 0, m_3, m_4)l_1 &\geq (l_1+l_2)(m_1+m_2)(m_3+m_4)l_1 \\ &\geq l_1 l_2 (m_1+m_2)(m_3+m_4), \end{aligned}$$

which is the desired inequality. □

For edges of $([G_y], \omega_y)$, we introduce a notion of *h-type* now. They are the edges which contribute to the above quantity $\min\{l_1, l_2\}$.

Definition 3.3. We say a pair of two distinct edges $\{e_1, e_2\}$ is of *h-type* if $\mathrm{Vert}\big[G_y^{\{e_1,e_2\}}\big] = \{P, Q\}$, where P and Q are two distinct points of $\big[G_y^{\{e_1,e_2\}}\big]$, and $\omega_y^{\{e_1,e_2\}} = 2P + 2Q$ (cf. Figure 4).

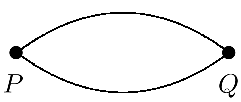

Figure 4: $\big[G_y^{\{e_1,e_2\}}\big]$ when $\{e_1, e_2\}$ is of h-type.

If $\{e_1, e_2\}$ is of h-type, then e_1 and e_2 sit in the same irreducible component of $[G_y]$ since otherwise, $\big[G_y^{\{e_1,e_2\}}\big]$ is a one-point sum of two circles. Taking account that $([G_y], \omega_y)$ is the polarized rigidified dual graph of a semistable curve of genus 3, we can easily see that the condition on the polarization is automatically satisfied and that there exists at most one irreducible components of $([G_y], \omega_y)$ containing a pair of h-type. If an irreducible component has a pair of h-type, its maximal model must be $[\mathbf{G}_1]$. It is clear from its configuration that $[\mathbf{G}_1]$ has at most one pair of h-type, hence there is at most one pair of h-type in $([G_y], \omega_y)$.

Let us put

$$h([\bar{G}_y]) := \begin{cases} \min\{l_{e_1}, l_{e_2} \mid \{e_1, e_2\} \text{ is a pair of h-type}\} & \\ & \text{if} \quad [G_y] \text{ has a pair of h-type}, \\ 0 & \text{otherwise}, \end{cases}$$

where l_e is the length of an edge e. Now an upper bound of the admissible constant can be given:

Proposition 3.4. *For the polarized metrized rigidified dual graph* $([\bar{G}_y], \omega_y)$ *of a semistable curve, we have*

$$\epsilon(\bar{G}_y, \omega_y) \leq \frac{32}{99}\delta_0(X_y) + \frac{5}{3}\delta_1(X_y) + \frac{4}{3}h([\bar{G}_y]).$$

Proof. Let $\{([\bar{G}_i], D_i)\}$ be the set of irreducible components of $([\bar{G}_y], \omega_y)$. If $([G_i], D_i)$ is a closed interval, then we have, as in [4],

$$\epsilon(\bar{G}_i, D_i) = \frac{5}{3} l_e,$$

where l_e is the length of the edge e of $[G_i]$. If $([G_i], D_i)$ is not a closed interval and does not have a pair of h-type, then $\mathbf{G}_2$ is a maximal model of it, and hence, by Lemma 3.2, Proposition 3.1 and Lemma 1.3, we have

$$\epsilon(\bar{G}_i, D_i) \leq \frac{32}{99} \sum_{e \in \mathrm{Ed}[G_i]} l_e.$$

If $([G_i], D_i)$ has a pair of h-type, then its maximal model is $\mathbf{G}_1$, and we have similarly

$$\epsilon(\bar{G}_i, D_i) \leq \frac{32}{99} \sum_{e \in \mathrm{Ed}[G_i]} l_e + \frac{4}{3} h([\bar{G}_y]).$$

Summing all them up and applying Proposition 1.2, we obtain our assertion. □

3.2. A lower bound of self-intersection of the relative dualizing sheaf

Let $f : X \to Y$ be a semistable curve of genus 3 as in Main Theorem. Put $\mathcal{R}_1 := f_* \omega_{X/Y}$ and $\mathcal{R}_2 := f_*({\omega_{X/Y}}^{\otimes 2})$, and let $S^2(\mathcal{R}_1)$ denote the second symmetric tensor product of $\mathcal{R}_1$. We assume that the generic fiber is not hyperelliptic. Then, the natural map

$$S^2(\mathcal{R}_1) \to \mathcal{R}_2$$

is surjective at the generic point of Y, and since $\operatorname{rk} S^2(\mathcal{R}_1) = \operatorname{rk} \mathcal{R}_2 = 6$, it is injective globally and has finite cokernel. For any $y \in Y$, *Horikawa index at* y, denoted by $\mathrm{Ind}(f, y)$, is defined as

$$\mathrm{length}_{\mathcal{O}_{Y,y}}(\mathrm{Coker}(S^2(\mathcal{R}_1) \to \mathcal{R}_2)_y).$$

Note that it vanishes except at finitely many points. Here is a lemma which is essentially proved in [7].

Lemma 3.5 (cf. [7]). *In the situation above, we have*

$$(\omega_{X/Y} \cdot \omega_{X/Y}) = \frac{1}{3}\delta(X/Y) + \frac{4}{3} \sum_{y \in Y} \mathrm{Ind}(f, y),$$

where $\delta(X/Y) := \delta_0(X/Y) + \delta_1(X/Y)$.

Proof. The paper [7] is not published, hence we give the proof for readers' convenience.

By Riemann-Roch theorem on Y, we have

$$\chi(\mathcal{R}_1) = 3\chi(\mathcal{O}_Y) + \deg(\mathcal{R}_1).$$

On the other hand, by Riemann-Roch theorem on X, we have

$$\chi(\omega_{X/Y}) = \chi(\mathcal{O}_X) + \frac{1}{2}(\omega_{X/Y} \cdot \omega_{X/Y} \otimes {\omega_X}^{-1}) = \chi(\mathcal{O}_X) + 4\chi(\mathcal{O}_Y).$$

Taking account of Lerray spectral sequence and Grothendieck duality $R^1 f_* \omega_{X/Y} \cong \mathcal{O}_Y$, we obtain

$$\deg(\mathcal{R}_1) = \chi(\mathcal{O}_X) + 2\chi(\mathcal{O}_Y).$$

Similarly, we obtain

$$\chi(\mathcal{R}_2) = \chi(\mathcal{O}_X) + (\omega_{X/Y} \cdot \omega_{X/Y}) + 8\chi(\mathcal{O}_Y).$$

Therefore, we have

$$\begin{aligned}
\sum_{y \in Y} \operatorname{Ind}(f, y) &= \chi(\mathcal{R}_2) - \chi(S^2(\mathcal{R}_1)) \\
&= \big(\chi(\mathcal{O}_X) + (\omega_{X/Y} \cdot \omega_{X/Y}) + 8\chi(\mathcal{O}_Y)\big) - \big(6\chi(\mathcal{O}_Y) + 4\deg(\mathcal{R}_1)\big) \\
&= (\omega_{X/Y} \cdot \omega_{X/Y}) - 3\deg(\mathcal{R}_1).
\end{aligned}$$

Using Noether's formula

$$\deg(\mathcal{R}_1) = \frac{(\omega_{X/Y} \cdot \omega_{X/Y}) + \delta(X/Y)}{12},$$

we obtain the desired formula. □

Remark 3.6.

(1) The proof needs locally freeness of $\mathcal{R}_1$ only, hence it holds without semistability assumption on f if we work over $\mathbb{C}$. The formula appears in [7] in the form

$$(\omega_{X/Y} \cdot \omega_{X/Y}) = 3\chi(\mathcal{O}_X) - 10\chi(\mathcal{O}_Y) + \sum_{y \in Y} \operatorname{Ind}(f, y).$$

(2) In [1], Konno defined Horikawa index for fibrations of higher genus, and generalized the formula in the lemma in the case of $k = \mathbb{C}$.

Thanks to this lemma, we are reduced to calculate $\operatorname{Ind}(f, y)$, which is completely a local problem. Let $f : \mathcal{C} \to S$ be a semistable curve of genus 3, where $S = \operatorname{Spec} R$ as before. We can define Horikawa index of f at the special point s of S as well.

Proposition 3.7. *If $f : \mathcal{C} \to S$ is non-hyperelliptic, then* $\operatorname{Ind}(f, s) \geq 2\delta_1(\mathcal{C}_s) + h([\bar{G}_s])$.

Before starting proof, we need some preparation. Let $\bar{f} : \bar{\mathcal{C}} \to S$ be the stable model of f and let $\pi : \mathcal{C} \to \bar{\mathcal{C}}$ be the contraction morphism. For a pair $\{P_1, P_2\}$ of distinct nodes of $\bar{\mathcal{C}}_s$, we say it is of *h-type* if the corresponding pair of edges in the dual graph $[G_s]$ is of h-type. We can easily see that $\{P_1, P_2\}$ is of h-type if and only if P_1 and P_2 are nodes of type 0 and the partial normalization at $\{P_1, P_2\}$ has exactly two connected components of genus 1. For $i = 1, 2$, the length of the corresponding edge to P_i is equal to

$$l_i := \text{the number of nodes in } \mathcal{C}_s \text{ mapped to } P_i \text{ by } \pi,$$

and hence, we have $h([\bar{G}_s]) = \min\{l_1, l_2\}$ by the definition.

Proof. We employ the notations just above. Let us consider the case where the stable model $\bar{\mathcal{C}}_s$ of $\mathcal{C}_s$ has a pair $\{P_1, P_2\}$ of nodes of h-type. Let $\bar{C}_0$ be the quasi-irreducible component of $\bar{\mathcal{C}}_s$ on which P_1 and P_2 lie. Let $\bar{D}_1$ and $\bar{D}_2$ be the closures of the connected components of $\bar{\mathcal{C}}_s \setminus \{P_1, P_2\}$. Let $\bar{C}_1$ and $\bar{C}_2$ be the quasi-irreducible components of genus 1 of $\bar{D}_1$ and $\bar{D}_2$ respectively. Note that there exists at most one node of type 1 of $\bar{\mathcal{C}}_s$ on each $\bar{D}_i$. Furthermore, we can see the following.

(1) If $\bar{D}_i$ does not have a node of type 1, then $\bar{C}_i = \bar{D}_i \subset \bar{C}_0$.

(2) If $\bar{D}_i$ has a node Q_i of type 1, then $\bar{C}_i \cap \bar{C}_0 = \{Q_i\}$ and there exist a smooth rational component on which P_1, P_2 and Q_i lie.

Here we define an injective set-theoretic map $\pi^\sharp : \mathrm{Comp}(\bar{\mathcal{C}}_s) \to \mathrm{Comp}(\mathcal{C}_s)$ by

$$\pi^\sharp(\bar{D}) := \text{the closure of } \pi^{-1}(\bar{D} \setminus \Sigma),$$

where $\Sigma := D \cap (\bar{\mathcal{C}}_s - D)$. We can easily check that $\pi^\sharp$ induces the map from $\mathrm{QIrr}(\bar{\mathcal{C}}_s)$ to $\mathrm{QIrr}(\mathcal{C}_s)$. We put $C_i := \pi^\sharp(\bar{C}_i)$ and further we put $d_{i,j} = d_{C_i}(C_j)$ to simplify the notations. Then, it is not difficult to see from the definitions and the above (1) and (2) that $d_{1,2} = d_{0,1} + d_{0,2} + h([\bar{G}_s])$. From the above (2), we can moreover see that $d_{0,i}$ is the number of nodes that are mapped by π to the only node Q_i of type 1 on $\bar{D}_i$ (if it exists), which implies that $d_{0,1} + d_{0,2}$ is equal to the number of nodes of type 1 on $\mathcal{C}_s$, namely, $\delta_1(\mathcal{C}_s)$. We have also $d_{1,2} = \delta_1(\mathcal{C}_s) + h([\bar{G}_s])$.

Let η_i be a non-zero section of ω_{C_i} for $i = 1, 2$, and let η_0 be a section of ω_{C_0} that does not vanish at any node of C_0. That is possible since ω_{C_0} is free from base points (cf. [4, Proposition 2.1.3]). Note that $\{\eta_0, \eta_1, \eta_2\}$ generates $H^0(\omega_{\mathcal{C}_s})$ if we regard each η_i canonically as a member of $H^0(\omega_{\mathcal{C}_s})$. We take a free basis $\{\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2\}$ of $f_*\omega_{\mathcal{C}/S}$ in Lemma 2.4 consisting of stepwise extensions of them. Let M be the image of the canonical map

$$\mathcal{R}_1 \otimes \mathcal{R}_1 \to \mathcal{R}_2,$$

where $\mathcal{R}_m = f_*\big({\omega_{\mathcal{C}/S}}^{\otimes m}\big)$ for $m = 1, 2$, and let $\tilde{\eta}_i\tilde{\eta}_j$ denote the image of $\tilde{\eta}_i \otimes \tilde{\eta}_j$ by this map. Then, M is generated by ${\tilde{\eta}_0}^2$, ${\tilde{\eta}_1}^2$, ${\tilde{\eta}_2}^2$, $\tilde{\eta}_0\tilde{\eta}_1$, $\tilde{\eta}_0\tilde{\eta}_2$ and $\tilde{\eta}_1\tilde{\eta}_2$. By Lemma 2.4, if we regard $\tilde{\eta}_1\tilde{\eta}_2$ as a section of ${\omega_{\mathcal{C}_s}}^{\otimes 2}$, the divisor $\mathrm{div}(\tilde{\eta}_1\tilde{\eta}_2) - d_{1,2}\mathcal{C}_s$ is effective, and hence $t^{-d_{1,2}}\tilde{\eta}_1\tilde{\eta}_2$ is a section of $\mathcal{R}_2$, where t

is a regular parameter of R. We can see that $t^{-d_{0,1}}\tilde{\eta}_0\tilde{\eta}_1$ and $t^{-d_{0,2}}\tilde{\eta}_0\tilde{\eta}_2$ sit in $\mathcal{R}_2$ as well. Therefore, we have

$$\begin{aligned}\mathrm{Ind}(f,s) &\geq \mathrm{length}\big(\langle \tilde{\eta}_0{}^2, \tilde{\eta}_1{}^2, \tilde{\eta}_2{}^2, t^{-d_{0,1}}\tilde{\eta}_0\tilde{\eta}_1, t^{-d_{0,2}}\tilde{\eta}_0\tilde{\eta}_2, t^{-d_{1,2}}\tilde{\eta}_1\tilde{\eta}_2\rangle/M\big)\\ &= \mathrm{length}\big(\langle t^{-d_{0,1}}\tilde{\eta}_0\tilde{\eta}_1, t^{-d_{0,2}}\tilde{\eta}_0\tilde{\eta}_2, t^{-d_{1,2}}\tilde{\eta}_1\tilde{\eta}_2\rangle/\langle \tilde{\eta}_0\tilde{\eta}_1, \tilde{\eta}_0\tilde{\eta}_2, \tilde{\eta}_1\tilde{\eta}_2\rangle\big)\\ &= d_{0,1} + d_{0,2} + d_{1,2}\\ &= 2\delta_1(\mathcal{C}_s) + h([\bar{G}_s]),\end{aligned}$$

where $\langle * \rangle$ stands for the submodule of $\mathcal{R}_2$ generated by $*$. Thus, we obtain our inequality in this case.

If $\bar{\mathcal{C}}_s$ does not have a pair of h-type, we can prove our inequality in the same way using a basis of $\mathcal{R}_1$ consisting of those which are stepwise extensions of global sections of the dualizing sheaf of the quasi-irreducible components of positive genus. That is rather simpler and we leave it to readers. □

Let us move on to the global case. Let X be a smooth projective surface over k, Y a smooth projective curve over k, and let $f : X \to Y$ be a generically smooth non-hyperelliptic semistable curve of genus 3 over Y. For any $y \in Y$, let $S := \operatorname{Spec} \mathcal{O}_{Y,y} \to Y$ be the morphism corresponding to the localization at y and let $f_S : X_S \to S$ denote the base change of f by this morphism. Then, Horikawa index of f at y is equal to that of f_S at the closed point of S by its definition. Accordingly, the following is immediate from Lemma 3.5 and Proposition 3.7.

Corollary 3.8. *If f is non-hyperelliptic, then we have*

$$(\omega_{X/Y} \cdot \omega_{X/Y}) \geq \frac{1}{3}\delta_0(X/Y) + 3\delta_1(X/Y) + \frac{4}{3}\sum_{y\in Y} h([\bar{G}_y]).$$

By virtue of Proposition 3.4, Corollary 3.8 and Key Fact in the introduction, we obtain Main Theorem.

4. Proof of Lemma 1.3

In this section, we will give proof of Lemma 1.3. First of all, we recall the situation of Lemma 1.3 and fix the notation. Let $[G]$ be a rigidified graph and let D be a polarization on $[G]$ with $\deg(D) \neq -2$. Numbering all the vertices and all the edges, we write $\{P_1, \ldots, P_m\} = \mathrm{Vert}[G]$ and $\{e_1, \ldots, e_n\} = \mathrm{Ed}[G]$, where m and n are the numbers of the vertices and that of the edges respectively. We assume that P_1 is a vertex on e_1. Let dm_G be a Lebesgue measure on G. We denote by $\bar{G}$ the metrized graph (G, dm_G). If a measure is given to a graph and the length of each edge e_i is l_i, then we can take a continuous map

$$t_i : I_i := [0, l_i] \to e$$

such that $t_i|_{(0,l_i)}$ is an isometry from $(0, l_i)$ to $e \setminus \{\text{the vertices on } e\}$. Here we assume that $t_1(0) = P_1$ for a notational reason. If we put $dy_i := t_i^* dm_G$, which is the canonical measure, then dm_G can be regarded as a collection $\{dy_i\}_{i=1}^n$.

Let $\pi : G \to H$ denote the contraction of e_1. Note that we have the canonical maps $\pi_* : \mathrm{Vert}[G] \to \mathrm{Vert}[H]$ and $\pi_* : \mathrm{Ed}[G] \setminus \{e_1\} \to \mathrm{Ed}[H]$. For P_i and e_j, we denote the images $\pi_*(P_i)$ and $\pi_*(e_j)$ by $\tilde{P}_i$ and $\tilde{e}_j$ respectively. H has the naturally induced measure dm_H by π. We also denote by $\tilde{t}_i$ the push-forward $\pi \circ t_i : I_i \to \tilde{e}_i$.

Here we introduce a new notation. For a $3n$-tuple $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) \in \mathbb{R}^{3n}$, we define a collection

$$\mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) = \{g_i\}_{i=1}^n$$

of quadratic functions on I_i by

$$g_i(y_i) := \alpha_i {y_i}^2 + \beta_i y_i + \gamma_i.$$

The members of $\mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n)$ can not be generally glued up to be a piecewise smooth function on $\bar{G}$, but if they are, we regard the collection of them as a function on G and write $\mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) \in F(\bar{G})$. Similarly, for a $3(n-1)$-tuple $(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n) \in \mathbb{R}^{3(n-1)}$, we define a collection

$$\mathcal{H}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n) := \{h_i\}_{i=2}^n$$

by

$$h_i(y_i) := \tilde{\alpha}_i y_i^2 + \tilde{\beta}_i y_i + \tilde{\gamma}_i.$$

We also write $\mathcal{H}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n) \in F(\bar{H})$ if they can be glued up to be a piecewise smooth function on $\bar{H}$.

The idea of the proof is simple. First we note that resistance is compatible with contractions: for any $1 \leq i, j \leq m$, we have

$$\lim_{l_1 \to 0} r_{\bar{G}}(P_i, P_j) = r_{\bar{H}}(\tilde{P}_i, \tilde{P}_j).$$

By [4, Lemma 4.1], therefore, it is enough to show

$$\lim_{l_1 \to 0} g_{(\bar{G},D)}(P_1, Q) = g_{(\bar{H},\tilde{D})}(\tilde{P}_1, \tilde{Q})$$

for any $Q \in \mathrm{Vert}[G]$. For this purpose, we would like to show that the function $g \circ t_i$ on I_i converges to $h \circ \tilde{t}_i$ for each $i > 1$, where we put $g(x) := g_{(\bar{G},D)}(P_1, x) - g_{(\bar{G},D)}(P_1, P_1)$ and $h(x) := g_{(\bar{H},\tilde{D})}(\tilde{P}_1, x) - g_{(\bar{H},\tilde{D})}(\tilde{P}_1, \tilde{P}_1)$. Since we know that $g \circ t_i$ and $h \circ \tilde{t}_i$ are quadratic functions on I_i, we can write

$$\begin{aligned} \{g \circ t_i\}_{i=1}^n &= \mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) \\ \{h \circ \tilde{t}_i\}_{i=2}^n &= \mathcal{H}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n). \end{aligned}$$

In the proof below, expressing the condition that they are Green's functions by matrices, we shall see that the $3(n-1)$-tuple $(\alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_n, \gamma_2, \ldots, \gamma_n)$ converges to $(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)$ as $l_1 \to 0$.

Let us start the proof. By virtue of Proposition 1.2, it is enough to prove our lemma for irreducible rigidified graphs, hence we assume the irreducibility of $[G]$. By the explicit formula [10, Lemma 3.7], we can see that if we write

$$\mu_{(\bar{G},D)} = \sum_{i=1}^{m} a_i \delta_{P_i} + \sum_{j=1}^{n} c_j \left(dm_G|_{e_j}\right),$$

then we have

$$\mu_{(\bar{H},\tilde{D})} = \sum_{i=1}^{m} a_i \delta_{\tilde{P}_i} + \sum_{j=2}^{n} \lim_{l_1 \to 0} c_j \left(dm_G|_{e_j}\right).$$

We put

$$\boldsymbol{c}(l_1) := (c_1, \ldots, c_n, a_1 - 1, a_2, \ldots, a_m, 0, \ldots, 0) \in \mathbb{R}^{3n}.$$

For $1 \le i \le m$ and $1 \le j \le n$, let $\sigma_{i,j}$ and $\tau_{i,j}$ be signatures defined in the following way.

$$\sigma_{i,j} = \begin{cases} 1 & \text{if} \quad t_j(0) = P_i, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_{i,j} = \begin{cases} 1 & \text{if} \quad t_j(l_j) = P_i, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $(\sigma_{i,j}, \tau_{i,j}) \neq (1,1)$ under the assumption of the irreducibility of $[G]$. The following matrices can be properly defined even if it is reducible, but the description will be more complicated.) First, let M' be an $(m, 2n)$-matrix defined as follows: for any i with $1 \le i \le m$, when $1 \le j \le n$, put

$$m'_{i,j} = -\tau_{i,j} \cdot 2l_j,$$

and when $n+1 \le j \le 2n$, put

$$m'_{i,j} = \sigma_{i,j-n} - \tau_{i,j-n}.$$

Second, for $1 \le k \le m$, we define a $(b_k - 1, 3n)$-matrix

$$M^{(k)} = \left(m^{(k)}_{i,j}\right)_{2 \le i \le b_k,\ 1 \le j \le 3n}.$$

Let $\left\{e_{\kappa_{k,1}}, \ldots, e_{\kappa_{k,b_k}}\right\}$ be the edges starting from P_k, where $\kappa_{k,1} < \cdots < \kappa_{k,b_k}$. When $1 \le j \le n$, put

$$m^{(k)}_{i,j} = \begin{cases} -\tau_{k,\kappa_{k,1}} {l_{\kappa_{k,1}}}^2 & \text{if} \quad j = \kappa_{k,1}, \\ \tau_{k,\kappa_{k,i}} {l_{\kappa_{k,i}}}^2 & \text{if} \quad j = \kappa_{k,i}, \\ 0 & \text{otherwise,} \end{cases}$$

when $n+1 \le j \le 2n$, put

$$m^{(k)}_{i,j} = \begin{cases} -\tau_{k,\kappa_{k,1}} l_{\kappa_{k,1}} & \text{if} \quad j - n = \kappa_{k,1}, \\ \tau_{k,\kappa_{k,1}} l_{\kappa_{k,i}} & \text{if} \quad j - n = \kappa_{k,i} \quad \text{and} \quad i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

and when $2n+1 \le j \le 3n$, put

$$m_{i,j}^{(k)} = \begin{cases} -1 & \text{if} \quad j-2n = \kappa_{k,1}, \\ 1 & \text{if} \quad j-2n = \kappa_{k,i} \quad \text{and} \quad i > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using these matrices defined above, we put

$$M(l_1) := \left(\begin{array}{c|c} 2I_n & 0 \\ \hline \multicolumn{1}{c}{M'} & \multicolumn{1}{|c}{0} \\ \hline \multicolumn{2}{c}{M^{(1)}} \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{M^{(m)}} \end{array} \right),$$

where I_n is the unit matrix of size n. We indicates by "(l_1)" that it is a matrix with a parameter l_1.

We can check straightforwardly that

$$\begin{cases} \mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) \in F(G), \\ \Delta(\mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n)) = \delta_{P_1} - \mu_{(\bar{G},D)}, \\ \mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n)(P_1) = 0, \end{cases}$$

i.e., $\mathcal{G}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) = g_{(\bar{G},D)}(P_1, x) - g_{(\bar{G},D)}(P_1, P_1)$ (via the identification of such a collection of functions with a function on G), if and only if

$$(1) \qquad \begin{cases} M(l_1)\ {}^t(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n) = {}^t\boldsymbol{c}(l_1), \\ \gamma_1 = 0. \end{cases}$$

(In checking that, we find that the matrix $(2I_n, 0)$ in $M(l_1)$ gives the condition on the second-order differential equation, the matrix $(M', 0)$ gives that on the first-order one, and each $M^{(i)}$ gives the continuity at P_i.) The unique existence of Green's function implies that (1) has a unique solution. Now suppose

$$\begin{cases} \mathcal{H}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n) \in F(H), \\ \Delta(\mathcal{H}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)) = \delta_{\tilde{P}_1} - \mu_{(\bar{H},\tilde{D})}, \\ \mathcal{H}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n, \tilde{\beta}_2, \ldots, \tilde{\beta}_n, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)(\tilde{P}_1) = 0. \end{cases}$$

Then, if we put

$$\begin{aligned} \tilde{\alpha}_1 &:= \frac{1}{2} \lim_{l_1 \to 0} c_1, \\ \tilde{\beta}_1 &:= -\sum_{j>1} \sigma_{1,j} \tilde{\beta}_j - \sum_{j>1} \bigl(-\tau_{1,j}(2\tilde{\alpha}_j l_j + \tilde{\beta}_j)\bigr) + a_1 - 1, \\ \tilde{\gamma}_1 &:= 0, \end{aligned}$$

we can check

$$\begin{cases} M(0)\ {}^t(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n, \tilde{\beta}_1, \dots, \tilde{\beta}_n, \tilde{\gamma}_1, \dots, \tilde{\gamma}_n) = {}^t\boldsymbol{c}(0), \\ \tilde{\gamma}_1 = 0. \end{cases}$$

Conversely, if

$$\begin{cases} M(0)\ {}^t(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n, \tilde{\beta}_1, \dots, \tilde{\beta}_n, \tilde{\gamma}_1, \dots, \tilde{\gamma}_n) = {}^t\boldsymbol{c}(0), \\ \tilde{\gamma}_1 = 0, \end{cases} \tag{2}$$

then we have

$$\begin{cases} \mathcal{H}(\tilde{\alpha}_2, \dots, \tilde{\alpha}_n, \tilde{\beta}_2, \dots, \tilde{\beta}_n, \tilde{\gamma}_2, \dots, \tilde{\gamma}_n) \in F(H), \\ \Delta(\mathcal{H}(\tilde{\alpha}_2, \dots, \tilde{\alpha}_n, \tilde{\beta}_2, \dots, \tilde{\beta}_n, \tilde{\gamma}_2, \dots, \tilde{\gamma}_n)) = \delta_{\tilde{P}_1} - \mu_{(\bar{H}, \tilde{D})}, \\ \mathcal{H}(\tilde{\alpha}_2, \dots, \tilde{\alpha}_n, \tilde{\beta}_2, \dots, \tilde{\beta}_n, \tilde{\gamma}_2, \dots, \tilde{\gamma}_n)(\tilde{P}_1) = 0. \end{cases}$$

From the unique existence of Green's function on $\bar{H}$, we can see that the system of linear equations (2) has a unique solution. Therefore, the solution of (1) converges to the solution of (2), which implies the convergence of Green's function on each edges. Thus, we obtain

$$\lim_{l_1 \to 0} g_{(\bar{G}, D)}(P_1, Q) = g_{(\bar{H}, \tilde{D})}(\tilde{P}_1, \tilde{Q})$$

for any $Q \in \mathrm{Vert}[G]$ and achieve the required conclusion.

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