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On the cohomology mod 2 of E_8 II

By

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Abstract

A quite simple method for determining the mod 2 cohomology of the compact, connected, simple, exceptional Lie group of rank 8 is given.

1. Introduction

In [7], the first named author gave a simple method for determining $H^*(\boldsymbol{E_8};\mathbb{F}_2)$ as an algebra over $\mathcal{A}_2$, the mod 2 Steenrod algebra. In Lemma 2.2 of [7], he used a closed connected subgroup $\boldsymbol{U}$ of local type $\boldsymbol{A_8}$ to prove $t_2^8 \neq 0$ where t_2 is the generator of $H^2(\Omega\boldsymbol{E_8};\mathbb{F}_2)$. Then, in Section 3 of [7], he used a closed connected subgroup $\boldsymbol{V}$ of local type $\boldsymbol{D_8}$ and the result of Floyd [6] about the fixed point set of a compact $\boldsymbol{Z}/2$-space to determine $H^*(\boldsymbol{E_8};\mathbb{F}_2)$.

The purpose of this paper is to give a method alternative to that in Section 3 of [7]. The method of this paper is simpler in the sense that we only use quite elementary facts and methods of homotopy theory. We use the natural map $Bi\colon B\boldsymbol{U} \to B\boldsymbol{E_8}$ and the Wu formula to eliminate the possibility that

$$H^*(\boldsymbol{E_8};\mathbb{F}_2) = \frac{\mathbb{F}_2[x_3,x_5,x_9]}{(x_3^{16},x_5^8,x_9^4)} \otimes \wedge(x_{15},x_{17},x_{23},x_{27},x_{59}),$$

where $i\colon \boldsymbol{U} \hookrightarrow \boldsymbol{E_8}$ is the inclusion.

2. A new method for determining $H^*(\boldsymbol{E_8};\mathbb{F}_2)$

We follow the notation and terminology of [7]. The subscript of an element of an algebra designates the degree unless otherwise stated.

Let $i\colon \boldsymbol{U} \to \boldsymbol{E_8}$ and $k\colon \tilde{\boldsymbol{E}}_{\boldsymbol{8}} \to \boldsymbol{E_8}$ be as in [7]. Let $g \in H^4(B\boldsymbol{E_8};\boldsymbol{Z}) \cong [B\boldsymbol{E_8}, K(\boldsymbol{Z},4)] \cong \boldsymbol{Z}$ be a generator. Let $B\tilde{\boldsymbol{E}}_{\boldsymbol{8}}$ and $B\tilde{\boldsymbol{U}}$ be the homotopy fibres of g and $g\circ(Bi)$ respectively. Then, we have the following homotopy commutative diagram

$$\begin{tikzcd}
\boldsymbol{E_8}/\boldsymbol{U} \arrow[r] \arrow[d, equal] & B\tilde{\boldsymbol{U}} \arrow[r, "B\tilde{i}"] \arrow[d, "Bk'"'] & B\tilde{\boldsymbol{E}}_{\boldsymbol{8}} \arrow[d, "Bk"] \\
\boldsymbol{E_8}/\boldsymbol{U} \arrow[r] & B\boldsymbol{U} \arrow[r, "Bi"'] & B\boldsymbol{E_8}
\end{tikzcd}$$

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(see [7]), where Bk and Bk' are the projections, $B\tilde{i}$ is a lift of $(Bi)\circ(Bk')$, and horizontal sequences are homotopy fiberings. Recall that $H^*(\boldsymbol{E_8}/\boldsymbol{U})$ is concentrated in even degrees.

By Lemma 2.2 of [7], we have $H^*(\Omega\tilde{\boldsymbol{E}}_{\boldsymbol{8}}) = \Delta(w_{14}, w_{22}, w_{26}, w_{28})$ for $* \le 30$. If $w_{14}^2 = w_{28}$, we can determine $H^*(\boldsymbol{E_8})$ as an algebra over $\mathcal{A}_2$ as desired by the argument of [7], p. 279. So, in the rest of this paper, we assume $w_{14}^2 = 0$ and deduce a contradiction.

We can see $H^*(\tilde{\boldsymbol{E}}_{\boldsymbol{8}}) = \wedge(y_{15}, y_{23}, y_{27})$ for $* \le 31$ and $H^*(B\tilde{\boldsymbol{E}}_{\boldsymbol{8}}) = \mathbb{F}_2[\alpha_{16}, \alpha_{24}, \alpha_{28}]$ for $* \le 32$. Recall that $H^*(B\boldsymbol{U}) = \mathbb{F}_2[c_2, c_3, \dots, c_9]$ where c_j is the j-th Chern class. (Note that $\deg c_j = 2j$.) Then, we have $H^*(B\tilde{\boldsymbol{U}}) = \mathbb{F}_2[u_6, u_{10}, u_{18}, \tilde{c}_4, \tilde{c}_6, \tilde{c}_7, \tilde{c}_8]$ for $* \le 32$, where $\tilde{c}_j = (Bk')^*(c_j)$ and $\mathrm{Sq}^4 u_6 = u_{10}$. We may put

$$(B\tilde{i})^*(\alpha_{16}) = p\tilde{c}_8 + q\tilde{c}_4^2 + \delta u_6 u_{10},$$

where p, q, and $\delta \in \mathbb{F}_2$. Note that p, q, or $\delta \ne 0$ since otherwise, α_{16} gives rise to a nonzero element of $H^{15}(\boldsymbol{E_8}/\boldsymbol{U}) = 0$. Applying Sq^4 to $(B\tilde{i})^*(\alpha_{16})$, we have $\delta = 0$. Then, applying $\mathrm{Sq}^4\mathrm{Sq}^8$, we have

$$(1) \qquad \begin{aligned}(B\tilde{i})^*(\mathrm{Sq}^4\mathrm{Sq}^8\alpha_{16}) &= \mathrm{Sq}^4(p\tilde{c}_8\tilde{c}_4 + q\tilde{c}_6^2)\\ &= p\tilde{c}_8\tilde{c}_6 + q\tilde{c}_7^2 \ne 0\end{aligned}$$

and hence we have $\mathrm{Sq}^8\alpha_{16} = \alpha_{24}$ and $\mathrm{Sq}^4\alpha_{24} = \alpha_{28}$. Then, we have $\mathrm{Sq}^8 y_{15} = y_{23}$ and $\mathrm{Sq}^4 y_{23} = y_{27}$, and we can easily see that

$$H^*(\boldsymbol{E_8}) = \frac{\mathbb{F}_2[x_3, x_5, x_9]}{(x_3^{16}, x_5^8, x_9^4)} \otimes \wedge(x_{15}, x_{17}, x_{23}, x_{27}, x_{59}),$$

where $\mathrm{Sq}^2 x_3 = x_5, \mathrm{Sq}^4 x_5 = x_9, \mathrm{Sq}^8 x_9 = x_{17}, \mathrm{Sq}^8 x_{15} = x_{23}$ and $\mathrm{Sq}^4 x_{23} = x_{27}$.

Note that we may assume x_{15} is primitive. Since if this is not the case, we can easily see that we may assume $\bar{\phi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5 + x_3^4 \otimes x_3$, where $\bar{\phi}$ is the reduced coproduct of $H^*(\boldsymbol{E_8})$. It follows, however, that $\bar{\phi}(x_{15}^2) \ne 0$, which is a contradiction.

Moreover, x_{15} is universally transgressive. Since if this is not the case, we can easily see that $H^*(B\boldsymbol{E_8}) = \mathbb{F}_2[z_4, z_6, z_7, z_{10}, z_{11}, z_{13}, z_{18}]/(z_4 z_6 z_7)$ for $* \le 18$. It follows, however, that $0 = \mathrm{Sq}^1(z_4 z_6 z_7) = z_4 z_7^2$, which is a contradiction.

Thus, x_{23} and x_{27} are also universally transgressive and hence

$$H^*(B\boldsymbol{E_8}) = \mathbb{F}_2[z_j, j = 4, 6, 7, 10, 11, 13, 16, 18, 19, 21, 24, 25, 28]$$

for $* < 60$ where $(Bi)^*(z_{\mathrm{odd}}) = 0, (Bi)^*(z_4) = c_2, (Bi)^*(z_6) = c_3 = \mathrm{Sq}^2 c_2$, $(Bi)^*(z_{10}) = c_5 = \mathrm{Sq}^4 c_3 + c_3 c_2, (Bi)^*(z_{18}) = c_9 + c_7 c_2 + c_6 c_3 + c_5 c_4 = \mathrm{Sq}^8 c_5$, and $(Bk)^*(z_j) = \alpha_j$ for $j = 16, 24, 28$. Moreover we may assume

$$(Bi)^*(z_{16}) = pc_8 + qc_4^2 + rc_6 c_2 + sc_4 c_2^2,$$

$z_{24} = \mathrm{Sq}^8 z_{16} + z_{16} z_4^2$, and $z_{28} = \mathrm{Sq}^4 z_{24}$, where p and q are those stated before and $r, s \in \mathbb{F}_2$. Recall that p or $q \ne 0$.

Applying Sq^2 to (1), we have $(B\tilde{i})^*(\mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^8\alpha_{16}) = p\tilde{c}_8\tilde{c}_7$, while clearly $\mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^8\alpha_{16} = 0$ by the dimensional reason. Hence we have $p = 0$ and $q = 1$. We have $(Bi)^*(\mathrm{Sq}^2 z_{16}) = r(c_7c_2 + c_6c_3) + sc_5c_2^2$. It follows that $\mathrm{Sq}^2 z_{16}$ is decomposable and hence $r = 0$. Moreover, we have $(Bi)^*(\mathrm{Sq}^4 z_{16}) = c_5^2 + s(c_6c_2^2 + c_4c_2^3 + c_4c_3^2)$. Since $\mathrm{Sq}^4 z_{16}$ is decomposable, $s = 0$.

Thus, we have $(Bi)^*(z_{16}) = c_4^2$ and hence $(Bi)^*(z_{24}) = c_6^2$ and $(Bi)^*(z_{28}) = c_7^2$. We can see that $(Bi)^*(\mathrm{Sq}^{16} z_{24}) = c_8^2c_2^2 + c_6^2c_4^2$. Since $\mathrm{Sq}^{16} z_{24}$ is decomposable, this is a contradiction.

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