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Spaces of polynomials without 3-fold real roots

By

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Abstract

Let $\mathrm{P}^d_n(\mathbb{R})$ denote the space consisiting of all monic polynomials $f(z) \in \mathbb{R}[z]$ of degree d which have no real roots of multplicity $\geq n$. In this paper we study the homotopy types of the spaces $\mathrm{P}^d_n(\mathbb{R})$ for the case $n = 3$.

1. Introduction

Let $\mathrm{P}^d_n(\mathbb{R})$ denote the space consisting of all monic real coefficients polynomials

$$f(z) = z^d + a_1 z^{d-1} + a_2 z^{d-2} + \cdots + a_{d-1} z + a_d \in \mathbb{R}[z],$$

which have no *real* roots of multiplicity $\geq n$ (but may have any complex roots of any multiplicity).

Let us consider the jet embedding $j^d_n : \mathrm{P}^d_n(\mathbb{R}) \to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{n-1} \simeq \Omega S^{n-1}$ given by

$$j^d_n(f)(t) = \begin{cases} [f(t) : f'(t) : \cdots : f^{(n-1)}(t)] & \text{if } t \in \mathbb{R} \\ [1 : 0 : 0 : \cdots\cdots\cdots : 0 : 0] & \text{if } t = \infty \end{cases} \qquad \text{for } t \in \mathbb{R} \cup \infty = S^1.$$

(Here $[d]_2 = 0$ or 1 according as d is even or odd.) For a connected CW complex X, let X_∞ denote the free monoid generated by $X - \{*\}$ with unit $* \in X$, where $* \in X$ is a fixed basepoint. It is well-known [8] that there is a natural homotopy equivalence $X_\infty \simeq \Omega\Sigma X$ and it is usually called the reduced product of X. For example, if $X = S^m$, the space ΩS^{m+1} may be identified with $(S^m)_\infty$ and it has the cell structure of the form

$$\Omega S^{m+1} \simeq (S^m)_\infty = S^m \cup e^{2m} \cup e^{3m} \cup e^{4m} \cup \cdots \cup e^{km} \cup e^{(k+1)m} \cup \cdots .$$

We denote by $J_k(\Omega S^{m+1})$ the James k-th stage filtration of $\Omega S^{m+1} \simeq (S^m)_\infty$,

$$J_k(\Omega S^{m+1}) = S^m \cup e^{2m} \cup e^{3m} \cup \cdots \cup e^{(k-1)m} \cup e^{km} \qquad (km\text{-skelton of } \Omega S^{m+1}).$$

We recall the following result.

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Theorem 1.1 ([10], [13], [16]).

(1) *If* $n \geq 3$, *the jet embedding* $j_n^d : \mathrm{P}_n^d(\mathbb{R}) \to \Omega S^{n-1}$ *is* $N(d,n)$*-connteced, where* $[x]$ *denotes the integer part of a real number* x *and we take* $N(d,n) = ([d/n]+1)(n-2)-1$.

(2) *In particular, when* $n \geq 4$, *there is a homotopy equivalence* $\mathrm{P}_n^d(\mathbb{R}) \simeq J_{[d/n]}(\Omega S^{n-1})$.

Remark. We say that a map $f : X \to Y$ is D-connected if the induced homomorphism $f_* : \pi_j(X) \to \pi_j(Y)$ is bijective when $j < D$ and surjective when $j = D$.

The homotopy type of $\mathrm{P}_n^d(\mathbb{R})$ is trivial when $n = 2$ and is well studied for the case $n \geq 4$. So in this paper we shall consider the case $n = 3$. Then it follows from the above theorem that $\mathrm{P}_3^d(\mathbb{R})$ and $J_{[d/3]}(\Omega S^2)$ may be homotopy equivalent and the authors submit the following problem.

Problem A. Does there exists a homotopy equivalence

$$\mathrm{P}_3^d(\mathbb{R}) \xrightarrow{\simeq} J_{[d/3]}(\Omega S^2)?$$

Of corse, Problem A clearly holds for $d = 1, 2$ and the main purpose of this paper is just to investigate whether such a homotopy equivalence exists or not if $d \geq 3$. The main result of this paper is as follows:

Theorem 1.2. *If* $d \leq 17$, *there exists a homotopy equivalence* $\mathrm{P}_3^d(\mathbb{R}) \simeq J_{[d/3]}(\Omega S^2)$. *So Problem* A *is affirmative when* $1 \leq d \leq 17$.

Although we do not know whether Problem A is true when $d \geq 18$, we can give the partial answer to this problem as follows.

Proposition 1.3. *There is a map* $f_d : \mathrm{P}_3^d(\mathbb{R}) \to J_{[d/3]}(\Omega S^2)$ *satisfying the following two conditions*

(i) $(f_d)_* : H_j(\mathrm{P}_3^d(\mathbb{R}), \mathbb{Z}) \xrightarrow{\cong} H_j(J_{[d/3]}(\Omega S^2), \mathbb{Z})$ *is an isomorphism for any integer* j.

(ii) f_d *is* $[d/3]$*-connected.*

The plan of this paper is as follows. In Section 2, we consider the subspace $\mathrm{P}_3^d \subset \mathrm{P}_3^d(\mathbb{R})$ and give the proof of Theorem 1.2 for $3 \leq d \leq 8$. In Section 3, we complete the proof of Theorem 1.2, and in Section 4 we prove Proposition 1.3.

2. The subspace P_3^d

Let P_3^d denote the subspace of $\mathrm{P}_3^d(\mathbb{R})$ consisting of all monic polynomials of the following form

$$f(z) = z^d + a_2 z^{d-2} + a_3 z^{d-3} + \cdots + a_{d-1} z + a_d \in \mathrm{P}_3^d(\mathbb{R}).$$

That is, if $f(z) \in \mathrm{P}_3^d(\mathbb{R})$, $f(z) \in \mathrm{P}_3^d$ if and only if its coefficient of z^{d-1} is zero. First, we recall the following two results.

Lemma 2.1 ([13]). *There is a deformation retract* $\mathrm{P}^d_3 \simeq \mathrm{P}^d_3(\mathbb{R})$.

Theorem 2.2 ([7], [10], [13]).

(1) *If* $d \geq 3$, *the induced homomorphism* $(j^d_3)_* : \pi_1(\mathrm{P}^d_3(\mathbb{R})) \xrightarrow{\cong} \pi_1(\Omega S^2) = \mathbb{Z}$ *is isomorphic.*

(2) *The stabilized map* $\lim_d j^d_3 : \lim_{d\to\infty} \mathrm{P}^d_3(\mathbb{R}) \xrightarrow{\simeq} \Omega S^2$ *is a homotopy equivalence.*

(3)

$$H^j(P^d_3(\mathbb{R}), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j = 0, 1, 2, 3, \dots, [d/3], \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, if $e_j \in H^j(\mathrm{P}^d_3(\mathbb{R}), \mathbb{Z}) \cong \mathbb{Z}$ *denotes the generator for* $1 \leq j \leq [d/3]$,

$$\begin{cases} (e_1)^2 = 0, \quad e_1 \cdot e_{2i} = e_{2i+1} & \textit{if} \quad 2i + 1 \leq [d/3], \\ e_{2i} \cdot e_{2j} = \binom{i+j}{i} e_{2(i+j)} & \textit{if} \quad 2(i + j) \leq [d/3]. \end{cases}$$

Let P^d denote the space consisting of all monic real coefficients polynomials

$$f(z) = z^d + a_2 z^{d-2} + a_3 z^{d-3} + \cdots + a_d \in \mathbb{R}[z]$$

of degree d. So there is a homeomorphism

$$\begin{array}{ccc} \mathrm{P}^d & \xrightarrow{\cong} & \mathbb{R}^{d-1} \\ f(z) = z^d + \sum_{j=2}^d a_j z^{d-j} & \longrightarrow & (a_2, a_3, \dots, a_d). \end{array}$$

We denote by Σ^d_3 the subspace of P^d consisting of all $f(z) \in \mathrm{P}^d$ such that $f(z)$ have at least one real root of multiplicity $\geq n$. Then $\mathrm{P}^d_3 = \mathrm{P}^d - \Sigma^d_3$.

Lemma 2.3. *There is a homotopy equivalence* $\mathrm{P}^d_3 \simeq S^1$ *for* $d = 3, 4, 5$.

Proof. Since the proof is similar, we prove the assertion only for the case $d = 4$. We note $\Sigma^4_3 = \{(z + \alpha)^3(z + A) : \alpha, A \in \mathbb{R}\}$ and that

$$z^4 + az^2 + bz + c = (z + \alpha)^3(z + A) \Leftrightarrow (A, a, b, c) = (-3\alpha, -6\alpha^2, -8\alpha^3, -3\alpha^4).$$

Hence $\Sigma^4_3 = \{z^4 - 6\alpha^2 z^2 - 8\alpha^3 z - 3\alpha^4 : \alpha \in \mathbb{R}\}$ and so

$$\begin{aligned} \mathrm{P}^4_3 = \mathrm{P}^4 - \Sigma^4_3 &\cong \mathbb{R}^3 - \{(-6\alpha^2, -8\alpha^3, -3\alpha^4) : \alpha \in \mathbb{R}\} \\ &\cong \mathbb{R}^3 - \mathbb{R} \simeq \mathbb{R}^2 - \{(0, 0)\} \simeq S^1. \end{aligned}$$

This completes the proof. □

Next, we recall the following result.

Lemma 2.4 (Y. G. Makhlin (1990), [13]). *There is a homotopy equivalence* $\mathrm{P}^d_3(\mathbb{R}) \simeq S^1 \vee S^2$ *for* $6 \leq d \leq 8$.

Because $\Omega S^2 \simeq S^1 \times \Omega S^3$, we may take $J_1(\Omega S^2) = S^1$ and $J_2(\Omega S^2) = S^1 \vee S^2$. So by Lemmas (2.2) and (2.3), we also obtain the following result.

Corollary 2.5. *If $1 \leq d \leq 8$, there exists a homotopy equivalence $\mathrm{P}^d_3(\mathbb{R}) \simeq J_{[d/3]}(\Omega S^2)$.*

3. The case $9 \leq d \leq 17$

First we recall the following two results.

Lemma 3.1 ([16]). *Let $s^d_3 : \mathrm{P}^d_3(\mathbb{R}) \to \mathrm{P}^{d+1}_3(\mathbb{R})$ denote the stabilization map given by adding a point from the edge as in* [16]. *Then the map $s^d_3 : \mathrm{P}^d_3(\mathbb{R}) \to \mathrm{P}^{d+1}_3(\mathbb{R})$ is $[d/3]$-connected.*

Lemma 3.2. *The fundamental group action on $\pi_k(\mathrm{P}^d_3(\mathbb{R}))$ is trivial for any $k < [d/3]$.*

Proof. This easily follows from (1) of Theorem 1.1. □

Proposition 3.3. *If $9 \leq d \leq 11$, there is a homotopy equivalence $\mathrm{P}^d_3(\mathbb{R}) \simeq S^1 \times S^2$.*

Proof. First consider the case $d = 9$. We note that

$$\begin{cases} H^*(\mathrm{P}^9_3(\mathbb{R});\mathbb{Z}) = E[e_1, e_2] \quad \text{(exterior algebra)}, \\ \text{where} \quad \deg(e_i) = i \quad (i = 1, 2) \quad \text{and} \quad \pi_1(\mathrm{P}^9_3(\mathbb{R})) = \mathbb{Z}. \end{cases}$$

Since $S^1 \vee S^2 \simeq \mathrm{P}^8_3(\mathbb{R}) \xrightarrow{s^8_3} \mathrm{P}^9_3(\mathbb{R})$ is 2-connected, it follows from Lemma 3.2 that there is a homotopy equivalence $\mathrm{P}^9_3(\mathbb{R}) \simeq S^1 \vee S^2 \cup_f e^3 = C_f$ for some $f \in \pi_2(S^1 \vee S^2) = \mathbb{Z}[t, t^{-1}]$, where $S^1 \xrightarrow{j_1} S^1 \vee S^2 \xleftarrow{j_2} S^2$ denote the inclusion maps and $[j_1, j_2] = t - 1$ ([14]).

Without loss of generalities, we may identify $\mathrm{P}^9_3(\mathbb{R}) = C_f = S^1 \vee S^2 \cup_f e^3$. We denote by the map $j : S^1 \vee S^2 \to \mathrm{P}^d_3(\mathbb{R})$ the natural inclusion. Consider the homotopy exact sequence

$$\begin{array}{ccccccc} \pi_3(\mathrm{P}^9_3(\mathbb{R}), S^1 \vee S^2) & \xrightarrow{\partial} & \pi_2(S^1 \vee S^2) & \xrightarrow{j_*} & \pi_2(\mathrm{P}^9_3(\mathbb{R})) & \longrightarrow & 0, \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ \mathbb{Z}[t, t^{-1}] \cdot \alpha_3 & & \mathbb{Z}[t, t^{-1}] & & \mathbb{Z}[t, t^{-1}]/(f) & & \end{array}$$

where $\alpha_3 : (D^3, S^2) \to (\mathrm{P}^9_3(\mathbb{R}), S^1 \vee S^2)$ denotes the characteristic map of the top cell and ∂ is a $\mathbb{Z}[t, t^{-1}]$-module homomorphism.

Since $\partial(\alpha_3) = f$, $\pi_2(\mathrm{P}^6_3(\mathbb{R})) = \mathbb{Z}[t, t^{-1}]/(f)$, where (f) denotes the ideal of $\mathbb{Z}[t, t^{-1}]$ generated by f. Moreover, because $(j^9_3)_*; \pi_2(\mathrm{P}^9_3(\mathbb{R})) \xrightarrow{\cong} \pi_2(\Omega S^2)$ is an isomorphism, π_1 acts on $\pi_2(\mathrm{P}^6_3(\mathbb{R}))$ trivially. Hence $j_*(t-1) = 0 \in \pi_2(\mathrm{P}^9_3(\mathbb{R}))$.

So $[j_1, j_2] = t-1 \in (f)$. Since clearly $f \neq 0$, $f = \pm t^m(t-1) = \pm t^m[j_1, j_2]$ for some integer $m \in \mathbb{Z}$.

We note that $\mathcal{E}(S^1 \vee S^2) \cong \{\pm 1\} \times \{\pm t^k : k \in \mathbb{Z}\}$, where $\mathcal{E}(X)$ denotes the group consisting of all homotopy classes of self-homotopy equivalences of a connected space X.

Then there is a homotopy equivalence $\theta \in \mathcal{E}(S^1 \vee S^2)$ such that $\theta \circ (t-1) = \pm t^m(t-1) = f$. Hence $\mathrm{P}^9_3(\mathbb{R}) = C_f \simeq S^1 \vee S^2 \cup_{t-1} e^3 = S^1 \vee S^2 \cup_{[j_1,j_2]} e^3 = S^1 \times S^2$. Hence the case $d = 9$ is proved. A similar method also proves the case $d = 10$ or $d = 11$, and this completes the proof. □

Proposition 3.4. *If* $12 \leq d \leq 14$*, there exists a homotopy equivalence* $\mathrm{P}^d_3(\mathbb{R}) \simeq J_{[d/3]}(\Omega S^2)$.

Proof. Since the proof is almost same, we give the proof for the case $d = 12$. We note that $s^{11}_3 : \mathrm{P}^{11}_3(\mathbb{R}) \to \mathrm{P}^{12}_3(\mathbb{R})$ is 3-connected and that $\mathrm{P}^{11}_3(\mathbb{R}) \simeq S^1 \times S^2$. Hence it follows from the cohomology structure of $H^*(\mathrm{P}^{12}_3(\mathbb{R}); \mathbb{Z})$ and Lemma 3.2 that there is a homotopy equivalecnce

$$\mathrm{P}^{12}_3(\mathbb{R}) \simeq (S^1 \times S^2) \cup_g e^4 = (S^1 \vee S^2 \cup_{[j_1,j_2]} e^3) \cup_g e^4$$

for some $g \in \pi_3(S^1 \times S^2) \cong \pi_3(S^2) = \mathbb{Z} \cdot \eta$, where $\eta \in \pi_3(S^2) \cong \mathbb{Z}$ denotes the Hopf map.

We recall the multiplicative stucture of $H^*(\mathrm{P}^{12}_3(\mathbb{R}); \mathbb{Z})$. Since $e_1 \cdot e_4 = e_5$ and $e_3 \cdot e_2 = 2e_5$, it follows from the Hopf invariant problem that there is a homotopy euivalence

$$\mathrm{P}^{12}_3(\mathbb{R}) \simeq (S^1 \times S^2) \cup \{*\} \times (S^2 \cup_{2\eta} e^4) \subset S^1 \times (S^2 \cup_{2\eta} e^4).$$

However, since there is a homotopy equivalence $\Omega S^2 \simeq S^1 \times \Omega S^3$, we may identify $J_4(\Omega S^2) \simeq (S^1 \times S^2) \cup (\{*\} \times (S^2 \cup_{2\eta} e^4))$. Hence there exists a homotopy equivalence $\mathrm{P}^{12}_3(\mathbb{R}) \simeq J_4(\Omega S^2)$. □

Proposition 3.5. *If* $15 \leq d \leq 17$*, there exists a homotopy equivalence* $\mathrm{P}^d_3(\mathbb{R}) \simeq J_5(\Omega S^2)$.

Proof. Because the proof is similar, it is sufficient to prove the case $d = 15$. Since $\Omega S^2 \simeq S^1 \times \Omega S^3$, we may identify $J_5(\Omega S^2) = S^1 \times J_2(\Omega S^3) = S^1 \times (S^2 \cup_{2\eta} e^4)$. So it suffces to show that there is a homotopy equivalence $\mathrm{P}^{15}_3(\mathbb{R}) \simeq S^1 \times (S^2 \cup_{2\eta} e^4)$.

First, we note that $s^{14}_3 : \mathrm{P}^{14}_3(\mathbb{R}) \to \mathrm{P}^{15}_3(\mathbb{R})$ is 4-connected. Hence it follows from the cohomology structure of $H^*(\mathrm{P}^{15}_3(\mathbb{R}), \mathbb{Z})$ and Lemma 3.2 that there exists a homotopy equivalence

$$\mathrm{P}^{15}_3(\mathbb{R}) \simeq J_4(\Omega S^2) \cup_h e^5 = \{(S^1 \times S^2) \cup (\{*\} \times (S^2 \cup_{2\eta} e^4))\} \cup_h e^5$$

for some $h \in \pi_4(J_4(\Omega S^2))$. Then because $e_1 \cdot e_4 = e_5$ and $e_2 \cdot e_3 = 2e_5$, using the solution of Hopf invariant one problem, there is a homotopy equivalence $\mathrm{P}^{15}_3(\mathbb{R}) \simeq S^1 \times (S^2 \cup_{2\eta} e^4)$ and this completes the proof. □

Proof of Theorem 1.2. The assertion easily follows from (2.5), (3.2), (3.3) and (3.4). □

Remark. We note that

$$J_m(\Omega S^2) = \begin{cases} S^1 \times J_k(\Omega S^3) & \text{if} \quad m = 2k+1, \\ (S^1 \times J_{k-1}(\Omega S^3)) \cup (\{*\} \times J_k(\Omega S^3)) & \text{if} \quad m = 2k, \end{cases}$$

and it has the cell-decomposition $J_k(\Omega S^3) = S^2 \cup_{2\eta} \cup e^4 \cup_\phi e^6 \cup e^8 \cup \cdots \cup e^{2k}$ (up to homotopy). For example, the attaching map $\phi \in \pi_5(S^2 \cup_{2\eta} e^4)$ of the cell e^6 cannot be detected by the structure of the cohomology ring and primary operations. So the similar proof of Theorem 1.2 does not work for the space $J_m(\Omega S^2)$ if $m \geq 6$ and Problem A is still open if $d \geq 18$.

4. Proof of Proposition 1.3

In this section we give the proof of Proposition 1.3.

Proof of Proposition 1.3. First, we note that π_1-action on $\pi_k(\mathrm{P}^d_3(\mathbb{R}))$ is trivial for each $k < [d/3]$ and that $j^d_3 : \mathrm{P}^d_3(\mathbb{R}) \to \Omega S^2$ is $[d/3]$-connected. Hence it follows from the structure of $H^*(\mathrm{P}^d_3(\mathbb{R}); \mathbb{Z})$ that $\mathrm{P}^d_3(\mathbb{R})$ has the cell-decomposition $\mathrm{P}^d_3(\mathbb{R}) \simeq F^d = S^1 \cup e^2 \cup e^3 \cup \cdots \cup e^{[d/3]-1} \cup e^{[d/3]}$ (up to homotopy). Let $u^d : F^d \xrightarrow{\simeq} \mathrm{P}^d_3(\mathbb{R})$ be the corresponding homotopy equivalence. Now consider the composite of maps

$$j^d_3 \circ u^d : F^d \underset{\simeq}{\xrightarrow{u^d}} \mathrm{P}^d_3(\mathbb{R}) \xrightarrow{j^d_3} \Omega S^2.$$

Using the cellular approximation theorem, there exists a cellular map $h_d : F^d \to \Omega S^2$ such that h_d is homotopic to the map $j^d_3 \circ u^d$. Because h_d is a cellular map, we may identify it with the map $h_d : F^d \to J_{[d/3]}(\Omega S^2)$.

Let $f_d : \mathrm{P}^d_3(\mathbb{R}) \to J_{[d/3]}(\Omega S^2)$ be the map of composite

$$f_d = h_d \circ v^d : \mathrm{P}^d_3(\mathbb{R}) \underset{\simeq}{\xrightarrow{v^d}} F^d \xrightarrow{h_d} J_{[d/3]}(\Omega S^2),$$

where v^d denotes the homotopy inverse of u^d.

An easy diagram chasing shows that f_d satisfies the following two conditions

(i) f_d is $[d/3]$-connected, and

(ii) $(f_d)_* : H_j(\mathrm{P}^d_3(\mathbb{R}); \mathbb{Z}) \to H_j(J_{[d/3]}(\Omega S^2); \mathbb{Z})$ is bijective when $j < [d/3]$ and surjective when $j = [d/3]$.

However, since $H_{[d/3]}(\mathrm{P}^d_3(\mathbb{R}); \mathbb{Z}) = H_{[d/3]}(J_{[d/3]}(\Omega S^2); \mathbb{Z}) = \mathbb{Z}$, $(f_d)_*$ is bijective when $j = [d/3]$, too. Moreover, $H_j(\mathrm{P}^d_3(\mathbb{R}); \mathbb{Z}) = H_j(J_{[d/3]}(\Omega S^2); \mathbb{Z}) = 0$ for any $j > [d/3]$. Hence $(f_d)_*$ is bijective for any j. □

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