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A direct proof of Moriwaki's inequality for semistably fibered surfaces and its generalization

By

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Introduction

Let X be a nonsingular projective surface over an algebraically closed field k, Y a nonsingular projective curve over k, and let $f : X \to Y$ be a generically smooth semistable curve of genus $g \geq 2$. In the paper [4], Moriwaki proved an inequality

$$(8g+4)\deg(f_*\omega_{X/Y}) \geq g\delta_0(X/Y) + \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i(X/Y)$$

under the assumption that the characteristic of k is zero, where $\delta_i(X/Y)$ is the number of nodes of type i in all fibers.

The purpose of this paper is to give another proof of Moriwaki's inequality and generalize it. First of all, let us recall his original proof in order to contrast ours with it.

Let us pretend that the smooth fine moduli space of stable curves with the universal family $\pi : \mathcal{C} \to \overline{M}_g$ exists in order to make the explanation simple. Let δ_i be the reduced divisor on $\overline{M}_g$ corresponding to the locus of the singular fibers with nodes of type i. We set $\overline{M}_g^\circ := \overline{M}_g \setminus \mathrm{Sing}(\delta_1 + \cdots + \delta_{[g/2]})$, $\mathcal{C}^\circ := \pi^{-1}(\overline{M}_g^\circ)$ and $\pi^\circ := \pi|_{\mathcal{C}^\circ}$. By modifying the kernel of the evaluation homomorphism

$$(\pi^\circ)^*\left((\pi^\circ)_*\left(\omega_{\mathcal{C}^\circ/\overline{M}_g^\circ}\right)\right) \to \omega_{\mathcal{C}^\circ/\overline{M}_g^\circ}$$

along singular fibers on $\overline{M}_g^\circ$, he constructed in [3] a reflexive sheaf F on $\mathcal{C}$ with the following properties.

(a) F is locally free on $\mathcal{C}^\circ$.

(b) $F|_{\pi^{-1}(y)} = \mathrm{Ker}(H^0(\omega_{\pi^{-1}(y)}) \otimes_k \mathcal{O}_{\pi^{-1}(y)} \to \omega_{\pi^{-1}(y)})$ for each y corresponding to a smooth curve.

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(c) $\mathrm{dis}_{\mathcal{C}/\overline{M}_g}(F) = (8g+4)c_1(\pi_*(\omega_{\mathcal{C}/\overline{M}_g})) - g\delta_0 - \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i$.

If F were locally free on whole $\mathcal{C}$, then pulling it back to X by the induced morphism $X \to \mathcal{C}$, we would obtain a locally free sheaf on E such that the restriction of it to the geometric generic fiber coincides with the restriction of $\mathrm{Ker}(f^*f_*\omega_{X/Y} \to \omega_{X/Y})$ and that

$$\mathrm{dis}(E) = (8g+4)\deg(f_*\omega_{X/Y}) - g\delta_0(X/Y) - \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i(X/Y),$$

and hence, by the semistability of E on the generic fiber (cf. [5]) and the relative Bogomolov inequality (cf. [3, Theorem 2.2.1]), we would reach the non-negativity of $\mathrm{dis}(E)$. It is however not a locally free sheaf actually, so that he used another kind of positivity on divisors, namely, weak positivity. It is stronger than numerical effectivity, and has an advantage that the weak positivity of a divisor restricted to outside of a closed subset of codimension more than one implies that of the divisor before restricted (cf. [4, Proposition 1.4]). Accordingly, it is sufficient to show that $\mathrm{dis}_{\mathcal{C}^\circ/\overline{M}_g^\circ}(F|_{\mathcal{C}^\circ})$ is weakly positive. In fact, he invented big machinery of the Bogomolov-inequality-type (cf. [4, Corollary 2.5]), which tells us that the semistability of a locally free sheaf on the fiber at a point assure the weak positivity of the discriminant of it at that point. In this way, he achieved the conclusion that $\mathrm{dis}_{\mathcal{C}/\overline{M}_g}(F)$ is weakly positive over M_g (cf. [4, Theorem 3.2]), pulled it back to Y and obtained Moriwaki's inequality.

As we have seen, he studied a sheaf and its discriminant divisor on the moduli space. In contrast, we shall deal with a semistable curve $f : X \to Y$ directly and shall modify $\mathrm{Ker}(f^*f_*\omega_{X/Y} \to \omega_{X/Y})$ along all the singular fibers. We shall in fact show the following theorem (cf. Theorem 2.1), which immediately leads to Moriwaki's inequality in characteristic zero.

Theorem A ($\mathrm{char}(k) \geq 0$). *There exists a locally free sheaf E on X such that the restriction of E to each smooth fiber coincides with the restriction of* $\mathrm{Ker}(f^*f_*\omega_{X/Y} \to \omega_{X/Y})$ *and*

$$\mathrm{dis}(E) = (8g+4)\deg(f_*\omega_{X/Y}) - g\delta_0(X/Y) - \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i(X/Y).$$

Our new approach is advantageous to Moriwaki's one in some aspects. One is that it is elementary: we do not need big tool like that Moriwaki invented in [4] or even the moduli space of curves. The other one is that we can make information hiding behind complicated singular fibers contribute to the inequality. We shall actually construct a locally free sheaf on X with the discriminant of smaller degree than E in Theorem A, by further elementary transformations along singular fibers (cf. Theorem 3.2). As corollaries, we shall also obtain inequalities to which certain pairs of nodes contribute (cf. Corollaries 3.3 and 3.4). We cannot expect to obtain them through the former approach, for the information of singular fibers corresponding to points outside $\overline{M}_g^\circ$ is lost in the nature of things.

This paper is organized as follows. In Section 1, we list notations and conventions that will be used in our argument. In Section 2, we give proof of Theorem A. In Section 3, we construct a locally free sheaf that leads a sharper inequality. We give remarks on the quotients of fibered surfaces by finite groups in Appendix A.

Finally, the author would like to express sincere gratitude to Prof. Moriwaki, who gave him useful comments (cf. Remark 3.12).

1. Notations, conventions and general remarks

Throughout this paper, we fix an algebraically closed field k.

1.1. When we write $f : X \to Y$, X is a nonsingular projective surface over k, Y is a nonsingular projective curve over k, and f is a generically smooth semistable curve of genus $g \geq 2$. We denote the set of critical values of f by $\mathrm{CV}(f)$. For a given $y \in \mathrm{CV}(f)$, we denote by $f : \mathcal{X} \to \mathcal{Y}$ the base change of $f : X \to Y$ by the canonical morphism $\mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$ and also denote the special fiber of $\mathcal{X} \to \mathcal{Y}$ by X_y. We always let $t \in \mathcal{O}_{\mathcal{Y}}$ stand for its local regular parameter.

1.2. A *nodal curve* is a reduced connected projective curve over an algebraically closed field with at most ordinary double points as singularities. We denote the arithmetic genus of a nodal curve Z by $p_a(Z)$. In this paper, "genus" always means "arithmetic genus".

1.3. We denote by $\mathrm{Sing}(X_y)$ the set of singular points of a closed fiber X_y of f. We denote the type of x by $\mathrm{tp}(x)$. Further, we denote by $\mathrm{Sing}_+(X_y)$ the set of nodes of X_y of positive type. Moreover, we set $\delta_i(X_y)$, $\delta_+(X_y)$ and $\delta_i(X/Y)$ as follows:

$$\begin{aligned}\delta_i(X_y) &:= \text{the number of the nodes of type } i \text{ in } X_y,\\ \delta_+(X_y) &:= \text{the number of the nodes of positive type in } X_y,\\ \delta_i(X/Y) &:= \textstyle\sum_{y\in \mathrm{CV}(f)} \delta_i(X_y).\end{aligned}$$

1.4. We define the discriminant $\mathrm{dis}(E)$ of a locally free sheaf E of rank r on a nonsingular projective surface over k to be the degree of a cycle class $2rc_2(E) - (r-1)c_1(E)^2$.

1.5. We defined in [6] the notion of quasi-irreducible components of X_y, which appear as the connected components of the partial normalization of X_y at all the nodes of positive type. We denote by $\mathrm{QIrr}(X_y)$ the set of quasi-irreducible components of X_y, and further we fix the following notations:

$$\begin{aligned}\mathrm{QIrr}^0(X_y) &:= \{C \in \mathrm{QIrr}(X_y) \mid p_a(C) = 0\},\\ \mathrm{QIrr}^+(X_y) &:= \{C \in \mathrm{QIrr}(X_y) \mid p_a(C) > 0\}.\end{aligned}$$

Note that $\#(\mathrm{QIrr}^+(X_y)) = \delta_+(X_y) + 1$. We usually regard a member of them as a reduced vertical divisor of X or $\mathcal{X}$.

1.6. We denote by ρ the canonical homomorphism $f_*\omega_{\mathcal{X}/\mathcal{Y}} \to H^0\big(X_y, \omega_{X_y}\big)$. We call the restriction of ρ to a subsheaf of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$ the canonical restriction.

Since f is supposed to be semistable, ρ is surjective by the base-change theorem. For any vertical divisor D on $\mathcal{X}$, the vector space $H^0(D,\omega_D)$ can be regarded as a subspace of $H^0(X_y,\omega_{X_y})$ via

$$\omega_D \cong \omega_{\mathcal{X}/\mathcal{Y}}(D-X_y)|_D \to \omega_{\mathcal{X}/\mathcal{Y}}|_{X_y} \cong \omega_{X_y}.$$

Note that $H^0(X_y,\omega_{X_y}) = \bigoplus_{C\in \mathrm{QIrr}^+(X_y)} H^0(C,\omega_C)$ through the above. For the base locus of ω_{X_y} and other facts, see [3, Proposition 2.1.3].

1.7. For a connected reduced vertical effective divisors C and D on $\mathcal{X}$, we defined in [6] the distance from C to D, denoted by by $d_C(D)$. Roughly speaking, it is the minimal number of nodes that we have to pass through when we move from C to D. Note $d_C(D) = d_D(C)$.

1.8. Let C be a connected reduced vertical effective divisor. A *stepwise extension* of a section $\eta_0 \in H^0(C,\omega_C)$ is a global section of $\omega_{\mathcal{X}/\mathcal{Y}}$ satisfying the following conditions:

(a) $\rho(\eta) = \eta_0$,

(b) $\eta \in f_*(\omega_{\mathcal{X}/\mathcal{Y}}(-\sum_{D\in\mathrm{Irr}(X_y)} d_C(D)D))$, where $\mathrm{Irr}(X_y)$ is the set of irreducible components.

A stepwise extension of η_0 exists by [6, Lemma 2.4].

1.9. Let α be a set of nodes of X_y such that $X_y \setminus \alpha$ has exactly two connected components, and let Z_1 and Z_2 be the closures of the two connected components. Let S be a non-empty subset of X_y with $S \not\subseteq \alpha$. We assume that $S \subset Z_1$ or $S \subset Z_2$. We define $Z^*_\alpha(S)$ by

$$Z^*_\alpha(S) := \begin{cases} Z_1 & \text{if} \quad S \subset Z_1, \\ Z_2 & \text{if} \quad S \subset Z_2. \end{cases}$$

Moreover, we put $Z_\alpha(S) := X_y - Z^*_\alpha(S)$.

2. First elementary transformation for Moriwaki's inequality

In this section, we give proof of the following theorem. As we have said in the introduction, Moriwaki's inequality follows from it immediately.

Theorem 2.1 ($\mathrm{char}(k) \geq 0$). *Let X be a nonsingular projective surface over k, Y a nonsingular projective curve over k, and let $f : X \to Y$ be a generically smooth semistable curve of genus $g \geq 2$. Then there exists a locally free sheaf E on X with the following property.*

(a) $E|_{f^{-1}(y)} = \mathrm{Ker}(H^0(f^{-1}(y),\omega_{f^{-1}(y)}) \otimes_k \mathcal{O}_{f^{-1}(y)} \to \omega_{f^{-1}(y)})$ *for any* $y \in Y \setminus \mathrm{CV}(f)$.

(b) $\mathrm{dis}(E) = (8g+4)\deg(f_*\omega_{X/Y}) - g\delta_0(X/Y) - \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i(X/Y)$.

Let us begin with preliminaries. For each $y \in \mathrm{CV}(f)$, we define divisors B_y, S_C, S^*_C and D_C supported in X_y as follows. For each $E \in \mathrm{QIrr}^0(X_y)$, let $m(E)$ be the minimum of the distances between E and the quasi-irreducible components of positive genus. We put $B_y := \sum_{E\in\mathrm{QIrr}^0(X_y)} m(E)E$, and for each $C \in \mathrm{QIrr}^+(X_y)$, put

$$S_C := \sum_{x \in \mathrm{Sing}_+(X_y)} Z_x(C), \quad S_C^* := \sum_{x \in \mathrm{Sing}_+(X_y)} Z_x^*(C), \quad D_C := S_C^* - B_y.$$

They are effective divisors. Here we note that there is another description of S_C and S_C^*. For each $C, C' \in \mathrm{QIrr}(X_y)$, we set

$$\mathrm{Sing}_+[C, C'] := \{x \in \mathrm{Sing}_+(X_y) \mid Z_x(C) \neq Z_x(C')\},$$

which is the set of nodes of positive type between two quasi-irreducible components C and C', and put $\delta_+^{[C,C']} := \#(\mathrm{Sing}_+[C, C'])$. Then, we can see from the definitions that

$$\text{(2.1.1)} \quad S_C = \sum_{C' \in \mathrm{QIrr}(X_y)} \delta_+^{[C,C']} C', \quad S_C^* = \delta_+(X_y) X_y - \sum_{C' \in \mathrm{QIrr}(X_y)} \delta_+^{[C,C']} C'.$$

For a given $y \in \mathrm{CV}(f)$, let us consider the localization $f : \mathcal{X} \to \mathcal{Y}$ at y (cf. 1.1). For each $C \in \mathrm{QIrr}^+(X_y)$, let $\mathcal{F}_C$ be a direct summand of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$ such that

(a) $\rho(\mathcal{F}_C) = H^0(C, \omega_C)$,

(b) $\mathcal{F}_C \subset f_*(\omega_{\mathcal{X}/\mathcal{Y}}(-S_C))$.

Note that it is of rank $h^0(\omega_C)$ and that $f_*\omega_{\mathcal{X}/\mathcal{Y}} = \bigoplus_{C \in \mathrm{QIrr}^+(X_y)} \mathcal{F}_C$. It is possible to take such $\mathcal{F}_C$'s: for each $C \in \mathrm{QIrr}^+(X_y)$, let $\mathcal{F}_C$ be the submodule generated by stepwise extensions of a basis of $H^0(C, \omega_C)$ for example. Then, it is obvious that it is a direct summand and satisfies the conditions (a), and taking account of the expression (2.1.1), we can see that it also satisfies (b). Throughout this section, $\mathcal{F}_C$ is supposed to satisfy those conditions.

Let us consider the following lemma before starting the proof.

Lemma 2.2. *For any $C \in \mathrm{QIrr}^+(X_y)$, the canonical homomorphism $\mathcal{F}_C \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X} \to \omega_{\mathcal{X}/\mathcal{Y}}(-S_C)$ induced by the evaluation homomorphism of $\omega_{\mathcal{X}/\mathcal{Y}}$ is surjective at $x \in X_y$ if either $x \in C$ or x lies on $E \in \mathrm{QIrr}^0(X_y)$ such that there is not a quasi-irreducible component of positive genus between C and E.*

Proof. If $x \in C$, then our assertion follows from the base-point-freeness of ω_C (cf. [3, Proposition 2.1.3]). Let us consider the other case.

Let $\eta \in \mathcal{F}_C$ be such a section that $\rho(\eta)$, as a section of ω_C, does not have a zero at any point of $C \cap (X_y - C)$. Let $\mathrm{div}(\eta)$ be the divisor of zero of η as a section of $\omega_{\mathcal{X}/\mathcal{Y}}$. It is sufficient to show that the support of an effective divisor $\mathrm{div}(\eta) - S_C$ is disjoint to E. We prove that by induction on $\delta_+^{[C,E]} = d_C(E)$. Suppose that $\delta_+^{[C,E]} = 1$. By the assumption on η, we can find that the support of $\mathrm{div}(\eta) - S_C$ does not contain E. On the other hand, letting l be the number of nodes of X_y lying on E, and taking account of the adjunction formula, we have $l - 2 = (\omega_{\mathcal{X}/\mathcal{Y}} \cdot E) = (\mathrm{div}(\eta) - S_C \cdot E) + (S_C \cdot E)$. Since $(S_C \cdot E) = l - 2$, we obtain $(\mathrm{div}(\eta) - S_C \cdot E) = 0$. Thus, we can see that $\mathrm{div}(\eta) - S_C$ is disjoint to E.

Now suppose it true up to $d-1$ for $d>1$ and suppose $\delta_+^{[C,E]}=d$. Let $E'\in \mathrm{QIrr}^0(X_y)$ be the previous one to E with respect to the distance from C. Since the support of $\mathrm{div}(\eta)-S_C$ is disjoint to E' by the induction hypothesis, it does not contain E. On the other hand, we find that $(\mathrm{div}(\eta)-S_C\cdot E)=0$ in a similar way to the above. Hence $\mathrm{div}(\eta)-S_C$ is disjoint to E, and thus we obtain our assertion. $\square$

Let us start the proof of Theorem 2.1. Our goal is to construct a locally free sheaf E with the required properties. To do that, we perform an elementary transformation.

The local isomorphism $(f^*f_*\omega_{X/Y})|_{\mathcal{X}}\cong\bigoplus_{C\in\mathrm{QIrr}^+(X_y)}\mathcal{F}_C\otimes_{\mathcal{O}_\mathcal{Y}}\mathcal{O}_\mathcal{X}$ gives rise to a surjective homomorphism

$$f^*f_*\omega_{X/Y}\to\bigoplus_{y\in\mathrm{CV}(f)}\bigoplus_{C\in\mathrm{QIrr}^+(X_y)}\mathcal{F}_C\otimes_{\mathcal{O}_\mathcal{Y}}\mathcal{O}_{D_C}=:G.$$

Putting F to be its kernel, which is locally free, we have an exact sequence

$$0\longrightarrow F\longrightarrow f^*f_*\omega_{X/Y}\longrightarrow G\longrightarrow 0. \tag{2.2.1}$$

Composing the canonical inclusion of F into $f^*f_*\omega_{X/Y}$ with the evaluation homomorphism $f^*f_*\omega_{X/Y}\to\omega_{X/Y}$, we obtain a homomorphism $\alpha:F\to\omega_{X/Y}$. We define E as $\mathrm{Ker}(\alpha)$. It satisfies the condition (a) in Theorem 2.1 as G vanishes on the smooth fibers.

Here we claim the following.

Lemma 2.3. *We have* $\mathrm{Im}(\alpha)=\omega_{X/Y}(-\sum_{y\in\mathrm{CV}(f)}B'_y)$*, where we put* $B'_y:=\delta_+(X_y)X_y-B_y$.

Proof. Fix an arbitrary $y\in\mathrm{CV}(f)$. Note that $B'_y=S_C+D_C$. Thus, it is easy to see $\mathrm{Im}(\alpha)\subset\omega_{X/Y}(-\sum_{y\in\mathrm{CV}(f)}B'_y)$ from the property (b) on $\mathcal{F}_C$ and the definitions of D_C and B'_y. We prove the other inclusion. For each $x\in X_y$, let $C\in\mathrm{QIrr}^+(X_y)$ be one of the nearest components, that is, a quasi-irreducible component of positive genus on which x lies, or one of the nearest components to an $E\in\mathrm{QIrr}^0(X_y)$ on which x lies. It is sufficient to show that the homomorphism $\mathcal{F}_C\otimes_\mathcal{Y}\mathcal{O}_\mathcal{X}\to\omega_{\mathcal{X}/\mathcal{Y}}(-S_C)$ induced by the evaluation homomorphism is surjective at x, which is nothing more than Lemma 2.2. $\square$

By the lemma above, we have an exact sequence

$$0\longrightarrow E\longrightarrow F\longrightarrow\omega_{X/Y}(-\textstyle\sum_{y\in\mathrm{CV}(f)}B'_y)\longrightarrow 0, \tag{2.3.1}$$

and the local freeness of E follows from that of F and $\omega_{X/Y}(-\sum_{y\in\mathrm{CV}(f)}B'_y)$.

Now we only have to calculate the discriminant of E. From the exact sequences (2.2.1) and (2.3.1), we have

$$\begin{aligned}\mathrm{ch}(E)&=f^*\mathrm{ch}(f_*\omega_{X/Y})-\mathrm{ch}(G)\\&\quad-\mathrm{ch}(\omega_{X/Y})\prod_{y\in\mathrm{CV}(f)}\mathrm{ch}(\mathcal{O}_X(B_y))\,\mathrm{ch}(\mathcal{O}_X(-\delta_+(X_y)X_y)).\end{aligned}$$

Taking account that $\mathcal{F}_C \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{D_C}$ is free $\mathcal{O}_{D_C}$-module of rank $p_a(C)$, we find

$$\mathrm{ch}(G) = \sum_{y \in \mathrm{CV}(f)} \sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C)([X] - \mathrm{ch}(\mathcal{O}_X(-D_C)),$$

and further, direct calculations tell us

$$\mathrm{ch}(\mathcal{O}_X(-D_C)) = [X] + (B_y - S_C^*) + \frac{1}{2}({B_y}^2) + \frac{1}{2}({S_C^*}^2) - (B_y \cdot S_C^*).$$

On the other hand, we can see that

$$\begin{aligned}\mathrm{ch}(\omega_{X/Y}) \prod_{y \in \mathrm{CV}(f)} \mathrm{ch}(\mathcal{O}_X(B_y)) = [X] + &\left(c_1(\omega_{X/Y}) + \sum_{y \in \mathrm{CV}(f)} B_y\right) \\ &+ \frac{1}{2}(c_1(\omega_{X/Y})^2) \\ &+ \sum_{y \in \mathrm{CV}(f)} \left(\frac{1}{2}({B_y}^2) + (c_1(\omega_{X/Y}) \cdot B_y)\right).\end{aligned}$$

Using these equalities, we can straightforwardly obtain

$$\begin{aligned}\mathrm{ch}(E) = (g-1) \cdot [X] + f^* c_1(f_* \omega_{X/Y}) - c_1(\omega_{X/Y}) \\ + \sum_{y \in \mathrm{CV}(f)} \left((g-1)B_y + \delta_+(X_y)X_y - \sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C) S_C^*\right) \\ - \frac{1}{2}(c_1(\omega_{X/Y})^2) + \sum_{y \in \mathrm{CV}(f)} \left(\frac{g-1}{2}({B_y}^2) + \delta_+(X_y)(c_1(\omega_{X/Y}) \cdot X_y)\right. \\ \left. - (c_1(\omega_{X/Y}) \cdot B_y) + \sum_{C \in \mathrm{QIrr}^+(X_y)} \left(\frac{p_a(C)}{2}({S_C^*}^2) - p_a(C)(B_y \cdot S_C^*)\right)\right),\end{aligned}$$

and thus,

$$\begin{aligned}\mathrm{dis}(E) = g(\omega_{X/Y} \cdot \omega_{X/Y}) + (4-4g)\deg(f_* \omega_{X/Y}) \\ - \sum_{y \in \mathrm{CV}(f)} 4g(g-1)\delta_+(X_y) \\ + \sum_{y \in \mathrm{CV}(f)} \left(\left(\left(\sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C) S_C^*\right)^2\right)\right. \\ \left. - (g-1) \sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C)({S_C^*}^2)\right) \\ + \sum_{y \in \mathrm{CV}(f)} 2\left(\omega_{X/Y} \cdot \sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C) S_C^*\right).\end{aligned} \tag{2.3.2}$$

Here we use the following to carry on the calculation.

Claim 2.3.3. $(S_C^* \cdot S_{C'}^*) = -\delta_+(X_y) + 2\delta_+^{[C,C']}$.

Proof. Taking account that $Z_x^*(C') = Z_x(C)$ for any $x \in \mathrm{Sing}_+[C,C']$, we have

$$(S_C^* \cdot S_{C'}^*) = \left(\sum_{x \in \mathrm{Sing}_+(X_y)} Z_x^*(C) \cdot \sum_{x \in \mathrm{Sing}_+[C,C']} Z_x(C) \right) + \left(\sum_{x \in \mathrm{Sing}_+(X_y)} Z_x^*(C) \cdot \sum_{x \in \mathrm{Sing}_+(X_y) \setminus \mathrm{Sing}_+[C,C']} Z_x^*(C) \right).$$

We can see $(Z_x^*(C) \cdot Z_{x'}(C)) = \epsilon_{x,x'}$ and $(Z_x^*(C) \cdot Z_{x'}^*(C)) = -\epsilon_{x,x'}$, where $\epsilon_{x,x'}$ is 1 or 0 according as $x = x'$ or $x \neq x'$ respectively. Thus, we obtain $(S_C^* \cdot S_{C'}^*) = -\delta_+(X_y) + 2\delta_+^{[C,C']}$ as desired. □

Expanding it and using Claim 2.3.3, we find

$$\left(\left(\sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C) S_C^* \right)^2 \right) = -g^2 \delta_+(X_y) + \sum_{C,C' \in \mathrm{QIrr}^+(X_y)} 2p_a(C)p_a(C')\delta_+^{[C,C']}.$$

Further, we have

$$\begin{aligned} &\sum_{C,C' \in \mathrm{QIrr}^+(X_y)} 2p_a(C)p_a(C')\delta_+^{[C,C']} \\ &\quad = \sum_{x \in \mathrm{Sing}_+(X_y)} \left(\sum_{C,C' \in \mathrm{QIrr}^+(X_y) \text{ with } Z_x(C) \neq Z_x(C')} 2p_a(C)p_a(C') \right) \\ &\quad = \sum_{x \in \mathrm{Sing}_+(X_y)} 4\,\mathrm{tp}(x)(g - \mathrm{tp}(x)) \\ &\quad = \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i(X_y). \end{aligned}$$

Thus, we obtain

$$\left(\left(\sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C) S_C^* \right)^2 \right) = -g^2 \delta_+(X_y) + \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i(X_y). \tag{2.3.4}$$

We can also immediately see by Claim 2.3.3

$$(g-1) \sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C)({S_C^*}^2) = -(g^2 - g)\delta_+(X_y). \tag{2.3.5}$$

Finally, by the adjunction formula,

$$(\omega_{X/Y} \cdot S_C^*) = 2 \sum_{x \in \mathrm{Sing}_+(X_y)} p_a(Z_x^*(C)) - \delta_+(X_y).$$

Since

$$\sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C) p_a(Z_x^*(C)) = \mathrm{tp}(x)^2 + (g - \mathrm{tp}(x))^2 = g^2 - 2\,\mathrm{tp}(x)(g - \mathrm{tp}(x))$$

for each $x \in \mathrm{Sing}_+(X_y)$, we find

$$\begin{aligned} &2 \sum_{C \in \mathrm{QIrr}^+(X_y)} p_a(C)(\omega_{X/Y} \cdot S_C^*) \\ &= 4 \sum_{x \in \mathrm{Sing}_+(X_y)} (g^2 - 2\,\mathrm{tp}(x)(g - \mathrm{tp}(x))) - 2g\delta_+(X_y) \\ &= (4g^2 - 2g)\delta_+(X_y) - \sum_{i=1}^{[\frac{g}{2}]} 8i(g-i)\delta_i(X_y). \end{aligned} \tag{2.3.6}$$

Taking account of (2.3.4) through (2.3.6) and Noether's formula, we obtain from (2.3.2)

$$\mathrm{dis}(E) = (8g+4)\deg(f_*\omega_{X/Y}) - g\delta_0(X/Y) - \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i(X/Y)$$

and thus we see that E has the expected discriminant.

3. Generalization of Moriwaki's inequality

In this section, we shall modify E in Theorem 2.1 and prove a generalized inequality. First of all, we formulate what will contribute the inequality.

Let $f : X \to Y$ be a semistable curve as before. We consider a family $\mathcal{P}_y$ of sets of two nodes of type 0 of X_y with the following property: $X_y \setminus \sigma$ has exactly two connected components for any $\sigma \in \mathcal{P}_y$, and the two nodes in $\tau \in \mathcal{P}_y \setminus \{\sigma\}$ belong to the same connected component of $X_y \setminus \sigma$. It is easy to see that if $\sigma \in \mathcal{P}_y$, then both the nodes belong to the same quasi-irreducible component. We say such $\mathcal{P}_y$ to be *favorably arranged.* Further we put $\mathcal{P} := \bigcup_{y \in \mathrm{CV}(f)} \mathcal{P}_y$ and call it a *favorably arranged family of* f. We fix such a set $\mathcal{P}_y$ and hence $\mathcal{P}$ throughout this section.

We can assign a number j to each $\sigma \in \mathcal{P}_y$ in the following way. Let $\nu : (X_y)_\sigma \to X_y$ be the partial normalization at the nodes in σ. Then $(X_y)_\sigma$ has exactly two connected components, and we put j to be the minimum of the arithmetic genera of them. We call this number j the *subtype* of σ. For each j with $1 \le j \le [(g-1)/2]$, we set $\xi_j(\mathcal{P}_y)$ to be the number of *pairs* in $\mathcal{P}_y$ of subtype j and we set $\xi_0(\mathcal{P}_y)$ to be the number of *nodes* of type 0 not in any pair in $\mathcal{P}_y$ of positive subtype. Note that $\delta_0(X_y) = \xi_0(\mathcal{P}_y) + 2\sum_{j=1}^{[(g-1)/2]} \xi_j(\mathcal{P}_y)$. Further we put $\xi_j(\mathcal{P}) := \sum_{y \in \mathrm{CV}(f)} \xi_j(\mathcal{P}_y)$.

Example 3.1 (char$(k) \geq 0$). Suppose that the generic fiber is a hyperelliptic curve. Let $\iota : X \to X$ be the hyperelliptic involution over Y and let $\pi : X \to X/\langle\iota\rangle$ be the quotient. Let $x \in X_y$ be a node of type 0 not fixed by ι. Then $\pi(x)$ is also a node of $(X/\langle\iota\rangle)_y$ and hence $(X/\langle\iota\rangle)_y \setminus \pi(x)$ is disconnected by Proposition A.2. Thus, we find that $X_y \setminus \{x, \iota(x)\}$ has exactly two connected components. We can see that the set $\mathcal{P}_y^{hyp}$ of all such pairs $\{x, \iota(x)\}$ is favorably arranged. Thus, we have a canonical favorably arranged family in the hyperelliptic case, and we simply write $\xi_j(X/Y)$ for $\xi_j(\mathcal{P}^{hyp})$.

We will give proof of the following theorem.

Theorem 3.2 (char$(k) \geq 0$). *Let X be a nonsingular projective surface over k, Y a nonsingular projective curve over k and $f : X \to Y$ be a generically smooth semistable curve of genus $g \geq 2$. Let $\mathcal{P}$ be a favorably arranged family of f. Then, there exists a locally free sheaf $\tilde{E}$ with the following properties.*

(a) $\tilde{E}|_{f^{-1}(y)} = \mathrm{Ker}(H^0(f^{-1}(y), \omega_{f^{-1}(y)}) \otimes_k \mathcal{O}_{f^{-1}(y)} \to \omega_{f^{-1}(y)})$ *for any* $y \in Y \setminus \mathrm{CV}(f)$.

(b)

$$\begin{aligned}\mathrm{dis}(\tilde{E}) &= (8g+4)\deg(f_*\omega_{X/Y}) - g\xi_0(\mathcal{P}) \\ &\quad - \sum_{j=1}^{[(g-1)/2]} 2(j+1)(g-j)\xi_j(\mathcal{P}) - \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i(X/Y).\end{aligned}$$

Taking account of the semistability of $\tilde{E}$ on the geometric generic fiber and Bogomolov's instability theorem, we can immediately obtain the following.

Corollary 3.3 (char$(k) = 0$). *With the same notation, we have*

$$\begin{aligned}(8g+4)\deg(f_*\omega_{X/Y}) \geq g\xi_0(\mathcal{P}) &+ \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j(\mathcal{P}) \\ &+ \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X/Y).\end{aligned}$$

If the generic fiber X_K is hyperelliptic, then $E|_{X_K}$ is a pull-back of a locally free sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(g-1)}$ on $\mathbb{P}^1$ by the double covering and hence is strongly semistable. Accordingly, by [4, Corollary 7.4] and Example 3.1, we obtain the following inequality.

Corollary 3.4 (char$(k) \geq 0$). *With the same notation, if f is hyperelliptic, then*

$$\begin{aligned}(8g+4)\deg(f_*\omega_{X/Y}) \geq g\xi_0(X/Y) &+ \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j(X/Y) \\ &+ \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X/Y).\end{aligned}$$

Although it is known in [1] that the equality holds in $\mathrm{char}(k) = 0$, it seems remarkable that the inequality holds even in $\mathrm{char}(k) = 2$. Here it should be remarked that we have to depend on Moriwaki's theory in [4] to obtain Corollary 3.4 in positive characteristic, for we do not know the Bogomolov instability theorem in that case.

The proof of Theorem 3.2 occupies the rest of this section.

3.1. Preliminaries

We introduce notation and terminology that will be used in the proof first, and prove preliminary results next.

Let us begin with the preparation of notations and the introduction of notions on subsets of $\mathcal{P}_y$. Let us consider a set $\mathcal{P}_y \cup \mathrm{Sing}_+(X_y)$ of sets of nodes of X_y. Let b be a nonsingular point of X_y. For any $\alpha \in \mathcal{P}_y \cup \mathrm{Sing}_+(X_y)$, the space $X_y \setminus \alpha$ consists of exactly two connected components, and one and only one of them contains b. From now on, we fix a nonsingular point b of X_y and simply write Z_α and Z_α^* for $Z_\alpha(\{b\})$ and $Z_\alpha^*(\{b\})$ respectively (cf. 1.9 for the notations). We further put A_σ (resp. A_σ^*) be the reduced subscheme or divisor of which support is $Z_\sigma \cap C_\sigma$ (resp. $Z_\sigma^* \cap C_\sigma$), where C_σ is the quasi-irreducible component containing $\sigma \in \mathcal{P}_y$.

Once such a base point b is fixed, we can introduce the following notion.

Definition 3.5. Let S be a subset of $\mathcal{P}_y \cup \mathrm{Sing}_+(X_y)$. A subset T of S is called an *admissible subset of* S, or said to be S-*admissible*, if it has the following property: for any $\alpha \in S$, if $Z_\alpha \subset Z_\beta$ for some $\beta \in T$, then $\alpha \in T$. When there is no danger of confusion, we simply say "admissible".

For each $\sigma \in \mathcal{P}_y$, a set $\tilde{\Gamma}_\sigma := \left\{\alpha \in \mathcal{P}_y \cup \mathrm{Sing}_+(X_y) \mid Z_\alpha \subset Z_\sigma\right\}$ is an important subset of $\mathcal{P}_y \cup \mathrm{Sing}_+(X_y)$. For each $C \in \mathrm{QIrr}^+(X_y)$, we denote $\{\sigma \in \mathcal{P}_y \mid \sigma \subset C\}$ by $\mathcal{P}_y \cap C$ and $\mathcal{P}_y \setminus (\mathcal{P}_y \cap C)$ by $\mathcal{P}_y \setminus C$ for simplicity. For any $C \in \mathrm{QIrr}^+(X_y)$ and for each $\sigma \in \mathcal{P}_y \cap C$, $\Gamma_\sigma := \{\tau \in \mathcal{P}_y \cap C \mid Z_\tau \subsetneq Z_\sigma\}$ is also an important subset of $\mathcal{P}_y \cap C$. They will often appear in what follows.

In Section 2, we denoted by $\mathcal{F}_C$ a certain kind of direct summand of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$. We also let the same symbol stand for a direct summand like that, but we require stronger condition of it in this section: for any $C \in \mathrm{QIrr}^+(X_y)$, $\mathcal{F}_C$ is a direct summand of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$ such that

(a) $\rho(\mathcal{F}_C) = H^0(C, \omega_C)$,

(b) $\mathcal{F}_C \subset f_*(\omega_{\mathcal{X}/\mathcal{Y}}(-\sum_{\alpha \in (\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C)))$.

Such $\mathcal{F}_C$ does exist since $\sum_{C' \in \mathrm{Irr}(X_y)} d_C(C')C' - \sum_{\alpha \in (\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C)$ is an effective divisor (cf. Section 2). We call the above properties for a direct summand of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$ the $\mathcal{P}_y$-*stepwise property with respect to* C. From now on, $\mathcal{F}_C$ is to have the $\mathcal{P}_y$-stepwise property with respect to C.

Finally, the following notion will be used in many steps of proofs of later assertions.

Definition 3.6. Let S be an admissible subset of $\mathcal{P}_y \cup \mathrm{Sing}_+(X_y)$. An element $\alpha \in \mathcal{P}_y \cup \mathrm{Sing}_+(X_y)$ is said to be S-*extremal* if $\alpha \in S$ and there is not an element $\beta \in S$ with $Z_\beta \subsetneq Z_\alpha$.

In order to prove Theorem 3.2, we further perform elementary transformations of E in Theorem 2.1. In the case of Section 2, we did not require special property of $\mathcal{F}_C$ other than the (weaker) stepwise property, but for further modification, we have to chose a suitable $\mathcal{F}_C$ equipped with appropriate subsheaves in addition (cf. Proposition 3.8). The following lemma helps us to construct such ones.

Lemma 3.7. *Let us fix any $C \in \mathrm{QIrr}^+(X_y)$ and any $\sigma \in \mathcal{P}_y \cap C$. For any $\eta \in \mathcal{F}_C$ that vanishes along Z_σ, there exists a section*

$$t\xi \in \left(t \left(\sum_{D \in \mathrm{QIrr}^+(X_y) \text{ with } D \subset Z_\sigma} \mathcal{F}_D + \mathcal{F}_C \right) \right) \cap f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in (\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C) \right) \right)$$

such that

$$\eta - t\xi \in f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in \tilde{\Gamma}_\sigma} Z_\alpha \right) \right) \cap f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in (\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C) \right) \right).$$

Proof. Let η' be a stepwise extension of $\rho(\eta) \in H^0(A^*_\sigma, \omega_{A^*_\sigma})$ (cf. Section 1). Since $\eta - \eta' \in t f_* \omega_{\mathcal{X}/\mathcal{Y}}$ and $f_* \omega_{\mathcal{X}/\mathcal{Y}} = \bigoplus_{D \in \mathrm{QIrr}^+(X_y)} \mathcal{F}_D$, there exist sections

$$\xi \in \sum_{D \in \mathrm{QIrr}^+(X_y) \text{ with } D \subset Z_\sigma} \mathcal{F}_D + \mathcal{F}_C$$

and

$$\xi' \in \sum_{D \in \mathrm{QIrr}^+(X_y) \text{ with } D \cap Z_\sigma = \emptyset} \mathcal{F}_D$$

such that $\eta = \eta' + t\xi + t\xi'$. We have

$$\eta - t\xi \in f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in ((\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)) \setminus \tilde{\Gamma}_\sigma} Z_\alpha(C) \right) \right)$$

as both η and $t\xi$ sit in it by the $\mathcal{P}_y$-stepwise property of such $\mathcal{F}_D$'s. On the other hand, we have

$$\eta - t\xi = \eta' + t\xi' \in f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in \tilde{\Gamma}_\sigma} Z_\alpha \right) \right)$$

as both η' and $t\xi'$ sit in it. Taking account that $Z_\alpha = Z_\alpha(C)$ for any $\alpha \in ((\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)) \cap \tilde{\Gamma}_\sigma$, we find consequently

$$\begin{aligned}
\eta - t\xi \in f_*&\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in\tilde{\Gamma}_\sigma} Z_\alpha\right)\right)\\
&\cap f_*\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in((\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y))\setminus\tilde{\Gamma}_\sigma} Z_\alpha(C)\right)\right)\\
&= f_*\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in\tilde{\Gamma}_\sigma} Z_\alpha\right)\right)\\
&\cap f_*\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C)\right)\right).
\end{aligned}$$

Finally, we have

$$t\xi \in f_*\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C)\right)\right)$$

as both η and $\eta - t\xi$ sit in it. Thus, we obtain our lemma. □

The following result plays a key role in our performing the elementary transformations.

Proposition 3.8. *For any $C \in \mathrm{QIrr}^+(X_y)$, there exist a direct summand $\mathcal{F}_C$ of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$, direct summands $\mathcal{F}_\sigma$ and $\mathcal{E}_\sigma$ of $\mathcal{F}_C$ for each $\sigma \in \mathcal{P}_y \cap C$, and direct summand $\mathcal{E}_{\tau,\sigma}$ of $\mathcal{F}_C$ for any (τ,σ) with $\tau \in \Gamma_\sigma$, satisfying the following conditions.*

(a) *$\mathcal{F}_C$ has the $\mathcal{P}_y$-stepwise property with respect to C.*

(b) *$\rho(\mathcal{F}_\sigma) = H^0(A^*_\sigma, \omega_{A^*_\sigma})$ and $\mathcal{F}_\sigma \subset f_*(\omega_{\mathcal{X}/\mathcal{Y}}(-\sum_{\alpha\in\tilde{\Gamma}_\sigma} Z_\alpha))$, and hence $\mathrm{rk}(\mathcal{F}_\sigma) = p_a(A^*_\sigma)$.*

(c) *$\mathcal{F}_C = \mathcal{E}_\sigma \oplus \mathcal{F}_\sigma$, $\mathcal{E}_\sigma = \mathcal{E}_\tau \oplus \mathcal{E}_{\tau,\sigma}$ and $\mathcal{F}_\tau = \mathcal{E}_{\tau,\sigma} \oplus \mathcal{F}_\sigma$.*

Proof. Let M be a maximal admissible subset of $\mathcal{P}_y \cap C$ with the following property: there exist a direct summand $\mathcal{F}^{(M)}$ of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$, direct summands $\mathcal{F}^{(M)}_\sigma$ and $\mathcal{E}^{(M)}_\sigma$ of $\mathcal{F}^{(M)}$ for each $\sigma \in M$, and those $\mathcal{E}^{(M)}_{\tau,\sigma}$ for each $\sigma, \tau \in M$ with $\tau \in \Gamma_\sigma$ satisfying the following conditions.

(M-a) $\mathcal{F}^{(M)}$ has the $\mathcal{P}_y$-stepwise property with respect to C.

(M-b) $\rho(\mathcal{F}^{(M)}_\sigma) = H^0(A^*_\sigma, \omega_{A^*_\sigma})$ and $\mathcal{F}^{(M)}_\sigma \subset f_*(\omega_{\mathcal{X}/\mathcal{Y}}(-\sum_{\alpha\in\tilde{\Gamma}_\sigma} Z_\alpha))$ for any $\sigma \in M$.

(M-c) For any $\sigma \in M$, we have $\mathcal{F}^{(M)} = \mathcal{E}^{(M)}_\sigma \oplus \mathcal{F}^{(M)}_\sigma$ and if $\tau \in \Gamma_\sigma \cap M$, then $\mathcal{F}^{(M)}_\tau = \mathcal{E}^{(M)}_{\tau,\sigma} \oplus \mathcal{F}^{(M)}_\sigma$ and $\mathcal{E}^{(M)}_\sigma = \mathcal{E}^{(M)}_\tau \oplus \mathcal{E}^{(M)}_{\tau,\sigma}$.

It is easy to see that $M \neq \emptyset$ if $\mathcal{P}_y \cap C \neq \emptyset$, hence we may assume $M \neq \emptyset$. Suppose that $M \neq \mathcal{P}_y \cap C$. Let $\upsilon \in (\mathcal{P}_y \cap C) \setminus M$ be such an element that $M' := \{\upsilon\} \cup M$ is an admissible subset of $\mathcal{P}_y \cap C$, and let $\mu \in M$ be the M-extremal element for υ. Since $\mathcal{F}^{(M)} = \mathcal{E}_\mu^{(M)} \oplus \mathcal{F}_\mu^{(M)}$, $\rho(\mathcal{F}_\mu^{(M)}) = H^0(A_\mu^*, \omega_{A_\mu^*})$ and $\upsilon \in \Gamma_\mu$, we can take a basis $\{u_{p_a(A_\mu^*)+1}, \dots, u_{p_a(C)}\}$ of $\mathcal{E}_\mu^{(M)}$ such that

$$\rho\left(\mathcal{F}_\mu^{(M)} + \mathcal{O}_{\mathcal{Y}} u_{p_a(A_\mu^*)+1} + \cdots + \mathcal{O}_{\mathcal{Y}} u_{p_a(A_\upsilon^*)}\right) = H^0(A_\upsilon^*, \omega_{A_\upsilon^*}) \subset H^0(C, \omega_C),$$

or equivalently such that any of $u_{p_a(A_\mu^*)+1}, \dots, u_{p_a(A_\upsilon^*)}$ vanishes along Z_υ as a global section of $\omega_{\mathcal{X}/\mathcal{Y}}$. By virtue of Lemma 3.7, for each j with $p_a(A_\mu^*) + 1 \leq j \leq p_a(A_\upsilon^*)$ there exist $u_j' \in \mathcal{F}^{(M)}$ and

$$\begin{aligned} t v_j \in & \left(t \sum_{D \in \mathrm{QIrr}^+(X_y) \text{ with } D \subset Z_\upsilon} \mathcal{F}_D \right) \\ & \cap f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in (\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C) \right) \right), \end{aligned}$$

where $\mathcal{F}_D$ is an arbitrary direct summand of $f_* \omega_{\mathcal{X}/\mathcal{Y}}$ with the $\mathcal{P}_y$-stepwise property with respect to D, such that

$$\begin{aligned} u_j - t u_j' - t v_j \in & f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in \tilde{\Gamma}_\upsilon} Z_\alpha \right) \right) \\ & \cap f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in (\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C) \right) \right). \end{aligned}$$

Since $\mathcal{F}^{(M)} = \mathcal{E}_\mu^{(M)} \oplus \mathcal{F}_\mu^{(M)}$ and since

$$\begin{aligned} \mathcal{F}_\mu^{(M)} \subset & f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in \tilde{\Gamma}_\upsilon} Z_\alpha \right) \right) \\ & \cap f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in (\mathcal{P}_y \setminus C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C) \right) \right) \end{aligned}$$

by (M-a) and (M-b), we can take each u_j' out of $\mathcal{E}_\mu^{(M)}$, and hence we replace each u_j by $u_j - t u_j'$ to obtain a new basis of $\mathcal{E}_\mu^{(M)}$. Thus, we may assume that we are given a basis $\{u_{p_a(A^*_\mu)+1}, \dots, u_{p_a(C)}\}$ of $\mathcal{E}^{(M)}_\mu$ such that

$$
\begin{aligned}
(3.8.1) \qquad u_j - tv_j \in f_* &\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in\tilde{\Gamma}_\upsilon} Z_\alpha\right)\right) \\
&\cap f_*\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C)\right)\right)
\end{aligned}
$$

for any j with $p_a(A^*_\mu)+1 \le j \le p_a(A^*_\upsilon)$.

Now, we put

$$
\tilde{u}_j := \begin{cases} u_j - tv_j & \text{if} \quad p_a(A^*_\mu)+1 \le j \le p_a(A^*_\upsilon), \\ u_j & \text{if} \quad p_a(A^*_\upsilon)+1 \le j \le p_a(C). \end{cases}
$$

We define a homomorphism $T' : \mathcal{E}^{(M)}_\mu \to f_*\omega_{\mathcal{X}/\mathcal{Y}}$ by $T'(u_j) = \tilde{u}_j$, and define T by

$$
T := T' \oplus I : \mathcal{E}^{(M)}_\mu \oplus \mathcal{F}^{(M)}_\mu \to f_*\omega_{\mathcal{X}/\mathcal{Y}},
$$

where I is the canonical inclusion map. Then, T is an isomorphism onto its image $\mathrm{Im}(T)$, to be which we defined $\mathcal{F}^{(M')}$. Since $\mathcal{F}^{(M)}$ is a direct summand of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$ with the $\mathcal{P}_y$-stepwise property and T is the canonical inclusion modulo

$$
\begin{aligned}
&\left(t \sum_{D\in\mathrm{QIrr}^+(X_y)\ \text{with}\ D\subset Z_\upsilon} \mathcal{F}_D\right) \\
&\qquad \cap f_*\left(\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C)\right)\right),
\end{aligned}
$$

$\mathcal{F}^{(M')}$ is also a direct summand of $f_*\omega_{\mathcal{X}/\mathcal{Y}}$ with the $\mathcal{P}_y$-stepwise property.

Now we put $\mathcal{F}^{(M')}_\sigma := T(\mathcal{F}^{(M)}_\sigma)$, $\mathcal{E}^{(M')}_{\tau,\sigma} := T(\mathcal{E}^{(M)}_{\tau,\sigma})$ and $\mathcal{E}^{(M')}_\sigma := T(\mathcal{E}^{(M)}_\sigma)$ for those $\sigma, \tau \in M$, put

$$
\begin{aligned}
\mathcal{E}^{(M')}_{\upsilon,\mu} &:= \mathcal{O}_\mathcal{Y}\tilde{u}_{p_a(A^*_\mu)+1} + \cdots + \mathcal{O}_\mathcal{Y}\tilde{u}_{p_a(A^*_\upsilon)}, \\
\mathcal{E}^{(M')}_{\upsilon,\sigma} &:= \mathcal{E}^{(M')}_{\upsilon,\mu} + \mathcal{E}^{(M')}_{\mu,\sigma}, \\
\mathcal{F}^{(M')}_\upsilon &:= \mathcal{E}^{(M')}_{\upsilon,\mu} + \mathcal{F}^{(M')}_\mu,
\end{aligned}
$$

and define $\mathcal{E}^{(M')}_\upsilon$ to be any direct summand of $\mathcal{E}^{(M')}_\mu$ complementary to $\mathcal{E}^{(M')}_{\upsilon,\mu}$. It is not difficult to see that they satisfies the condition (M'-c) from the definitions. We check the condition (M'-b).

In the case of $Z_\mu \subset Z_\sigma$, since $\mathcal{F}^{(M)}_\sigma$ is a subsheaf of $\mathcal{F}^{(M)}_\mu$ and $T|_{\mathcal{F}^{(M)}_\mu}$ is the canonical inclusion on $\mathcal{F}^{(M)}_\mu$, we have $\mathcal{F}^{(M')}_\sigma = \mathcal{F}^{(M)}_\sigma$. Thus $\mathcal{F}^{(M')}_\sigma$ has the properties in (M'-b).

Let us look at $\mathcal{F}_\sigma^{(M)}$ with $Z_\mu \cap Z_\sigma = \emptyset$. In this case, taking account that $Z_\upsilon \cap Z_\sigma = \emptyset$, we find that $Z_\alpha(D) = Z_\alpha$ for any $D \in \mathrm{QIrr}^+(X_y)$ contained in Z_υ and for any $\alpha \in \tilde{\Gamma}_\sigma$. Hence we see $\sum_{\alpha \in (\mathcal{P}_y \setminus D) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(D) - \sum_{\alpha \in \tilde{\Gamma}_\sigma} Z_\alpha$ is an effective divisor for such D, and that implies

$$\left(t \sum_{D \in \mathrm{QIrr}^+(X_y) \text{ with } D \subset Z_\upsilon} \mathcal{F}_D \right) \subset f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in \tilde{\Gamma}_\sigma} Z_\alpha \right) \right).$$

Therefore T is the canonical inclusion modulo

$$\begin{aligned}(t f_* \omega_{\mathcal{X}/\mathcal{Y}}) \cap f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in \tilde{\Gamma}_\sigma} Z_\alpha \right) \right) \\ \cap f_* \left(\omega_{\mathcal{X}/\mathcal{Y}} \left(- \sum_{\alpha \in (\mathcal{P}_y \cap C) \cup \mathrm{Sing}_+(X_y)} Z_\alpha(C) \right) \right)\end{aligned}$$

and hence $\rho(\mathcal{F}_\sigma^{(M')}) = H^0(A_\sigma^*, \omega_{A_\sigma^*})$ and $\mathcal{F}_\sigma^{(M')}$ remains in the submodule $f_*(\omega_{\mathcal{X}/\mathcal{Y}}(-\sum_{\alpha \in \tilde{\Gamma}_\sigma} Z_\alpha))$. Thus $\mathcal{F}^{(M')}$ has the properties in (M'-b).

Finally, from the definition of $\tilde{u}_j$'s and (3.8.1), it is easy to check that $\mathcal{F}_\upsilon^{(M')}$ has the properties in (M'-b).

Thus, we have constructed those subsheaves which satisfy the required conditions for M', which contradicts to the maximality of M. □

In what follows, we always assume that $\mathcal{F}_C$ is that in Proposition 3.8 and use the notations in it.

3.2. Second elementary transformations

Now we are ready to carry out further elementary transformations to construct a locally free sheaf $\tilde{E}$ in Theorem 3.2. As the first step, we construct a sequence of locally free sheaves and homomorphisms, fixing our eyes upon a given quasi-irreducible component.

Fix an arbitrary $C \in \mathrm{QIrr}^+(X_y)$. Put $m := \#\mathcal{P}_y \cap C$ and let

$$\emptyset = P_0 \subset P_1 \subset \cdots \subset P_m = \mathcal{P}_y \cap C$$

be a sequence of admissible subsets of $\mathcal{P}_y \cap C$ with $\#P_i = i$ for every i. We denote the element of $P_i \setminus P_{i-1}$ by σ_i. Let $\mathcal{F}_C$ be that in Proposition 3.8. We construct inductively the following things: locally free subsheaves $\mathcal{H}_{P_i}$ of $\mathcal{F}_C \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$ for any i with $0 \le i \le m$ having the following Property 3.9, and surjective homomorphisms ϕ_{P_i} from $\mathcal{H}_{P_{i-1}}$ to a free $\mathcal{O}_{Z_{\sigma_i}}$-module of rank $p_a(A_{\sigma_i}) + 1$ for any i with $1 \le i \le m$, such that $\mathrm{Ker}(\phi_{P_i}) = \mathcal{H}_{P_i}$.

Property 3.9. For a given $\sigma \in (\mathcal{P}_y \cap C) \setminus P_i$, let $\tau_1, \ldots, \tau_l$ be the elements of P_i such that $Z_\sigma \subsetneq Z_{\tau_l} \subsetneq \cdots \subsetneq Z_{\tau_1}$ and $\{\tau_1, \ldots, \tau_l\} = \{\tau \in P_i \mid Z_\sigma \subsetneq Z_\tau\}$. If $l \neq 0$, there exists an open neighborhood U_σ of Z_σ such that

$$\mathcal{H}_{P_i}|_{U_\sigma} = \left(t^l \mathcal{E}_{\tau_l} + t^{l-1} \mathcal{E}_{\tau_l, \tau_{l-1}} + \cdots + t \mathcal{E}_{\tau_2, \tau_1} + \mathcal{F}_{\tau_1} \right) \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{U_\sigma}.$$

First we put $\mathcal{H}_{P_0} := \mathcal{F}_C \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}$, and we can see that it has the property (since the condition is empty). Suppose that we have $\mathcal{H}_{P_i}$ with the Property 3.9 and ϕ_{P_i} with the required property up to i. If $i < m$, we construct the $(i+1)$-th ones in the following way. Let $\tau_1, \dots, \tau_l$ be those in 3.9 for σ_{i+1} and let $U_{\sigma_{i+1}}$ be an open neighborhood of $Z_{\sigma_{i+1}}$ as in 3.9 for P_i. By the conditions on $\mathcal{E}_{\tau_{j+1},\tau_j}$'s and $\mathcal{F}_{\tau_1}$ in Proposition 3.8, we find that for any $\eta \in t^l \mathcal{E}_{\tau_l} + t^{l-1}\mathcal{E}_{\tau_l,\tau_{l-1}} + \cdots + t\mathcal{E}_{\tau_2,\tau_1} + \mathcal{F}_{\tau_1}$, the rational section η/t^l of $\omega_{\mathcal{X}/\mathcal{Y}}$ is regular around $Z_{\sigma_{i+1}}$, and hence we may assume that it is regular over $U_{\sigma_{i+1}}$ by shrinking it if necessary. That implies, by the projection formula, there exists a vertical effective divisor D disjoint to $Z_{\sigma_{i+1}}$ such that

$$\mathcal{H}_{P_i}|_{U_{\sigma_{i+1}}} \subset f_*(\omega_{\mathcal{X}/\mathcal{Y}}(D)) \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{U_{\sigma_{i+1}}}(-l(X_y \cap U_{\sigma_{i+1}})).$$

By the natural homomorphisms $\mathcal{H}_{P_i} \to \mathcal{H}_{P_i}|_{U_{\sigma_{i+1}}}$ and

$$\begin{aligned} &f_*(\omega_{\mathcal{X}/\mathcal{Y}}(D)) \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{U_{\sigma_{i+1}}}(-l(X_y \cap U_{\sigma_{i+1}})) \\ &\qquad \to H^0(Z_{\sigma_{i+1}}, \omega_{\mathcal{X}/\mathcal{Y}}|_{Z_{\sigma_{i+1}}}) \otimes_k \mathcal{O}_{U_{\sigma_{i+1}}}(-l(X_y \cap U_{\sigma_{i+1}}))|_{Z_{\sigma_{i+1}}}, \end{aligned}$$

we have a homomorphism

$$\mathcal{H}_{P_i} \to H^0(Z_{\sigma_{i+1}}, \omega_{\mathcal{X}/\mathcal{Y}}|_{Z_{\sigma_{i+1}}}) \otimes_k \mathcal{O}_{U_{\sigma_{i+1}}}(-l(X_y \cap U_{\sigma_{i+1}}))|_{Z_{\sigma_{i+1}}}.$$

Moreover, we can see that the image of the homomorphism

$$H^0(Z_{\sigma_{i+1}}, \omega_{\mathcal{X}/\mathcal{Y}}|_{Z_{\sigma_{i+1}}}) \to H^0(A_{\sigma_{i+1}}, \omega_{\mathcal{X}/\mathcal{Y}}|_{A_{\sigma_{i+1}}})$$

coincides with $H^0(A_{\sigma_{i+1}}, \omega_{A_{\sigma_{i+1}}}(\sigma_{i+1}))$ which is regarded in a canonical way as a linear subspace of $H^0(A_{\sigma_{i+1}}, \omega_{\mathcal{X}/\mathcal{Y}}|_{A_{\sigma_{i+1}}})$. Taking account that $\mathcal{O}_{Z_{\sigma_{i+1}}} \cong \mathcal{O}_{U_{\sigma_{i+1}}}(-l(X_y \cap U_{\sigma_{i+1}}))|_{Z_{\sigma_{i+1}}}$, we thus define a homomorphism

$$\phi_{P_{i+1}} : \mathcal{H}_{P_i} \to H^0(A_{\sigma_{i+1}}, \omega_{A_{\sigma_{i+1}}}(\sigma_{i+1})) \otimes_k \mathcal{O}_{Z_{\sigma_{i+1}}}.$$

We put $\mathcal{H}_{P_{i+1}} := \mathrm{Ker}(\phi_{P_{i+1}})$. Let us show that it has the property 3.9 and that $\phi_{P_{i+1}}$ is surjective. We fix any $\sigma \in (\mathcal{P}_y \cap C) \setminus P_{i+1}$.

If $Z_\sigma \cap Z_{\sigma_{i+1}} = \emptyset$, then $\phi_{P_{i+1}}$ is a trivial homomorphism around Z_σ, and hence $\mathcal{H}_{P_{i+1}}$ obviously has the required property.

Let us consider the case of $Z_\sigma \subset Z_{\sigma_{i+1}}$. In this case, we may handle them only over $U_{\sigma_{i+1}}$ and we put $\psi := \phi_{P_{i+1}}|_{U_{\sigma_{i+1}}}$ for simplicity. We note that $\mathcal{E}_{\tau_l} = \mathcal{E}_{\sigma_{i+1}} \oplus \mathcal{E}_{\sigma_{i+1},\tau_l}$ first. From the definition of $\phi_{P_{i+1}}$ and the conditions on $\mathcal{E}_{\sigma_{i+1},\tau_l}$, $\mathcal{E}_{\tau_{j+1},\tau_j}$'s and $\mathcal{F}_{\tau_1}$ again, we can see that

$$\psi((t^l \mathcal{E}_{\sigma_{i+1},\tau_l} + t^{l-1}\mathcal{E}_{\tau_l,\tau_{l-1}} + \cdots + t\mathcal{E}_{\tau_2,\tau_1} + \mathcal{F}_{\tau_1}) \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{U_{\sigma_{i+1}}}) = 0.$$

Let us consider the restriction ψ' of ψ to $t^l \mathcal{E}_{\sigma_{i+1}} \otimes \mathcal{O}_{U_{\sigma_{i+1}}}$. By the definition of $\phi_{P_{i+1}}$, the map ψ' can be described as

$$t^l \eta \otimes f \mapsto \eta \otimes f \mapsto r(\eta) \otimes f|_{Z_{\sigma_{i+1}}},$$

where r is the homomorphism $\mathcal{E}_{\sigma_{i+1}} \twoheadrightarrow H^0(A_{\sigma_{i+1}}, \omega_{A_{\sigma_{i+1}}}(\sigma_{i+1}))$ induced by the natural restriction, and we find accordingly that ψ' is and hence $\phi_{P_{i+1}}$ is surjective. Moreover, taking account that

$$\mathrm{rk}_{\mathcal{O}_\mathcal{Y}}(\mathcal{E}_{\sigma_{i+1}}) = \dim_k H^0(A_{\sigma_{i+1}}, \omega_{A_{\sigma_{i+1}}}(\sigma_{i+1})),$$

we see that ψ' is nothing more than the restriction homomorphism

$$t^l\mathcal{E}_{\sigma_{i+1}} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{U_{\sigma_{i+1}}} \to t^l\mathcal{E}_{\sigma_{i+1}} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{Z_{\sigma_{i+1}}}.$$

We have therefore $\mathrm{Ker}(\psi') = t^l\mathcal{E}_{\sigma_{i+1}} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{U_{\sigma_{i+1}}}(-(Z_{\sigma_{i+1}}))$, and hence

$$\begin{aligned} \text{(3.9.2)} \quad \mathcal{H}_{P_{i+1}}|_{U_{\sigma_{i+1}}} &= t^l\mathcal{E}_{\sigma_{i+1}} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{U_{\sigma_{i+1}}}(-Z_{\sigma_{i+1}}) \\ &\quad + \left(t^l\mathcal{E}_{\sigma_{i+1},\tau_l} + t^{l-1}\mathcal{E}_{\tau_l,\tau_{l-1}} + \cdots + t\mathcal{E}_{\tau_2,\tau_1} + \mathcal{F}_{\tau_1}\right) \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{U_{\sigma_{i+1}}}. \end{aligned}$$

Accordingly, shrinking U_σ so that $Z_{\sigma_{i+1}} \cap U_\sigma = X_y \cap U_\sigma$ if necessary, we obtain

$$\begin{aligned} \mathcal{H}_{P_{i+1}}|_{U_\sigma} = &\left(t^{l+1}\mathcal{E}_{\sigma_{i+1}} + t^l\mathcal{E}_{\sigma_{i+1},\tau_l} + t^{l-1}\mathcal{E}_{\tau_l,\tau_{l-1}} + \cdots + t\mathcal{E}_{\tau_2,\tau_1} + \mathcal{F}_{\tau_1}\right) \\ &\otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_{U_\sigma}. \end{aligned}$$

Since $\{\tau_1, \ldots, \tau_l, \sigma_{i+1}\} = \{\tau \in P_{i+1} \mid Z_\sigma \subset Z_\tau\}$, that is the property 3.9 in the case of $Z_\sigma \subset Z_{\sigma_{i+1}}$. Thus we check that they are the $(i+1)$-th ones.

Here we note the following lemma.

Lemma 3.10. *For any i, the image of $\mathcal{H}_{P_i} \subset f_*\omega_{\mathcal{X}/\mathcal{Y}} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X} \to \omega_{\mathcal{X}/\mathcal{Y}}$ is contained in*

$$\omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C) - \sum_{\sigma\in P_i} Z_\sigma\right).$$

Moreover, if $x \in C$ or if x lies on such a component $E \in \mathrm{QIrr}^0(X_y)$ that there is not a quasi-irreducible component of positive genus between C and E, then they coincide with each other at x.

Proof. We prove our assertion by induction on i. If $i = 0$, it is obvious from Lemma 2.2, for $\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C)$ coincides with S_C around such a point x. Suppose it up to i. Put, for simplicity

$$\mathcal{L}_i := \omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C) - \sum_{\sigma\in P_i} Z_\sigma\right).$$

We have the homomorphism $\mathcal{H}_{P_i} \to \mathcal{L}_i$ surjective at x by the induction hypothesis. Since, outside $Z_{\sigma_{i+1}}$, $\mathcal{H}_{P_{i+1}}$ coincides with $\mathcal{H}_{P_i}$ and $\mathcal{L}_{i+1}$ coincides with $\mathcal{L}_i$, we only have to show it over $U_{\sigma_{i+1}}$, but it can be easily seen from

(3.9.2), the properties on $\mathcal{E}_{\sigma_{i+1}}$, $\mathcal{E}_{\sigma_{i+1},\tau_l}$, $\mathcal{E}_{\tau_j,\tau_{j-1}}$'s and $\mathcal{F}_{\tau_1}$ in Proposition 3.8, and the induction hypothesis. $\square$

Let us move on to the construction of $\tilde{E}$ now. We do that in an inductive way. Let us focus on one $y \in \mathrm{CV}(f)$. Put $n := \#\mathcal{P}_y$ and let

$$\emptyset = \mathcal{P}_y(0) \subset \mathcal{P}_y(1) \subset \cdots \subset \mathcal{P}_y(n) = \mathcal{P}_y$$

be a ascending sequence of admissible subsets of $\mathcal{P}_y$ with $\#\mathcal{P}_y(i) = i$. We denote the only element of $\mathcal{P}_y(i) \setminus \mathcal{P}_y(i-1)$ by σ_i. For a locally free subsheaf F_i of $f^*f_*\omega_{X/Y}$, let us consider conditions:

($\spadesuit_i$) we have

$$F_i|_{\mathcal{X}} = \bigoplus_{C \in \mathrm{QIrr}^+(X_y)} \mathcal{H}_{\mathcal{P}_y(i) \cap C} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}\left(-D_C - \sum_{\sigma \in \mathcal{P}_y(i) \text{ with } C \subset Z_\sigma} Z_\sigma\right),$$

($\heartsuit_i$) the image L_i of the homomorphism $F_i \subset f^*f_*\omega_{X/Y} \to \omega_{X/Y}$ is an invertible sheaf.

Suppose that we are given a locally free sheaf F_0 that satisfies ($\spadesuit_0$) and ($\heartsuit_0$). We define the following things inductively: locally free sheaves F_i satisfying ($\spadesuit_i$), sheaves G_i supported in X_y and surjective homomorphisms $\Phi_i : F_{i-1} \to G_i$, for all i with $1 \le i \le n$, and after that, we show that F_i satisfies ($\heartsuit_i$) for any i. Suppose that we have them up to i. In order to construct the $(i+1)$-th ones, we define, for each $C \in \mathrm{QIrr}^+(X_y)$, a homomorphism ψ_C whose sauce is

$$\mathcal{H}_{\mathcal{P}_y(i) \cap C} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}\left(-D_C - \sum_{\sigma \in \mathcal{P}_y(i) \text{ with } C \subset Z_\sigma} Z_\sigma\right)$$

as follows. For $C \in \mathrm{QIrr}^+(X_y)$ with $C \cap Z_{\sigma_{i+1}} = \emptyset$, let ψ_C be the trivial homomorphism

$$\mathcal{H}_{\mathcal{P}_y(i) \cap C} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}\left(-D_C - \sum_{\sigma \in \mathcal{P}_y(i) \text{ with } C \subset Z_\sigma} Z_\sigma\right) \to 0.$$

For $C \in \mathrm{QIrr}^+(X_y)$ with $C \subset Z_{\sigma_{i+1}}$, taking account that $\mathcal{P}_y(i) \cap C = \emptyset$ hence $\mathcal{H}_{\mathcal{P}_y(i) \cap C} = \mathcal{F}_C \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}$ by the admissibility of $\mathcal{P}_y(i)$, let ψ_C be the restriction homomorphism

$$\begin{aligned}\mathcal{F}_C \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} &\left(-D_C - \sum_{\sigma \in \mathcal{P}_y(i) \text{ with } C \subset Z_\sigma} Z_\sigma\right) \\ &\to \mathcal{F}_C \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}\left(-D_C - \sum_{\sigma \in \mathcal{P}_y(i) \text{ with } C \subset Z_\sigma} Z_\sigma\right)\Bigg|_{Z_{\sigma_{i+1}}}.\end{aligned}$$

For $C \in \mathrm{QIrr}^+(X_y)$ with $\sigma_{i+1} \subset C$, let ψ_C be the homomorphism $\phi_{\mathcal{P}_y(i+1)\cap C}$ tensored with $\mathcal{O}_{\mathcal{X}}(-D_C - \sum_{\sigma\in\mathcal{P}_y(i) \text{ with } C\subset Z_\sigma} Z_\sigma)$. Put

$$G_{i+1} := \bigoplus_{C\in\mathrm{QIrr}^+(X_y)} \mathrm{target}(\psi_C),$$

where target(ψ_C) is the target of ψ_C. Via the identification of $(\spadesuit_i)$, we can define a surjective homomorphism $\Phi_{i+1} : F_i \to G_{i+1}$ by

$$\Phi_{i+1}|_{\mathcal{X}} := (\psi_C)_{C\in\mathrm{QIrr}^+(X_y)} : \bigoplus_{C\in\mathrm{QIrr}^+(X_y)} \mathrm{sauce}(\psi_C) \to \bigoplus_{C\in\mathrm{QIrr}^+(X_y)} \mathrm{target}(\psi_C).$$

Then, we can see that $\mathrm{Ker}(\Phi_{i+1})|_{\mathcal{X}}$ is the direct sum of

$$\bigoplus_{C\in\mathrm{QIrr}^+(X_y) \text{ with } C\cap Z_{\sigma_{i+1}}=\emptyset} \mathcal{H}_{\mathcal{P}_y(i)\cap C} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}\left(-D_C - \sum_{\sigma\in\mathcal{P}_y(i) \text{ with } C\subset Z_\sigma} Z_\sigma\right),$$

$$\bigoplus_{C\in\mathrm{QIrr}^+(X_y) \text{ with } C\subset Z_{\sigma_{i+1}}} \mathcal{H}_{\mathcal{P}_y(i)\cap C} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}\left(-D_C - \sum_{\sigma\in\mathcal{P}_y(i+1) \text{ with } C\subset Z_\sigma} Z_\sigma\right),$$

and

$$\mathcal{H}_{\mathcal{P}_y(i+1)\cap C_{i+1}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}\left(-D_{C_{i+1}} - \sum_{\sigma\in\mathcal{P}_y(i) \text{ with } C_{i+1}\subset Z_\sigma} Z_\sigma\right),$$

where C_{i+1} is the quasi-irreducible component including σ_i. Thus we have

$$\mathrm{Ker}(\Phi_{i+1})|_{\mathcal{X}} = \bigoplus_{C\in\mathrm{QIrr}^+(X_y)} \mathcal{H}_{\mathcal{P}_y(i+1)\cap C} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}\left(-D_C - \sum_{\sigma\in\mathcal{P}_y(i+1) \text{ with } C\subset Z_\sigma} Z_\sigma\right).$$

Now we define F_{i+1} as $\mathrm{Ker}(\Phi_{i+1})$. It satisfies the condition $(\spadesuit_{i+1})$, and thus, we have constructed F_i satisfying $(\spadesuit_i)$ for any i. To see that it satisfies the condition $(\heartsuit_i)$, the following lemma is sufficient, for F_i is coincides with F_0 outside of X_y and $(\heartsuit_0)$ is assumed.

Lemma 3.11. *L_i coincides with $\omega_{X/Y}(-B'_y - \sum_{\sigma\in\mathcal{P}_y(i)} Z_\sigma)$ around X_y.*

Proof. It is not difficult to see $L_i|_{\mathcal{X}} \subset \omega_{\mathcal{X}/\mathcal{Y}}(-B'_y - \sum_{\sigma\in\mathcal{P}_y(i)} Z_\sigma)$ from $(\spadesuit_i)$ and the former part of Lemma 3.10. We consider the other inclusion. For any $x \in X_y$, let $C \in \mathrm{QIrr}^+(X_y)$ be a nearest one to x (cf. the proof of Lemma 2.3). Then, since $\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C)$ coincides with S_C around x, we can see, as in the proof of Lemma 2.3, that it is sufficient to show

$$\mathcal{H}_{\mathcal{P}_y(i)\cap C} \to \omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C) - \sum_{\sigma\in\mathcal{P}_y(i) \text{ with } C\not\subseteq Z_\sigma} Z_\sigma\right)$$

is surjective at x. Taking account of the choice of C, we can see that if $C \cap Z_\sigma = \emptyset$, then $x \notin Z_\sigma$, and hence the problem is the surjectivity of

$$\mathcal{H}_{\mathcal{P}_y(i)\cap C} \to \omega_{\mathcal{X}/\mathcal{Y}}\left(-\sum_{\alpha\in(\mathcal{P}_y\setminus C)\cup\mathrm{Sing}_+(X_y)} Z_\alpha(C) - \sum_{\sigma\in\mathcal{P}_y(i)\cap C} Z_\sigma\right)$$

at x. But it is nothing more than the latter part of Lemma 3.10. □

Thus we have constructed the desired ones.

Let E_i be the kernel of $F_i \to L_i$. It is locally free. By the above construction and Lemma 3.11, we have the following diagram, in which any line is exact.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E_i \arrow[r] & F_i \arrow[r] \arrow[d] & L_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_{i-1} \arrow[r] & F_{i-1} \arrow[r] \arrow[d] & L_{i-1} \arrow[r] \arrow[d] & 0 \\
 & & G_i \arrow[d] & L_{i-1}|_{Z_{\sigma_i}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

From that we have $\mathrm{ch}(E_i) = \mathrm{ch}(E_{i-1}) - \mathrm{ch}(G_i) + \mathrm{ch}(L_{i-1}|_{Z_{\sigma_i}})$. By the definition, we see

$$\begin{aligned}
\mathrm{ch}(G_i) &= \sum_{C\in\mathrm{QIrr}^+(X_y) \text{ with } C\subset Z_{\sigma_i}} p_a(C)\left(Z_{\sigma_i} - \frac{1}{2}({Z_{\sigma_i}}^2)\right) \\
&\quad + (p_a(A_{\sigma_i}) + 1)\left(Z_{\sigma_i} - \frac{1}{2}({Z_{\sigma_i}}^2)\right) \\
&= (p_a(Z_{\sigma_i}) + 1)\left(Z_{\sigma_i} - \frac{1}{2}({Z_{\sigma_i}}^2)\right),
\end{aligned}$$

and by Lemma 3.11, we compute to have

$$\mathrm{ch}(L_{i-1}|_{Z_{\sigma_i}}) = Z_{\sigma_i} + (c_1(\omega_{X/Y}) \cdot Z_{\sigma_i}) - \frac{1}{2}({Z_{\sigma_i}}^2).$$

We see accordingly

$$\mathrm{ch}(E_i) = \mathrm{ch}(E_{i-1}) - p_a(Z_{\sigma_i})Z_{\sigma_i} + \frac{p_a(Z_{\sigma_i})}{2}({Z_{\sigma_i}}^2) + (c_1(\omega_{X/Y}) \cdot Z_{\sigma_i})$$

and hence

$$c_1(E_i)^2 = c_1(E_{i-1})^2 + p_a(Z_{\sigma_i})^2({Z_{\sigma_i}}^2) + 2p_a(Z_{\sigma_i})(c_1(\omega_{X/Y}) \cdot Z_{\sigma_i}).$$

Taking account that $\deg(c_1(\omega_{X/Y}) \cdot Z_{\sigma_i}) = 2p_a(Z_{\sigma_i})$ and $({Z_{\sigma_i}}^2) = -2$, we have

$$\begin{aligned}\operatorname{dis}(E_i) &= \deg(-2(g-1)(\operatorname{ch}(E_i))_{(2)} + c_1(E_i)^2) \\ &= \operatorname{dis}(E_{i-1}) - 2p_a(Z_{\sigma_i})p_a(Z^*_{\sigma_i}),\end{aligned}$$

where $(\operatorname{ch}(E_i))_{(2)}$ is the codimension 2 component of $\operatorname{ch}(E_i)$. Summing all them up through $1 \le i \le n$, we obtain

$$\operatorname{dis}(E_n) = \operatorname{dis}(E_0) - \sum_{j=1}^{[\frac{g-1}{2}]} 2j(g-1-j)\xi_j(\mathcal{P}_y).$$

Let F be the locally free sheaf in Section 2 that is constructed from direct summands $\mathcal{F}_C$'s in Proposition 3.8. Then $F_0 := F$ satisfies ($\spadesuit_0$) and ($\heartsuit_0$), and E_0 is nothing more than E constructed in Section 2. Thus, beginning with this F_0 and proceeding that argument for all the singular fibers, we finally obtain a locally free sheaf $\tilde{E}$ such that

$$\operatorname{dis}(\tilde{E}) = \operatorname{dis}(E) - \sum_{j=1}^{[\frac{g-1}{2}]} 2j(g-1-j)\xi_j(\mathcal{P}).$$

Combined with Theorem 2.1 (b), that leads

$$\begin{aligned}\operatorname{dis}(\tilde{E}) &= (8g+4)\deg(f_*\omega_{X/Y}) \\ &\qquad - g\xi_0(\mathcal{P}) - \sum_{j=1}^{[\frac{g-1}{2}]} 2(j+1)(g-j)\xi_j(\mathcal{P}) - \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i(X/Y).\end{aligned}$$

Theorem 3.2 (a) is trivial from the construction. Thus, we complete the proof of Theorem 3.2.

Remark 3.12. Moriwaki knew how to modify the kernel of $f^*f_*\omega_{X/Y} \to \omega_{X/Y}$ around such a singular fiber X_y that has exactly two nodes of type 0 and $\mathcal{P}_y$ consists of the pair of them, which he communicated to the author in a master's course. In that case, we do not need any special preparations for the elementary transformations, and the argument is considerably simple.

Appendix. The quotient fibration of a hyperelliptic fibration

Let S be a connected locally noetherian regular scheme of dimension 1 and let $f : X \to S$ be a generically smooth semistable curve with the normal total space X. Let G be a finite subgroup of $\operatorname{Aut}_S(X)$. Let Y be the quotient X/G, and let $g : Y \to S$ be the structure morphism.

Lemma A.1 (cf. [7]). *Any geometric fiber of g is reduced.*

Proof. Since the finite-group-quotient is compatible with the flat base-change, we may assume that S is the spectrum of a discrete valuation ring (R, tR, k) with the algebraically closed residue field k (cf. [2, Theorem 29.1]) and X is a normal noetherian integral scheme over R. Then Y is also a normal integral scheme flat over S. By the condition (S_2) on Y and the flatness of g, the special fiber Y_s satisfies the condition (S_1), and by the reducedness of the fibers of f, a divisor Y_s is reduced as a Cartier divisor, hence it satisfies the condition (R_0) as a scheme. Thus Y_s is reduced. □

Now suppose that f is hyperelliptic, i.e., there exists an S-automorphism $\iota : X \to X$ of order 2 such that the quotient of the geometric generic fiber by $G := \langle \iota \rangle$ is isomorphic to $\mathbb{P}^1$.

Proposition A.2. *Under the assumption above, any singular fiber of g is a nodal curve of genus* 0.

Proof. We may assume that S is the spectrum of a discrete valuation ring (R, tR, k) with the algebraically closed residue field. By the compatibility, the geometric generic fiber must be the smooth rational curve and hence, from the flatness of g, the arithmetic genus of the special fiber is 0. That implies any irreducible component of the special fiber is a smooth rational curve and for any singular point $y \in Y_s$, the space $Y_s \setminus \{y\}$ is not connected. If $\#\pi^{-1}(y) = 2$, then π is étale over y hence it is a node of Y_s. Suppose that $\pi^{-1}(y)$ consists of one point, namely $\{x\}$. Then, $X_s \setminus \{x\}$ has at most two connected components, and hence so does $Y_s \setminus \{y\}$. That implies any singular point of Y_s is a intersection point of the two irreducible components. Again since the arithmetic genus is 0, the two direction of components must be transversal, and thus it is a node. □

See [7] for more general statements on finite-group-quotients.

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References

[1] M. Cornalba and J. Harris, Divisor classes associated to families of stable varieties, with application to the moduli space of curves, Ann. Sci. École Norm. Sup., **21** (1988), 455–475.

[2] H. Matsumura, Commutative ring theory, Cambridge studies in Adv. Math. 8.

[3] A. Moriwaki, A sharp slope inequality for general stable fibrations of curves, J. Reine Angew. Math., **480** (1996), 177–195.

[4] A. Moriwaki, Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves, J. of AMS, **11** (1998), 569–600.

[5] A. Paranjape and S. Ramanan, On the canonical ring of an algebraic curve, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi NAGATA, vol. II, 1987, pp. 501–516.

[6] K. Yamaki, Geometric Bogomolov's conjecture for curves of genus 3 over function fields, J. Math. Kyoto. Univ., **42** (2002), 57–81.

[7] M. Saïdi, Abhyankar's conjecture II: the use of semi-stable curves, in Courbes semi-stables et groupe foundamental en géométie algébrique, Progress in Math. 187, pp. 249–265.