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Toward a generalization of strong approximation theorem to a general PF field

By

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Abstract

We aim to generalize Eichler's strong approximation theorem, which is known for a division algebra D over a global field K, to the case that K is a general PF field. First we show by an example that the generalized theorem is false for $SL_1(D)$. But if we replace $SL_1(D)$ by the commutator group $[D^\times, D^\times]$, the generalizaion may be possible. Though its validity is not yet known, in this paper we decompose the generalized theorem into two parts, one of which can be formulated in a more general case that K is the quotient field of a Dedekind domain. Further, we prove the equivalence of four approximaion properties (a) $\sim$ (a$'''$), which was open in our previous paper.

1. Preliminary discussions

Let R be a Dedekind domain and K be its quotient field. A prime ideal p of R induces a valuation on K, which is called the p-adic valuation or a prime valuation. We denote the completion of K (resp. R) in this valuation with K_p (resp. R_p). The adele ring K_A is defined as $K_A = \bigcup_S \left(\prod_{p\in S} K_p \times \prod_{p\notin S} R_p\right)$, where S runs over all finite sets of prime ideals. The weak topology is the weakest topology in which all projections $(x_p) \mapsto x_p$ are continuous. The strong topology is the weakest group topology which is stronger than the weak topology and makes $R_A = \prod_p R_p$ open.

The idele group $K_A^\times$ is defined as $K_A^\times = \bigcup_S \left(\prod_{p\in S} K_p^\times \times \prod_{p\notin S} R_p^\times\right)$, ($^\times$ means the multiplicative group of all inversible elements). The weak topology is the restriction of that of K_A, and the strong topology is the one which makes $R_A^\times = \prod_p R_p^\times$ open.

Let D be a central division algebra over K, and Γ be a full R-order in D. The adele ring D_A and the idele group $D_A^\times$ (both are non-commutative) are defined as $D_A = \bigcup_S \left(\prod_{p\in S} D_p \times \prod_{p\notin S} \Gamma_p\right)$ and $D_A^\times = \bigcup_S \left(\prod_{p\in S} D_p^\times \times \prod_{p\notin S} \Gamma_p^\times\right)$, where $D_p = D \otimes_K K_p$ and $\Gamma_p = \Gamma \otimes_R R_p$. The weak and the strong topologies are defined similarly as above, using $\Gamma_A = \prod_p \Gamma_p$ instead of $R_A = \prod_p R_p$.

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K is called a PF field, if it admits the product formula. This means that adding some non-prime valuations to the set of all p-adic valuations, we get the formula $\prod_v v(x) = 1$ for any $x \in K^\times$, where v runs over all prime and non-prime valuations. Artin proved ([1]) that a PF field is nothing but an algebraic number field or an algebraic function field of one variable over some coefficient field. A global field is a PF field such that the completion K_v is locally compact. This is nothing but an algebraic number field or an algebraic function field over a finite field.

A central division algebra D over K splits over some finite extension L of K, namely $D_L = D \otimes_K L$ is isomorphic with the full matrix ring over L. The reduced norm $\mathfrak{n}_{D/K}(x)$ is defined as $\mathfrak{n}_{D/K}(x) = \det x$ regarding $x \in D$ as an element of $D_L \simeq M_n(L)$. Another definition is as follows. For a maximal subfield M of D containing x, we get $\mathfrak{n}_{D/K}(x) = N_{M/K}(x)$ where $N_{M/K}$ is the norm operator. The reduced norm $\mathfrak{n}_{D/K}$ is a group homomorphism from $D^\times$ to $K^\times$. Its kernel is denoted with $SL_1(D)$. Evidently the commutator subgroup $[D^\times, D^\times]$ is contained in $SL_1(D)$. If K is a global field, we have $SL_1(D) = [D^\times, D^\times]$. In general this equality does not hold, and the reduced Whitehead group $SK_1(D) = SL_1(D)/[D^\times, D^\times]$ is a target of research. (see [2])

Regarding D as a K-vector space, $\mathfrak{n}_{D/K}$ is a polynomial function, so that it can be extended naturally as a group homomorphism from $D_A^\times$ to $K_A^\times$. This enables us to define $SL_1(D_A)$.

Eichler inquired when $SL_1(D)$ is dense in $SL_1(D_A)$ in the strong topology. ($D^\times$ is imbedded diagonally in $D_A^\times$). For an algebraic number field K, he found the necessary and sufficient condition is that $D_v = D \otimes_K K_v$ is not a division algebra for some non-prime valuation v. This condition is called Eichler condition, and his result is called Eichler's strong approximation theorem. Soon his result was generalized to a global field with positive characteristic, namely to an algebraic function field over a finite field. Afterwards, the theorem was generalized for a wider class of algebraic groups apart from $SL_1(D)$. Now we know the complete result for any semi-simple algebraic group (see [4]), but only over a global field.

In this paper our interest is concerned with a generalization to a non-global K. Instead, the group is limited in the classical $SL_1(D)$ or $[D^\times, D^\times]$.

2. Counter example

In this section we shall show by an example that the strong approximation theorem does not hold for $SL_1(D)$. In general $SL_1(D)$ does not have even the weak approximation property, as Platonov proved as a byproduct of his study of the reduced Whitehead group. Here we shall sketch the outline of his argument, and discuss the relation to the strong approximation theorem.

Let K be a field, K_v be its completion in a valuation v over K. Let D be a division algebra over K. The weak approximation proprety of $SL_1(D)$ means that $SL_1(D)$ is dense in $SL_1(D_v)$. But it is known that $[D_v^\times, D_v^\times]$ is an open subgroup of $SL_1(D_v)$ (see [2]), so that the closure $\overline{SL_1(D)}$ of $SL_1(D)$

is nothing but $SL_1(D)[D_v^\times, D_v^\times]$. (Since $D^\times$ is dense in $D_v^\times$, the commutator subgroup $[D^\times, D^\times]$ has the weak approximation property, so we have $\overline{SL_1(D)} \supset [D_v^\times, D_v^\times]$). This means that $SL_1(D_v)/\overline{SL_1(D)} \simeq SK_1(D_v)/\operatorname{Im}\varphi$, where φ is the homomorphism $SK_1(D) \to SK_1(D_v)$ induced by the imbedding $D \hookrightarrow D_v$.

Therefore we get the desired counter example if we show that φ is not surjective. Especially it is sufficient if $|SK_1(D)| < |SK_1(D_v)|$, both being finite.

Let k be an algebraic number field, and $K = k((y, z))$ be the formal power series field of two variables over k. Assume that k has a primitive n-th root of 1. Consider a valuation w on k. Take cyclic extensions $L_1 = k(\sqrt[n]{\alpha})$ and $L_2 = k(\sqrt[n]{\pi})$ such that $L_1 k_w$ is unramified and $L_2 k_w$ is totally ramified over k_w. These induce also cyclic extensions of K, so we can construct the following cyclic algebras D_1 and D_2 over K.

$$D_1 = A(K(\sqrt[n]{\alpha}), y), \quad D_2 = A(K(\sqrt[n]{\pi}), z)$$

(This means $D_1 = \bigoplus_{j=0}^{n-1} K(\sqrt[n]{\alpha})u^j$, $u^n = y$, $\xi u = u\xi^\sigma$ $({}^\forall \xi \in K(\sqrt[n]{\alpha}))$ where σ is a generator of $\mathrm{Gal}(k(\sqrt[n]{\alpha})/k)$. D_2 is defined similarly).

Put $D = D_1 \otimes_K D_2$ and $D(x) = D \otimes_K K(x)$ where $K(x)$ is the rational function field over K. $D(x)$ is a division algebra over $K(x)$, and gives a counter example of the weak approximation property, as explained below.

k_w contains an infinite Galois extension of k, so that it contains a finite Galois extension of k with arbitrarily large dimension over k. Let $t(x)$ be an irreducible polynomial over k defining such an Galois extension. $t(x)$ is irreducible also over $K = k((y, z))$, and induces a valuation $v_{t(x)}$ on $K(x)$. Let $K(x)_{t(x)}$ be the completion of $K(x)$ in the valuation $v_{t(x)}$. Its residue field is isomorphic to $k_{t(x)} = k[x]/t(x)k[x]$. The weak appproximation property means that $SL_1(D(x))$ is dense in $SL_1(D(x)_{t(x)})$ where $D(x)_{t(x)} = D(x) \otimes_{K(x)} K(x)_{t(x)}$.

The reduced Whitehead group can be expressed in terms of the Brauer group as follows (see [2] and [3]).

(1) $SK_1(D(x)_{t(x)}) \simeq Br(k_{t(x)}L_1L_2/k_{t(x)})/Br(k_{t(x)}L_1/k_{t(x)})Br(k_{t(x)}L_2/k_{t(x)})$

(2) $SK_1(D(x)) \simeq Br(L_1L_2/k)/Br(L_1/k)Br(L_2/k)$

The latter group (2) is a finite group, while the former group (1) is also finite but its order is not bounded as the degree of $t(x)$ tends to infinity. (see [3] in detail). So if $\deg t$ is large enough, we have $|SK_1(D(X)_{t(x)})| > |SK_1(D(x))|$ and this gives a counter example for the weak approximation property of $SL_1(D(x))$.

This is the outline of Platonov's argument. Next we shall explain how it provides a counter example for the strong approximation theorem.

Since $SL_1(D(x))$ does not have the weak approximation property, it does not have the strong approximation property neither. On the other hand, if we regard $K(x)$ as the rational function field over K, $D(x)$ does not satisfy Eichler condition (which we denote with (ec)).

However, regarding $K(x)$ as an algebraic function field, $D(x)$ can satisfy (ec) as explained below. In general, for infinitely many irreducible polinomials $p(x)$, the completions $D(x)_{p(x)}$ (in the corresponding valuation $v_{p(x)}$) are not division algebras. (The proof is in [6]). Take one of such $p(x)$.

$K(x)$ is a finite extension of $K(p(x))$. (Namely for $L = K(X)$, we have $K(x) = L(x')$, $Xp(x') = 1$). Then $K(x)$ can be regarded as an algebraic function field and as such $v_{p(x)}$ is a non-prime valuation, so $D(x)$ satisfies (ec). In other words, if we do not insist on the rational function field and allow to consider an algebraic function field, any valuation can be non-prime. In this sense, the above example is a counter example for the strong approximation theorem.

3. Decomposition of the theorem into two parts

As mentioned in Section 2, the strong approximation theorem does not hold for $SL_1(D)$. But if we replace $SL_1(D)$ by the commutator subgroup $[D^\times, D^\times]$, the possibility of the generalization of the theorem to the case of a general PF field still remains. The known results ([5] and [6]) strongly suggests this possibility.

Though we do not yet know whether the generalized theorem holds or not, in this section we shall decompose the theorem into two parts, one of which can be formulated for a more general quotient field of an arbitrary Dedekind domain.

Before starting the above argument, we shall restate four approximation properties for $[D^\times, D^\times]$, which are mentioned in [6] in connection with the criteria of the cancellation of lattices over orders.

Let K be the quotient field of a Dedekind domain R. Let D be a central division algebra over K. We consider four approximation properties (a) $\sim$ (a‴) as follows.

(a) $[D^\times, D^\times]$ is dense in $[D_A^\times, D_A^\times]$ (in the strong topology defined in Section 1).

(a′) $[D_A^\times, D_A^\times]$ is contained in the closure of $D^\times$.

(a″) $[D_A^\times, D_A^\times]$ is contained in the closure of $R_A^\times D^\times$.

(a‴) $[D_A^\times, D_A^\times]$ is contained in the closure of $K_A^\times D^\times$.

($R_A^\times$ and $K_A^\times$ are canonically imbedded in $D_A^\times$). Evidently we have (a) $\Rightarrow$ (a′) $\Rightarrow$ (a″) $\Rightarrow$ (a‴).

Let K be a PF field and consider the following Eichler condition (ec).

(ec) There exists a non-prime valuation v such that D_v is not a division algebra.

The strong approximation theorem claims that (ec) is equivalent with (a).

In [6], we already proved (a″) $\Rightarrow$ (ec) for a general PF field. So the unsolved assertion is (ec) $\Rightarrow$ (a). The proof of (a″) $\Rightarrow$ (ec) can be applied evidently to (a) $\Rightarrow$ (ec). Regarding this proof as such, we can decompose it into two parts: $\neg$(ec) $\Rightarrow$ (d) and (d) $\Rightarrow$ $\neg$(a), where $\neg$ means the negation of the proposition, and (d) means

(d) $[D^\times, D^\times]$ is discrete in the strong topology.

For the sake of convenience, we restate $\neg$(ec) and $\neg$(a) below.

$\neg$(ec): For any non-prime valuation v, D_v is a division algebra.

$\neg$(a): $[D^\times, D^\times]$ is not dense in $[D_A^\times, D_A^\times]$.

(d) $\Rightarrow$ $\neg$(a) is evident. The proof of $\neg$(ec) $\Rightarrow$ (d) is done using the product formula and the fact that $[D_v^\times, D_v^\times]$ is bounded for a division algebra $D_v^\times$. (When K is a global field, $[D_v^\times, D_v^\times]$ is compact. For a general PF field K, $[D_v^\times, D_v^\times]$ is not compact, but its boundedness is sufficient for the proof of $\neg$(ec) $\Rightarrow$ (d)).

The strong approximation theorem (ec) $\Rightarrow$ (a) holds true, if both (ec) $\Rightarrow$ $\neg$(d) and $\neg$(d) $\Rightarrow$ (a) hold true. Because $\neg$(ec) $\Rightarrow$ (d) and (d) $\Rightarrow$ $\neg$(a) are known to be true, any counter example of either of (ec) $\Rightarrow$ $\neg$(d) or $\neg$(d) $\Rightarrow$ (a) provides a counter example of (ec) $\Rightarrow$ (a). In this sense, the strong approximation theorem is decomposed into two statements:

(ec) $\Rightarrow$ $\neg$(d) and $\neg$(d) $\Rightarrow$ (a).

Note that the second half $\neg$(d) $\Rightarrow$ (a) does not involve non-prime valuations. So this part can be regarded as a statement for a more general case of the quotient field of an arbitrary Dedekind domain.

When K is a global field, the strong approximation theorem holds true. Its proof essentially uses the fact that the compactness of $[D_v^\times, D_v^\times]$ is equivalent to D_v being a division algebra. The replacement of the compactness by the boundedness destroys the validity of the proof when K is a general PF field. Some more detailed arguments will be needed, as to whether the generalized strong approximation theorem is true or not.

4. Equivalence of four approximation properties

One of the clues of solving the problem is the non-central simplicity of $[D_v^\times, D_v^\times]$ when D_v is not a division algebra. Though this clue is not effective for the proof of the strong approximation theorem, it works well for the proof of the equivalence of four approximation properties, which was an open problem in our previous [6]. We shall explain this below.

First we shall give a lemma on normal subgroups of a product group.

Lemma. *A normal subgroup N of a product group $G \times G'$ can be written in the following way. For normal subgroups N_1, N_2 (resp. N_1', N_2') of G (resp. G') such that $[G, N_1] \subset N_2 \subset N_1$ (resp. $[G', N_1'] \subset N_2' \subset N_1'$) and for an isomorphism ϕ of N_1/N_2 onto N_1'/N_2', we have*

$$N = \bigcup_{\overline{x} \in N_1/N_2} \overline{x} \times \phi(\overline{x}),$$

where the N_2-coset $\overline{x}$ (resp. N_2'-coset $\phi(\overline{x})$) is regarded as a subset of G (resp. G').

Proof. Let π (resp. π') be the projection of $G \times G'$ onto G (resp. G'). Put $N_1 = \pi(N)$ and $N_2 \times (1) = N \cap (G \times (1))$. They are normal subgroups of G, and $N_2 \subset N_1$.

For $(x, x') \in N$ and $y \in G$, we have $N \ni [(x, x'), (y, 1)] = ([x, y], [x', 1]) = ([x, y], 1)$, so that we get $[G, N_1] \subset N_2$.

We apply similar discussions to G'.

For $x \in N_1$, $(x, x'_1) \in N$ and $(x, x') \in N$ imply $x'_1 x'^{-1} \in N'_2$, so that the section of N with respect to x is a N'_2-coset.

For $x_1, x_2 \in N_1$, the corresponding section coincide if and only if x_1 and x_2 belong to the same N_2-coset. Thus a mapping $\phi : N_1/N_2 \to N'_1/N'_2$ is induced.

Applying similar discussions by switching G and G', we see that ϕ is bijective. Then N must be in the form of the theorem as a subset of $G \times G'$. Since N is a subgroup, ϕ must be a group isomorphism.

Conversely, the subset N of $G \times G'$ in the form of the theorem is evidently a subgroup.

Any conjugate by an element of G (resp G') does not change the N_2-coset (resp. N'_2-coset), since $[G, N_1] \subset N_2$ (resp. $[G', N'_1] \subset N'_2$). Therefore N is a normal subgroup of $G \times G'$. $\square$

Remark. If G is non-central simple (namely if G does not have a proper normal subgroup not contained in the center Z), we have $N_1 = N_2 = G$ or $N_2 \subset N_1 \subset Z$. (We assume that $[G, G] \not\subset Z$).

Theorem. *Four conditions* (a) $\sim$ (a$'''$) *given in Section* 3 *are equivalent.*

Proof. It is sufficient to show (a$'''$) $\Rightarrow$ (a). First decompose the set P of all prime ideals of R into two parts:

$$P_1 = \{p \in P | D_p \text{ is not a division algebra}\},$$
$$P_2 = \{p \in P | D_p \text{ is a division algebra}\}.$$

Correspondingly, the idele group $D_A^\times$ can be written as the direct product

$$D_A^\times = D_{AP_1}^\times \times D_{AP_2}^\times,$$

where $D_{AP_1}^\times = \bigcup_S \left(\prod_{p \in S} D_p^\times \times \prod_{p \in P_1 \setminus S} \Gamma_p^\times \right)$, S being a finite subset of P_1. $D_{AP_2}^\times$ is defined similarly.

Let H be the closure of $[D^\times, D^\times]$ in $[D_A^\times, D_A^\times]$ in the strong topology. Our goal is to show $H = [D_A^\times, D_A^\times]$ under the assumption (a$'''$).

First we shall show that H is a normal subgroup of $[D_A^\times, D_A^\times]$ under (a$'''$). It is because the normalizer of $[D^\times, D^\times]$ contains both $D^\times$ and $K_A^\times$ (since $K_A^\times$ is the center of $D_A^\times$), so that the normalizer of H contains the closure of $K_A^\times D^\times$ which contains $[D_A^\times, D_A^\times]$ by (a$'''$). $\square$

Proposition 1. $\forall p \in P_1$, $H \supset [D_p^\times, D_p^\times] \times (1)$.

Proof. Decompose P as the sum of p and others, and we have

$$[D_A^\times, D_A^\times] = [D_p^\times, D_p^\times] \times [D_{AP\setminus p}^\times, D_{AP\setminus p}^\times].$$

Applying the remark to the lemma to this direct product, we see $N_1 = N_2 = [D_p^\times, D_p^\times]$, where $N_1 = \pi_p(H)$ and $N_2 \times (1) = H \cap ([D_p^\times, D_p^\times] \times (1))$. Note that for $p \in P_1$, $[D_p^\times, D_p^\times]$ is non-central simple and $N_1 \supset [D^\times, D^\times]$ is not contained in the center of $[D_p^\times, D_p^\times]$. $\square$

Proposition 2. $H \supset [D_{AP_1}^{\times}, D_{AP_1}^{\times}] \times (1)$.

Proof. From Proposition 1, we see

$$H \supset \prod_{p \in S} [D_p^{\times}, D_p^{\times}] \times (1)$$

for a finite subset S of P_1. However, the closure of $\bigcup_S \left(\prod_{p \in S} [D_p^{\times}, D_p^{\times}] \times (1) \right)$ is nothing but $[D_{AP_1}^{\times}, D_{AP_1}^{\times}] \times (1)$. (Take any element $x = (x_p)$ in $[D_{AP_1}^{\times}, D_{AP_1}^{\times}]$. For a finite subset S of P_1, consider the idele $x_S = (x_{Sp})$ such that $x_{Sp} = x_p$ for $p \in S$ and $x_{Sp} = 1$ for $p \in P_1 \setminus S$. Evidently x_S converges to x in the strong topology). □

Proposition 3. $H = [D_{AP_1}^{\times}, D_{AP_1}^{\times}] \times [D_{AP_2}^{\times}, D_{AP_2}^{\times}] = [D_A^{\times}, D_A^{\times}]$.

Proof. Applying the lemma to the direct product

$$[D_{AP_1}^{\times}, D_{AP_1}^{\times}] \times [D_{AP_2}^{\times}, D_{AP_2}^{\times}] = [D_A^{\times}, D_A^{\times}],$$

we get $H = [D_{AP_1}^{\times}, D_{AP_1}^{\times}] \times H_2$ since $N_2 = [D_{AP_1}^{\times}, D_{AP_1}^{\times}]$ by Proposition 2. The closedness of H in the product group implies the closedness of H_2 in $[D_{AP_2}^{\times}, D_{AP_2}^{\times}]$. H_2 contains $\pi'([D^{\times}, D^{\times}])$, namely the diagonal imbedding of $[D^{\times}, D^{\times}]$ in $[D_{AP_2}^{\times}, D_{AP_2}^{\times}]$. So, if $[D_{AP_2}^{\times}, D_{AP_2}^{\times}]$ satisfies the strong approximation property (a), we get $H_2 = [D_{AP_2}^{\times}, D_{AP_2}^{\times}]$. □

Proposition 4. *If D_p is a division algebra for every valuation p, then the strong approximation property* (a) *is satisfied* (*independently of the condition* (a‴).)

Proof. For almost all p, Γ_p is a maximal R_p-order. But since D_p is a division algebra, the maximal R_p-order is unique and equals to R_{D_p}. (The valuation p on K_p extends uniquely on D_p, and its (non-commutative) valuation ring is R_{D_p}).

Then we have $[D_p^{\times}, D_p^{\times}] \subset SL_1(D_p) \subset R_{D_p}^{\times} = \Gamma_p^{\times}$. (Denoting the reduced norm $D_p \to K_p$ as $\mathfrak{n}$, we have $SL_1(D_p) = \ker \mathfrak{n}$, $R_{D_p} = \mathfrak{n}^{-1}(R_p)$, and $R_{D_p}^{\times} = \mathfrak{n}^{-1}(R_p^{\times})$).

Therefore, the strong topology and the weak topology coincide on $[D_A^{\times}, D_A^{\times}]$. (The fundamental neighbourhood of unity is given by $\left(\prod_{p \in S} U_p \times \prod_{p \notin S} D_p^{\times} \right) \cap D_A^{\times}$ in the weak topology and $\prod_{p \in S} U_p \times \prod_{p \notin S} \Gamma_p^{\times}$ in the strong topology, where U_p is a neighbourhood of unity in $D_p^{\times}$. But on the set $[D_A^{\times}, D_A^{\times}]$, the condition $x_p \in \Gamma_p^{\times}$ holds automatically for $p \notin S$ if S is large enough.)

Since $[D^{\times}, D^{\times}]$ is dense in $[D_A^{\times}, D_A^{\times}]$ in the weak topology (the weak approximation property), this coincidence of two topologies assures the strong approximation property. □

Remark. Without the condition (a$'''$), (a) can be proved if only Proposition 1 is true. (The discussions after Proposition 1 do not use the fact that H is a normal subgroup).

More weakly, the following condition (s) is sufficient for the strong approximation property (a).

(s) $\forall p \in P_1$, $\exists x \in [D_p^\times, D_p^\times] \setminus K_p$, $(x, 1) \in H$.

($(x, 1)$ is the idele whose p-component is x and other components are 1.) (s) means that $N_2 = H \cap ([D_p^\times, D_p^\times] \times (1))$ is not contained in the center, so it implies Proposition 1 since $[D_p^\times, D_p^\times]$ is non-central simple. N_2 is a normal subgroup of $[D_p^\times, D_p^\times]$ even if H is not so, because the normalizer of N_2 contains $[D^\times, D^\times]$ and weakly closed in $[D_p^\times, D_p^\times]$, so it equals to $[D_p^\times, D_p^\times]$ by the weak approximation property. (The normalizer of N_2 is weakly closed, because $x_j \to x$ in $D_p^\times$ implies $(x_j, 1) \to (x, 1)$ in $D_A^\times$ in the strong topology).

Indeed, for the case of a global field and for the case of the rational function field over real field, the strong approximation theorem was proved by checking the condition (s). This is a very effective tool, but it may be difficult to find such $x \in [D_p^\times, D_p^\times] \setminus K_p$ concretely for the case of a general K. Still it provides a hint for the proof of the generalized strong approximation theorem.

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References

[1] E. Artin, Algebraic Numbers and Algebraic Functions I, Gordon and Breach, 1951.

[2] V. Platonov, The Tannaka-Artin Problem and Reduced K-theory, Math. USSR Izvestija, **10** (1976), 211–243.

[3] V. Platonov, Reduced K-theory and Approximation in Algebraic Groups, Trudi Mat. Inst. Steklov, **142** (1976), 198–207; Engl. trans. Proc. Steklov Inst. Math. Issue, **3** (1979), 213–224.

[4] V. Platonov and A. Rapinchuk, Algebraic Groups and Number Theory, Chapter 7 Academic Press, 1994.

[5] A. Yamasaki, Strong Approximation Theorem for Division Algebras over $\mathbb{R}(X)$, J. Math. Soc. Japan, **49**-3 (1997), 455–467.

[6] A. Yamasaki, Cancellation of Lattices and Approximation Properties of Division Algebras, J. Math. Kyoto Univ., **36**-4 (1996), 857–867.