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Explicit lower bound of the Ricci tensor on free loop algebras

By

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1. Introduction

In this article we will give an explicit expression of the Ricci tensors and their lower bounds associated with the "Levi-Civita connection" on the free loop group over a compact Lie group. We will equip various H_1-Hilbert norms on the free loop algebra. So various Ricci tensors associated with them are defined. We will calculate the lower bound of those Ricci tensors quite explicitly in terms of the Fourier series.

The lower bound of the Ricci tensor is so important that it appears in the logarithmic Sobolev inequality with respect to the heat kernel measures on loop groups, i.e.,

$$\int_{\mathcal{L}(G)} f^2 \log \frac{f^2}{\|f\|^2_{L^2(\nu^T)}} d\nu^T \leq \frac{2(e^{CT}-1)}{C} \int_{\mathcal{L}(G)} \|\nabla f\|^2 d\nu^T, \tag{1.1}$$

where $-C$ is the lower bound of the Ricci tensor associated with the given H^1-metric and $d\nu^T$ is the heat kernel measure at time $T > 0$ associated with the given H^1-metric. This kind of logarithmic Sobolev inequality was first proved by Driver and Lohrenz [3] on pinned loop groups with the usual metric

$$\|X\|^2 = \int_0^1 |X'(t)|^2 dt = \sum_{n \in \mathbf{Z}} (2\pi n)^2 |\hat{X}(n)|^2. \tag{1.2}$$

The Ricci tensor for the pinned loop algebra case was explicitly obtained by Freed [5] for the metric

$$\|X\|^2 = \sum_{n \in \mathbf{Z}} n^2 |\hat{X}(n)|^2.$$

The lower bound of the Ricci tensor for this metric is 1 if we assume (2.1). (In fact, the definitions in Freed [5] are slightly different from those in [3], [1] and

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this paper. See Remark 3.4). The logarithmic Sobolev inequality of type (1.1) on free loop groups was first proved by Carson [1], [2], in which the norm is defined by

$$\|X\|^2 = \int_0^1 |X(t)|^2 dt + \int_0^1 |X'(t)|^2 dt.$$

After Carson, Inahama [6] did in terms of Fourier series. However, the explicit expression of the lower bound of the Ricci tensor is not known. We will compute it in this article.

Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra. We assume that the Ad_G-invariant inner product of $\mathfrak{g}$ is given by minus of the Killing form (see assumption (2.1)). The Hilbert spaces we use in this article are the Sobolev space of differential order one whose norms are given by

$$\|X\|^2_{H_1^\delta} = \sum_{n \in \mathbf{Z}} (1 + \delta^2 n^2) |\hat{X}(n)|^2.$$

Here $\delta > 0$ is a parameter. When $\delta = 2\pi$, $\|X\|_{H_1^\delta}$ coincides with the norm in Carson [1], [2]. and when $\delta = 1$, $\|X\|_{H_1^\delta}$ coincides with the norm in Inahama [6].

For each given Hilbert norm, the "Levi-Civita covariant derivative" operator, the curvature, the Ricci tensor are defined as in (2.3), (2.4) and (2.5). In Theorem 2.5 we will obtain an explicit expression of the Ricci tensor by means of the residue theorem. In Theorem 3.1 and Corollary 3.2 we will obtain the lower bound of the Ricci tensor as explicitly as possible. For example, the lower bound for the case $\delta = 1$ is $-7\pi \coth(\pi)/80$ and that for the case $\delta = 2\pi$ is $-16^{-1} \coth(1/2)(8\pi^2 - 1)(1 + 4\pi^2)^{-1}(1 + \pi^2)^{-1}$.

2. Ricci tensor expressed in terms of fourier series

Let G be a compact Lie group of dimension d with its Lie algebra $\mathfrak{g}$ which is endowed with an Ad_G-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Since G is compact, such an inner product always exists. We will assume for simplicity that minus of the Killing form is equal to the inner product, i.e.,

$$\langle a, b \rangle_{\mathfrak{g}} = \sum_{i=1}^{d} \langle [e_i, a], [e_i, b] \rangle_{\mathfrak{g}}, \tag{2.1}$$

where $\{e_i\}_{i=1}^d$ are an (in fact any) orthonormal basis of $\mathfrak{g}$. It is well-known that if $\mathfrak{g}$ is semisimple, then the Killing form $-K\langle a, b \rangle = \mathrm{Trace}(\mathrm{ad}(a)\mathrm{ad}(b))$ is strictly negative-definite and $K\langle a, b \rangle$ becomes an inner product which satisfies (2.1). We can easily modify our results in this paper for the cases in which (2.1) is not satisfied (see Remark 3.3). Since we use complex Fourier series we prepare complexification $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{g}$. As usual we define the conjugation by $\overline{a + b\sqrt{-1}} = a - b\sqrt{-1}$ and the Hermitian inner product on $\mathfrak{g}^{\mathbf{C}}$ by $\langle a + b\sqrt{-1}, a' + b'\sqrt{-1} \rangle_{\mathfrak{g}^{\mathbf{C}}} = \langle a + b\sqrt{-1}, \overline{a' + b'\sqrt{-1}} \rangle_{\mathfrak{g}}$, where $a, b, a', b' \in \mathfrak{g}$.

Let

$$\mathcal{L}(\mathfrak{g}) = \{l : [0,1] \to \mathfrak{g}|\ l \text{ is continuous and } l(0) = l(1)\}$$

be the continuous free loop algebra and let

$$\hat{X}(n) = \int_0^1 X(t)e^{-2\pi\sqrt{-1}nt}dt \in \mathfrak{g}^{\mathbf{C}}$$

be the n-th Fourier coefficient of $X \in \mathcal{L}(\mathfrak{g})$. That X is $\mathfrak{g}$-valued is equivalent to $\hat{X}(-n) = \overline{\hat{X}(n)}$ for all $n \in \mathbf{Z}$. We define Hilbert spaces as a linear subspace of $\mathcal{L}(\mathfrak{g})$ by

$$H_1^\delta = \left\{ X \in \mathcal{L}(\mathfrak{g}) \,\middle|\, X \text{ is } \mathfrak{g}\text{-valued and } \|X\|_{H_1^\delta}^2 = \sum_{n\in\mathbf{Z}} (1+\delta^2 n^2)\|\hat{X}(n)\|_{\mathfrak{g}^{\mathbf{C}}}^2 < \infty \right\} \tag{2.2}$$

for $\delta > 0$. Obviously all H_1^δ-norms are equivalent and all H^δ are the same set. It is well-known that $[X,Y] \in H_1^\delta$ for $X, Y \in H_1^\delta$, where $[X,Y](t) = [X(t), Y(t)]$. If we set $\delta = 2\pi$, then we can easily see that $\|X\|_{H_1^{2\pi}}^2 = \int_0^1 \|X(t)\|_{\mathfrak{g}}^2 dt + \int_0^1 \|X'(t)\|_{\mathfrak{g}}^2 dt$ and therefore this definition of $H_1^{2\pi}$-norm is the same as that in Carson [1], [2]. Note also that H_1^1-norm above coincides with the special case in Malliavin [7] or in Inahama [6] and that a constant multiple of $H_1^{1/\epsilon}$-norm coincides with the norm in Fang [4].

Let us define $D^{(\delta)} = D$ by

$$D_X Y = \frac{1}{2}(\mathrm{ad}_X Y - \mathrm{ad}_X^* Y - \mathrm{ad}_Y^* X) \tag{2.3}$$

for $X, Y \in H_1^\delta$. Here $\mathrm{ad}_X Y = [X,Y]$ and ad_X^* denotes the adjoint operator of $\mathrm{ad}_X : H_1^\delta \to H_1^\delta$.

Proposition 2.1. *Let $D^{(\delta)} = D$ be as above. Then the linear map $Y \mapsto DY$ is a bounded operator from H_1^δ to $H_1^{\delta *} \otimes H_1^\delta$. Moreover, D is torsion free in the sense that $D_X Y - D_Y X = [X,Y]$ for any $X, Y \in H_1^\delta$ and metric compatible in the sense that $(D_X Y, Z)_{H_1^\delta} = -(Y, D_X Z)_{H_1^\delta}$ for any $X, Y, Z \in H_1^\delta$.*

Proof. See Inahama [6] for the Hilbert-Schmidt property when $\delta = 1$. See also Freed [5] and Carson [1]. The modification for general $\delta > 0$ is easy. So we omit the proof. □

As usual the curvature $R = R^{(\delta)}$ is defined by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{2.4}$$

for $X, Y, Z \in H_1^\delta$. The Ricci tensor $\mathrm{Ric} = \mathrm{Ric}^{(\delta)}$ is defined by

$$\mathrm{Ric}(X, W) = \sum_{Z:good} \Big(R(X,Z)Z, W \Big)_{H_1^\delta}, \tag{2.5}$$

where $\sum_{Z:good}$ denote the sum over "good basis" $\{Z\}$ of H^δ. Since we cannot expect that the sum in the right hand side of (2.5) converges for any orthonormal basis of H_1^δ, we allow only "good basis". (See Driver and Lorenz [3]). Though the definitions of "good basis" in [3], [1], [6] are different, the definitions of $\mathrm{Ric}(X, W)$ coincide. Note that the sum in (2.5) does not depend on the choice of "good basis." See [3], [1], [6] for proofs.

Carson [1] is the first paper which contains the explicit expression of the Ricci tensor on free loop algebra.

Proposition 2.2 (Carson [1]). *Let $\delta = 2\pi$. Then the Ricci tensor is well-defined and we have*

$$\begin{aligned}\mathrm{Ric}^{(2\pi)}(X, X) = &\int_0^1 \int_0^1 \left\{ \frac{5}{4} G^2(s,t) + F^2(s,t) \right\} \langle X(s), X(t) \rangle_{\mathfrak{g}} ds dt \\ &- \int_0^1 G(s,s) \langle X(s), X(s) \rangle_{\mathfrak{g}} ds.\end{aligned}$$

Here

$$\begin{aligned} G(s,t) &= \frac{1}{2} \sinh(1/2)^{-1} \cosh(r(s,t)), \\ F(s,t) &= \frac{1}{2} \sinh(1/2)^{-1} \sinh(r(s,t)) \end{aligned}$$

and

$$r(s,t) = \begin{cases} s - t - \dfrac{1}{2} & (\textit{if} \quad 0 \le t < s), \\ s - t + \dfrac{1}{2} & (\textit{if} \quad s \le t \le 1). \end{cases}$$

Proof. See Carson [1], [2]. □

For general $\delta > 0$, we have the following expression of the Ricci tensor in terms of the Fourier series.

Lemma 2.3. *The Ricci tensors are well-defined and we have*

$$\begin{aligned}&\mathrm{Ric}^{(\delta)}(X, X) \\ &\quad = \sum_{l \in \mathbf{Z}} (1 + \delta^2 l^2) \|\hat{X}(l)\|^2_{\mathfrak{g}^{\mathbf{C}}} \sum_{n \in \mathbf{Z}} \frac{1 - 4\delta^2 l n}{4(1+\delta^2 l^2)(1+\delta^2 n^2)(1+\delta^2 (n-l)^2)}.\end{aligned}$$

In particular, $\mathrm{Ric}^{(\delta)}$ *is bounded in the sense that there exists a positive constant* c_δ *such that*

$$|\mathrm{Ric}^{(\delta)}(X,X)| \leq c_\delta \|X\|^2_{H_1^\delta}$$

for any $X \in H_1^\delta$.

Proof. In Inahama [6] it is shown that, for $\delta = 1$,

$$\mathrm{Ric}^{(1)}(X,X) = \sum_{l\in\mathbf{Z}} (1+l^2)\|\hat{X}(l)\|^2_{\mathfrak{g}^{\mathbf{C}}} \sum_{n\in\mathbf{Z}} \frac{T_{l+n,-n} - T_{l,-n}T_{n,l-n}}{1+n^2},$$

where

$$T_{l,n} = \frac{1}{2}\Big(1 + \frac{-(1+l^2)+(1+n^2)}{1+(l+n)^2}\Big) = \frac{1+2n(l+n)}{2(1+(l+n)^2)}.$$

Hence we have

$$\begin{aligned}
&T_{l+n,-n} - T_{l,-n}T_{n,l-n} \\
&= \frac{1-2nl}{2(1+l^2)} - \frac{1-2n(l-n)}{2(1+(l-n)^2)} \cdot \frac{1+2(l-n)l}{2(1+l^2)} \\
&= \frac{(1-2nl)2\{1+(l-n)^2\} - \{1-2n(l-n)\}\{1+2(l-n)l\}}{4(1+l^2)(1+(l-n)^2)} \\
&= \frac{2-4nl+2(l-n)^2-4nl(l-n)^2-1-2l(l-n)+2n(l-n)+4nl(l-n)^2}{4(1+l^2)(1+(l-n)^2)} \\
&= \frac{1-4nl}{4(1+l^2)(1+(l-n)^2)}.
\end{aligned}$$

The boundedness of the Ricci tensor is easily verified. This proves the lemma for $\delta = 1$. For general $\delta > 0$, we have only to multiply all the integers by δ. $\square$

In order to obtain more explicit expression for the Ricci tensor than in Lemma 2.3, we need a lemma of the complex function theory. For a meromorphic function F, we denote the residue of F at $\zeta \in \mathbf{C}$ by $\mathrm{Res}(F;\zeta)$.

Lemma 2.4. *Let* $f(z)$ *and* $g(z)$ *be polynomials such that* $\deg f + 2 \leq \deg g$ *and let* $F(z) = f(z)/g(z)$. *Suppose further that none of the poles of* F *is an integer. Then we have*

$$\sum_{n\in\mathbf{Z}} F(n) = -\sum_{\zeta} \mathrm{Res}(F\cdot w;\zeta),$$

where the sum on the right hand side is taken all over the poles of F *and* $w(z) = \pi/\tan(\pi z)$.

Proof. This is a simple application of residue theorem. Note that the poles of w are $\mathbf{Z}$ and they are all the single poles with $\mathrm{Res}(w; n) = 1 \quad (n \in \mathbf{Z})$. Hence the poles of $F \cdot w$ are disjoint union of the poles of F and $\mathbf{Z}$ and $\mathrm{Res}(F \cdot w; n) = F(n) \quad (n \in \mathbf{Z})$.

Let $r > 0$ be a positive number such that all the poles of F are included by the quadrangle $D_r = \{z = x + y\sqrt{-1} \in \mathbf{C} | |x| \le r, |y| \le r\}$. We denote the boundary of D_r by $C(r)$. Then by the residue theorem we have, for $N \ge r$,

$$\frac{1}{2\pi\sqrt{-1}} \oint_{C(N+1/2)} F(z)w(z)dz = \sum_{\zeta} \mathrm{Res}(F \cdot w; \zeta) + \sum_{|n| \le N} F(n),$$

where the line integral in the left hand side is counter-clockwise.

On the other hand, by the condition of the degrees of f and g, there exists a constant $r' > 0$ and $c > 0$ such that $|F(z)| \le c|z|^{-2}$ for any $z \notin D_{r'}$. Note also that $|w(z)| \le 2\pi$ for $z \in C(N + 1/2)$. Therefore,

$$\begin{aligned} \Big| \oint_{C(N+1/2)} F(z)w(z)dz \Big| &\le \oint_{C(N+1/2)} |F(z)||w(z)||dz| \\ &\le c|N + 1/2|^{-2} \cdot 2\pi \cdot 8(N + 1/2) \to 0 \end{aligned}$$

as $N \to \infty$. This proves the lemma. □

Using Lemmas 2.3 and 2.4, we will prove the following theorem.

Theorem 2.5. *The Ricci tensor on the loop group is written more explicitly as follows;*

$$\begin{aligned} \mathrm{Ric}^{(\delta)}(X, X) = \|\hat{X}(0)\|_{\mathfrak{g}}^2 \cdot \Big(\frac{\pi \coth(\pi/\delta)}{8\delta} + \frac{\pi^2}{8\delta^2 \sinh^2(\pi/\delta)} \Big) \\ + \sum_{l \ne 0} (1 + \delta^2 l^2) \|\hat{X}(l)\|_{\mathfrak{g}^{\mathbf{C}}}^2 \cdot \frac{1 - 2\delta^2 l^2}{2\delta(1 + \delta^2 l^2)(4 + \delta^2 l^2)} \pi \coth(\pi/\delta). \end{aligned}$$

Hence, though the Ricci tensor is not negative definite, it is "negative definite except finite dimensional directions".

Proof. For $l \ne 0$, we set

$$F(z) = \frac{1 - 4\delta^2 l z}{(1 + \delta^2 z^2)(1 + \delta^2 (z - l)^2)}.$$

Then poles of F are $\pm\delta^{-1}\sqrt{-1}$, $l \pm \delta^{-1}\sqrt{-1}$ and they are all single. The residue of $F \cdot w$ at $z = \delta^{-1}\sqrt{-1}$ is easily calculated as follows;

$$
\begin{aligned}
\operatorname{Res}(F \cdot w|\delta^{-1}\sqrt{-1}) &= F(z)w(z)(z-\delta^{-1}\sqrt{-1})\Big|_{z=\delta^{-1}\sqrt{-1}} \\
&= \frac{1-4\delta l\sqrt{-1}}{2\delta\sqrt{-1}\{1+\delta^2(\delta^{-1}\sqrt{-1}-l)^2\}} w(\delta^{-1}\sqrt{-1}) \\
&= \frac{1-4\delta l\sqrt{-1}}{2\delta\sqrt{-1}(-2\delta l\sqrt{-1}+\delta^2 l^2)} w(\delta^{-1}\sqrt{-1}) \\
&= \frac{(1-4\delta l\sqrt{-1})(2\sqrt{-1}+\delta l)}{2\sqrt{-1}\delta^2 l(\delta^2 l^2+4)} w(\delta^{-1}\sqrt{-1}) \\
&= \frac{9\delta l+(2-4\delta^2 l^2)\sqrt{-1}}{2\sqrt{-1}\delta^2 l(\delta^2 l^2+4)} \times \{-\sqrt{-1}\pi\coth(\pi/\delta)\} \\
&= -\frac{9\pi\coth(\pi/\delta)}{2\delta(\delta^2 l^2+4)} + \frac{(4\delta^2 l^2-2)\pi\coth(\pi/\delta)}{2\delta^2 l(\delta^2 l^2+4)}\sqrt{-1}.
\end{aligned} \tag{2.6}
$$

Similarly we have

$$
\operatorname{Res}(F \cdot w|-\delta^{-1}\sqrt{-1}) = -\frac{9\pi\coth(\pi/\delta)}{2\delta(\delta^2 l^2+4)} - \frac{(4\delta^2 l^2-2)\pi\coth(\pi/\delta)}{2\delta^2 l(\delta^2 l^2+4)}\sqrt{-1}. \tag{2.7}
$$

We obtain the residue of $F \cdot w$ at $z = l + \delta^{-1}\sqrt{-1}$ as follows;

$$
\begin{aligned}
&\operatorname{Res}(F \cdot w|l+\delta^{-1}\sqrt{-1}) \\
&\quad= F(z)w(z)(z-l-\delta^{-1}\sqrt{-1})\Big|_{z=l+\delta^{-1}\sqrt{-1}} \\
&\quad= \frac{1-4\delta^2 l(l+\delta^{-1}\sqrt{-1})}{\{1+\delta^2(l+\delta^{-1}\sqrt{-1})^2\}2\delta^{-1}\sqrt{-1}} w(l+\delta^{-1}\sqrt{-1}) \\
&\quad= \frac{1-4\delta^2 l^2-4\delta l\sqrt{-1}}{2\sqrt{-1}\delta^2 l(\delta l+2\sqrt{-1})} w(l+\delta^{-1}\sqrt{-1}) \\
&\quad= \frac{(1-4\delta^2 l^2-4\delta l\sqrt{-1})(\delta l-2\sqrt{-1})}{2\sqrt{-1}\delta^2 l(\delta^2 l^2+4)} w(l+\delta^{-1}\sqrt{-1}) \\
&\quad= \frac{-\delta l(4\delta^2 l^2+7)+(4\delta^2 l^2-2)\sqrt{-1}}{2\sqrt{-1}\delta^2 l(\delta^2 l^2+4)} \times \{-\sqrt{-1}\pi\coth(\pi/\delta)\} \\
&\quad= \frac{(4\delta^2 l^2+7)\pi\coth(\pi/\delta)}{2\delta(\delta^2 l^2+4)} - \frac{(2\delta^2 l^2-1)\pi\coth(\pi/\delta)}{\delta^2 l(\delta^2 l^2+4)}\sqrt{-1}.
\end{aligned} \tag{2.8}
$$

Similarly we have

$$
\begin{aligned}
&\operatorname{Res}(F \cdot w|l-\delta^{-1}\sqrt{-1}) \\
&\quad= \frac{(4\delta^2 l^2+7)\pi\coth(\pi/\delta)}{2\delta(\delta^2 l^2+4)} + \frac{(2\delta^2 l^2-1)\pi\coth(\pi/\delta)}{\delta^2 l(\delta^2 l^2+4)}\sqrt{-1}.
\end{aligned} \tag{2.9}
$$

By (2.6), (2.7), (2.8), (2.9) and Lemma 2.4 we have

(2.10)
$$\begin{aligned}\sum_{n\in\mathbf{Z}}\frac{1-4\delta^2 ln}{(1+\delta^2n^2)(1+\delta^2(n-l)^2)} &= -\operatorname{Res}(F\cdot w|\delta^{-1})-\operatorname{Res}(F\cdot w|-\delta^{-1})\\ &\quad -\operatorname{Res}(F\cdot w|l+\delta^{-1})-\operatorname{Res}(F\cdot w|l-\delta^{-1})\\ &=\frac{2(1-2\delta^2l^2)}{\delta(\delta^2l^2+4)}\pi\coth(\pi/\delta).\end{aligned}$$

Thus we have checked the second term in the right hand side of the equation in the theorem.

Next we will compute the first term. Set $G(z)=(1+\delta^2z^2)^{-2}$. It is easy to see that G has two double poles at $z=\pm\delta^{-1}$. Noting that

$$\begin{aligned}w'(\sqrt{-1}/\delta)&=-\pi^2\frac{1}{\sin^2(\pi\sqrt{-1}/\delta)}\\ &=-\pi^2\Big(\frac{2\sqrt{-1}}{e^{\sqrt{-1}\pi\sqrt{-1}/\delta}-e^{-\sqrt{-1}\pi\sqrt{-1}/\delta}}\Big)^2\\ &=\pi^2\Big(\frac{2}{e^{\pi/\delta}-e^{-\pi/\delta}}\Big)^2\\ &=\frac{\pi^2}{\sinh^2(\pi/\delta)},\end{aligned}$$

we obtain

$$\begin{aligned}\operatorname{Res}(G\cdot w|\delta^{-1}\sqrt{-1})&=\frac{d}{dz}\Big|_{z=\delta^{-1}\sqrt{-1}}\{G(z)w(z)(z-\delta^{-1}\sqrt{-1})^2\}\\ &=\frac{1}{\delta^4}\frac{d}{dz}\Big|_{z=\delta^{-1}\sqrt{-1}}\Big\{\frac{w(z)}{(z+\delta^{-1}\sqrt{-1})^2}\Big\}\\ &=\frac{1}{\delta^4}\left(\frac{-2w(\delta^{-1}\sqrt{-1})}{(2\delta^{-1}\sqrt{-1})^3}+\frac{w'(\delta^{-1}\sqrt{-1})}{(2\delta^{-1}\sqrt{-1})^2}\right)\\ &=-\frac{\pi\coth(\pi/\delta)}{4\delta}-\frac{\pi^2}{4\delta^2\sinh^2(\pi/\delta)}.\end{aligned}\tag{2.11}$$

Similarly we have

$$\operatorname{Res}(G\cdot w|-\delta^{-1}\sqrt{-1})=-\frac{\pi\coth(\pi/\delta)}{4\delta}-\frac{\pi^2}{4\delta^2\sinh^2(\pi/\delta)}.\tag{2.12}$$

By (2.11), (2.12) and Lemma 2.4 we have

$$\sum_{n\in\mathbf{Z}}\frac{1}{(1+\delta^2n^2)^2}=\frac{\pi\coth(\pi/\delta)}{2\delta}+\frac{\pi^2}{2\delta^2\sinh^2(\pi/\delta)}.$$

Thus we have proved Theorem 2.5. $\square$

3. Lower bound of the Ricci tensor

In this section we will compute explicitly the lower bound of the Ricci tensor from Theorem 2.5. In order to do so, it is sufficient to consider the maximum of

$$\frac{2\delta^2 l^2 - 1}{(1+\delta^2 l^2)(4+\delta^2 l^2)}$$

as a function of $l = 1, 2, \ldots$ for exch fixed $\delta > 0$. Let us define

$$h(x) = \frac{2x-1}{(1+x)(4+x)} = \frac{3}{4+x} - \frac{1}{1+x},$$

for $x \geq 0$. We can easily see that

$$\begin{aligned} h'(x) &= -\frac{3}{(4+x)^2} + \frac{1}{(1+x)^2} \\ &= \frac{-2x^2 + 2x + 13}{(4+x)^2(1+x)^2}. \end{aligned}$$

Hence $h(x)$ increases if $x \in (0, (1+3\sqrt{3})/2)$ and decreases if $x \in ((1+3\sqrt{3})/2, \infty)$. We set

$$\lfloor x \rfloor = \max\{n \in \mathbf{Z} | n \leq x\}$$

for $x \in \mathbf{R}$ and

$$l_0 = \left\lfloor \delta^{-1} \sqrt{\frac{1+3\sqrt{3}}{2}} \right\rfloor. \tag{3.1}$$

Hence we have

$$\max_{l=1,2,\ldots} \frac{2\delta^2 l^2 - 1}{(1+\delta^2 l^2)(4+\delta^2 l^2)} = h(\delta^2 l_0^2) \vee h(\delta^2 (l_0+1)^2). \tag{3.2}$$

This is also valid when $l_0 = 0$ because $h(0) < 0$ and $h(\delta^2) > 0$.

From Theorem 2.5 and (3.2), we have the next theorem.

Theorem 3.1. *Let $\delta > 0$ and l_0 be as in* (3.1). *Let us denote the lower bound of the Ricci tensor by $-C^{(\delta)}$, i.e.,*

$$-C^{(\delta)} = \inf_{X \neq 0} \frac{\mathrm{Ric}^{(\delta)}(X, X)}{\|X\|^2_{H_1^\delta}}.$$

(*We easily see from Theorem* 2.3 *that $-C^{(\delta)} > -\infty$ for any $\delta > 0$*). *Then we have*

$$C^{(\delta)} = \frac{\pi \coth(\pi/\delta)}{2\delta} \{h(\delta^2 l_0^2) \vee h(\delta^2 (l_0+1)^2)\}.$$

Corollary 3.2. *We have*

$$C^{(1)} = \frac{7\pi\coth(\pi)}{80},$$
$$C^{(2\pi)} = \frac{\coth(1/2)(8\pi^2-1)}{16(1+4\pi^2)(1+\pi^2)}.$$

Proof. When $\delta = 1$, we can easily see that $l_0 = \lfloor\{(1+3^{3/2})/2\}^{1/2}\rfloor = 1$. We have $h(1) = 1/10$ and $h(2^2) = 7/40$. Hence we have $C^{(1)} = 7\pi\coth(\pi)/80$ by Theorem 3.1. When $\delta = 2\pi$, $l_0 = \lfloor(2\pi)^{-1}\{(1+3^{3/2})/2\}^{1/2}\rfloor = 0$ and

$$h((2\pi)^2\cdot 1) = \frac{8\pi^2-1}{4(1+4\pi^2)(1+\pi^2)}.$$

Hence

$$C^{(2\pi)} = \frac{\coth(1/2)(8\pi^2-1)}{16(1+4\pi^2)(1+\pi^2)}.$$

Thus we have proved Corollary 3.2. □

Remark 3.3. In this paper we assumed equation (2.1). However, the modification to the general case is easy. If we do not assume (2.1), then Theorem 2.5, for example, is modified as follows:

$$\begin{aligned}\mathrm{Ric}^{\delta}(X,X) &= K^{\mathbf{C}}\langle\hat{X}(0),\hat{X}(0)\rangle\left(\frac{\pi\coth(\pi/\delta)}{8\delta}+\frac{\pi^2}{8\delta^2\sinh^2(\pi/\delta)}\right)\\ &\quad+\sum_{l\neq 0}(1+\delta^2l^2)K^{\mathbf{C}}(\hat{X}(l),\hat{X}(l))\\ &\quad\cdot\frac{1-2\delta^2l^2}{2\delta(1+\delta^2l^2)(4+\delta^2l^2)}\pi\coth(\pi/\delta).\end{aligned}$$

Here $-K^{\mathbf{C}}$ is the Hermitian form which is obtained as the natural extension of the Killing form. Similarly Theorem 3.1 is modified as follows:

$$C^{(\delta)} = \|K\|\frac{\pi\coth(\pi/\delta)}{2\delta}\{h(\delta^2l_0^2)\vee h(\delta^2(l_0+1)^2)\}.$$

Here $-K$ is the Killing form of $\mathfrak{g}$ and

$$\|K\| = \sup\{K\langle a,a\rangle | a\in\mathfrak{g}, \|a\|_{\mathfrak{g}} = 1\}.$$

Remark 3.4. Let us consider the following metrics

$$\|X\|^2_{\delta^{-1}H_1^{\delta}} = \sum_{n\in\mathbf{Z}}(\delta^{-2}+n^2)\|\hat{X}(n)\|^2$$

as in Fang [4]. As $\delta\to\infty$, $\|X\|_{\delta^{-1}H_1^{\delta}}$ goes to the usual H^1-norm for the pinned loop algebra by (1.2) (at least formally). We will consider the Ricci tensor for

$\delta^{-1}H^{\delta}$-metric. By direct calculation the operator D defined as in (2.3) remains invariant under the multiplication of constant δ^{-1} and so does the Ricci tensor defined as in (2.5). Since the lower bound of the Ricci is defined as in Theorem 3.1, we can easily see that the new lower bound is δ^2-times the original one. For sufficiently large δ, l_0 in equation (3.1) is zero. Hence by Theorem 3.1 the new lower bound $-\tilde{C}^{\delta}$ of the Ricci tensor is written as

$$\tilde{C}^{\delta} = \delta^2 C^{\delta} = \frac{\pi}{\delta}\coth(\pi/\delta)\cdot\frac{\delta^2(2\delta^2-1)}{2(1+\delta^2)(4+\delta^2)}.$$

We can easily see that $-\tilde{C}^{\delta}$ goes to 1 as $\delta \to \infty$, which in a sense coincides with the result in Freed [5]. Freed showed that the lower bound of the Ricci tensor on the pinned loop algebra for H_1^1-norm given by $\|X\|^2 = \sum_{n\neq 0} n^2\|\hat{X}(n)\|^2$ is -1. (Note that the definition of the Ricci tensor in [5] is different from ours by multiplication of -1 and that time interval in [5] is $[0, 2\pi]$. However, it does not matter and modification is easy).

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References

[1] T. R. Carson, Logarithmic Sobolev inequality for the free loop group, Ph. D. thesis, University of California, San Diego.

[2] T. R. Carson, A logarithmic Sobolev inequality for the free loop group, C. R. Acad. Sci. Paris Ser. I Math., **326**-2 (1998), 223–228.

[3] B. K. Driver and T. Lohrenz, Logarithmic Sobolev inequalities for pinned loop groups, J. Funct. Anal., **140** (1996), 381–448.

[4] S. Fang, Ricci tensors on some infinite dimensional Lie algebras, J. Funct. Anal., **161** (1999), 132–151.

[5] D. S. Freed, The geometry of loop groups, J. Differential Geom., **28** (1988), 223–276.

[6] Y. Inahama, Logarithmic Sobolev inequality on free loop groups for heat kernel measures associated with the general Sobolev spaces, J. Funct. Anal., **179** (2001), 170–213.

[7] P. Malliavin, Hypoellipticity in infinite dimensions, in "Diffusion Process and Related Problems in Analysis" (M. A. Pinsky, ed.), Vol. I, Birkhäuser, Basel, 1989/1991.