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Families of Galois closure curves for plane quartic curves

By

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Abstract

For a smooth quartic plane curve C we show an existence of a family of Galois closure curves $\varphi : S \longrightarrow C$, where S is a nonsingular projective surface and $\varphi^{-1}(P)$ is isomorphic to the Galois closure curve C_P for a general point $P \in C$. Moreover we determine the types of singular fibers. As a corollary we can say that C_P is not isomorphic to C_Q if P is close to Q.

1. Introduction

Let the ground field of our discussions be the complex number field. Let C be a smooth curve of degree d (≥ 4) in the projective plane $\mathbb{P}^2$ and P be a point in C. Then we have a projection $\pi_P : C \longrightarrow l$ with center P, where l is a line not passing through P. This projection induces the extension of fields ${\pi_P}^* : k(l) \hookrightarrow k(C) = K$ with $[K : k(l)] = d-1$. The structure of this extension does not depend on the choices of l, but on P. So we denote $k(l)$ by K_P. There rise several questions under this situation. We have studied them from several points of view, see [4], [8] and also [9]. Especially we have considered the following: let L_P be the Galois closure of this extension K/K_P.

Definition. Let C_P be the nonsingular projective model of L_P. We call C_P the Galois closure curve at $P \in C$ and let $g(P)$ be the genus of C_P. In the case where K/K_P is a Galois extension, we call P a Galois point for C.

We have studied the Galois group $G_P := Gal(L_P/K_P)$, the genus $g(P)$ and the number of Galois points etc. (cf. [8])

The motive for this research is the interest to know how the curve C_P varies when P moves on C. In the case where $d = 4$ we can answer the problem raised from this motive. We can construct a family $\varphi : S \longrightarrow C$ satisfying that $\varphi^{-1}(P) \cong C_P$ for a general point $P \in C$ and study the structure of this fiber space. Then we consider whether C_P is not isomorphic to C_Q if $P \neq Q$.

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2. Statement of results

Unless otherwise mentioned, we assume that C is a smooth quartic curve. We have found several facts by the study under the above frame work, see [4]. Especially the Galois group G_P is isomorphic to the symmetric group of degree three if P is a general point, and there exist at most four Galois points.

Let T_P denote the tangent line to C at P and $I(C_1, C_2; Q)$ denote the intersection number of C_1 and C_2 at Q. The point $P \in C$ is said to be an i-flex ($i = 1, 2$), where $i = I(C, T_P; P) - 2$. We have the fact $\sum_{Q \in C}\{I(C, T_Q; C) - 2\} = 24$ (cf. [3]).

Note that a Galois point is a 2-flex (but not the converse). Since the center of the projection is in C, we make the following definition. Let $\nu_i = \nu_i(P)$, where $i = 1, 2$, be the number of lines l satisfying that $l \ni P$ and $I(C, l; Q) = i + 1$ for some $Q \in C$, where $Q \neq P$. Then put $m = \nu_1$ [resp. $\nu_1 + 1$] if P is not a flex [resp. is a 1-flex], and $n = \nu_2$ [resp. $\nu_2 + 1$] if P is a 1-flex [resp. 2-flex]. Applying the Riemann-Hurwitz formula to the covering $\pi_P : C \longrightarrow l$, we get $m + 2n = 10$, moreover we have $g(P) = 10 - n$ if P is not a Galois point, where $0 \leq n \leq 4$ (cf. [4]). Let Σ be the set of points P in C satisfying the following condition (1) or (2):

(1) P is a 2-flex.

(2) $Q(\neq P)$ is a 1-flex and $T_Q \ni P$.

Clearly the number of elements of Σ is at most 24.

Let (D_1, D_2) [resp. D^2] denote the intersection number of D_1 and D_2 [resp. the self-intersection number of D] on some surface.

Under these situations our main theorem can be stated as follows:

Theorem 2.1. *There exist a nonsingular projective minimal surface of general type S and a morphism $\varphi : S \longrightarrow C$ with the following properties*:

(1) *If $P \in C \setminus \Sigma$, then $\varphi^*(P) \cong C_P$ and $g(P) = 10$.*

(2) *If $P \in \Sigma$, then the assertions are stated in three sub cases separately.*

(2-1) *If P is not a 2-flex, then $\varphi^*(P)$ is an irreducible curve with n-pieces of nodes and $g(P) = 10 - n$.*

(2-2) *If P is a 2-flex but not a Galois point, then $\varphi^*(P) = \Delta + E$ as divisors, where Δ and E are irreducible curves with the following properties*:

(i) *$g(\Delta) = 10 - n$ and Δ has $(n-1)$-pieces of nodes.*

(ii) *$\Delta^2 = -2$ and E is a (-2)-curve.*

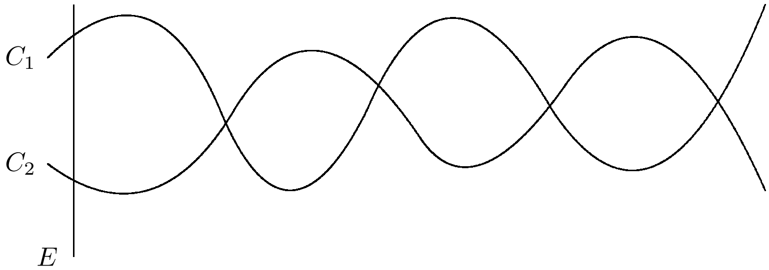
(iii) *Δ and E meet at in two points, where they have normal crossings.*

(2-3) *If P is a Galois point, then $\varphi^*(P) = C_1 + C_2 + E$ as divisors with the following properties*:

(i) *$C_i \cong C$ $(i = 1, 2)$, $(C_i)^2 = -5$ and E is a (-2)-curve.*

(ii) *$\varphi^*(P)$ is a divisor with normal crossings.*

(iii) *$(C_1, C_2) = 4$, $(C_i, E) = 1$ $(i = 1, 2)$.*



Especially, if C is a general quartic curve, then the cardinality of Σ is 24, and for the point $P \in \Sigma$, $\varphi^*(P)$ is an irreducible curve with one node and its genus is 9. Furthermore the canonical bundle of S is ample.

As a corollary we have the following assertion which has a close relation to the one in [5, Theorem 2.3.1]

Corollary 2.1. *For any $P \in C$, there exists an open neighborhood U_P of P satisfying that C_Q is not isomorphic to $C_{Q'}$ if Q and Q' belong to U_P and $Q \neq Q'$.*

This assertion does not necessarily hold true if Q' is not near to Q. Indeed, we have the following.

Remark 1. Suppose C has a non-trivial automorphism σ. Then, it is a restriction of a projective transformation. It is easy to see that $C_Q \cong C_{\sigma(Q)}$, however $\sigma(Q) \neq Q$ for a general point Q.

Remark 2. We mention a uniqueness of S in the following sense: let $\varphi' : S' \longrightarrow C$ be a surjective morphism satisfying that S' is a smooth projective minimal algebraic surface and ${\varphi'}^{-1}(P) \cong C_P$ for a general point $P \in C$. Suppose that φ' is induced from the projection π_P, i.e., S' is the nonsingular projective minimal model of the minimal splitting variety (see [7], for the definition) for the triple covering $p_1 : C \times C \longrightarrow C \times \mathbb{P}^1$, where $p_1(P, Q) = (P, \pi_P(Q))$. Then, there exists an isomorphism $\psi : S' \longrightarrow S$ satisfying that $\varphi \cdot \psi = \varphi'$.

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3. Proofs

Let $(X : Y : Z)$ be a set of homogeneous coordinates on $\mathbb{P}^2$ and put $x = X/Z$, $y = Y/Z$. Let C_0 be the affine part $Z \neq 0$ of C and $f(x, y) = 0$ be the defining equation. In this first paragraph and two lemmas below, we do not assume that the degree d of C is four. Let $y = t(x - u) + v$ be a line passing through a point $P = (u, v) \in \mathbb{A}^2$ and let $D = D(t, u, v)$ be the discriminant of $f(x, t(x - u) + v)$ with respect to x. Then the degree of D with respect to

t is $c = d(d-1)$, which is the class of C. Similarly the (total) degree of D with respect to u and v is $c = d(d-1)$. Let Φ be the homogeneous equation of the dual curve $\widehat{C}$ of C and put $D'(t,u,v) = \Phi(t,-1,-ut+v)$. The following assertion is shown in [8, Lemma 3].

Lemma 3.1. *We have $D = D'$.*

Moreover, in case $P = (u,v)$ is in C_0, we let $\Psi(t,u,v)$ be the discriminant of $F(x,t,u,v) := f(x, t(x-u)+v)/(x-u)$ with respect to x. Define $\bar{f}(t,u,v)$ to be the value of $F(x,t,u,v)$ at $x = u$. Then we have the following.

Lemma 3.2. $(-1)^{d-1}(\bar{f}(t,u,v))^2\Psi(t,u,v) = \Phi(t,-1,-ut+v)$.

Proof. The proof is done by direct computation of the discriminant and is essentially the same as in the proof of [8, Lemma 12]. □

Hereafter we assume that $d = 4$. Let B_0 be the divisor defined by Ψ in $\mathbb{A}^1 \times C_0$. Take the other affine parts of $S_0 := \mathbb{P}^1 \times C$ and define the similar divisor as B_0. Let B be the divisor on S_0 obtained from patching such ones. It is easy to see that the support of $B \setminus B_0$ consists of finitely many points. Moreover, putting $t = T_1/T_0, u = U/W$ and $v = V/W$, we define $\widetilde{D}(T_0,T_1;U,V,W) := T_0^{12}W^{12}D(t,u,v)$.

Claim 1. The divisor B is even, i.e., $\mathcal{O}_{S_0}(B) = \mathcal{L}^{\otimes 2}$ for some $\mathcal{L} \in \mathrm{Pic}(S_0)$.

Proof. We can find a polynomial G satisfying that $\widetilde{D}/G^2$ defines a rational function on $\mathbb{P}^1 \times \mathbb{P}^2$ and the restriction of $\widetilde{D}/G^2$ to $\mathbb{P}^1 \times C$ defines also a rational function on $\mathbb{P}^1 \times C$. The right hand side of the equation in Lemma 3.2 is obtained by restricting D to $\mathbb{A}^1 \times C_0$. The same fact holds true on each affine part, hence we infer that B is even. □

Let X_1 be the affine surface defined by $f(u,v) = 0$ and $F(x,t,u,v) = 0$ in $\mathbb{A}^2 \times \mathbb{A}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2$. Let $\overline{X}_1$ be the projective closure of X_1 in $\mathbb{P}^2 \times \mathbb{P}^2$. Then it is easy to see that the second projection induces a fiber space $\overline{X}_1 \longrightarrow C$, each fiber of which is isomorphic to C. Hence X_1 is irreducible. Let S_1 be a nonsingular projective relative minimal surface which is birational to X_1.

Claim 2. S_1 is isomorphic to $C \times C$.

Proof. Since the function field $k(S_1)$ is equal to $k(X_1) = k(x,t,u,v)$, where x,t,u and v satisfy $f(u,v) = F(x,t,u,v) = 0$ and $t = (y-v)/(x-u)$, we have $k(x,t,u,v) = k(x,y,u,v)$. Thus S_1 and $C \times C$ are birationally equivalent to each other. They are not ruled surfaces, hence they are isomorphic to each other. □

Claim 3. There exists a finite morphism $p_1 : S_1 \longrightarrow S_0$ of degree three.

Proof. Define $p_1 : C \times C \longrightarrow l \times C \cong \mathbb{P}^1 \times C$ by $p_1(Q,P) = (\pi_P(Q), P)$. Since $\pi_P : C \longrightarrow l$ is a triple covering for each P, the assertion is clear. $\square$

Claim 4. The divisor B is reduced and its singularity is isomorphic to the one defined by $y^2 + x^{3k} = 0$, where $k = 1, 2$.

Proof. Since C has only a finitely many flexes, for a general $(a,b) = (a : b : 1) \in C$ [resp. general $\tau \in \mathbb{P}^1$], $\Psi(t,a,b)$ [resp. $\Psi(\tau,u,v)$] has no multiple factors. Therefore B is reduced. Similarly we infer that the point $(t_0, u_0, v_0) \in B$ is a singular point if and only if $P = (u_0, v_0) \in C$ and the line $l_0 : y = t_0(x - u_0) + v_0$ satisfy one of the following conditions:

(1) $l_0 = T_Q$, where $Q \neq P$ and Q is a flex of C.

(2) $l_0 = T_P$ and P is a 2-flex.

We make use of Lemma 3.2 to determine the type of the singularity. The singularity is a local property, so we can assume as follows:

In the case (1), $P = (0,0)$, $f(x,y) = x + ay + f_2 + f_3 + f_4$ and $f(x,0) = x(bx+1)^3$, where f_i is a homogeneous polynomial of degree i and $b \neq 0$.

In the case (2), $P = (0,0)$, $f(x,y) = y + f_2 + f_3 + f_4$ and $f(x,0) = cx^4$, where $c \neq 0$.

First we treat the case (1). We consider the singularity at $t = u = v = 0$. By direct computation we have $\bar{f}(t,u,v) = 1 + at + (*)$, where $(*)$ vanishes at $(0,0,0)$. Since $(-1/b, 0) = (1 : 0 : -b)$ is a 1-flex of C and $Y = 0$ is the tangent line to C at the flex, we see that the dual curve $\widehat{C}$ of C has a $(2,3)$-cusp at $(0 : 1 : 0) \in \widehat{\mathbb{P}^2}$ and the tangent line to $\widehat{C}$ at the cusp is given by $X - bZ = 0$. Therefore, putting $\xi = X/Y, \eta = Z/Y$, we have the defining equation of $\widehat{C}$ is $\widehat{f}(\xi,\eta) = (\xi - b\eta)^2 + \widehat{f}_3 + \cdots + \widehat{f}_{12}$, where $\widehat{f}_i$ is the homogeneous polynomial of degree i. Since $(0,0) \in \widehat{C}$ is a $(2,3)$-cusp, if we put $\widehat{f}_3(\xi,\eta) = \sum_{i+j=3} a_{ij}\xi^i\eta^j$, then we have $\widehat{f}_3(b,1) = \sum_{i+j=3} a_{ij}b^i \neq 0$. Thus we obtain $\Phi(X,Y,Z) = (X - bZ)^2Y^{10} + \widehat{f}_3(X,Z)Y^9 + \cdots$. Near $(0,0)$ the curve C can be expressed locally as $u = \phi(v)$, where $\phi = \phi(v)$ is a holomorphic function at $v = 0$ and $\phi(0) = 0$. Therefore we have $\Phi(t, -1, -\phi t + v) = \{(t - bv) + b\phi t\}^2 - \widehat{f}_3(t, -\phi t + v) + \cdots$. If we take a new system of coordinates (s,v), where $s = t - bv$, this becomes

$$\begin{aligned}&\{s + b\phi(s + bv)\}^2 - \widehat{f}_3(s + bv, -\phi(s + bv) + v) + \cdots \\ &= s^2 - \sum_{i+j=3} a_{ij}(s + bv)^i v^j + (\text{higer terms}).\end{aligned}$$

Since $\bar{f}(t,u,v)$ is a unit in the convergent power series ring $\mathbb{C}\{t,v\}$ and $\sum_{i+j=3} a_{ij}b^i \neq 0$, this implies that the singularity is the $(2,3)$-cusp.

Next we treat the case (2). We consider the singularity at $t = u = v = 0$. Since $f(x,y)$ can be expressed as $y + cx^4 + g(x,y)y$, where $c \neq 0$ and $g(0,0) = 0$, we have $\bar{f}(t,u,v) = t + 4cu^3 + tg(u,v) + \bar{g}(t,u,v)v$, where $\bar{g}(t,u,v)$ is the value of $\{g(x, t(x-u) + v) - g(u,v)\}/(x-u)$ at $x = u$. Since $(0 : 0 : 1)$ is a 2-flex of C and $Y = 0$ is the tangent line to C at the flex, the dual curve $\widehat{C}$ has a

$(3,4)$-cusp at $(0:1:0) \in \widehat{\mathbb{P}^2}$ and the tangent line to $\widehat{C}$ at the cusp is given by $Z=0$. Therefore, putting $\xi = X/Y, \eta = Z/Y$, we have the defining equation of $\widehat{C}$ is $\widehat{f}(\xi,\eta) = \eta^3 + \widehat{f}_4 + \cdots + \widehat{f}_{12}$, where $\widehat{f}_4(\xi,0) = a\xi^4 \neq 0$. Thus we obtain $\Phi(X,Y,Z) = Y^9Z^3 + \widehat{f}_4(X,Z)Y^8 + \cdots$. Near $(0,0)$ the curve C can be expressed locally as $v = \phi(u) = -cu^4 + \cdots$, where $\phi = \phi(u)$ is a holomorphic function at $u=0$ and $c \neq 0$. Therefore we have $\Phi(t,-1,-ut+v) = -(-ut+\phi)^3 + \widehat{f}_4(t,-ut+\phi) + \cdots$, which is equal to $-\{t + 4cu^3 + tg(u,v) + \bar{g}(t,u,v)\phi\}^2\Psi(t,u,v)$. Since $\widehat{f}_4(t,-ut+\phi) = at^4 + \cdots$ $(a \neq 0)$, the multiplicity of the singular point is two. Perform the quadratic transformation $t = su,\ u = u$. After dividing by u^2 both sides, we obtain $-u^4(-s+\phi_2)^3 + u^2\widehat{f}_4(s,-ut+\phi_3) + \cdots = -(s + 4cu^2 + sg(u,\phi) + \bar{g}\phi)^2\Psi(su,u,\phi(u))$, where we put $\phi_2 = \phi/u^2$ and $\phi_3 = \phi/u$. Therefore $\Psi(su,u,\phi)$ is divisible by u^2. Put $\Psi_1(s,u) = \Psi(su,u,\phi)/u^2$. Then, we get $-u^2(-s+\phi_2)^3 + \widehat{f}_4(s,-us+\phi_3) + \cdots = -(s + 4cu^2 + sg + \bar{g}\phi_3)^2\Psi_1(s,u)$. Once more perform the quadratic transformation $s = ru,\ u = u$. After dividing by u^2 both sides, we obtain

$$-u^3(-r+\phi_1)^3 + u^2\widehat{f}_4(r,-ur+\phi_2) + \cdots = -(r + 4cu + rg + \bar{g}\phi_2)^2\Psi_1(ru,u),$$

where $\phi_1 = \phi(u)/u^3$. Thus $\Psi_1(ru,u)$ is divisible by u^2. Putting $\Psi_2(r,u) = \Psi_1(ru,u)/u^2$, we obtain $-u(-r+\phi_1)^3 + \widehat{f}_4(r,-ur+\phi_2) + \cdots = -(r + 4cu + rg + \bar{g}\phi_2)^2\Psi_2(r,u)$. Let $Q(r,u)$ be the leading form of $\Psi_2(r,u)$, i.e., the order two part of $\Psi_2(r,u)$. Then we have $u(r+cu)^3 + ar^4 = -(r+4cu)^2Q$. This implies $ac = 3^3/4^4$. Therefore we obtain $Q = ar^2 + (5/32)ur + (1/16)cu^2$, which has distinct factors. This implies that the singularity is of A_5-type. $\square$

Remark 3. If C is a general quartic, then the divisor B is irreducible.

Proof. Suppose that B has an irreducible decomposition $B = B_1 + \cdots + B_r$. Then, each B_i is an ample divisor, because B_i is not a fiber of the projection $\pi_1 : S_0 \longrightarrow \mathbb{P}^1$ nor the one $\pi_2 : S_0 \longrightarrow C$. Hence B is connected. Since the singularities of B are only $(2,3)$-cusps, B cannot be reducible. $\square$

Following Definitions 1.1 and 1.2 in [7], we define S_2 to be the discriminant variety of S_0. By Claim 1 we see S_2 is the double covering of S_0 branched along B. Let S_3 be the "minimal splitting variety" of S_1, then we have the following commutative diagram:

$$\begin{tikzcd}
 & S_3 \arrow[dl, "p_2"'] \arrow[dr, "q_2"] \arrow[dd, "p"] & \\
S_1 \arrow[dr, "p_1"'] & & S_2 \arrow[dl, "q_1"] \\
 & S_0 &
\end{tikzcd}$$

Clearly S_3 is a surface of general type. Note that these are objects over C. Let $\varphi_i : S_i \longrightarrow C$ be the structure morphism $(0 \le i \le 3)$. It is easy to see that if $P \in C \setminus \Sigma$, then ${\varphi_3}^*(P)$ is "the minimal splitting variety" of $\pi_P : C \longrightarrow l$, i.e., it is the Galois closure curve C_P.

Claim 5. There exists an automorphism σ on S_3 satisfying that $S_3/\langle\sigma\rangle \cong S_2$. Moreover it has fixed points which are just $q_2^{-1}(Sing(S_2))$, where $Sing(S_2)$ denotes the set of singular points of S_2.

Proof. The proofs of the former and latter statements are the same as in the one of [7, Proposition 1.3] and [7, Proposition 3.4] respectively. □

By Claim 4 the singularities of S_2 are of type A_{3k-1}, where $k = 1$ or 2. By Claim 5 we see that if R_0 is a singular point on S_2 of type A_1, then $q_2^{-1}(R_0)$ is a smooth one on S_3. While, in the case where $k = 2$, the corresponding point on S_3 is a quotient singular one. We inspect these in more detail. The quotient map q_2 can be expressed as follows near the ramification point $R = q_2^{-1}(R_0)$. In the case where $k = 1$, the point R is nonsingular and there exist open neighborhood U of R with coordinates (x, y) and an automorphism σ_0 satisfying that $\sigma_0(x, y) = (\omega x, \omega^{-1} y)$, where ω is a primitive cubic root of 1. Then q_2 is the quotient map $U \longrightarrow U/\langle\sigma_0\rangle$. On the other hand, in the case where $k = 2$, it is easy to see that the point R is a quotient singular point. The morphism q_2 coincides with the quotient map $U/\langle{\sigma_1}^3\rangle \longrightarrow U/\langle\sigma_1\rangle$, where $\sigma_1(x, y) = (\zeta x, \zeta^{-1} y)$ and ζ is a primitive 6-th root of 1. Therefore R is the singular point of type A_1. Moreover, the automorphism σ in Claim 5 preserves each fiber of $\varphi_3 : S_3 \longrightarrow C$, i.e., we have $\varphi_3 \cdot \sigma = \varphi_3$.

Now we consider the singular fibers of φ. First we treat the case where P is not a Galois point. Since S_2 is locally defined by $z^2 - \Psi(t, u, v) = f(u, v) = 0$ and φ_2 is given by $\varphi_2(z, t, u, v) = (u, v)$, we infer that near a singular point S_2 can be expressed as $z^2 = y^2 + x^3$ [resp. $z^2 = y^2 + x^6$] if P is not a 2-flex [resp. 2-flex]. Looking in the proof of Claim 4, we infer that φ_2 can be given by $\varphi_2(x, y, z) = x$. Therefore, if P is not a 2-flex, the singularity of the fiber is a node. On the contrary, if P is a 2-flex, then S_3 has the singularity of type A_1. Resolving this singularity, we obtain the assertion in the theorem.

Second we treat the case of the singular fiber over a Galois point. If $P = (a, b)$ is a Galois point, then the discriminant $\Psi(t, a, b)$ is a complete square $\prod_{i=1}^{5}(t - \alpha_i)^2$. Therefore $\varphi_2^*(P)$ is locally expressed as $z^2 - \prod_{i=1}^{5}(t - \alpha_i)^2 = 0$. Hence it is a reducible curve $\Gamma_1 + \Gamma_2$, where $\Gamma_i \cong \mathbb{P}^1$, Γ_1 and Γ_2 meet in at five points. Since $\Delta_i = q_2^{-1}(\Gamma_i)$ is a component of $p_2^{-1}(C \times P)$, we infer that Δ_i is isomorphic to C $(i = 1, 2)$, Δ_1 and Δ_2 meet in at five points. Moreover at the point corresponding to the 2-flex S_3 has an A_1-singularity. By resolving the singularity, we obtain the singular fiber as in the theorem. Thus we have proved (1), (2) and (3). In the case where C is a general quartic, there exists no 2-flex, hence no Galois point. Therefore the number of 1-flexes is 24, each singular fiber has one node and S_3 is smooth. Since $p_2 : S = S_3 \longrightarrow S_1 = C \times C$ is a finite double covering, there exists no (-2)-curve on S. Hence K_S is ample. Thus we complete the proof of the theorem.

Since $\varphi : S \longrightarrow C$ is a semi-stable fibration, we have a holomorphic map $\rho : C \longrightarrow \overline{\mathfrak{H}_{10}/\Gamma_{10}}$, which is the Satake's compactification of the period domain (cf. [1]). Since ρ is not constant, the corollary holds true.

The proof of the assertion of Remark 3.1 is clear. Since S' is birational to the minimal splitting variety of the triple covering and is the minimal model, there exists such the isomorphism.

Finally we raise problems.

Problems. (1) Do the similar study as in Theorem 2.1 in the case where C is a smooth curve of degree $d \geq 5$.

(2) Let C be a smooth plane curve of degree $d \geq 3$ and P a point in $\mathbb{P}^2$. For the projection $\pi_P : C \longrightarrow l$ consider the Galois closure curve C_P at P. Then we will obtain similarly a smooth threefold V and a morphism $\varphi : V \longrightarrow \mathbb{P}^2$, whose fiber over P is isomprphic to C_P for a general point P. Study the structure of V and singular fibers of φ. In this case, if $P \in C$, then $\varphi^*(P)$ becomes a singular fiber. Are these singular fibers semi-stable, too?

Remark 4. Concerning (2) we have the following remarks.

For all $P \in \mathbb{P}^2 \setminus C$ the Galois group G_P is "constant", i.e., is isomorphic to the full symmetric group of degree d if C is general (cf. [2]). However the moduli of the Galois closure curves will vary. In fact, Sakai [6] has shown that in the case where C is a cubic and the points are general, C_P and C_Q is not isomorphic to each other if P and Q are close.

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