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Homotopy normality of Lie groups and the adjoint action

By

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Abstract

Homotopy normality of Lie groups is studied by using the adjoint action on the space of based loops.

1. Introduction

The notion of homotopy normality of topological groups was introduced by James and McCarty in the different ways. (See [13] and [22].) The concept of homotopy normality is closely related to that of Samelson products of homotopy groups. In fact, if H is homotopy normal in G in the sense of McCarty where (G, H) is a pair of topological groups, then all mixed Samelson products from $\pi_k(H) \times \pi_l(G, H)$ to $\pi_{k+l}(G, H)$ vanish and H is homotopy normal in G in the sense of James. If H is homotopy normal in G in the sense of James, then all Samelson products from $i_*\pi_k(H) \times \pi_l(G)$ to $\pi_{k+l}(G)$ lie in the image of i_* where $i\colon H \hookrightarrow G$ is the inclusion and $i_*\colon \pi_*(H) \to \pi_*(G)$ is induced from i. Note that we may consider homotopy normality not only of a pair of topological groups but also of a homomorphism of topological groups. Moreover, for a prime p, we may consider mod p homotopy normality by localizing at p.

The purpose of this paper is to study mod p homotopy normality of homomorphisms of Lie groups by a new method. In [6], Furukawa used the Hopf algebra structure of the mod p cohomology of a Lie group G to study homotopy normality of a Lie group H in G. We modify his method and use the mod p homology map of the adjoint action of G on ΩG, the space of based loops on G, and the Hopf algebra structure of the mod p homology of ΩG. (See [17], [7], [8], [9], [10], and [11].) We can prove some new results (see Theorem 5.1 and Section 6) and we can reprove many of the results of [4], [5], [6], [19], and [20] concerning mod p homotopy normality of homomorphisms of Lie groups. Note that Kudou and Yagita in [19] and [20] used the Morava K-theory of homology and cohomology but we only use the ordinary theory.

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2. Method

Let $f\colon H \to G$ be a homomorphism of topological groups and p a prime. In the following, assume that all spaces and maps have been localized at p. Recall that the self adjoint action $\mathrm{ad}\colon G \times G \to G$ of G is defined by $\mathrm{ad}(g_1, g_2) = g_1 g_2 g_1^{-1}$ and the commutator map $\mathrm{co}\colon G \times G \to G$ of G by $\mathrm{co}(g_1, g_2) = g_1 g_2 g_1^{-1} g_2^{-1}$ for $g_1, g_2 \in G$. We define that f is *mod p homotopy normal* (mod p homotopy normal in the sense of James) if there exist a homotopy $\gamma_t\colon G \wedge H \to G$ and a continuous map $\gamma'\colon G \wedge H \to H$ such that $\gamma_1 = f \circ \gamma'$ and $\gamma_0(g \wedge h) = \mathrm{co}(g, f(h))$ for $g \in G$ and $h \in H$. (See [13], [6], and [21]. Let denote $\pi(g, h)$ by $g \wedge h$ for the natural projection $\pi\colon G \times H \to G \wedge H$.) It is easy to see that if f is mod p homotopy normal, then we have a homotopy $\nu_t\colon G \times H \to G$ and a continuous map $\nu'\colon G \times H \to H$ such that $\nu_1 = f \circ \nu'$ and $\nu_0(g, h) = \mathrm{ad}(g, f(h))$ for $g \in G$ and $h \in H$.

Denote the mod p (ordinary) homology and cohomology by $H_*(\ )$ and $H^*(\ )$ respectively. If $f\colon H \to G$ is mod p homotopy normal, then we have the following commutative diagram.

(2.1)
$$\begin{tikzcd}
H^*(G) \otimes H^*(H) & H^*(G) \otimes H^*(G) \arrow[l, "1\otimes f^*"'] & H^*(G) \arrow[l, "\mathrm{ad}^*(\text{resp. } \mathrm{co}^*)"'] \arrow[d, "f^*"] \\
 & & H^*(H) \arrow[ull, "(\nu')^*(\text{resp. } (\gamma'\circ\pi)^*)"]
\end{tikzcd}$$

Thus, if we can find an element $x \in H^*(G)$ such that $f^*(x) = 0$ and $(1 \otimes f^*) \circ \mathrm{ad}^*(x) \neq 0$, then we know that f is not mod p homotopy normal. Note that we can compute ad^* using the Hopf algebra structure of $H^*(G)$. In [6], Furukawa used this argument to show non (mod p) homotopy normality in various cases.

We modify his argument as follows. Assume that $f\colon H \to G$ is mod p homotopy normal. Then, we have a homotopy $N_t\colon G \times \Omega H \to \Omega G$ and a continuous map $N'\colon G \times \Omega H \to \Omega H$ defined by $N_t(g, l)(s) = \nu_t(g, l(s))$ and $N'(g, l)(s) = \nu'(g, l(s))$, respectively, for $g \in G$, $l \in \Omega H$, and $s \in [0, 1]$. Similarly, we have a homotopy $\Gamma_t\colon \Omega G \times H \to \Omega G$ and a continuous map $\Gamma'\colon \Omega G \times H \to \Omega H$ defined by $\Gamma_t(l, h)(s) = \gamma_t(l(s) \wedge h)$ and $\Gamma'(l, h)(s) = \gamma'(l(s) \wedge h)$, respectively, for $l \in \Omega G$, $h \in H$, and $s \in [0, 1]$.

Define the adjoint action $\mathrm{Ad}\colon G \times \Omega G \to \Omega G$ of G on ΩG by $\mathrm{Ad}(g, l)(s) = \mathrm{ad}(g, l(s))$ and the commutator map $\mathrm{Co}\colon \Omega G \times G \to \Omega G$ of G with ΩG by $\mathrm{Co}(l, g)(s) = \mathrm{co}(l(s), g)$ for $g \in G$, $l \in \Omega G$, and $s \in [0, 1]$. Given a Hopf algebra A, let $\mathrm{P}A$ and $\mathrm{Q}A$ denote the primitives and the indecomposables respectively. Then, we have the following commutative diagrams.

(2.2)
$$\begin{tikzcd}
H_*(G) \otimes H_*(\Omega H) \arrow[r, "1\otimes(\Omega f)_*"] \arrow[drr, "N'_*"'] & H_*(G) \otimes H_*(\Omega G) \arrow[r, "\mathrm{Ad}_*"] & H_*(\Omega G) \arrow[r, two heads] & \mathrm{Q}H_*(\Omega G) \\
 & & H_*(\Omega H) \arrow[u, "(\Omega f)_*"] \arrow[r, two heads] & \mathrm{Q}H_*(\Omega H) \arrow[u, "\mathrm{Q}(\Omega f)_*"']
\end{tikzcd}$$

(2.3)

$$\begin{tikzcd}
H_*(\Omega G)\otimes H_*(H) \arrow[r, "1\otimes f_*"] \arrow[rrd, "\Gamma'_*"'] & H_*(\Omega G)\otimes H_*(G) \arrow[r, "\mathrm{Co}_*"] & H_*(\Omega G) \arrow[r, two heads] & \mathrm{Q}H_*(\Omega G) \\
 & & H_*(\Omega H) \arrow[u, "(\Omega f)_*"'] \arrow[r, two heads] & \mathrm{Q}H_*(\Omega H) \arrow[u, "\mathrm{Q}(\Omega f)_*"']
\end{tikzcd}$$

Note that since

$$\mathrm{Co} = m \circ (1 \times \mathrm{Ad}) \circ (1 \times 1 \times \iota) \circ (1 \times \mathrm{tw}) \circ (\Delta \times 1),$$

where $\Delta\colon \Omega G \to \Omega G \times \Omega G$ is the diagonal map, $\mathrm{tw}\colon \Omega G \times G \to G \times \Omega G$ is the twisting map, $\iota\colon \Omega G \to \Omega G$ is the inverse map, and $m\colon \Omega G \times \Omega G \to \Omega G$ is the multiplication map, we can compute Co_* using Ad_* and the Hopf algebra structure of $H_*(\Omega G)$. In particular, we can easily see that

$$\mathrm{Co}_*(t \otimes x) = (-1)^{|t||x|+1}\mathrm{Ad}_*(x \otimes t)$$

for $t \in \mathrm{P}H_*(\Omega G)$ and $x \in H_*(G)$ with $|x| > 0$.

Similarly we define that $f\colon H \to G$ is *strongly mod p homotopy normal* (mod p homotopy normal in the sense of McCarty) if f is mod p homotopy normal and if there exists a homotopy $\xi_t\colon H \wedge H \to H$ such that $\gamma_t \circ (f \wedge 1) = f \circ \xi_t$. (See [22].) Then, if f is strongly mod p homotopy normal and if f is monomorphic, we have the following commutative diagrams.

(2.4)

$$\begin{tikzcd}
H^*(G)\otimes H^*(H) \arrow[d, "f^*\otimes 1"'] & H^*(G)\otimes H^*(G) \arrow[l, "1\otimes f^*"'] & H^*(G) \arrow[l, "\mathrm{ad}^*(\text{resp. }\mathrm{co}^*)"'] \arrow[d, "f^*"] \\
H^*(H)\otimes H^*(H) & & H^*(H) \arrow[ll, "\mathrm{ad}^*(\text{resp. }\mathrm{co}^*)"] \arrow[llu, "(\nu')^*(\text{resp. }(\gamma'\circ\pi)^*)"']
\end{tikzcd}$$

(2.5)

$$\begin{tikzcd}
H_*(G)\otimes H_*(\Omega H) \arrow[r, "1\otimes(\Omega f)_*"] \arrow[rrd, "N'_*"'] & H_*(G)\otimes H_*(\Omega G) \arrow[r, "\mathrm{Ad}_*"] & H_*(\Omega G) \arrow[r, two heads] & \mathrm{Q}H_*(\Omega G) \\
H_*(H)\otimes H_*(\Omega H) \arrow[u, "f_*\otimes 1"] \arrow[rr, "\mathrm{Ad}_*"'] & & H_*(\Omega H) \arrow[u, "(\Omega f)_*"'] \arrow[r, two heads] & \mathrm{Q}H_*(\Omega H) \arrow[u, "\mathrm{Q}(\Omega f)_*"']
\end{tikzcd}$$

(2.6)

$$\begin{tikzcd}
H_*(\Omega G)\otimes H_*(H) \arrow[r, "1\otimes f_*"] \arrow[rrd, "\Gamma'_*"'] & H_*(\Omega G)\otimes H_*(G) \arrow[r, "\mathrm{Co}_*"] & H_*(\Omega G) \arrow[r, two heads] & \mathrm{Q}H_*(\Omega G) \\
H_*(\Omega H)\otimes H_*(H) \arrow[u, "(\Omega f)_*\otimes 1"] \arrow[rr, "\mathrm{Co}_*"'] & & H_*(\Omega H) \arrow[u, "(\Omega f)_*"'] \arrow[r, two heads] & \mathrm{Q}H_*(\Omega H) \arrow[u, "\mathrm{Q}(\Omega f)_*"']
\end{tikzcd}$$

We can use (2.2), (2.3), (2.4), (2.5), and (2.6) to show non (strong) mod p homotopy normality in various cases as Furukawa used (2.1).

The following lemma is easily proved.

Lemma 2.1. *Let $f\colon H \to G$ be a homomorphism of topological groups where H is connected and G is 1-connected. Let $q\colon H' \to H$ be the universal covering over H and set $f' = f \circ q\colon H' \to G$, which is also a homomorphism of topological groups. Then, if f is mod p homotopy normal, so is f'.*

3. Homomorphisms of Lie groups and regular sequences

For a compact, connected Lie group K and a prime p, let $\mathrm{T}H^*(K) = \mathrm{T}H^*(K;\mathbb{F}_p)$ denote the set of the transgressive elements in the mod p cohomology Serre spectral sequence associated with the fibering $K \to K/T \to \mathrm{B}T$ where T is a maximal torus of K. (See Toda [26] and Ishitoya-Kono-Toda [12].) Denote by τ the transgression of such a spectral sequence. Recall that $\mathrm{T}H^*(K)$ is invariant under $\mathcal{A}_p$, the mod p Steenrod algebra, and that for a homomorphism $k\colon K_1 \to K_2$ between compact, connected Lie groups, we have $k^*(\mathrm{T}H^*(K_2)) \subset \mathrm{T}H^*(K_1)$. Moreover, recall that we may assume that all algebra generators of $H^*(K)$ lie in $\mathrm{T}H^*(K)$. Set $\wp^j = \mathrm{Sq}^{2j}$ if $p = 2$.

Proposition 3.1.

(1) *Let $i\colon H \hookrightarrow G$ be a pair of compact, connected Lie groups and p a prime such that $\pi_1(H)_{(p)} = \pi_1(G)_{(p)} = 0$. Suppose that there exists an element $x_3 \in H^3(G)$ such that $\wp^{p^{m-1}} \cdots \wp^p \wp^1 x_3 \neq 0$. Further suppose that* rank $G -$ rank $H \leq m$. *Then, $i^*(x_3) \neq 0$.*

(2) *Let H and G be compact, connected Lie groups, p a prime, and $C_p = \mathbb{Z}/p$ a central subgroup of H such that $\pi_1(H)_{(p)} = \pi_1(G)_{(p)} = 0$ and that $i\colon H/C_p \hookrightarrow G$ is a Lie pair. Suppose that there exists an element $x_3 \in H^3(G)$ such that $\wp^{p^{m-1}} \cdots \wp^p \wp^1 x_3 \neq 0$. Further suppose that* rank $G -$ rank $H < m$. *Then, $q^* \circ i^*(x_3) \neq 0$ where $q\colon H \to H/C_p$ is the projection.*

Proof. (1) Put $n =$ rank G and $n' =$ rank H. Pick a maximal torus T^n of G which contains a maximal torus $T^{n'}$ of H. Consider the following commutative diagram:

$$\begin{tikzcd} G \arrow[r] & G/T^n \arrow[r] & \mathrm{B}T^n \\ H \arrow[r] \arrow[u, "i"] & H/T^{n'} \arrow[r] \arrow[u] & \mathrm{B}T^{n'} \arrow[u, "\Phi"'] \end{tikzcd} \tag{3.1}$$

where the horizontal sequences are fiberings and the vertical maps are natural ones. We have $H^*(\mathrm{B}T^n) = \mathbb{F}_p[T_1, T_2, \ldots, T_n]$ as an algebra where $|T_j| = 2$. We may assume that Ker $\Phi^* = (T_1, T_2, \ldots, T_{n-n'})$. Pick a basis $\{a_j\}_{j=1}^n$ over $\mathbb{F}_p$ of $\mathrm{T}H^{\mathrm{odd}}(G)$ such that $|a_j| \leq |a_{j+1}|$ for any j. For each j, pick a representative $b_j \in H^*(\mathrm{B}T^n)$ of $\tau(a_j)$. Then, according to Toda [26], $\{b_j\}_{j=1}^n$ is a regular sequence in $H^*(\mathrm{B}T^n)$. By the hypothesis, we may assume that it contains a subsequence $\{b_1, \wp^1 b_1, \ldots, \wp^{p^{m-1}} \cdots \wp^1 b_1\}$ where b_1 is a representative of $\tau(x_3)$. Now, suppose that $i^*(x_3) = 0$. By naturality of the transgressions in the diagram (3.1) and by $\pi_1(H)_{(p)} = 0$, this implies that $\Phi^*(b_1) = 0$. It follows that the regular sequence $\{b_1, \wp^1 b_1, \ldots, \wp^{p^{m-1}} \cdots \wp^1 b_1\}$ is in the ideal $\mathrm{Ker}\ \Phi^* = (T_1, T_2, \ldots, T_{n-n'})$. Since $n - n' \le m$, this is a contradiction. (See Baum [2] and Quillen [25].) Hence we have $i^*(x_3) \neq 0$.

(2) Similarly, put $n = \mathrm{rank}\ G$ and $n' = \mathrm{rank}\ H = \mathrm{rank}\ H/C_p$. Pick a maximal torus T^n of G which contains a maximal torus $T^{n'}$ of H/C_p. Put $\tilde{T}^{n'} = q^{-1}(T^{n'})$ which is a maximal torus of H. Consider the following commutative diagram:

$$\begin{array}{ccccc} G & \longrightarrow & G/T^n & \longrightarrow & \mathrm{B}T^n \\ {\scriptstyle i\circ q}\uparrow & & \uparrow & & \uparrow{\scriptstyle \Phi} \\ H & \longrightarrow & H/\tilde{T}^{n'} & \longrightarrow & \mathrm{B}\tilde{T}^{n'} \end{array},$$

where the horizontal sequences are fiberings and the vertical maps are natural ones. We have $H^*(\mathrm{B}T^n) = \mathbb{F}_p[T_1, T_2, \ldots, T_n]$ as an algebra where $|T_j| = 2$. The map Φ is decomposable into the natural maps $\mathrm{B}\tilde{T}^{n'} \xrightarrow{\Phi_1} \mathrm{B}T^{n'} \xrightarrow{\Phi_2} \mathrm{B}T^n$ and we may assume that $\mathrm{Ker}\ \Phi_2^* = (T_1, T_2, \ldots, T_{n-n'})$ and that $\mathrm{Ker}\ \Phi_1^*$ is generated by one generator. Hence we may assume that $\mathrm{Ker}\ \Phi^* = (T_1, T_2, \ldots, T_{n-n'+1})$. Then, the same argument as that in (1) can be applied to the map $i \circ q \colon H \to G$ and hence we have $q^* \circ i^*(x_3) \neq 0$. $\square$

Remark 1. We can prove results such as Proposition 3.1 also by using Lie algebras and root systems of Lie groups. (See Mimura-Toda [24].)

4. Furukawa's method

In this section, we formulate Furukawa's method [6] in a particular case. We refer to it in Section 5. The proof is omitted.

Theorem 4.1. *Set $p = 3$. Let $f \colon H \to E_8$ be a homomorphism of Lie groups where H is compact and connected. If one of*
(1) $\wp^1 f^* H^3(E_8) \neq 0$ *and* $\mathrm{T}H^{27}(H) = 0$,
(2) $\wp^3 \wp^1 f^* H^3(E_8) \neq 0$ *and* $\mathrm{T}H^{39}(H) = 0$
is satisfied, then f is not mod 3 homotopy normal.

5. Homotopy normality of maximal subgroups of maximal rank

In this section, we prove the following theorem. (For the detail of maximal subgroups of maximal rank, see Borel-Siebenthal [3] and Mimura-Toda [24].)

Theorem 5.1. *Let p be a prime and G a compact, 1-connected, simple, exceptional Lie group which has integral p-torsion. Let $H \overset{f}{\hookrightarrow} G$ be a maximal subgroup of maximal rank. Then, H is not mod p homotopy normal in G if $(G, H, p) \neq (F_4, \mathrm{Spin}(9), 2)$ and $\mathrm{Spin}(9)$ is not strongly mod 2 homotopy normal in F_4.*

Remark 2. The last assertion was proved first in Furukawa [4].

The rest of this section is devoted to the proof of Theorem 5.1. Let generators of $H^*(E_8;\mathbb{F}_3)$ be as in Kono-Mimura [18]. (We use the symbol x_j instead of e_j in [18].)

First consider the case $(G,H,p)=(E_8,\mathrm{Ss}(16),3)$. By Proposition 3.1.1, we have $f^*(x_3)\neq 0$. Hence we have $f^*(x_{19})=\wp^3\wp^1 f^*(x_3)\neq 0$. By $\mathrm{T}H^{39}(\mathrm{Ss}(16))=0$ and Theorem 4.1, the assertion follows. Similarly, we can show the assertion for the following cases:

$$(E_8,H,5),\ H\neq\frac{\mathrm{SU}(5)\times\mathrm{SU}(5)}{\mathbf{Z}/5},\ \left(E_8,\frac{\mathrm{SU}(5)\times\mathrm{SU}(5)}{\mathbf{Z}/5},3\right),\ \left(E_8,\frac{\mathrm{SU}(9)}{\mathbf{Z}/3},2\right),$$
$$\left(E_8,\frac{\mathrm{SU}(5)\times\mathrm{SU}(5)}{\mathbf{Z}/5},2\right),\ \left(E_8,\frac{E_6\times\mathrm{SU}(3)}{\mathbf{Z}/3},2\right).$$

Next, consider the case $(G,H,p)=(E_8,(\mathrm{SU}(9))/(\mathbf{Z}/3),3)$. Let $q\colon \mathrm{SU}(9)\to(\mathrm{SU}(9))/(\mathbf{Z}/3)$ be the universal covering. By Proposition 3.1.2, we have $q^*\circ f^*(x_3)\neq 0$. Hence we have $q^*\circ f^*(x_7)=\wp^1 q^*\circ f^*(x_3)\neq 0$. By $\mathrm{T}H^{27}(\mathrm{SU}(9))=0$ and Theorem 4.1, $f\circ q$ is not mod 3 homotopy normal. By Lemma 2.1, the assertion follows. Similarly, we can show the assertion for the following cases:

$$\left(E_8,\frac{\mathrm{SU}(5)\times\mathrm{SU}(5)}{\mathbf{Z}/5},5\right),\ \left(E_6,\frac{\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3)}{\mathbf{Z}/3},3\right),$$
$$\left(F_4,\frac{\mathrm{SU}(3)\times\mathrm{SU}(3)}{\mathbf{Z}/3},3\right),\ (E_8,\mathrm{Ss}(16),2),\ \left(E_7,\frac{\mathrm{SU}(8)}{\mathbf{Z}/2},2\right).$$

Next, consider the case $(G,H,p)=(E_8,(E_6\times\mathrm{SU}(3))/(\mathbf{Z}/3),3)$. Let $q\colon E_6\times\mathrm{SU}(3)\to(E_6\times\mathrm{SU}(3))/(\mathbf{Z}/3)$ be the universal covering and $i\colon E_6\hookrightarrow E_6\times\mathrm{SU}(3)$ the natural inclusion. Then, applying Proposition 3.1.1 to the inclusion $f\circ q\circ i\colon E_6\hookrightarrow E_8$, we have $i^*\circ q^*\circ f^*(x_3)\neq 0$. Hence we have $i^*\circ q^*\circ f^*(x_7)=\wp^1 i^*\circ q^*\circ f^*(x_3)\neq 0$. It follows that $q^*\circ f^*(x_7)\neq 0$. By $\mathrm{T}H^{27}(E_6\times\mathrm{SU}(3))=0$ and Theorem 4.1, $f\circ q$ is not mod 3 homotopy normal. By Lemma 2.1, the assertion follows. Similarly, we can show the assertion for the following cases:

$$\left(E_8,\frac{E_7\times\mathrm{SU}(2)}{\mathbf{Z}/2},3\right),\ \left(E_7,\frac{\mathrm{Spin}(12)\times\mathrm{SU}(2)}{\mathbf{Z}/2},3\right),\ \left(E_7,\frac{\mathrm{SU}(3)\times\mathrm{SU}(6)}{\mathbf{Z}/3},3\right),$$
$$\left(E_7,\frac{E_6\times T}{\mathbf{Z}/3},3\right),\ \left(E_6,\frac{\mathrm{SU}(2)\times\mathrm{SU}(6)}{\mathbf{Z}/2},3\right),\ \left(F_4,\frac{\mathrm{Sp}(3)\times\mathrm{Sp}(1)}{\mathbf{Z}/2},3\right),$$
$$\left(E_8,\frac{E_7\times\mathrm{SU}(2)}{\mathbf{Z}/2},2\right),\ \left(E_7,\frac{\mathrm{Spin}(12)\times\mathrm{SU}(2)}{\mathbf{Z}/2},2\right),\ \left(E_7,\frac{\mathrm{SU}(3)\times\mathrm{SU}(6)}{\mathbf{Z}/3},2\right),$$
$$\left(E_7,\frac{E_6\times T}{\mathbf{Z}/3},2\right),\ \left(E_6,\frac{\mathrm{Spin}(10)\times T}{\mathbf{Z}/4},2\right),\ \left(E_6,\frac{\mathrm{SU}(2)\times\mathrm{SU}(6)}{\mathbf{Z}/2},2\right).$$

For the cases $(G,H,p)=(E_7,(\mathrm{SU}(8))/(\mathbf{Z}/2),3)$, $(E_6,(\mathrm{Spin}(10)\times T)/(\mathbf{Z}/4),3)$, and $(F_4,\mathrm{Spin}(9),3)$, the assertion follows from the following theorem with the technique as above. Given a Hopf algebra A, let $\bar{x} \in \mathrm{Q}A$ denote the indecomposable class of an element $x \in A$.

Theorem 5.2. *Set $p = 3$. Let $f\colon H \to G$ be a homomorphism of Lie groups where H is compact and connected, $\pi_1(H)_{(3)} = 0$, and $G = F_4, E_6$, or E_7. If $\wp^1 f^* H^3(G) \neq 0$, if $H_*(H; \mathbb{Z})$ is 3-torsion free, and if $\mathrm{Q}H^{23}(H; \mathbb{Q}) = 0$, then f is not mod 3 homotopy normal.*

Proof. Assume that we have the diagram (2.2) where G and H are as above. (Of course, localize all spaces and maps at 3.) Let generators of $H_*(\Omega G)$ be as in Hamanaka-Hara [10]. Also let $y_8 \in H_8(G)$ be as in [10]. We can pick $t_6' \in H_6(\Omega H)$ such that $(\Omega f)_*(t_6') = t_6 \in H_6(\Omega G)$. Then, by [10], we have

$$\overline{\mathrm{Ad}_* \circ (1 \otimes (\Omega f)_*)(y_8^2 \otimes t_6')} = \overline{y_8^2 * t_6} = \bar{t}_{22}.$$

By the hypothesis, we have $\mathrm{Q}H_{22}(\Omega H) = 0$. This is a contradiction and the theorem follows. $\square$

Similarly, for the cases $(G, H, p) = (E_6, (\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3))/(\mathbf{Z}/3), 2)$, $(F_4, (\mathrm{SU}(3) \times \mathrm{SU}(3))/(\mathbf{Z}/3), 2)$, and $(G_2, \mathrm{SU}(3), 2)$, we can show the assertion by considering the relation $\overline{y_6 * b_4} = \bar{b}_{10}$ in $\mathrm{Q}H_*(\Omega G) = \mathrm{Q}H_*(\Omega G; \mathbb{F}_2)$ where $G = E_6, F_4$, or G_2. (For the notation, see Hamanaka [7].)

Next, consider the case $(G, H, p) = (F_4, (\mathrm{Sp}(3) \times \mathrm{Sp}(1))/(\mathbf{Z}/2), 2)$. Let $q\colon \mathrm{Sp}(3) \times \mathrm{Sp}(1) \to (\mathrm{Sp}(3) \times \mathrm{Sp}(1))/(\mathbf{Z}/2)$ be the universal covering. Then, we can apply a method similar to the above to the map $f \circ q\colon \mathrm{Sp}(3) \times \mathrm{Sp}(1) \to F_4$ by considering the relation $y_6 * b_2 = b_4^2$ in $H_*(\Omega F_4)$. (For the notation, see Hamanaka [7].) Thus, $f \circ q$ is not mod 2 homotopy normal. By Lemma 2.1, the assertion follows. The case $(G_2, \mathrm{SO}(4), 2)$ is similar.

Finally, we prove that $\mathrm{Spin}(9)$ is not strongly mod 2 homotopy normal in F_4. We use the result of Hamanaka [9]. Set $p = 2$. Let $b_j, b_j' \in H_*(\Omega\mathrm{Spin}(9))$ be as in [9]. Put $z_2 = b_1, z_4 = b_2, z_6 = b_3 + b_2 b_1, z_{10} = b_5' + b_4' b_1$, and $z_{14} = b_7' + b_6' b_1 + b_5' b_2 + b_4' b_3$. Then, we have

$$H_*(\Omega\mathrm{Spin}(9)) = \wedge(z_2) \otimes \mathbb{F}_2[z_4, z_6, z_{10}, z_{14}]$$

as an algebra where $|z_j| = j$ and z_j is primitive if $j \neq 4$. Let $y_6 \in H_*(\mathrm{Spin}(9))$ be as in [9]. Then, we have

$$\mathrm{Co}_*(z_6 \otimes y_6) = y_6 * z_6 = z_6^2.$$

We can easily see that $(\Omega f)_*(z_6) = 0$. (For the detail of $H_*(\Omega F_4)$, see Kono-Kozima [16].) Then, we can prove the assertion by using the diagram (2.6).

6. Other applications

We close this paper by showing some more applications.

Furukawa in [5] showed that the natural inclusion $G_2 \hookrightarrow F_4$ is not strongly mod 2 homotopy normal by calculating Samelson products of homotopy groups.

Then, Kudou and Yagita in [20] showed that $G_2 \hookrightarrow F_4$ is not mod 2 homotopy normal by using the Morava K-theory. We can reprove this result by using the relation

$$\mathrm{Co}_*(b_{22} \otimes y_6) = y_6 * b_{22} = b_{14}^2$$

in $H_*(\Omega F_4) = H_*(\Omega F_4; \mathbb{F}_2)$ and the diagram (2.3). (For the notation, see Hamanaka [7].)

Kudou and Yagita in [19] asked whether the natural inclusions $G_2 \hookrightarrow F_4$ and $\mathrm{Spin}(9) \hookrightarrow F_4$ are mod 3 homotopy normal or not. We showed in Theorem 5.1 that the latter is not mod 3 homotopy normal. We can show that the former is also not mod 3 homotopy normal. In fact, we have the following theorem.

Theorem 6.1. *Set $p = 3$. Let $f\colon H \to G$ be a homomorphism of Lie groups where H is compact and connected, $\pi_1(H)_{(3)} = 0$, and $G = F_4, E_6$, or E_7. If $f^*H^3(G) \neq 0$, if $H_*(H;\mathbb{Z})$ is 3-torsion free, and if $\mathrm{Q}H^7(H;\mathbb{Q}) = \mathrm{Q}H^{19}(H;\mathbb{Q}) = 0$, then f is not mod 3 homotopy normal.*

We can prove this theorem by using the relation $y_8^2 * t_2 = \pm t_6^3$ in $H_*(\Omega G)$ and the diagram (2.2). (For the notation, see Hamanaka-Hara [10].) Also we have the following theorem.

Theorem 6.2. *Set $p = 3$. Let $f\colon H \to E_8$ be a homomorphism of Lie groups where H is compact and connected, and $\pi_1(H)_{(3)} = 0$. If $f^*H^3(E_8) \neq 0$, if $H_*(H;\mathbb{Z})$ is 3-torsion free, and if $\mathrm{Q}H^{23}(H;\mathbb{Q}) = 0$, then f is not mod 3 homotopy normal.*

We can prove this theorem by using the relation $y_{20} * t_2 = \pm t_{22}$ in $H_*(\Omega E_8)$ and the diagram (2.2). (For the notation, see [10].) For example, $\mathrm{SU}(2), \mathrm{SU}(3)$, $\mathrm{Spin}(4)$, and G_2 are not mod 3 homotopy normal in E_8.

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