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Vorticity existence of an ideal incompressible fluid in $B^0_{\infty,1}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$

By

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Abstract

We prove a local (in time) unique vorticity existence for the Euler equation of an ideal incompressible fluid in a critical Besov space $\mathbf{B}^0_{\infty,1}(\mathbb{R}^3) \cap \mathbf{L}^p(\mathbb{R}^3)$ with the initial vorticity $\omega_0 \in \mathbf{B}^0_{\infty,1}(\mathbb{R}^3) \cap \mathbf{L}^p(\mathbb{R}^3)$ for some $1 < p < 3$.

1. Introduction

In this paper, we consider the non-stationary Euler equations of an ideal incompressible fluid

$$
\begin{aligned}
\frac{\partial}{\partial t}\, u_j &= -\sum_{i=1}^{n} u_i\, \partial_i u_j - \partial_j p, \quad 1 \le j \le n, \\
\operatorname{div} u &= \sum_{j=1}^{n} \partial_j u_j = 0, \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\tag{1.1}
$$

Here $u(x,t) = (u_1, u_2, \dots, u_n)$ is the Eulerian velocity of a fluid flow and $\partial_i \equiv \frac{\partial}{\partial x_i}$. For $n = 3$, by taking curl on (1.1), we get the vorticity equation

$$
\frac{\partial}{\partial t}\, \omega(x,t) = -(u,\nabla)\omega + (\omega,\nabla)u, \qquad \omega = \operatorname{curl} u. \tag{1.2}
$$

Existence and uniqueness theories of (2 or 3 dimensional) solutions of the Euler equations are studied by many mathematicians and physicists. L. Lichtenstein, N. Gunther and Wolibner started the subject on Hölder classes. D. Ebin and J. Marsden, J. Bourguignon and H. Brezis, R. Temam, T. Kato and G. Ponce studied this subject on Sobolev spaces. The work on the Euler equation in Besov spaces has been done by M. Vishik in [12] and [13]. In [12], he proved

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that the solution $u(t)$ constructed stays globally in $\mathbf{B}^{s+1}_{p,1}$ by developing an elegant logarithmic inequality, where the initial velocity u_0 belongs to $\mathbf{B}^{s+1}_{p,1}$ with the condition $sp = n = 2$. He has also proven the uniqueness of the n-dimensional Euler equations ($n \geq 2$) with vorticity bounded in Besov-type spaces that contain essentially unbounded functions ([13]).

In this paper, we investigate the question of local existence of the solution to the 3-D vorticity equation (1.2) and prove that the vorticity $\omega(x,t)$ stays locally in $\mathbf{B}^0_{\infty,1}(\mathbb{R}^3) \cap \mathbf{L}^p(\mathbb{R}^3)$ if the initial vorticity $\omega_0(x)$ is in $\mathbf{B}^0_{\infty,1}(\mathbb{R}^3) \cap \mathbf{L}^p(\mathbb{R}^3)$.

The Besov space $B^0_{\infty,1}(\mathbb{R}^n)$ is seemed of particular interest in the sense that these function spaces are very close to the space $C(\mathbb{R}^n)$ of continuous functions, but on which pseudo-differential operators (with rough symbols) act. The following is our main result.

Theorem 1.1. *Let $1 < p < 3$. Assume that $\omega_0 \in \mathbf{B}^0_{\infty,1}(\mathbb{R}^3) \cap \mathbf{L}^p(\mathbb{R}^3)$. There exists a positive constant $T > 0$ such that the initial value problem for the vorticity equation* (1.2) *with initial vorticity $\omega(x,0) = \omega_0(x)$ has a unique solution $\omega(x,t) \in L^\infty([0,T]; \mathbf{B}^0_{\infty,1}(\mathbb{R}^3) \cap \mathbf{L}^p(\mathbb{R}^3))$.*

The proof of the main theorem is divided into two parts—Sections 3 and 4. A collection of a-priori estimates which is used in the proof of the main theorem is located in Section 4 after the proof of the main theorem (Section 3). The essential tools for the estimates are Bony's para-product formula and the Littlewood-Paley decomposition. In Section 3, we use a *compactness argument* ([13], [14]) to get a local existence of the solution rather than a standard *iterative algorithm* ([6]), mainly because the priori estimates used in the proof are much dependent on the fact that the vector fields in arguments are divergence free. To complete the compactness argument, an Osgood-type ordinary differential inequality is solved (page 7).

We close this section with some remarks on the Biot-Savart's law that explains the relationship between the divergence free velocity $u(x,t)$ and its vorticity $\omega(x,t)$;

$$u(x,t) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \frac{1}{4\pi}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 \frac{1}{|x|}* & \partial_2 \frac{1}{|x|}* & \partial_3 \frac{1}{|x|}* \\ \omega_1(x,t) & \omega_2(x,t) & \omega_3(x,t) \end{vmatrix}, \tag{1.3}$$

where $*$ represents the convolution operation

$$(f * g)(x) = \int_{\mathbb{R}^3} f(x-y)g(y)dy.$$

We denote the right hand side of (1.3) as $\mathcal{K} * \omega$ (with a little notational abuse).

Notation. Throughout this paper, C denotes various real positive constants.

2. Littlewood-Paley analysis

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class. We choose a radial function $\widehat{\Phi} \in C_0^\infty(\mathbb{R}^n)$ and a radial function $\widehat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that the following properties are satisfied:

- $\operatorname{supp} \widehat{\Phi} \subset \left\{ |\xi| \leq \frac{5}{6} \right\}$;
- $\operatorname{supp} \widehat{\varphi} \subset \left\{ \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \right\}$;
- $\widehat{\Phi}(\xi) + \sum_{j=0}^{\infty} \widehat{\varphi}_j(\xi) = 1, \qquad \xi \in \mathbb{R}^n$

where $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ (that is, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$), and $\hat{f}$ represents the Fourier transform of $f \in \mathcal{S}$ defined by

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) \equiv \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Notation. Let $f \in \mathcal{S}'$. Then

$$\begin{aligned}
& \Delta_{-1} f = \widehat{\Phi}(D) f = \Phi * f. \\
\text{For } j \geq 0, \quad & \Delta_j f = \widehat{\varphi}_j(D) f = \varphi_j * f. \\
\text{For } j \leq -2, \quad & \Delta_j f = 0. \\
\text{For } k \in \mathbb{Z}, \quad & S_k f = \sum_{j \leq k} \Delta_j f.
\end{aligned}$$

Definition 2.1. Assume $s \in \mathbb{R}$, and $1 \leq p \leq \infty$. For $1 \leq q < \infty$, the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined by

$$f \in B^s_{p,q}(\mathbb{R}^n) \Leftrightarrow \sum_{j \in \mathbb{Z}} \|2^{js}\Delta_j f\|^q_{L^p} < \infty,$$

and $B^s_{p,\infty}(\mathbb{R}^n)$ are defined by

$$f \in B^s_{p,\infty}(\mathbb{R}^n) \Leftrightarrow \sup_{j \in \mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^p} < \infty.$$

Hereafter, the corresponding spaces of vector-valued functions will be denoted by bold faced symbols. For example, we denote the product space $L^p(\mathbb{R}^n)^d$ by $\mathbf{L}^p(\mathbb{R}^n)$ and the corresponding triple Besov spaces $B^s_{p,q}(\mathbb{R}^3)$ by $\mathbf{B}^s_{p,q}(\mathbb{R}^3) \equiv B^s_{p,q}(\mathbb{R}^3)^3$. The Besov spaces $\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^n)$ are known as Lipschitz spaces $\mathbf{\Lambda}^s(\mathbb{R}^n)$ and also, equivalent to the classical Hölder Spaces $\mathbf{C}^s(\mathbb{R}^n)$ (see, e.g., [6, p. 26] or [9]). We use the notation $\mathbf{\Lambda}^s(\mathbb{R}^n)$ for the Besov spaces $\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^n)$, i.e., $\mathbf{\Lambda}^s(\mathbb{R}^n) \equiv \mathbf{B}^s_{\infty,\infty}(\mathbb{R}^n)$.

We define *Bony's para-product formula* which decomposes the product $f \cdot g$ of two functions f and g into three parts:

$$f \cdot g = T_f g + T_g f + R(f, g),$$

where $T_f g$ is Bony's "para-product" of f and g defined by

$$T_f g \equiv \sum_{i \le j-2} \Delta_i f \Delta_j g = \sum_j S_{j-2} f \Delta_j g$$

and $R(f, g)$ denotes the remainder of the para-product

$$R(f, g) \equiv \sum_{|i-j| \le 1} \Delta_i f \Delta_j g.$$

Remark 1. We have the following expression:

$$\begin{aligned}
&(S_{j-2}u, \nabla)\Delta_j \omega - \Delta_j (u, \nabla)\omega \\
&\quad = -\sum_{i=1}^{3} \Delta_j T_{\partial_i \omega} u_i + \sum_{i=1}^{3} [T_{u_i}\partial_i, \Delta_j]\omega - \sum_{i=1}^{3} T_{u_i - S_{j-2}u_i} \partial_i \Delta_j \omega \\
&\qquad - \sum_{i=1}^{3} \{\Delta_j R(u_i, \partial_i \omega) - R(S_{j-2}u_i, \Delta_j \partial_i \omega)\}.
\end{aligned} \tag{2.1}$$

In fact, we expand $(S_{j-2}u, \nabla)\Delta_j \omega - \Delta_j (u, \nabla)\omega$ by Bony's para-product formula to get the right hand side of (2.1) plus zero terms. The details can be found in [12, p. 204].

The following Remark shows the relationship between the role of the scaling factor 2^{js} and the role of the differentiation index s in $B^s_{p,q}$. The proof of the remark can be found in [6, p. 16].

Remark 2 (Bernstein's Lemma). Assume $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \le r\}$. Then there exists a constant $C = C(s)$ such that

$$\begin{aligned}
\|f\|_{L^{p_1}} &\le C r^{n\left(\frac{1}{p} - \frac{1}{p_1}\right)} \|f\|_{L^p}, \quad 1 \le p \le p_1 \le \infty, \\
\|D^s f\|_{L^p} &\le C r^s \|f\|_{L^p}, \quad 1 \le p \le \infty.
\end{aligned}$$

For $f \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty]$ and $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\}$, there exists a constant $C(s)$ such that

$$C^{-1} 2^{js} \|f\|_{L^p} \le \|D^s f\|_{L^p} \le C\, 2^{js} \|f\|_{L^p}.$$

Remark 3. Let $1 < p < n$ and $\omega = \operatorname{curl} u$. Then we have

1. $\|\Delta_{-1} u\|_{\mathbf{L}^\infty} \le C \|\Delta_{-1}\omega\|_{\mathbf{L}^p}$,
2. $\|\Delta_j u\|_{\mathbf{L}^\infty} \le C 2^{-j} \|\Delta_j \omega\|_{\mathbf{L}^\infty}, \qquad j \ge 0$,
3. $\|\nabla u\|_{\mathbf{L}^\infty} \le C(\|\omega\|_{\mathbf{B}^0_{\infty,1}} + \|\omega\|_{\mathbf{L}^p})$.

Proof. The proof of the first inequality can be found in [13]. Take $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near the origin. Then

$$\begin{aligned}
\|\Delta_{-1} u\|_{\mathbf{L}^\infty} &\le \|(\chi \mathcal{K}) * \Delta_{-1}\omega\|_{\mathbf{L}^\infty} + \|((1-\chi)\mathcal{K}) * \Delta_{-1}\omega\|_{\mathbf{L}^\infty} \\
&\le \|\chi \mathcal{K}\|_{L^1} \|\Delta_{-1}\omega\|_{\mathbf{L}^\infty} + \|(1-\chi)\mathcal{K}\|_{L^q} \|\Delta_{-1}\omega\|_{\mathbf{L}^p} \\
&\le C \|\Delta_{-1}\omega\|_{\mathbf{L}^p},
\end{aligned}$$

where $\chi\mathcal{K} \in L^1$, $(1-\chi)\mathcal{K} \in L^q$ and $\frac{1}{p}+\frac{1}{q}=1$. The second inequality follows from the fact stated at the bottom of [5, p. 676]. Using these estimates, we get

$$\begin{aligned}
\|\nabla u\|_{\mathbf{L}^\infty} &\le \|\Delta_{-1}\nabla u\|_{\mathbf{L}^\infty} + \sum_{j=0}^{\infty}\|\Delta_j \nabla u\|_{\mathbf{L}^\infty} \\
&\le C\left(\|\omega\|_{\mathbf{L}^p} + \sum_{j=0}^{\infty}\|\Delta_j\omega\|_{\mathbf{L}^\infty}\right) \\
&\le C(\|\omega\|_{\mathbf{L}^p} + \|\omega\|_{\mathbf{B}^0_{\infty,1}}).
\end{aligned}$$

□

3. The proof of the main theorem

Let $1<p<3$, and we are given initial vorticity $\omega|_{t=0}=\omega_0 \in \mathbf{B}^0_{\infty,1}\cap\mathbf{L}^p$. In order to prove that the vorticity $\omega(t)$ (representing the solution of the vorticity equation (1.2)) stays (locally) in the function space $\mathbf{B}^0_{\infty,1}\cap\mathbf{L}^p$, we define a sequence $\{\omega_n\}_{n\in\mathbb{N}}$ of vector fields depending on time by means of the following restrictions on each initial vector field:

$$\begin{aligned}
&\omega_n|_{t=0} = S_n\omega_0, \qquad n=1,2,3,\dots \\
&\frac{\partial}{\partial t}\,\omega_n(x,t) = -(u_n,\nabla)\omega_n + (\omega_n,\nabla)u_n, \\
&u_n(x,t) = \mathcal{K}*\omega_n(x,t).
\end{aligned} \tag{3.1}$$

Then $\omega_n(0) \in \mathbf{\Lambda}^s\cap\mathbf{L}^p$ for any $s\in(0,\infty)$. Therefore, for fixed $s\in(0,\infty)$, according to classical results (see [6]), for each n, there exist a maximal time $T_n^*\in(0,\infty]$ and a solution ω_n to the equation (3.1) in $L^\infty_{loc}([0,T_n^*);\mathbf{\Lambda}^s\bigcap\mathbf{L}^p)$.

First, we will find a positive lower bound T_1 with $0<T_1\le T_n^*$, where the constant T_1 depends on ω_0 but not on n, and show that the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T_1];\mathbf{B}^0_{\infty,1}\bigcap\mathbf{L}^p)$. In fact, we will see that the positive number T_1 depends only on $\|\omega_0\|_{\mathbf{B}^0_{\infty,1}\cap\mathbf{L}^p}$ and find a continuous positive non-decreasing function $\lambda(t)$ satisfying: for $t\le T_1$

$$\|\omega_n(t)\|_{\mathbf{B}^0_{\infty,1}\cap\mathbf{L}^p} \le \lambda(t), \quad \text{for all } n\in\mathbb{N}.$$

The function $\lambda(\cdot)$ defined on (at least) $[0,T_1]$ is a solution of the following ordinary differential equation:

$$\frac{d}{dt}\lambda = C_0\lambda^2, \qquad \lambda(0) = C_0\|\omega_0\|_{\mathbf{B}^0_{\infty,1}\cap\mathbf{L}^p},$$

where the constant $C_0>0$ will be chosen later. Then from the well-known Beale-Kato-Majda type blow-up criterion ([6]) stating that

$$T_n^*<\infty \Rightarrow \int_0^{T_n^*}\|\omega_n(\tau)\|_{\mathbf{L}^\infty}d\tau = \infty,$$

and the fact that $\|\omega_n(\tau)\|_{\mathbf{L}^\infty} \le \|\omega_n(\tau)\|_{\mathbf{B}^0_{\infty,1}}$, we observe that $T_1 > 0$ is a lower bound of $\{T_n^* : n \in \mathbb{N}\}$.

Taking Δ_j operator on both sides of (3.1) and adding $(S_{j-2}u_n, \nabla)\Delta_j\omega_n$ on both sides (in order to count Remark 1), we have

$$\begin{aligned}\frac{\partial}{\partial t}\,\Delta_j\omega_n &+ (S_{j-2}u_n, \nabla)\Delta_j\omega_n \\ &= (S_{j-2}u_n, \nabla)\Delta_j\omega_n - \Delta_j(u_n, \nabla)\omega_n + \Delta_j(\omega_n, \nabla)u_n.\end{aligned}$$

By considering the *trajectory flow* $\{X_j^n(x,t)\}$ *along* $S_{j-2}u_n$ defined by the solution of the ordinary differential equations

$$\begin{cases}\dfrac{\partial}{\partial t}X_j^n(x,t) = (S_{j-2}u_n)(X_j^n(x,t),t),\\ X_j^n(x,0) = x,\end{cases}$$

(we observe that $\operatorname{div} S_{j-2}u_n = 0$ implies $x \mapsto X_j^n(x,t)$ is a volume preserving mapping) we get

$$\begin{aligned}\|\Delta_j\omega_n(t)\|_{\mathbf{L}^\infty} \le \|\Delta_j S_n\omega_0\|_{\mathbf{L}^\infty} &+ \int_0^t \|\Delta_j((\omega_n, \nabla)u_n)\|_{\mathbf{L}^\infty}d\tau \\ &+ \int_0^t \|(S_{j-2}u_n, \nabla)\Delta_j\omega_n - \Delta_j((u_n, \nabla)\omega_n)\|_{\mathbf{L}^\infty}d\tau.\end{aligned}$$

Or

$$\begin{aligned}\|\omega_n(t)\|_{\mathbf{B}^0_{\infty,1}} \le \|S_n\omega_0\|_{\mathbf{B}^0_{\infty,1}} &+ \int_0^t \|(\omega_n, \nabla)u_n\|_{\mathbf{B}^0_{\infty,1}}d\tau \\ &+ \int_0^t \sum_{j\ge -1} \|(S_{j-2}u_n, \nabla)\Delta_j\omega_n - \Delta_j((u_n, \nabla)\omega_n)\|_{\mathbf{L}^\infty}d\tau.\end{aligned}$$

From Proposition 4.1, 4.2 (which will be found in Section 4), we get an estimate:

$$\|\omega_n(t)\|_{\mathbf{B}^0_{\infty,1}} \le C\|\omega_0\|_{\mathbf{B}^0_{\infty,1}} + C\int_0^t \|u_n\|_{\mathbf{B}^1_{\infty,1}}\|\omega_n(\tau)\|_{\mathbf{B}^0_{\infty,1}}d\tau. \tag{3.2}$$

On the other hand, using the trajectory flow $\{Y_n(x,t)\}$ along u_n we obtain from equation (3.1)

$$|\omega_n(Y_n(x,t),t)| \le |S_n\omega_0(x)| + \int_0^t |((\omega_n\cdot\nabla)u_n)(Y_n(x,\tau),\tau)|d\tau. \tag{3.3}$$

By taking $\mathbf{L}^p$-norm on both sides of (3.3) and using the fact that $x \longmapsto Y_n(x,t)$ is volume preserving, we have

$$\begin{aligned}\|\omega_n(t)\|_{\mathbf{L}^p} &\le \|S_n\omega_0\|_{\mathbf{L}^p} + \int_0^t \|\nabla u_n(\tau)\|_{\mathbf{L}^\infty}\|\omega_n(\tau)\|_{\mathbf{L}^p}d\tau \\ &\le \sup_n \|S_n\omega_0\|_{\mathbf{L}^p} + C\int_0^t \|u_n\|_{\mathbf{B}^1_{\infty,1}}\|\omega_n\|_{\mathbf{L}^p}\,d\tau.\end{aligned} \tag{3.4}$$

Let $\mathbf{V}$ denote the space $\mathbf{B}^0_{\infty,1}(\mathbb{R}^3) \cap \mathbf{L}^p(\mathbb{R}^3)$. Then putting (3.2) and (3.4) together and counting Remark 3, we get

$$\begin{aligned} \|\omega_n(t)\|_{\mathbf{V}} &\leq C\|\omega_0\|_{\mathbf{V}} + C\int_0^t \|u_n\|_{\mathbf{B}^1_{\infty,1}} \|\omega_n(\tau)\|_{\mathbf{V}} d\tau \\ &\leq C_0\|\omega_0\|_{\mathbf{V}} + C_0\int_0^t \|\omega_n(\tau)\|^2_{\mathbf{V}} d\tau, \end{aligned} \tag{3.5}$$

for some constant C_0. By virtue of Gronwall's inequality this leads to

$$\sup_{0\leq\tau\leq t} \|\omega_n(\tau)\|_{\mathbf{V}} \leq C_0\|\omega_0\|_{\mathbf{V}} \exp\left\{C_0\int_0^t \sup_{0\leq\tau'\leq\tau} \|\omega_n(\tau')\|_{\mathbf{V}} d\tau\right\}. \tag{3.6}$$

Let $\lambda(\cdot)$ satisfy the following ordinary differential equation:

$$\frac{d}{dt}\lambda = C_0\lambda^2, \qquad \lambda(0) = C_0\|\omega_0\|_{\mathbf{V}}, \tag{3.7}$$

and let

$$\lambda_1(t) \equiv C_0\|\omega_0\|_{\mathbf{V}} \exp\left\{C_0\int_0^t \sup_{0\leq\tau'\leq\tau} \|\omega_n(\tau')\|_{\mathbf{V}} d\tau\right\}.$$

Then from (3.6) and the definition of $\lambda_1(t)$, we can notice that

$$\frac{d}{dt}\lambda_1 \leq C_0\lambda_1^2, \qquad \lambda_1(0) = C_0\|\omega_0\|_{\mathbf{V}}. \tag{3.8}$$

The time $T_1 > 0$ is chosen to be less than the blow-up time for (3.7). Then by solving the separable ordinary differential inequality (3.8), we observe that $\lambda_1(t) \leq \lambda(t)$ for $t \in [0, T_1]$. Indeed, (3.8) leads to $-\frac{d}{dt}\left(\frac{1}{\lambda_1}\right) \leq C_0$. This implies that for $t \in [0, T_1]$, $\lambda_1(t) \leq \dfrac{C_0\|\omega_0\|_{\mathbf{V}}}{1 - tC_0^2\|\omega_0\|_{\mathbf{V}}} = \lambda(t)$. Hence we get

$$\|\omega_n(t)\|_{\mathbf{V}} \leq \lambda(t), \quad t \in [0, T_1] \tag{3.9}$$

for all $n \in \mathbb{N}$, that is, the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty([0, T_1]; \mathbf{V})$, and we also see that $T_1 > 0$ is a lower bound of $\{T_n^* : n \in \mathbb{N}\}$.

Next, we will find a strictly positive time T_2 (depending on $\|\omega_0\|_{\mathbf{V}}$) such that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0, T_2]; \mathbf{B}^0_{\infty,1}(\mathbb{R}^3))$. To do this, we subtract the two relations on the corresponding Euler equations of the vorticity equations (3.1) to get

$$\begin{aligned} &\frac{\partial}{\partial t}(u_{n+1} - u_n) + (u_{n+1}, \nabla)(u_{n+1} - u_n) + (u_{n+1} - u_n, \nabla)u_n \\ &\qquad\qquad = \nabla p_n - \nabla p_{n+1}, \\ &(u_{n+1} - u_n)|_{t=0} = \Delta_{n+1}u_0. \end{aligned} \tag{3.10}$$

Taking Δ_j operator, and adding $(S_{j-2}u_{n+1}, \nabla)\Delta_j(u_{n+1} - u_n)$ on both sides of (3.10), we have

$$\begin{aligned}
&\|(u_{n+1} - u_n)(t)\|_{\mathbf{B}^0_{\infty,1}} \leq \|\Delta_{n+1}u_0\|_{\mathbf{B}^0_{\infty,1}} \\
&+ \int_0^t \sum_{j\geq -1} \|(S_{j-2}u_{n+1}, \nabla)\Delta_j(u_{n+1} - u_n) - \Delta_j(u_{n+1}, \nabla)(u_{n+1} - u_n)\|_{\mathbf{L}^\infty} d\tau \\
&+ \int_0^t \|(u_{n+1} - u_n, \nabla)u_n\|_{\mathbf{B}^0_{\infty,1}} d\tau + \int_0^t \|\nabla p_n - \nabla p_{n+1}\|_{\mathbf{B}^0_{\infty,1}} d\tau.
\end{aligned}$$

We notice that

$$\nabla p_n - \nabla p_{n+1} = \pi(u_{n+1} - u_n, u_{n+1}) + \pi(u_n, u_{n+1} - u_n),$$

where we define

$$\pi(u_1, u_2) \equiv \sum_{i,j=1}^{3} \nabla\Delta^{-1}\partial_i u_1^j \partial_j u_2^i.$$

Hence, by Proposition 4.1, 4.2, 4.3, Remark 3 and estimate (3.9), we get

$$\begin{aligned}
\|u_{n+1} - u_n\|_{L^\infty([0,T];\mathbf{B}^0_{\infty,1})} \leq\ & \|\Delta_{n+1}u_0\|_{\mathbf{B}^0_{\infty,1}} \\
& + C_1 T\lambda(T)\|u_{n+1} - u_n\|_{L^\infty([0,T];\mathbf{B}^0_{\infty,1})} \\
\leq\ & 2^{-n}\|\Delta_{n+1}u_0\|_{\mathbf{B}^1_{\infty,1}} \\
& + C_1 T\lambda(T)\|u_{n+1} - u_n\|_{L^\infty([0,T];\mathbf{B}^0_{\infty,1})},
\end{aligned}$$

for $T \leq T_1$ and some constant $C_1 > 0$. Choosing $T_2 > 0$ small enough to ensure $T_2 \leq \min\left\{T_1, \frac{1}{2C_1\lambda(T_1)}\right\}$, we get

$$\|u_{n+1} - u_n\|_{L^\infty([0,T_2];\mathbf{B}^0_{\infty,1})} \leq 2^{-n+1}\|\Delta_{n+1}u_0\|_{\mathbf{B}^1_{\infty,1}}.$$

This implies that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3))$. Hence there exists a strong limit u of the sequence $\{u_n\}_{n\in\mathbb{N}}$ in the space $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3)) \bigcap C([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3))$. Indeed,"classical" solutions used in our approximation scheme are continuous with values in $\mathbf{B}^0_{\infty,1}$.

Since the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1} \bigcap \mathbf{L}^p)$ (in particular, it is bounded in $L^\infty([0,T_2];\mathbf{L}^p(\mathbb{R}^3))$) there exists a weak*-limit ω for a subsequence of the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ in $L^\infty([0,T_2];\mathbf{L}^p(\mathbb{R}^3))$. Also, since $u_n = \mathcal{K} * \omega_n$, it follows from the Hardy-Littlewood-Sobolev inequality that (possibly after choosing a subsequence)

$$u_n \rightharpoonup u \quad \text{weak}^* \text{ in } L^\infty([0,T_2];\mathbf{L}^{\frac{3p}{3-p}}(\mathbb{R}^3)). \tag{3.11}$$

(Here the weak-$*$ limit u in (3.11) is the same as the strong limit of the Cauchy sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3))$.) Therefore $(u - \mathcal{K} * \omega)(x,t)$ is a harmonic vector field in x that belongs to $L^\infty([0,T_2];\mathbf{L}^{\frac{3p}{3-p}}(\mathbb{R}^3))$. We conclude that $u = \mathcal{K} * \omega$. (We borrowed this argument from [14].) Also, since the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3))$, it weak*-converges (up to a subsequence) to ω in $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3))$. Hence we conclude that ω stays in $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}\bigcap\mathbf{L}^p)$.

We claim that u satisfies the Euler system (1.1). Since the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3))$, it converges strongly to u in $L^\infty([0,T_2]\times\mathbb{R}^3)$. For test functions $\psi,\phi\in\mathcal{S}(\mathbb{R}^3)$ with $\operatorname{div}\phi = 0$ and $\theta\in C_0^\infty([0,T_2])$, we have from the Euler equation corresponding to (3.1)

$$\begin{aligned}\langle u_n(0),\phi\rangle\theta(0) + \int_0^{T_2}\langle u_n(\tau),\phi\rangle\frac{\partial}{\partial t}\theta(\tau)d\tau \\ + \int_0^{T_2}\langle u_n(\tau),(u_n(\tau),\nabla)\phi\rangle\theta(\tau)d\tau = 0, \\ \langle u_n,\nabla\psi\rangle = 0.\end{aligned}$$

We can observe

$$\begin{aligned}\langle u_n(0),\phi\rangle = \langle S_n u_0,\phi\rangle &\longrightarrow \langle u_0,\phi\rangle, \\ \int_0^{T_2}\langle u_n(\tau),\phi\rangle\frac{\partial}{\partial t}\theta(\tau)d\tau &\longrightarrow \int_0^{T_2}\langle u(\tau),\phi\rangle\frac{\partial}{\partial t}\theta(\tau)d\tau, \\ \int_0^{T_2}\langle u_n(\tau),(u_n(\tau),\nabla)\phi\rangle\theta(\tau)d\tau &\longrightarrow \int_0^{T_2}\langle u(\tau),(u(\tau),\nabla)\phi\rangle\theta(\tau)d\tau, \\ \text{and}\qquad 0 = \langle u_n,\nabla\psi\rangle &\longrightarrow \langle u,\nabla\psi\rangle = 0\end{aligned}$$

as $n\to\infty$. Therefore the limit u satisfies the Euler equation and the initial condition

$$\begin{cases}\dfrac{\partial}{\partial t}\,u + (u,\nabla)u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0.\end{cases}$$

We may continue to use this argument until the value $\|\omega(T^*)\|_{\mathbf{B}^0_{\infty,1}\cap\mathbf{L}^p}$ blows up, i.e., $\lim\limits_{t\uparrow T^*}\|\omega(t)\|_{\mathbf{B}^0_{\infty,1}\cap\mathbf{L}^p} = \infty$. This completes the proof of the local existence.

In order to prove the uniqueness, consider u and v two solutions of the system (1.1) such that their vorticities curl u and curl v belong to the space $L^\infty([0,T_2];\mathbf{B}^0_{\infty,1}(\mathbb{R}^3)\bigcap\mathbf{L}^p(\mathbb{R}^3))$ with the same initial function. By subtraction, we observe that the vector field $u-v$ obeys

$$\begin{aligned}\frac{\partial}{\partial t}\,(u-v) + (u,\nabla)(u-v) = (v-u,\nabla)v + \pi(u,u) - \pi(v,v), \\ (u-v)|_{t=0} = 0,\end{aligned}$$

where $\pi(u, v)$ is defined in page 8. By Proposition 4.1, 4.2, 4.3 and estimate (3.9), we get as before

$$\|u - v\|_{\mathbf{B}^0_{\infty,1}} \leq C \int_0^T (\|u\|_{\mathbf{V}} + \|v\|_{\mathbf{V}}) \|u - v\|_{\mathbf{B}^0_{\infty,1}} d\tau.$$

By virtue of Gronwall's inequality, this implies that $u = v$. The uniqueness of solution for (1.2) in $L^\infty_{loc}([0, T^*); \mathbf{B}^0_{\infty,1} \bigcap \mathbf{L}^p)$ is now proved. □

4. A priori estimates

In this section, we will state a collection of a-priori estimates which was used in the proof of the main theorem. Those a-priori estimates are discussed in arbitrary dimension $n \geq 2$ containing the case of the dimension 3 which was essentially needed in the proof of the main Theorem.

Thanks to the Bony's para-product formula, we have the following estimate.

Proposition 4.1. *Let u and ω be divergence free vector fields. We have*:

$$\|(\omega, \nabla)u\|_{\mathbf{B}^0_{\infty,1}} \leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \|\omega\|_{\mathbf{B}^0_{\infty,1}}.$$

Proof. Since ω is a divergence free vector field, we notice that

$$\sum_{i=1}^n \omega^i \, \partial_i u^k = \sum_{i=1}^n \partial_i(\omega^i u^k)$$

for $k = 1, 2, \ldots, n$. Then applying Bony's formula, we get

$$\sum_{i=1}^n \Delta_j(\omega^i \partial_i u) = \sum_{i=1}^n \{\partial_i \Delta_j T_{\omega^i} u + \Delta_j T_{\partial_i u} \omega^i + \partial_i \Delta_j R(\omega^i, u)\}. \tag{4.1}$$

The second term in the right side of (4.1) follows from the observation that

$$\begin{aligned}\sum_{i=1}^n \partial_i \Delta_j T_u \omega^i &= \sum_{j'} \Delta_j \sum_{i=1}^n \partial_i (S_{j'-2} u \, \Delta_{j'} \omega^i) \\ &= \sum_{j'} \Delta_j \sum_{i=1}^n (S_{j'-2} \partial_i u \, \Delta_{j'} \omega^i) = \sum_{i=1}^n \Delta_j T_{\partial_i u} \omega^i,\end{aligned}$$

since $\operatorname{div} \omega = 0$. To estimate the first term in the right side of (4.1) we notice

$$\sum_{i=1}^n \partial_i \Delta_j T_{\omega^i} u = \sum_{i=1}^n \sum_{j'=1}^\infty \partial_i \Delta_j \{S_{j'-2}(\omega^i) \Delta_{j'} u\}, \tag{4.2}$$

since $S_{j'-2}(\omega^i) = 0$ for $j' \leq 0$. Hence from the fact that

$$\operatorname{supp} \mathcal{F}(S_{j'-2}(\omega^i) \Delta_{j'} u) \subset \{2^{j'-1} \leq |\xi| \leq 2^{j'+2}\}, \tag{4.3}$$

and by the Bernstein's Lemma (Remark 2), we get

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} \partial_i \Delta_j T_{\omega^i} u \right\|_{\mathbf{L}^\infty} &\leq C \sum_{i=1}^{n} \sum_{\substack{|j-j'|\leq 3, \\ j'\geq 1}} \|S_{j'-2}\omega^i\|_{\mathbf{L}^\infty} (2^j \|\Delta_{j'} u\|_{\mathbf{L}^\infty}) \\
&\leq C\|\omega\|_{\mathbf{B}^0_{\infty,1}} \sum_{|j-j'|\leq 3} 2^{j'} \|\Delta_{j'} u\|_{\mathbf{L}^\infty}.
\end{aligned} \tag{4.4}
$$

Similarly, we have

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} \Delta_j T_{\partial_i u} \omega^i \right\|_{\mathbf{L}^\infty} &\leq C \sum_{i=1}^{n} \sum_{\substack{|j-j'|\leq 3, \\ j'\geq 1}} \|S_{j'-2}\partial_i u\|_{\mathbf{L}^\infty} \|\Delta_{j'} \omega^i\|_{\mathbf{L}^\infty} \\
&\leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \sum_{|j-j'|\leq 3} \|\Delta_{j'} \omega\|_{\mathbf{L}^\infty}.
\end{aligned} \tag{4.5}
$$

Next, we estimate the third part of the right-side of the identity (4.1). Considering the supports of functions in the third term of (4.1), we may assume that $j-3\leq j'$. Hence we have

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} \partial_i \Delta_j R(\omega^i, u) \right\|_{\mathbf{L}^\infty} &\leq C \sum_{i,k=1}^{n} \sum_{\substack{|j'-j''|\leq 1, \\ j-j'\leq 3}} 2^j \|\Delta_j(\Delta_{j'}\omega^i \Delta_{j''} u^k)\|_{\mathbf{L}^\infty} \\
&\leq C \sum_{\substack{|j'-j''|\leq 1, \\ j-j'\leq 3}} \|\Delta_{j'}\omega\|_{\mathbf{L}^\infty} 2^{j-j'} 2^1 (2^{j''} \|\Delta_{j''} u\|_{\mathbf{L}^\infty}) \\
&\leq C \sum_{j-j'\leq 3} \|\Delta_{j'}\omega\|_{\mathbf{L}^\infty} 2^{j-j'} (\|u\|_{\mathbf{B}^1_{\infty,1}}) \\
&\leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \sum_{j-j'\leq 3} 2^{j-j'} \|\Delta_{j'}\omega\|_{\mathbf{L}^\infty}.
\end{aligned}
$$

From this, we get

$$
\begin{aligned}
\sum_{j=-1}^{\infty} \left\| \sum_{i=1}^{n} \partial_i \Delta_j R(\omega^i, u) \right\|_{\mathbf{L}^\infty} &\leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \sum_{m\geq -3} 2^{-m} \sum_{j=-1}^{\infty} \|\Delta_{j+m}\omega\|_{\mathbf{L}^\infty} \\
&\leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \left(\sum_{m\geq -3} 2^{-m} \right) \|\omega\|_{\mathbf{B}^0_{\infty,1}} \\
&\leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \|\omega\|_{\mathbf{B}^0_{\infty,1}}.
\end{aligned} \tag{4.6}
$$

By combining (4.4), (4.5) and (4.6), we get

$$
\|(\omega, \nabla)u\|_{\mathbf{B}^0_{\infty,1}} \leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \|\omega\|_{\mathbf{B}^0_{\infty,1}}.
$$

$\square$

From Remark 1, we write

$$(S_{j-2}u,\nabla)\Delta_j v-\Delta_j(u,\nabla)v=\sum_{m=1}^{4}\mathbf{R}_{\mathbf{j}}^{\mathbf{m}}(u,v),$$

where we set

$$\begin{aligned}
\mathbf{R}_{\mathbf{j}}^{\mathbf{1}}(u,v)&=-\sum_{i=1}^{n}\Delta_j T_{\partial_i v}u^i=-\sum_{i=1}^{n}\sum_{j'=-\infty}^{\infty}\Delta_j\{S_{j'-2}(\partial_i v)\Delta_{j'}u^i\},\\
\mathbf{R}_{\mathbf{j}}^{\mathbf{2}}(u,v)&=\sum_{i=1}^{n}[T_{u^i}\partial_i,\Delta_j]v,\\
\mathbf{R}_{\mathbf{j}}^{\mathbf{3}}(u,v)&=-\sum_{i=1}^{n}T_{u^i-S_{j-2}u^i}\partial_i\Delta_j v,\ \text{ and}\\
\mathbf{R}_{\mathbf{j}}^{\mathbf{4}}(u,v)&=-\sum_{i=1}^{n}\{\Delta_j R(u^i,\partial_i v)-R(S_{j-2}u^i,\Delta_j\partial_i v)\}.
\end{aligned}$$

We have learned the following Lemma from M. Vishik.

Lemma 4.1. *For any divergence free vector fields u,v, we have the following estimates*:

$$\begin{aligned}
\|\mathbf{R}_{\mathbf{j}}^{\mathbf{1}}(u,v)\|_{\mathbf{L}^\infty}&\le C\sum_{|j-j'|\le 3}\|S_{j'-2}\nabla v\|_{\mathbf{L}^\infty}\|\Delta_{j'}u\|_{\mathbf{L}^\infty},\\
\|\mathbf{R}_{\mathbf{j}}^{\mathbf{2}}(u,v)\|_{\mathbf{L}^\infty}&\le C\sum_{|j-j'|\le 3}\|S_{j'-2}\nabla u\|_{\mathbf{L}^\infty}\|\Delta_{j'}v\|_{\mathbf{L}^\infty},\\
\|\mathbf{R}_{\mathbf{j}}^{\mathbf{3}}(u,v)\|_{\mathbf{L}^\infty}&\le C\sum_{|j-j'|\le 1}(\|\Delta_j\nabla u\|_{\mathbf{L}^\infty}+\|\Delta_{-1}u\|_{\mathbf{L}^\infty})\|\Delta_{j'}v\|_{\mathbf{L}^\infty},\\
\|\mathbf{R}_{\mathbf{j}}^{\mathbf{4}}(u,v)\|_{\mathbf{L}^\infty}&\le C\sum_{|j-j'|\le 3}\sum_{|j'-j''|\le 1}(\|\Delta_{j'}\nabla u\|_{\mathbf{L}^\infty}+\|\Delta_{-1}u\|_{\mathbf{L}^\infty})\|\Delta_{j''}v\|_{\mathbf{L}^\infty}\\
&\quad+C\sum_{j'\ge j-3}2^{j-j'}\sum_{|j'-j''|\le 1}\|\Delta_{j'}\nabla u\|_{\mathbf{L}^\infty}\|\Delta_{j''}v\|_{\mathbf{L}^\infty}.
\end{aligned}$$

Proof. $\underline{\text{Estimate for } \mathbf{R}_{\mathbf{j}}^{\mathbf{1}}(u,v)}$: By the same arguments used in (4.2) and (4.3), we get

$$\begin{aligned}
\|\mathbf{R}_{\mathbf{j}}^{\mathbf{1}}(u,v)\|_{\mathbf{L}^\infty}&\le C\sum_{i=1}^{n}\sum_{|j-j'|\le 3,\ j'\ge 1}\|S_{j'-2}(\partial_i v)\|_{\mathbf{L}^\infty}\|\Delta_{j'}u^i\|_{\mathbf{L}^\infty}\\
&\le C\sum_{|j-j'|\le 3}\|S_{j'-2}\nabla v\|_{\mathbf{L}^\infty}\|\Delta_{j'}u\|_{\mathbf{L}^\infty}.
\end{aligned}$$

Estimate for $\mathbf{R_j^2}(u,v)$: From the fact that u is divergence free, we observe

$$
\begin{aligned}
\mathbf{R_j^2}(u,v) &= \sum_{i=1}^{n}[T_{u^i}\partial_i, \Delta_j]v \\
&= \sum_{i=1}^{n}\sum_{|j-j'|\le 3}\{S_{j'-2}u^i\Delta_{j'}(\partial_i\Delta_j v) - \Delta_j(S_{j'-2}u^i\partial_i\Delta_{j'}v)\} \\
&= \sum_{i=1}^{n}\sum_{|j-j'|\le 3}\int \varphi_j(x-y)\{S_{j'-2}u^i(x) - S_{j'-2}u^i(y)\}\partial_i\Delta_{j'}v(y)dy \\
&= \sum_{i=1}^{n}\sum_{|j-j'|\le 3} 2^{j(n+1)}\int \partial_i\varphi(2^j(x-y))\{S_{j'-2}u^i(x) - S_{j'-2}u^i(y)\}\Delta_{j'}v(y)dy \\
&= \sum_{i=1}^{n}\sum_{|j-j'|\le 3} 2^{j(n+1)}\int \partial_i\varphi(2^j(x-y)) \\
&\qquad\qquad \left(\sum_{m=1}^{n}\int_0^1 S_{j'-2}\partial_m u^i(x+\tau(y-x))(x^m-y^m)d\tau\right)\Delta_{j'}v(y)dy \\
&= \sum_{i,m=1}^{n}\sum_{|j-j'|\le 3}\int \partial_i\varphi(z)\left(\int_0^1 S_{j'-2}\partial_m u^i(x-\tau 2^{-j}z)z^m d\tau\right)\Delta_{j'}v(x-2^{-j}z)dz.
\end{aligned}
$$

Hence we get

$$
\|\mathbf{R_j^2}(u,v)\|_{\mathbf{L}^\infty} \le C\sum_{i=1}^{n}\sum_{|j-j'|\le 3}\sum_{m=1}^{n}\|S_{j'-2}\partial_m u^i\|_{\mathbf{L}^\infty}\|\Delta_{j'}v\|_{\mathbf{L}^\infty}.
$$

Estimate for $\mathbf{R_j^3}(u,v)$: Note that

$$
\begin{aligned}
\mathbf{R_j^3}(u,v) &= -\sum_{i=1}^{n} T_{u^i - S_{j-2}u^i}\partial_i\Delta_j v \\
&= -\sum_{i=1}^{n}\sum_{|j-j'|\le 1} S_{j'-2}(u^i - S_{j-2}u^i)\Delta_{j'}(\partial_i\Delta_j v) \\
&= -\sum_{i=1}^{n}\sum_{|j-j'|\le 1} S_{j'-2}\left(\sum_{m=j-1}^{\infty}\Delta_m u^i\right)\partial_i\Delta_{j'}\Delta_j v \\
&= -\sum_{i=1}^{n}\sum_{|j-j'|\le 1} S_{j'-2}\left(\sum_{m=j-1}^{j}\Delta_m u^i\right)\partial_i\Delta_{j'}\Delta_j v.
\end{aligned}
$$

Therefore we have

$$\begin{aligned}
\|\mathbf{R_j^3}(u,v)\|_{\mathbf{L}^\infty} &\le \sum_{i=1}^{n}\sum_{|j-j'|\le 1}\|S_{j'-2}(\Delta_{j-1}u^i+\Delta_j u^i)\|_{\mathbf{L}^\infty}\|\partial_i(\Delta_{j'}\Delta_j v)\|_{\mathbf{L}^\infty}\\
&\le C\sum_{i=1}^{n}\sum_{|j-j'|\le 1}(\|\Delta_{j-1}u^i\|_{\mathbf{L}^\infty}+\|\Delta_j u^i\|_{\mathbf{L}^\infty})\|\partial_i(\Delta_j\Delta_{j'}v)\|_{\mathbf{L}^\infty}\\
&\le C\sum_{|j-j'|\le 1}\{2^{-j+1}\|\Delta_{j-1}\nabla u\|_{\mathbf{L}^\infty}+2^{-j}\|\Delta_j\nabla u\|_{\mathbf{L}^\infty}\}2^j\|\Delta_{j'}v\|_{\mathbf{L}^\infty}\\
&\qquad + C\|\Delta_{-1}u\|_{\mathbf{L}^\infty}\sum_{|j-j'|\le 1}\|\Delta_{j'}v\|_{\mathbf{L}^\infty}\\
&\le C\sum_{|j-j'|\le 1}(\|\Delta_j\nabla u\|_{\mathbf{L}^\infty}+\|\Delta_{-1}u\|_{\mathbf{L}^\infty})\|\Delta_{j'}v\|_{\mathbf{L}^\infty}.
\end{aligned}$$

Estimate for $\mathbf{R_j^4}(u,v)$: We divide $\mathbf{R_j^4}(u,v)$ into two parts as following:

$$\begin{aligned}
\mathbf{R_j^4}(u,v) &= -\sum_{i=1}^{n}\{\Delta_j R(u^i,\partial_i v)-R(S_{j-2}u^i,\Delta_j\partial_i v)\}\\
&= -\sum_{i=1}^{n}\{\Delta_j\partial_i R(u^i-S_{j-2}u^i,v)\}\\
&\qquad\qquad -\sum_{i=1}^{n}\{\Delta_j R(S_{j-2}u^i,\partial_i v)-R(S_{j-2}u^i,\Delta_j\partial_i v)\}\\
&= \mathbf{R_j^{4,1}}(u,v)+\mathbf{R_j^{4,2}}(u,v).
\end{aligned}$$

The first term $\mathbf{R_j^{4,1}}(u,v)$ can be considered with two subparts:

$$\begin{aligned}
\mathbf{R_j^{4,1}}(u,v) &= -\sum_{i=1}^{n}\partial_i\Delta_j\sum_{|j'-j''|\le 1}\Delta_{j'}(u^i-S_{j-2}u^i)\Delta_{j''}v\\
&= -\sum_{i=1}^{n}\partial_i\Delta_j\sum_{|1+j''|\le 1}\Delta_{-1}(u^i-S_{j-2}u^i)\Delta_{j''}v\\
&\qquad\qquad -\sum_{i=1}^{n}\partial_i\Delta_j\sum_{|j'-j''|\le 1, j'\ge 0}\Delta_{j'}(u^i-S_{j-2}u^i)\Delta_{j''}v\\
&\equiv \mathbf{R_j^{4,1,1}}(u,v)+\mathbf{R_j^{4,1,2}}(u,v).
\end{aligned}$$

In order to estimate $\mathbf{R_j^{4,1,1}}(u,v)$, we first note that

$$\operatorname{supp}\mathcal{F}(\Delta_{-1}(u^i-S_{j-2}u^i)) = \operatorname{supp}\mathcal{F}\left(\Delta_{-1}\left(\sum_{m=j-1}^{\infty}\Delta_m u^i\right)\right)\neq\varnothing$$

if and only if $j = -1, 0, 1$. Hence we have

$$\|\mathbf{R_j^{4,1,1}}(u,v)\|_{\mathbf{L}^\infty} \le C \sum_{i=1}^{n} \sum_{j''=-1}^{0} \|\Delta_{-1}(u^i - S_{j-2}u^i)\|_{\mathbf{L}^\infty} \|\Delta_{j''}v\|_{\mathbf{L}^\infty}$$

$$\le \begin{cases} C \displaystyle\sum_{j''=-1}^{0} \|\Delta_{-1}u\|_{\mathbf{L}^\infty} \|\Delta_{j''}v\|_{\mathbf{L}^\infty} & \text{if } j = -1, 0, 1, \\ 0 & \text{if } j \ge 2. \end{cases} \tag{4.7}$$

Considering the supports of functions in $\mathbf{R_j^{4,1,2}}(u,v)$, we may assume that $j - 3 \le j'$. Hence we get

(4.8)

$$\|\mathbf{R_j^{4,1,2}}(u,v)\|_{\mathbf{L}^\infty} \le C \sum_{i=1}^{n} 2^j \left\| \Delta_j \sum_{|j'-j''|\le 1, j'\ge 0} \Delta_{j'}(u^i - S_{j-2}u^i)\Delta_{j''}v \right\|_{\mathbf{L}^\infty}$$

$$\le C \sum_{i=1}^{n} \sum_{j'\ge j-3} 2^{j-j'} \sum_{|j'-j''|\le 1, j'\ge 0} 2^{j'} \|\Delta_{j'}u^i\|_{\mathbf{L}^\infty} \|\Delta_{j''}v\|_{\mathbf{L}^\infty}$$

$$\le C \sum_{j'\ge j-3} 2^{j-j'} \sum_{|j'-j''|\le 1} \|\Delta_{j'}\nabla u\|_{\mathbf{L}^\infty} \|\Delta_{j''}v\|_{\mathbf{L}^\infty}.$$

Now we will take care of $\mathbf{R_j^{4,2}}(u,v)$.

$$\mathbf{R_j^{4,2}}(u,v) = -\sum_{i=1}^{n} \sum_{j-3\le j'\le j-1} \sum_{|j'-j''|\le 1} [\Delta_j,\ \Delta_{j'}S_{j-2}u^i]\Delta_{j''}\partial_i v.$$

To take a closer look at $[\Delta_j,\ \Delta_{j'}S_{j-2}u^i]\Delta_{j''}\partial_i v$, we use the explicit representation of Δ_j:

$$\begin{aligned}
&[\Delta_j,\ \Delta_{j'}S_{j-2}u^i]\Delta_{j''}\partial_i v \\
&\quad = 2^{jn} \int \varphi(2^j(x-y))\{\Delta_{j'}S_{j-2}u^i(y) - \Delta_{j'}S_{j-2}u^i(x)\}\partial_i \Delta_{j''}v dy \\
&\quad = 2^{j(n+1)} \int \partial_i\varphi(2^j(x-y))\{\Delta_{j'}S_{j-2}u^i(y) - \Delta_{j'}S_{j-2}u^i(x)\}\Delta_{j''}v dy \\
&\quad = 2^{j(n+1)} \int \partial_i\varphi(2^j(x-y)) \\
&\qquad\qquad \left\{ \sum_{m=1}^{n} \int_0^1 \Delta_{j'}S_{j-2}\partial_m u^i(x + \tau(y-x))(y_m - x_m)d\tau \right\} \Delta_{j''}v dy \\
&\quad = -\sum_{m=1}^{n} \int \partial_i\varphi(z) \\
&\qquad\qquad \left\{ \int_0^1 \Delta_{j'}S_{j-2}\partial_m u^i(x - 2^{-j}\tau z)z_m d\tau \right\} \Delta_{j''}v(x - 2^{-j}z)dz.
\end{aligned}$$

So we have

$$\begin{aligned}\|[\Delta_j,\ \Delta_{j'}S_{j-2}u^i]\Delta_{j''}\partial_i v\|_{\mathbf{L}^\infty} &\le C\sum_{m=1}^{n}\|\Delta_{j'}S_{j-2}\partial_m u^i\|_{\mathbf{L}^\infty}\|\Delta_{j''}v\|_{\mathbf{L}^\infty}\\ &\le C\|\Delta_{j'}S_{j-2}\nabla u\|_{\mathbf{L}^\infty}\|\Delta_{j''}v\|_{\mathbf{L}^\infty}.\end{aligned}$$

Then,

$$\|\mathbf{R}_{\mathbf{j}}^{\mathbf{4,2}}(u,v)\|_{\mathbf{L}^\infty} \le C\sum_{|j-j'|\le 3}\sum_{|j'-j''|\le 1}\|\Delta_{j'}\nabla u\|_{\mathbf{L}^\infty}\|\Delta_{j''}v\|_{\mathbf{L}^\infty}. \tag{4.9}$$

From (4.7), (4.8), and (4.9), we have the estimate for $\|\mathbf{R}_{\mathbf{j}}^{\mathbf{4}}(u,v)\|_{\mathbf{L}^\infty}$. $\square$

From the Lemma above, we get the following Propositions.

Proposition 4.2. *Let u and ω be divergence free vector fields. We have*:

$$\sum_{j\ge -1}\|(S_{j-2}u,\nabla)\Delta_j\omega - \Delta_j((u,\nabla)\omega)\|_{\mathbf{L}^\infty} \le C\|u\|_{\mathbf{B}^1_{\infty,1}}\|\omega\|_{\mathbf{B}^0_{\infty,1}}.$$

Proof. From Lemma 4.1, we get

$$\begin{aligned}\|\mathbf{R}_{\mathbf{j}}^{\mathbf{1}}(u,\omega)\|_{\mathbf{L}^\infty} &\le C\sum_{|j-j'|\le 3}\|S_{j'-2}(\nabla\omega)\|_{\mathbf{L}^\infty}\|\Delta_{j'}u\|_{\mathbf{L}^\infty}\\ &\le C\sum_{|j-j'|\le 3}2^{j'-1}\|S_{j'-2}\omega\|_{\mathbf{L}^\infty}\|\Delta_{j'}u\|_{\mathbf{L}^\infty}\\ &\le C\sum_{|j-j'|\le 3}\left(\sum_{j''=0}^{j'-2}\|\Delta_{j''}\omega\|_{\mathbf{L}^\infty}\right)2^{j'}\|\Delta_{j'}u\|_{\mathbf{L}^\infty}\\ &\le C\|\omega\|_{\mathbf{B}^0_{\infty,1}}\sum_{|j-j'|\le 3}2^{j'}\|\Delta_{j'}u\|_{\mathbf{L}^\infty};\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\|\mathbf{R}_{\mathbf{j}}^{\mathbf{2}}(u,\omega)\|_{\mathbf{L}^\infty} &\le C\sum_{|j-j'|\le 3}\|S_{j'-2}\nabla u\|_{\mathbf{L}^\infty}\|\Delta_{j'}\omega\|_{\mathbf{L}^\infty}\\ &\le C\|u\|_{\mathbf{B}^1_{\infty,1}}\sum_{|j-j'|\le 3}\|\Delta_{j'}\omega\|_{\mathbf{L}^\infty}.\end{aligned} \tag{4.11}$$

Also, we have

$$\begin{aligned}\|\mathbf{R}_{\mathbf{j}}^{\mathbf{3}}(u,\omega)\|_{\mathbf{L}^\infty} &\le C(\|\nabla u\|_{\mathbf{L}^\infty}+\|\Delta_{-1}u\|_{\mathbf{L}^\infty})\sum_{|j-j'|\le 1}\|\Delta_{j'}\omega\|_{\mathbf{L}^\infty}\\ &\le C\|u\|_{\mathbf{B}^1_{\infty,1}}\sum_{|j-j'|\le 1}\|\Delta_{j'}\omega\|_{\mathbf{L}^\infty},\end{aligned} \tag{4.12}$$

and

(4.13)
$$\begin{aligned}\|\mathbf{R_j^4}(u,\omega)\|_{\mathbf{L}^\infty} \leq C \sum_{|j-j'|\leq 3} \sum_{|j'-j''|\leq 1} (\|\Delta_{j'}\nabla u\|_{\mathbf{L}^\infty} + \|\Delta_{-1}u\|_{\mathbf{L}^\infty})\|\Delta_{j''}\omega\|_{\mathbf{L}^\infty}\\ + C \sum_{j'\geq j-3} 2^{j-j'} \sum_{|j'-j''|\leq 1} \|\Delta_{j'}\nabla u\|_{\mathbf{L}^\infty}\|\Delta_{j''}\omega\|_{\mathbf{L}^\infty}\\ \leq C\|u\|_{\mathbf{B}^1_{\infty,1}} \sum_{|j-j'|\leq 5} \|\Delta_{j'}\omega\|_{\mathbf{L}^\infty}\\ + C\|u\|_{\mathbf{B}^1_{\infty,1}} \sum_{j'\geq j-3} 2^{j-j'} \sum_{|j'-j''|\leq 1} \|\Delta_{j''}\omega\|_{\mathbf{L}^\infty}.\end{aligned}$$

We observe that

(4.14)
$$\begin{aligned}\sum_{j\geq -1}\sum_{j'\geq j-3} 2^{j-j'} \sum_{j''=j'-1}^{j'+1} \|\Delta_{j''}\omega\|_{\mathbf{L}^\infty} &\leq C \sum_{m\geq -3} 2^{-m}\left(3\sum_{j\geq -1}\|\Delta_{j+m}\omega\|_{\mathbf{L}^\infty}\right)\\ &\leq C\left(\sum_{m\geq -3} 2^{-m}\right)\|\omega\|_{\mathbf{B}^0_{\infty,1}}\\ &\leq C\|\omega\|_{\mathbf{B}^0_{\infty,1}}.\end{aligned}$$

By putting estimates (4.10), (4.11), (4.12), (4.13) and (4.14) together we get

$$\begin{aligned}\sum_{j\geq -1}\|(S_{j-2}u,\nabla)\Delta_j\omega - \Delta_j((u,\nabla)\omega)\|_{\mathbf{L}^\infty} &\leq \sum_{j\geq -1}\sum_{m=1}^{4}\|\mathbf{R_j^m}(u,\omega)\|_{\mathbf{L}^\infty}\\ &\leq C\|u\|_{\mathbf{B}^1_{\infty,1}}\|\omega\|_{\mathbf{B}^0_{\infty,1}}.\end{aligned}$$

□

We now present the estimates for the potential term.

Proposition 4.3. *Let $1 < p < n$, and let u_i, $i = 1,2$ be divergence free vector fields and $\omega_i = \operatorname{curl} u_i$, $i = 1,2$. We have the following estimate*:

$$\|\pi(u_1,u_2)\|_{\mathbf{B}^0_{\infty,1}} \leq C(\|\omega_1\|_{\mathbf{B}^0_{\infty,1}} + \|\omega_1\|_{\mathbf{L}^p})\|u_2\|_{\mathbf{B}^0_{\infty,1}}, \tag{4.15}$$

or

$$\|\pi(u_1,u_2)\|_{\mathbf{B}^0_{\infty,1}} \leq C\|u_1\|_{\mathbf{B}^0_{\infty,1}}(\|\omega_2\|_{\mathbf{B}^0_{\infty,1}} + \|\omega_2\|_{\mathbf{L}^p}), \tag{4.16}$$

where we set

$$\pi(u_1,u_2) \equiv \sum_{i,j=1}^{n} \nabla\Delta^{-1}\partial_i u_1^j \partial_j u_2^i = \nabla\Delta^{-1}\operatorname{div}((u_1,\nabla)u_2).$$

Proof. The lack of the isometric inequality for the case of $j = -1$ in the Bernstein's Lemma (Remark 2) forces to consider two cases.
Case 1: $j \geq 0$. We observe that

$$\operatorname{div}((S_{j-2}u_1, \nabla)\Delta_j u_2) = \sum_{m,l=1}^{n} S_{j-2}\partial_m u_1^l \Delta_j \partial_l u_2^m,$$

due to the fact that $\operatorname{div} u_2 = 0$. So, we have

$$\begin{aligned}
&\|\Delta_j \operatorname{div}((u_1, \nabla)u_2)\|_{\mathbf{L}^\infty} \\
&\leq \|\operatorname{div}\{\Delta_j(u_1, \nabla)u_2 - (S_{j-2}u_1, \nabla)\Delta_j u_2\}\|_{\mathbf{L}^\infty} + \|\operatorname{div}(S_{j-2}u_1, \nabla)\Delta_j u_2\|_{\mathbf{L}^\infty} \\
&\leq 2^j \|\Delta_j(u_1, \nabla)u_2 - (S_{j-2}u_1, \nabla)\Delta_j u_2\|_{\mathbf{L}^\infty} + \sum_{m,l=1}^{n} \|S_{j-2}\partial_m u_1^l \Delta_j \partial_l u_2^m\|_{\mathbf{L}^\infty}.
\end{aligned}$$

On the other hand, since

$$\Delta_j\ \pi(u_1, u_2) = \mathcal{F}^{-1}(i\xi|\xi|^{-2}\mathcal{F}(\Delta_j\ \operatorname{div}(u_1, \nabla)u_2)),$$

we have for $j \geq 0$

$$\|\Delta_j\ \pi(u_1, u_2)\|_{\mathbf{L}^\infty} \leq C2^{-j}\|\Delta_j \operatorname{div}((u_1, \nabla)u_2)\|_{\mathbf{L}^\infty}.$$

Hence from the fact that

$$\begin{aligned}
&\sum_{m,l=1}^{n}\sum_{j=0}^{\infty} 2^{-j}\|S_{j-2}\partial_m u_1^l \Delta_j \partial_l u_2^m\|_{\mathbf{L}^\infty} \\
&\qquad\qquad \leq C \sum_{m,l=1}^{n}\sum_{j=0}^{\infty} \|S_{j-2}\partial_m u_1\|_{\mathbf{L}^\infty}(2^{-j}\|\Delta_j \partial_l u_2^m\|_{\mathbf{L}^\infty}) \\
&\qquad\qquad \leq C\|u_1\|_{\mathbf{B}^1_{\infty,1}}\|u_2\|_{\mathbf{B}^0_{\infty,1}},
\end{aligned}$$

and Proposition 4.2, we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} \|\Delta_j\ \pi(u_1, u_2)\|_{\mathbf{L}^\infty} &\leq C\sum_{j=0}^{\infty} 2^{-j}\|\Delta_j \operatorname{div}((u_1, \nabla)u_2)\|_{\mathbf{L}^\infty} \\
&\leq C\sum_{j=0}^{\infty} \|\Delta_j(u_1, \nabla)u_2 - (S_{j-2}u_1, \nabla)\Delta_j u_2\|_{\mathbf{L}^\infty} \\
&\quad + C\sum_{m,l=1}^{n}\sum_{j=0}^{\infty} 2^{-j}\|S_{j-2}\partial_m u_1^l \Delta_j \partial_l u_2^m\|_{\mathbf{L}^\infty} \\
&\leq C\|u_1\|_{\mathbf{B}^1_{\infty,1}}\|u_2\|_{\mathbf{B}^0_{\infty,1}}.
\end{aligned} \tag{4.17}$$

Since $\pi(u_1, u_2) = \pi(u_2, u_1)$, we also get

$$\sum_{j=0}^{\infty} \|\Delta_j\ \pi(u_1, u_2)\|_{\mathbf{L}^\infty} \leq C\|u_1\|_{\mathbf{B}^0_{\infty,1}}\|u_2\|_{\mathbf{B}^1_{\infty,1}}. \tag{4.18}$$

Case 2: $j = -1$. Let $q = \frac{np}{n-p}$. Using the Bernstein's inequality and the boundedness of the Riesz Transforms on $\mathbf{L}^q$, we have

$$\begin{aligned} \|\Delta_{-1}\pi(u_1,u_2)\|_{\mathbf{L}^\infty} &= \|\Delta_{-1}\nabla\Delta^{-1}\operatorname{div}((u_1,\nabla)u_2)\|_{\mathbf{L}^\infty} \\ &\le C\|\nabla\Delta^{-1}\operatorname{div}\Delta_{-1}((u_1,\nabla)u_2)\|_{\mathbf{L}^q} \\ &\le C\|\Delta_{-1}\operatorname{div}(u_1\otimes u_2)\|_{\mathbf{L}^q} \\ &\le C\|\Delta_{-1}(u_1\otimes u_2)\|_{\mathbf{L}^q} \\ &\le C\|u_1\|_{\mathbf{L}^q}\|u_2\|_{\mathbf{L}^\infty} \\ &\le C\|\omega_1\|_{\mathbf{L}^p}\|u_2\|_{\mathbf{L}^\infty}. \end{aligned} \tag{4.19}$$

The last inequality follows from the Hardy-Littlewood-Sobolev inequality. Since $\pi(u_1,u_2) = \pi(u_2,u_1)$, we also have

$$\|\Delta_{-1}\pi(u_1,u_2)\|_{\mathbf{L}^\infty} \le C\|\omega_2\|_{\mathbf{L}^p}\|u_1\|_{\mathbf{B}^0_{\infty,1}}. \tag{4.20}$$

The estimates (4.17)and (4.19) imply inequality (4.15), and the estimates (4.18) and (4.20) imply inequality (4.16). □

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