J. Math. Kyoto Univ. (JMKYAZ)
46-3 (2006), 657–692

On $SL(2) - GL(n)$ strange duality

By

Takeshi Abe

Abstract

We prove $SL(2)$-$GL(n)$ strange duality conjecture for a general smooth projective irreducible curve.

1. Introduction

Let C be a smooth projective irreducible curve of genus g over an algebraically closed field k of characteristic zero. Let $SU(r, L)$ be the moduli space of semistable vector bundles of rank r, whose determinant is isomorphic to a fixed line bundle L on C. Let $U(n, e)$ be the moduli space of semistable vector bundles of rank n and degree e on C. Let

$$\tau : SU(r, \mathcal{O}_C) \times U(n, n(g-1)) \to U(rn, rn(g-1))$$

be the tensor product map. On $U(rn, rn(g-1))$, there is a natural divisor

$$\Theta = \left\{ F \in U(rn, rn(g-1)) \,\middle|\, \mathrm{H}^0(C, F) \neq 0 \right\}.$$

The pull-back $\tau^*\mathcal{O}(\Theta)$ of the line bundle $\mathcal{O}(\Theta)$ by τ is isomorphic to $\mathcal{M} \boxtimes \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are line bundles on $SU(r, \mathcal{O}_C)$ and $U(n, n(g-1))$ respectively. The pull-back of the canonical section $\mathbf{1} \in \mathcal{O}(\Theta)$ defines an element of the vector space $\mathrm{H}^0(SU(r, \mathcal{O}_C), \mathcal{M}) \otimes \mathrm{H}^0(U(n, n(g-1)), \mathcal{N})$, so we obtain a linear map (up to $k^\times$-multiple)

$$\vartheta_{n,r} : \mathrm{H}^0\left(U(n, n(g-1)), \mathcal{N}\right)^\vee \to \mathrm{H}^0\left(SU(r, \mathcal{O}_C), \mathcal{M}\right).$$

The following conjecture is known as $SL(r)$-$GL(n)$ strange duality conjecture:

Conjecture 1.0.1. *The map $\vartheta_{n,r}$ is an isomorphism.*

It is known that $\dim \mathrm{H}^0(U(n, n(g-1)), \mathcal{N}) = \dim \mathrm{H}^0(SU(r, \mathcal{O}_C), \mathcal{M})$ (cf. [B2, §8]). Thus the above conjecture is equivalent to the following conjecture.

2000 *Mathematics Subject Classification*(*s*). 14H60, 14D20
Received February 7, 2006
Revised July 28, 2006

Conjecture 1.0.2. *The vector space* $\mathrm{H}^0(SU(r,\mathcal{O}_C),\mathcal{M})$ *is spanned by divisors* Θ_G *for* $G \in U(n, n(g-1))$*, where* $\Theta_G := \{E \in SU(r,\mathcal{O}_C) | \mathrm{H}^0(C, E \otimes G) \neq 0\}$.

The above equivalent conjectures are known to be true in the following cases:

- $n = 1$ ([BNR]),
- $r = 2$, $n = 2$ and C has no vanishing thetanull ([B1]),
- $r = 2$, $n = 4$ and C has no vanishing thetanull ([vG-P]),
- $r = 2$, n is even, $n \geq 2g - 4$ and C is general ([L]).

(See the last paragraph of this introduction for the recent result.)

In [D-T], Donagi and Tu generalized Conjecture 1.0.1 to bundles of arbitrary degree as follows. Let r, n be positive integers and let d, e be integers such that $re + nd + rn(1-g) = 0$. Fix a line bundle L of degree d on C. Let $\tau : SU(r,L) \times U(n,e) \to U(rn, rn(g-1))$ be the tensor product map. Just as the case $L = \mathcal{O}_C$, we have $\tau^*\mathcal{O}(\Theta) = \mathcal{M} \boxtimes \mathcal{N}$ for some line bundles $\mathcal{M}$ and $\mathcal{N}$ on $SU(r,L)$ and $U(n,e)$ respectively, and we have a linear map $\vartheta_{n,r} : \mathrm{H}^0(U(n,e),\mathcal{N})^\vee \to \mathrm{H}^0(SU(r,L),\mathcal{M})$. We have

Conjecture 1.0.3. $\vartheta_{n,r}$ *is an isomorphism.*

Now the generalization of Conjecture 1.0.2 to bundles of arbitrary degree is straightforward.

Conjecture 1.0.4. *The vector space* $\mathrm{H}^0(SU(r,L),\mathcal{M})$ *is spanned by divisors* Θ_G *for* $G \in U(n,e)$*, where* $\Theta_G := \{E \in SU(r,L) | \mathrm{H}^0(C, E \otimes G) \neq 0\}$.

These two conjectures are also equivalent.

The purpose of this paper is to show that Conjecture 1.0.4 holds true for a general curve C of genus $g \geq 1$ if $r = 2$.

The strategy we employ in this paper is degeneration of C. We degenerate a smooth projective irreducible curve C to a nodal curve and prove Conjecture 1.0.4 for $r = 2$ by induction on the genus g. Let us explain how to do this in more detail. In order to make the induction process work, we first formulate a generalization of Conjecture 1.0.4 for parabolic bundles on a pointed curve in Section 2. In Section 3, we deal with the case $r = 2$. In Subsection 3.2, we prove the generalization of Conjecture 1.0.4 for an m-pointed curve of genus one for $m \leq 2$. This is the starting step of the induction. In Subsection 3.5, we degenerate a one-pointed smooth projective irreducible curve C of genus $g \geq 2$ to a reducible one-pointed nodal curve that is a union of a one-pointed smooth curve C_1 of genus one and a smooth curve C_2 of genus $g - 1$. Taking the intersection point of C_1 and C_2 into account, we have two-pointed curve C_1 and one-pointed C_2. The main theorem (Theorem 3.5.1) in Subsection 3.5 says that if the generalization of Conjecture 1.0.4 holds true for these pointed curves, then it holds for the one-pointed curve C. In this way we prove the parabolic generalization of Conjecture 1.0.4 for a general one-pointed curve C if $r = 2$. (The pointless case can be regarded as a special case of a pointed case.)

In this paper we use stack formulation of moduli of bundles. Our main reference is [L-M], but for reader's convenience, we included some (probably well-known) facts on algebraic stacks in Section 4. In Subsection 4.2 we gathered some facts on a compactification of SL_2.

After I finished this work, I learned that Prakash Belkale [Bel] proved Conjecture 1.0.2 (equivalently Conjecture 1.0.1) for arbitrary r and n for a general curve. He also uses degeneration argument, but the methods are a little different from those in this paper.

Notation and Convention.

- If $\mathcal{V}$ is a vector bundle on a scheme T, then $\mathrm{Grass}^r(\mathcal{V})$ denotes the Grassmannian of rank r quotient bundles of $\mathcal{V}$. $\mathbb{P}(\mathcal{V})$ denotes $\mathrm{Grass}^1(\mathcal{V})$, and $\mathbb{P}_*(\mathcal{V})$ denotes $\mathrm{Grass}^{\mathrm{rank}\mathcal{V}-1}(\mathcal{V})$.
- If $T \to S$ is a morphism of schemes (or stacks) and $*$ is an object (for example, scheme, stack, morphism, sheaf etc.) over S, then the base-change of $*$ over T is denoted by a subscript T like $(*)_T$ or $*_T$.
- All rings appearing in this paper are commutative and noetherian. All schemes appearing in this paper are locally noetherian.

2. Parabolic strange duality conjecture

2.1. Basic definitions

Definition 2.1.1. Let r,n be positive integers.

(1) A Young diagram Λ is said to be of type $\leq(r,n)$ if the number of rows is less than or equal to r and the number of columns is less than or equal to n.

(2) As in [Ful], the conjugate of a Young diagram Λ (i.e. the Young diagram obtained by interchanging the rows and columns of Λ) is denoted by $\widetilde{\Lambda}$.

(3) If Λ is a Young diagram, $|\Lambda|$ denotes the number of the boxes in Λ.

Notation 2.1.2. Let r and n be positive integers and Λ be a Young diagram of type $\leq(r,n)$.

(1) For $1 \leq j \leq r$, $\lambda_i(\Lambda)$ denotes the number of the boxes in the i-th row. Put $\lambda_0(\Lambda) = n$.

(2) Put $l(\Lambda) := \#\{\lambda_0(\Lambda), \lambda_1(\Lambda), \dots, \lambda_r(\Lambda)\} - 1$.

(3) Put in order the distinct numbers in $\{\lambda_0(\Lambda), \dots, \lambda_r(\Lambda)\}$ as $n = \gamma_0(\Lambda) > \gamma_1(\Lambda) > \cdots > \gamma_{l(\Lambda)}(\Lambda)$.

(4) For $1 \leq i \leq l(\Lambda)$, put $d_i(\Lambda) := r + 1 - \min\{j|\lambda_j(\Lambda) = \gamma_i(\Lambda)\}$. (Then we have $r \geq d_1(\Lambda) > \cdots > d_{l(\Lambda)}(\Lambda) > 0$.)

(5) Λ^* denotes the Young diagram of type $\leq (r,n)$ such that $\lambda_i(\Lambda^*) = n - \lambda_{r-i}(\Lambda)$.

(6) If T is a scheme and $\mathcal{V}$ is a locally free $\mathcal{O}_T$-module of rank r, then we let $\mathcal{F}lag_\Lambda$ denote the contravariant functor from the category of T-schemes to the category of sets, which associates to a T-scheme $f: U \to T$ the set of filtrations $f^*\mathcal{V} \supseteq \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_{l(\Lambda)}$ such that $\mathcal{V}_i$ is a rank $d_i(\Lambda)$ subbundle of $f^*\mathcal{V}$. (We say that such filtrations are of type Λ.) $Flag_\Lambda$ denotes a projective

T-scheme that represents $\mathcal{F}lag_\Lambda$. By convention we understand that $\mathcal{V}_0 = f^*\mathcal{V}$ and $\mathcal{V}_{l(\Lambda)+1} = 0$.

Remark 2.1.3. If Λ is a Young diagram of type $\leq (r,n)$, then $\widetilde{\Lambda}$ is a Young diagram of type $\leq (n,r)$, and we have $l(\Lambda) = l(\widetilde{\Lambda})$, $d_i(\widetilde{\Lambda}) = n - \gamma_{l(\Lambda)+1-i}(\Lambda)$ and $\gamma_i(\widetilde{\Lambda}) = r - d_{l(\Lambda)+1-i}(\Lambda)$.

Definition 2.1.4. Let Λ be a Young diagram of type $\leq (r,n)$. Put $l := l(\Lambda) = l(\widetilde{\Lambda})$. Let T be a scheme, and $\mathcal{V}$, $\mathcal{W}$ be locally free $\mathcal{O}_T$-modules of rank r and n respectively. The T-morphism $\mu_\Lambda : Flag_\Lambda(\mathcal{V}) \times_T Flag_{\widetilde{\Lambda}}(\mathcal{W}) \to \mathrm{Grass}^{|\Lambda|}(\mathcal{V}\otimes\mathcal{W})$ is defined as follows. Let $f : U \to T$ be a T-scheme. If $\mathbb{F}(f^*\mathcal{V}) = (f^*\mathcal{V} \supseteq \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_l)$ and $\mathbb{F}(f^*\mathcal{W}) = f^*\mathcal{W} \supseteq \mathcal{W}_1 \supset \cdots \supset \mathcal{W}_l)$ are filtrations of $f^*\mathcal{V}$ and $f^*\mathcal{W}$ of type Λ and $\widetilde{\Lambda}$ respectively, then its image by μ_Λ is the quotient $f^*(\mathcal{V}\otimes\mathcal{W}) \to f^*(\mathcal{V}\otimes\mathcal{W})/\sum_{i=1}^{l}\mathcal{V}_i\otimes\mathcal{W}_{l+1-i}$, where $\sum_{i=1}^{l}\mathcal{V}_i\otimes\mathcal{W}_{l+1-i}$ means the subsheaf of $f^*(\mathcal{V}\otimes\mathcal{W})$ generated by $\mathcal{V}_i\otimes\mathcal{W}_{l+1-i}$ $(1\leq i\leq l)$, which one can check is a subbundle of $f^*(\mathcal{V}\otimes\mathcal{W})$ of rank $rn-|\Lambda|$. We denote this quotient by $\mu_\Lambda(\mathbb{F}(f^*\mathcal{V}),\mathbb{F}(f^*\mathcal{W})) : f^*(\mathcal{V})\otimes f^*(\mathcal{W}) \to \mathcal{Q}(\mathbb{F}(f^*\mathcal{V}),\mathbb{F}(f^*\mathcal{W}))$.

Lemma 2.1.5. *Let the notation as in Definition* 2.1.4. *Let π be the projections to T from $Flag_\Lambda(\mathcal{V})$, $Flag_{\widetilde{\Lambda}}(\mathcal{W})$ etc. Let $\pi^*(\mathcal{V}\otimes\mathcal{W}) \to \mathcal{Q}$ be the universal quotient on* $\mathrm{Grass}^{|\Lambda|}(\mathcal{V}\otimes\mathcal{W})$, *and let $\pi^*\mathcal{V} \supseteq \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_l$ and $\pi^*\mathcal{W} \supseteq \mathcal{W}_1 \supset \cdots \supset \mathcal{W}_l$ be the universal filtrations on $Flag_\Lambda(\mathcal{V})$ and $Flag_{\widetilde{\Lambda}}(\mathcal{W})$ respectively. Let $\mathcal{L}_\Lambda$ and $\mathcal{L}_{\widetilde{\Lambda}}$ be the line bundles $\bigotimes_{j=0}^{l}\det(\mathcal{V}_j/\mathcal{V}_{j+1})^{\otimes\gamma_j(\Lambda)}$ and $\bigotimes_{j=0}^{l}\det(\mathcal{W}_j/\mathcal{W}_{j+1})^{\otimes\gamma_j(\widetilde{\Lambda})}$ on $Flag_\Lambda(\mathcal{V})$ and $Flag_{\widetilde{\Lambda}}(\mathcal{W})$ respectively. Then we have a natural isomorphism*

$$\det\mu_\Lambda^*(\mathcal{Q}) \simeq \mathcal{L}_\Lambda \boxtimes \mathcal{L}_{\widetilde{\Lambda}}. \tag{2.1}$$

Proof.

$$\begin{aligned}
\det\mu_\Lambda^*(\mathcal{Q}) &\simeq \det\left(\pi^*(\mathcal{V}\otimes\mathcal{W})/\sum_{s+t\geq l+1}\mathcal{V}_s\otimes\mathcal{W}_t\right)\\
&\simeq \bigotimes_{i=0}^{l}\det\Big((\mathcal{V}_i/\mathcal{V}_{i+1})\otimes(\pi^*\mathcal{W}/\mathcal{W}_{l-i+1})\Big)\\
&\simeq \bigotimes_{i=0}^{l}\Big((\det\mathcal{V}_i/\mathcal{V}_{i+1})^{\otimes n-d_{l-i+1}(\widetilde{\Lambda})}\otimes(\det\pi^*\mathcal{W}/\mathcal{W}_{l-i+1})^{\otimes d_i(\Lambda)-d_{i+1}(\Lambda)}\Big)\\
&\simeq \mathcal{L}_\Lambda\boxtimes\left(\bigotimes_{i=0}^{l}(\det\pi^*\mathcal{W}/\mathcal{W}_{l-i+1})^{(-\gamma_{l+1-i}(\widetilde{\Lambda})+\gamma_{l-i}(\widetilde{\Lambda}))}\right)\\
&\simeq \mathcal{L}_\Lambda\boxtimes\mathcal{L}_{\widetilde{\Lambda}}.
\end{aligned}$$

□

Next we shall define moduli stack of (quasi-)parabolic bundles.

Definition 2.1.6. Fix positive integers r and n. Let S be a scheme and $\pi : \mathcal{C} \to S$ be a smooth projective morphism such that the geometric fibers are irreducible curves of genus g. Let $s_i : S \to \mathcal{C}$ be a section of π ($i = 1, \ldots, m$, possibly $m = 0$). Let $\Lambda_1, \ldots, \Lambda_m$ be Young diagrams of type $\leq (r, n)$. Let d be an integer. Assume that $s_i(S) \cap s_j(S) = \emptyset$ for $i \neq j$.

(1) The S-stack $\mathcal{PU}(r, d; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m)$, or $\mathcal{PU}$ for short, is defined as follows. If $f : T \to S$ is an affine scheme over S, an object of the groupoid $\mathcal{PU}(T)$ is $(\mathcal{E}, \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$, where $\mathcal{E}$ is an r-bundle on $\mathcal{C} \times_S T$ of degree d on each fiber and $\mathbb{F}((s_i)_T^*\mathcal{E}) = ((s_i)_T^*\mathcal{E} \supseteq F_1((s_i)_T^*\mathcal{E}) \supset \cdots \supset F_{l(\Lambda_i)}((s_i)_T^*\mathcal{E}))$ is a filtration of $(s_i)_T^*\mathcal{E}$ of type Λ_i. The isomorphisms in $\mathcal{PU}(T)$ are isomorphisms of r-bundles compatible with filtrations.

(2) If $\mathcal{P}ic(\mathcal{C}/S)$ denotes the Picard stack of $\mathcal{C}/S$, we have a morphism $\det : \mathcal{PU}(r, d; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m) \to \mathcal{P}ic(\mathcal{C}/S)$ which sends $(\mathcal{E}, \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$ to $\det \mathcal{E}$. A line bundle $\mathcal{L}$ on $\mathcal{C}$ of degree d on each fiber induces $\sigma : S \to \mathcal{P}ic(\mathcal{C}/S)$. The S-stack $\mathcal{PSU}(r, \mathcal{L}; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m)$, or $\mathcal{PSU}$ for short, is defined to be the product stack $\mathcal{PU} \times_{\mathcal{P}ic} S$ of $\mathcal{PU} \xrightarrow{\det} \mathcal{P}ic$ and $S \xrightarrow{\sigma} \mathcal{P}ic(\mathcal{C}/S)$.

2.1.7. To fix notation, we recall atlases of the stacks $\mathcal{PU}$ and $\mathcal{PSU}$. Let $\mathcal{O}(1)$ be a π-ample line bundle on $\mathcal{C}$. For a positive integer N, we have the open stack $\mathcal{PU}^{(N)}$ (or $\mathcal{PSU}^{(N)}$) of $\mathcal{PU}$ (or $\mathcal{PSU}$) such that an object $(\mathcal{E}, \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$, (or $(\mathcal{E}, \det \mathcal{E} \simeq \mathcal{L}_T, \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$) is in $\mathcal{PU}^{(N)}$ (or $\mathcal{PSU}^{(N)}$) if and only if $\mathrm{H}^1(\mathcal{C}_t, \mathcal{E}_t(N)) = 0$ and $\mathcal{E}_t(N)$ is globally generated for each $t \in T$. Let $H^{(N)}$ (or $SH^{(N)}$) be the scheme such that, for $T \to S$, T-valued points of $H^{(N)}$ (or $SH^{(N)}$) are $(\mathcal{E}, \mathcal{O}_{\mathcal{C}_T}^{\oplus h(N)} \to \mathcal{E}(N), \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$ (or $(\mathcal{E}, \det \mathcal{E} \simeq \mathcal{L}_T, \mathcal{O}_{\mathcal{C}_T}^{\oplus h(N)} \to \mathcal{E}(N), \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$), where $h(N) = d + r(N \deg \mathcal{O}(1) + 1 - g)$ and $\mathcal{O}_{\mathcal{C}_T}^{\oplus h(N)} \to \mathcal{E}(N)$ is a surjective homomorphism that induces isomorphisms of vector spaces $k(t)^{\oplus h(N)} \simeq \mathrm{H}^0(\mathcal{C}_t, \mathcal{E}_t(N))$ for all $t \in T$. Then $\tau : H^{(N)} \to \mathcal{PU}^{(N)}$ (or $\tau : SH^{(N)} \to \mathcal{PSU}^{(N)}$) gives an atlas. Taking disjoint union, we have an atlas $\tau : \coprod_N H^{(N)} \to \mathcal{PU}$ (or $\tau : \coprod_N SH^{(N)} \to \mathcal{PSU}$). It is known that there exists N_0 depending only on r, d and g such that $H^{(N)}$ and $SH^{(N)}$ are non-empty, smooth and geometrically irreducible over S for $\forall N \geq N_0$ (cf. [Le]).

Definition 2.1.8. We use the notation in Definition 2.1.6. If $\lambda_1(\Lambda) - \lambda_r(\Lambda) < n$, then we define an open substack $\mathcal{PU}(r, d; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m)^{ss} \subset \mathcal{PU}(r, d; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m)$ as follows. Let T be an affine S-scheme and $\underline{\mathcal{E}} = (\mathcal{E}, \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$ be an object of $\mathcal{PU}(T)$. If $\lambda_1(\Lambda_i) = n$, then $(s_i)_T^*\mathcal{E} \supsetneqq F_1((s_i)_T^*\mathcal{E})$. Thus $(s_i)_T^*\mathcal{E}$ has $l(\Lambda_i)$ filters. We choose

$$0 < 1 - \frac{\gamma_1(\Lambda_i)}{n} < \cdots < 1 - \frac{\gamma_{l(\Lambda_i)}(\Lambda_i)}{n}$$

as parabolic weights at s_i. If $\lambda_1(\Lambda_i) < n$, then $(s_i)_T^*\mathcal{E} = F_1((s_i)_T^*\mathcal{E})$. Thus $(s_i)_T^*\mathcal{E}$ has $l(\Lambda_i) - 1$ filters. We choose

$$0 < \frac{\gamma_1(\Lambda_i) - \gamma_2(\Lambda_i)}{n} < \cdots < \frac{\gamma_1(\Lambda_i) - \gamma_{l(\Lambda_i)}(\Lambda_i)}{n}$$

as parabolic weights at s_i.

Then we can consider parabolic semistability for $\underline{\mathcal{E}}$ with respect to these parabolic weights. (See [M-S] for parabolic semistability.) $\underline{\mathcal{E}}$ is an object of $\mathcal{PU}^{ss}(T)$ if and only if it is parabolic semistable on each geometric fiber over T.

$\mathcal{PSU}^{ss} \subset \mathcal{PSU}$ is defined similarly.

Similarly as in the paragraph 2.1.7, for $N > 0$, $\mathcal{PU}^{ss(N)}$, $\mathcal{PSU}^{ss(N)}$ and their atlases $H^{ss(N)}$, $SH^{ss(N)}$ are defined.

Lemma 2.1.9. *Let k be an algebraically closed field and C be a smooth projective irreducible curve over k. Let $s_1, \dots s_m$ be distinct closed points of C and $\Lambda_1, \dots, \Lambda_m$ be Young diagrams of type $\leq (r, n)$. Let $\mathcal{L}$ be a line bundle on C. Then we have an isomorphism*

$$k \xrightarrow{\sim} \mathrm{H}^0 \left(\mathcal{PSU}\left(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m\right), \mathcal{O}_{\mathcal{PSU}}\right).$$

Proof. The morphism $\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m) \to \mathcal{PSU}(r, \mathcal{L}; C)$, which associates $\mathcal{E}$ to $(\mathcal{E}, \{\mathbb{F}(s_i^*\mathcal{E})\}_{i=1}^m)$, is a $\prod_{i=1}^m Flag_{\Lambda_i}$-bundle. Thus the pull-back morphism $\mathrm{H}^0(\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m), \mathcal{O}) \leftarrow \mathrm{H}^0(\mathcal{PSU}(r, \mathcal{L}; C), \mathcal{O})$ is an isomorphism. Therefore we only need to prove the lemma for some $m \geq 0$ and $\Lambda_1, \dots, \Lambda_m$. Let us choose $\Lambda_1, \dots, \Lambda_m$ for some m so that $\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)^{ss} \neq 0$. Let PSU be the coarse moduli space of $\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)^{ss}$. We have

$$\mathrm{H}^0(\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)^{ss}, \mathcal{O}) \simeq \mathrm{H}^0(PSU, \mathcal{O}_{PSU}) \simeq k$$

because PSU is an irreducible variety. The restriction map

$$\mathrm{H}^0(\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m), \mathcal{O}) \to \mathrm{H}^0(\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)^{ss}, \mathcal{O})$$

is injective because $\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)^{ss}$ is open dense in $\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)$. This proves the lemma. $\square$

Definition 2.1.10. We use the notation in Definition 2.1.6. By abuse of notation, we denote by s_i the universal section $(s_i)_{\mathcal{PSU}} : \mathcal{PSU} \to \mathcal{C} \times_S \mathcal{PSU}$. Let $(\mathcal{E}, \{\mathbb{F}(s_i^*\mathcal{E})\}_{i=1}^m)$ be the universal bundle, where $\mathbb{F}(s_i^*\mathcal{E}) = (s_i^*\mathcal{E} \supseteq F_1(s_i^*\mathcal{E}) \supset \cdots \supset F_{l(\Lambda_i)}(s_i^*\mathcal{E}))$. The line bundle $\Xi_{\mathcal{PSU}}^{(n)}$ on $\mathcal{PSU}$ is defined to be

$$(\det \mathbb{R}pr_{\mathcal{PSU}*}\mathcal{E})^{\otimes(-n)} \otimes \bigotimes_{i=1}^{m} \bigotimes_{j=0}^{l(\Lambda_i)} \det \left(F_j(s_i^*\mathcal{E}) / F_{j+1}(s_i^*\mathcal{E})\right)^{\otimes \gamma_j(\Lambda_i)}. \tag{2.2}$$

Remark 2.1.11. Let $S = \mathrm{Spec} k$ with k an algebraically closed field of characteristic zero. Let $sl_r(k) \supset h$ be the diagonal Cartan subalgebra. Let $\epsilon_i^* : h \to k$ be the weight given by $\mathrm{diag}(a_1, \dots, a_r) \mapsto a_i$. (Note that $\epsilon_1^* + \cdots \epsilon_r^* = 0$.) For the Young diagram Λ_i, put $\mu(\Lambda_i) := \sum_{j=1}^r \lambda_j(\Lambda_i) \cdot \epsilon_j^*$. By [P] and [B-L, §9], if $0 \leq d \leq r$, then we have

$$\dim \mathrm{H}^0 \left(\mathcal{PSU}\left(r, \mathcal{L}; \Lambda_1, \dots, \Lambda_m\right), \Xi_{\mathcal{PSU}}^{(n)}\right) = N_g(\mu_0, \mu_1, \dots \mu_m),$$

where $\mu_0 := n\sum_{j=1}^{r-d}\epsilon_i^*$ and $\mu_i := \mu(\Lambda_i)$ for $1 \le i \le m$, and $N_g(\mu_0, \mu_1, \dots, \mu_m)$ is the dimension of conformal block (of level n) (See [B3]).

We will use the following important fact known as factorization rules. Let $\Lambda_1, \dots, \Lambda_{m_1}, \Gamma_1, \dots, \Gamma_{m_2}$ be Young diagrams of type $\le (r, n)$. Put $\mu_i := \mu(\Lambda_i)$ and $\gamma_i := \mu(\Gamma_i)$. If $g = g_1 + g_2$, then we have

$$\begin{aligned} &N_g(\mu_1, \dots, \mu_{m_1}, \gamma_1, \dots, \gamma_{m_2}) \\ &\quad = \sum_{\Lambda} N_{g_1}(\mu_1, \dots, \mu_{m_1}, \mu(\Lambda)) N_{g_2}(\gamma_1, \dots, \gamma_{m_2}, \mu(\Lambda^*)), \end{aligned} \tag{2.3}$$

where Λ runs through Young diagrams of type $\le (r, n)$ with $\lambda_r(\Lambda) = 0$.

Remark 2.1.12. Let C,$s_1, \dots, s_m$ and $\Lambda_1, \dots, \Lambda_m$ be as in Lemma 2.1.9. Let PSU be the coarse moduli space of $\mathcal{PSU}(r, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)^{ss}$. If the rational number e defined by the equation (2.4) is an integer, the line bundle $\Xi^{(n)}_{\mathcal{PSU}}$ on $\mathcal{PSU}$ descends to a line bundle $\Xi^{(n)}_{PSU}$ on PSU (see [P, Théorème 3.3]). We have $\dim \mathrm{H}^0(\mathcal{PSU}, \Xi^{(n)}_{\mathcal{PSU}}) = \dim \mathrm{H}^0(PSU, \Xi^{(n)}_{PSU})$.

If ζ_r denotes the r-th root of unity, every object of $\mathcal{PSU}$ has ζ_r-multiplication as an automorphism, which induces an action on the vector space $\mathrm{H}^0(\mathcal{PSU}, \Xi^{(n)}_{\mathcal{PSU}})$. If e is not an integer, this action is not trivial. This means that $\mathrm{H}^0(\mathcal{PSU}, \Xi^{(n)}_{\mathcal{PSU}}) = 0$ if e is not an integer.

Definition 2.1.13. Let S be a scheme and $\pi : \mathcal{C} \to S$ be a flat projective morphism such that geometric fibers are reduced connected curves of arithmetic genus g and let N be an integer. Let $\mathcal{F}$ be the universal bundle on $C \times_S \mathcal{U}(N, N(g-1); \mathcal{C}/S)$, where $\mathcal{U}(N, N(g-1); \mathcal{C}/S)$ is the moduli stack of rank N torsion-free sheaves of degree $N(g-1)$, or equivalently $\chi = 0$, on $\mathcal{C}/S$.

(1) The line bundle $\Theta_{\mathcal{U}}$ on $\mathcal{U}(N, N(g-1); \mathcal{C}/S)$ is defined to be $(\det \mathbb{R}pr_{\mathcal{U}*}\mathcal{F})^{\vee}$.

(2) The canonical global section σ_Θ of $\Theta_{\mathcal{U}}$ is defined as follows. Let T be an affine scheme over S and let $\mathcal{F}'$ be an object of $\mathcal{U}(N, N(g-1), \mathcal{C}/S)(T)$. We can find an exact sequence $0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{F}' \to 0$ of coherent sheaves on $\mathcal{C} \times_S T$ such that $\mathrm{H}^0(\mathcal{G}_t) = 0$ for $\forall t \in T$. (In fact, if $\mathcal{O}(1)$ is a π-ample line bundle on $\mathcal{C}$, take $\mathcal{G} = (\pi^*\pi_*(\mathcal{F}'(a)))(-a)$ for large $a > 0$.) Then $R^1 pr_{T*}\mathcal{H}$ and $R^1 pr_{T*}\mathcal{G}$ are vector bundles of the same rank. The homomorphism $R^1 pr_{T*}\mathcal{H} \to R^1 pr_{T*}\mathcal{G}$ induces a section $\sigma_\Theta(\mathcal{F}')$ of $(\det R^1 pr_{T*}\mathcal{G}) \otimes (\det R^1 pr_{T*}\mathcal{H})^{\vee} \simeq \Theta_{\mathcal{U}}(\mathcal{F}')$, where $\Theta_{\mathcal{U}}(\mathcal{F}')$ is the pull-back of the line bundle $\Theta_{\mathcal{U}}$ on $\mathcal{U}(N, N(g-1); \mathcal{C}/S)$ by the object $\mathcal{F}'$. The global section σ_Θ of $\Theta_{\mathcal{U}}$ is defined by the assignment $\mathcal{F}' \mapsto \sigma_\Theta(\mathcal{F}')$. (Although we made a choice of an exact sequence in the definition, σ_Θ is defined well.)

Remark 2.1.14. In Definition 2.1.13 (2), $\sigma_\Theta(\mathcal{F}')$ vanishes at $t \in T$ if and only if $\mathrm{H}^0(\mathcal{F}'_t) \ne 0$.

2.1.15. Fix positive integers r and n. Let $\pi : \mathcal{C} \to S$, s_i, Λ_i $(1 \le i \le m)$ be as in Definition 2.1.6 and let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ of degree d on each fiber. Let e be the rational number determined by

$$re + nd + rn(1-g) = \sum_{i=1}^{m} |\Lambda_i|. \tag{2.4}$$

If e is an integer, then for $\underline{\mathcal{G}} = (\mathcal{G}, \{\mathbb{F}(s_i^*\mathcal{G})\}_{i=1}^m)$, where $\mathcal{G}$ is a vector bundle of rank n on $\mathcal{C}$ of degree e on each fiber and $\mathbb{F}(s_i^*\mathcal{G}) = (s_i^*\mathcal{G} \supseteq F_1(s_i^*\mathcal{G}) \supset \cdots \supset F_{l(\widetilde{\Lambda_i})}(s_i^*\mathcal{G}))$ is a filtration of $s_i^*\mathcal{G}$ of type $\widetilde{\Lambda_i}$, we define the morphism $\varphi_{\underline{\mathcal{G}}} : \mathcal{PSU}(r, \mathcal{L}; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m) \to \mathcal{U}(rn, rn(g-1); \mathcal{C}/S)$ as follows. Let $f : T \to S$ be an affine S-scheme and $\underline{\mathcal{E}} = (\mathcal{E}, \det \mathcal{E} \simeq \mathcal{L}_T, \{\mathbb{F}((s_i)_T^*\mathcal{E})\}_{i=1}^m)$ be an object of $\mathcal{PSU}(r, \mathcal{L}; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m)(T)$. $\mathbb{F}((s_i)_T^*\mathcal{E})$ and $f^*\mathbb{F}(s_i^*\mathcal{G})$ are filtrations of $(s_i)_T^*\mathcal{E}$ and $f^*s_i^*\mathcal{G}(= (s_i)_T^*\mathcal{G}_T)$ of type Λ_i and $\widetilde{\Lambda_i}$, hence by Definition 2.1.4 we have

$$\mu_{\Lambda_i}(\mathbb{F}((s_i)_T^*\mathcal{E}), f^*\mathbb{F}(s_i^*\mathcal{G})) : (s_i)_T^* (\mathcal{E} \otimes \mathcal{G}_T) \to \mathcal{Q}(\mathbb{F}((s_i)_T^*\mathcal{E}), f^*\mathbb{F}(s_i^*\mathcal{G})).$$

On $\mathcal{C} \times_S T$, we have the surjection

$$\beta : \mathcal{E} \otimes \mathcal{G}_T \twoheadrightarrow \bigoplus_{i=1}^{m} (s_i)_{T*}(s_i)_T^* \mathcal{Q}(\mathbb{F}((s_i)_T^*\mathcal{E}), f^*\mathbb{F}(s_i^*\mathcal{G})).$$

Then we define $\varphi_{\underline{\mathcal{G}}}$ to associate $\mathrm{Ker}(\beta)$ to $\underline{\mathcal{E}}$.

Now assume that $S = \mathrm{Spec}k$ with k an algebraically closed field. We have an isomorphism $\alpha : \varphi_{\underline{\mathcal{G}}}^*\Theta_{\mathcal{U}} \xrightarrow{\sim} \Xi_{\mathcal{PSU}}^{(n)}$ (See the proof of Lemma 3.5.8). Although this isomorphism is not canonical, it is determined up to $k^\times$-multiple by Lemma 2.1.9. By abuse of notation, the global section $\alpha(\varphi_{\underline{\mathcal{G}}}^*\sigma_\Theta)$ of $\Xi_{\mathcal{PSU}}^{(n)}$ is denoted by $\varphi_{\underline{\mathcal{G}}}^*\sigma_\Theta$, which is, of course, determined up to $k^\times$-multiple.

Definition 2.1.16. We say that $(SD)_{r,n}$ holds for $\mathcal{PSU}(r, \mathcal{L}; \mathcal{C}/\mathrm{Spec}k; \Lambda_1, \ldots, \Lambda_m)$ if e is not an integer, or if e is an integer and there exist a finite number of objects $\underline{\mathcal{G}}_b = (\mathcal{G}_b, \{\mathbb{F}(s_i^*\mathcal{G}_b)\}_{i=1}^m)$ of $\mathcal{PU}(n, e; \mathcal{C}/\mathrm{Spec}k; \widetilde{\Lambda_1}, \ldots, \widetilde{\Lambda_m})(\mathrm{Spec}k)$ such that the set $\{\varphi_{\underline{\mathcal{G}}_b}^*\sigma_\Theta\}$ spans the vector space $\mathrm{H}^0(\mathcal{PSU}, \Xi_{\mathcal{PSU}}^{(n)})$.

Remark 2.1.17. It is easily seen that if $|\Lambda_1| = 0$ or rn, then $(SD)_{r,n}$ holds for $\mathcal{PSU}(r, \mathcal{L}; \mathcal{C}/\mathrm{Spec}k; \Lambda_1, \Lambda_2, \ldots, \Lambda_m)$ if and only if $(SD)_{r,n}$ holds for $\mathcal{PSU}(r, \mathcal{L}; \mathcal{C}/\mathrm{Spec}k; \Lambda_2, \ldots, \Lambda_m)$.

2.2. Invariance of (SD) by field base-change

Fix positive integers r and n. Let $\pi : \mathcal{C} \to S = \mathrm{Spec}k$ with k an algebraically closed field, $\mathcal{L}$, s_i and Λ_i $(1 \leq i \leq m)$ be as in the paragraph 2.1.15. Let K be an algebraically closed field over k. Put $\mathcal{C}_K := \mathcal{C} \times_{\mathrm{Spec}k} \mathrm{Spec}K$ and $\mathcal{L}_K := \mathcal{L}_{\mathrm{Spec}K}$.

The following proposition says that the property $(SD)_{r,n}$ does not depend on the choice of an algebraically closed field.

Proposition 2.2.1. *$(SD)_{r,n}$ holds for $\mathcal{PSU}(r,\mathcal{L};\mathcal{C};\Lambda_1,\dots,\Lambda_m)$ if and only if $(SD)_{r,n}$ holds for $\mathcal{PSU}(r,\mathcal{L}_K;\mathcal{C}_K;\Lambda_1,\dots,\Lambda_m)$.*

Proof. We may only consider the case that e is an integer. Take the atlas $\tau : H^{(N)} \to \mathcal{PU}(n,e;\mathcal{C};\widetilde{\Lambda_1},\dots,\widetilde{\Lambda_m})^{(N)}$ described in the paragraph 2.1.7. Let $\underline{\mathcal{G}} = (\mathcal{G},\mathbb{F}((s_i)^*_{H^{(N)}}\mathcal{G}))$ be the n-bundle on $\mathcal{C}\times_{\mathrm{Spec}k} H^{(N)}$ together with filtrations of $(s_i)^*_{H^{(N)}}\mathcal{G}$ of type $\widetilde{\Lambda_i}$ that induces τ. By the paragraph 2.1.15, we have a morphism of $H^{(N)}$-stacks

$$\varphi_{\underline{\mathcal{G}}} : \mathcal{PSU}(r,\mathcal{L}_{H^{(N)}};\mathcal{C}_{H^{(N)}}/H^{(N)};\Lambda_1,\dots) \to \mathcal{U}(rn,rn(g-1);\mathcal{C}_{H^{(N)}}/H^{(N)}).$$

We can find a finite affine covering $\mathrm{Spec}B := U^{(N)} := \coprod_b U_b^{(N)} \to H^{(N)}$ such that the morphism $\varphi_{\underline{\mathcal{G}}_{U^{(N)}}}$ of $U^{(N)}$-stacks induces an isomorphism $\varphi^*_{\underline{\mathcal{G}}_{U^{(N)}}}\Theta_{\mathcal{U}} \simeq \Xi^{(n)}_{\mathcal{PSU}}$. The section $\varphi^*_{\underline{\mathcal{G}}_{U^{(N)}}}\sigma_\Theta$ induces the B-module homomorphism $\theta : B \to \mathrm{H}^0(\mathcal{PSU}_{U^{(N)}},\Xi^{(n)})$. For a positive integer A, we put

$$\mathrm{Spec}B' := U^{(N,A)} := \overbrace{U^{(N)}\times_{H^{(N)}}\cdots\times_{H^{(N)}}U^{(N)}}^{A\text{times}},$$

and let $q_a : U^{(N,A)} \to U^{(N)}$ be the a-th projection. The pull-back of θ by q_a gives rise to B'-homomorphism $\theta_a : B' \to \mathrm{H}^0(\mathcal{PSU}_{U^{(N,A)}},\Xi^{(n)})$. Let $\theta^{(N,A)} : B'^{\oplus A} \to \mathrm{H}^0(\mathcal{PSU}_{U^{(N,A)}},\Xi^{(n)})$ map $(x_1,\dots,x_A)$ to $\sum_{a=1}^A \theta_a(x_a)$. Let $U^{(N,A)\circ}$ be the subset of $U^{(N,A)}$ consisting of points $u\in U^{(N,A)}$ such that $\theta^{(N,A)}\otimes_{B'} k(u)$ is surjective. By Proposition 4.1.2 (2), H^0-base change theorem holds in our situation, so $\mathrm{H}^0(\mathcal{PSU}_{U^{(N,A)}},\Xi^{(n)})$ is a finitely generated B'-algebra. Hence $U^{(N,A)\circ}$ is open in $U^{(N)}$. Moreover $(SD)_{r,n}$ holds for $\mathcal{PSU}(r,\mathcal{L};\mathcal{C}/\mathrm{Spec}k;\Lambda_1,\dots,\Lambda_m)$ if and only if $U^{(N,A)\circ}\neq\emptyset$ for some $N>0$ and $A>0$. This last condition is invariant under the base change $\mathrm{Spec}K \to \mathrm{Spec}k$. This completes the proof. $\square$

2.3. Openness of (SD)

Proposition 2.3.1. *Let $\pi : \mathcal{C}\to S = \mathrm{Spec}R$, s_i and Λ_i $(1\le i\le m)$ be as in Definition 2.1.6, let $\mathcal{L}$ be a line bundle of degree d on each fiber. Let S' be the subset of S such that $x\in S$ is in S' if and only if $(SD)_{r,n}$ holds for $\mathcal{PSU}(r,\mathcal{L}_{\bar{x}};\mathcal{C}_{\bar{x}};\Lambda_1,\dots,\Lambda_m)$, where $\bar{x} := \mathrm{Spec}\overline{k(x)}$. Put $\mathcal{PSU} := \mathcal{PSU}(r,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)$. Assume that the following condition (♠) holds.*

(♠) *$\mathrm{H}^0(\mathcal{PSU},\Xi^{(n)}_{\mathcal{PSU}})$ is a finitely generated R-module, and for $T=\mathrm{Spec}R'$ with R' an R-algebra, the natural morphism $\mathrm{H}^0(\mathcal{PSU},\Xi^{(n)}_{\mathcal{PSU}})\otimes_R R' \to \mathrm{H}^0(\mathcal{PSU}_T,\Xi^{(n)}_{\mathcal{PSU}_T})$ is an isomorphism.*
Then S' is an open subset of S.

Proof. This proposition is proved by a similar argument as in the proof of Proposition 2.2.1. Take the atlas $H^{(N)} \to \mathcal{PU}(n,e;\mathcal{C}/S;\widetilde{\Lambda_1},\dots,\widetilde{\Lambda_m})^{(N)}$. Let

$\underline{\mathcal{G}}$, Spec$B = U^{(N)} = \coprod_b U_b \to H^{(N)}$,

$$\varphi_{\underline{\mathcal{G}}} : \mathcal{PSU}(r, \mathcal{L}_{U^{(N)}}; \mathcal{C}_{U^{(N)}}/U^{(N)}; \Lambda_1, \dots, \Lambda_m) \to \mathcal{U}(rn, rn(g-1); \mathcal{C}_{U^{(N)}}/U^{(N)}),$$

$U^{(N,A)}$,

$$\theta^{(N,A)} : B'^{\oplus A} \to \mathrm{H}^0(\mathcal{PSU}(r, \mathcal{L}_{U^{(N,A)}}; \mathcal{C}_{U^{(N,A)}}/U^{(N,A)}; \Lambda_1, \dots, \Lambda_m), \Xi^{(n)})$$

and $U^{(N,A)\circ}$ be as in the proof of Proposition 2.2.1. (But this time $U^{(N)} \to H^{(N)}$ is an étale covering.) Let $g^{(N,A)}$ denote the projection $U^{(N,A)} \to S$. Then $g^{(N,A)}$ is an open map because $H^{(N)} \to S$ is smooth and $U^{(N)} \to H^{(N)}$ is étale. By (♠), $U^{(N,A)\circ}$ is open and

$$S' = \bigcup_{N,\, A} g^{(N,A)}(U^{(N,A)\circ}).$$

This completes the proof. □

3. The case $r = 2$

3.1. Elementary transformation

Let C be a smooth projective irreducible curve over an algebraically closed field k and let $\mathcal{L}$ be a line bundle on C of degree d. Let $s_1, \dots, s_m$ be distinct closed points of C and let $\Lambda_1, \dots, \Lambda_m$ be Young diagrams of type $\leq (2, n)$. Let $A_{a,b}$ $(a \geq b)$be the Young diagram of type $\leq (2, n)$ having a boxes in the first row and b boxes in the second row. Let $(\mathcal{E}, \{\mathbb{F}((s_i)^*_{\mathcal{PSU}}\mathcal{E})\}_{i=1}^m)$ be the universal bundle with filtrations over $\mathcal{PSU}(2, \mathcal{L}; C; \Lambda_1, \dots, \Lambda_m)$. Now assume that $\Lambda_1 = A_{n,a}$ with $0 < a < n$. Note that the rank of $F_1((s_1)^*_{\mathcal{PSU}}\mathcal{E})(\subset (s_1)^*_{\mathcal{PSU}}\mathcal{E})$ is one. Put

$$\mathcal{E}' := \mathrm{Ker}\left(\mathcal{E} \to (s_1)_{\mathcal{PSU}*}\left((s_1)^*_{\mathcal{PSU}}\mathcal{E}/F_1((s_1)^*_{\mathcal{PSU}}\mathcal{E})\right)\right). \tag{3.1}$$

Let $\mathbb{F}((s_1)^*_{\mathcal{PSU}}\mathcal{E}')$ be the filtration

$$(s_1)^*_{\mathcal{PSU}}\mathcal{E}' = F_1((s_1)^*_{\mathcal{PSU}}\mathcal{E}') \supset F_2((s_1)^*_{\mathcal{PSU}}\mathcal{E}') = \mathrm{Ker}\left((s_1)^*_{\mathcal{PSU}}\mathcal{E}' \to (s_1)^*_{\mathcal{PSU}}\mathcal{E}\right)$$

of $(s_1)^*_{\mathcal{PSU}}\mathcal{E}'$ of type $A_{a,0}$. Put $\mathbb{F}((s_i)^*_{\mathcal{PSU}}\mathcal{E}') := \mathbb{F}((s_2)^*_{\mathcal{PSU}}\mathcal{E})$ for $i \geq 2$. Then, by associating $(\mathcal{E}', \{\mathbb{F}((s_i)^*_{\mathcal{PSU}}\mathcal{E}')\}_{i=1}^m)$ to $(\mathcal{E}, \{\mathbb{F}((s_i)^*_{\mathcal{PSU}}\mathcal{E})\}_{i=1}^m)$, we have an isomorphism elm of stacks

$$\mathcal{PSU}\left(2, \mathcal{L}; C; A_{n,a}, \Lambda_2, \dots, \Lambda_m\right) \xrightarrow{elm} \mathcal{PSU}\left(2, \mathcal{L}(-s_1); C; A_{a,0}, \Lambda_2, \dots, \Lambda_m\right). \tag{3.2}$$

Let $\underline{\mathcal{G}} = (\mathcal{G}, \{\mathbb{F}(s_i^*\mathcal{G})\}_{i=1}^m)$ be an object of $\mathcal{PU}(n, e; C; \widetilde{A_{n,a}}, \widetilde{\Lambda_2}, \dots, \widetilde{\Lambda_m})(\mathrm{Spec}k)$, where $\mathbb{F}(s_i^*\mathcal{G}) = (s_i^*\mathcal{G} \supset F_1(s_i^*\mathcal{G}))$. Then $\underline{\mathcal{G}}$ can be regarded as an object of $\mathcal{PU}(n, e; C; \widetilde{A_{a,0}}, \widetilde{\Lambda_2}, \dots, \widetilde{\Lambda_m})(\mathrm{Spec}k)$.

Lemma 3.1.1. *The following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{PSU}\,(2, \mathcal{L}; C; A_{n,a}, \Lambda_2, \ldots, \Lambda_m) & \xrightarrow{\varphi_{\mathcal{G}}} & \mathcal{U}(2n, 2n(g-1); C) \\ \wr\downarrow elm & & \| \\ \mathcal{PSU}\,(2, \mathcal{L}(-s_1); C; A_{a,0}, \Lambda_2, \ldots, \Lambda_m) & \xrightarrow{\varphi_{\mathcal{G}}} & \mathcal{U}(2n, 2n(g-1); C) \end{array} \tag{3.3}$$

Proof. For simplicity assume that $m = 1$. We have the commutative diagram of sheaves on $C \times \mathcal{PSU}\,(2, \mathcal{L}; C; A_{n,a})$;

(3.4)

$$\begin{array}{ccc} \mathcal{E} \otimes (\mathcal{G})_{\mathcal{PSU}} & \xrightarrow{f} & (s_1)^*_{\mathcal{PSU}}\mathcal{E} \otimes (s_1^*\mathcal{G})_{\mathcal{PSU}} / F_1((s_1)^*_{\mathcal{PSU}}\mathcal{E}) \otimes (F_1(s_1^*\mathcal{G}))_{\mathcal{PSU}} \\ \uparrow & & \uparrow \\ \mathcal{E}' \otimes (\mathcal{G})_{\mathcal{PSU}} & \xrightarrow{f'} & ((s_1)^*_{\mathcal{PSU}}\mathcal{E}' / F_2((s_1)^*_{\mathcal{PSU}}\mathcal{E}')) \otimes (s_1^*\mathcal{G}/F_1(s_1^*\mathcal{G}))_{\mathcal{PSU}}\,. \end{array}$$

Then the proposition follows from $\mathrm{Ker}(f) = \mathrm{Ker}(f')$. □

Proposition 3.1.2. *$(SD)_{2,n}$ holds for $\mathcal{PSU}(2, \mathcal{L}; C; A_{n,a}, \Lambda_2, \ldots, \Lambda_m)$ if and only if $(SD)_{2,n}$ holds for $\mathcal{PSU}(2, \mathcal{L}(-s_1); C; A_{a,0}, \Lambda_2, \ldots, \Lambda_m)$.*

Proof. For simplicity we assume that $m = 1$. If $0 < a < n$, then this is a direct consequence of Lemma 3.1.1. Assume that $a = 0$. Let $\mathcal{E}'$ be as in the equation 3.1. By associating $\mathcal{E}'$ to $(\mathcal{E}, \mathbb{F}((s_1)^*_{\mathcal{PSU}}\mathcal{E}))$, we have the $\mathbb{P}^1$-bundle

$$\mathcal{PSU}(2, \mathcal{L}; C; A_{n,0}) \xrightarrow{h} \mathcal{PSU}(2, \mathcal{L}(-s_1); C),$$

and $h^*\Xi^{(n)}_{\mathcal{PSU}(2,\mathcal{L}(-s_1))} \simeq \Xi^{(n)}_{\mathcal{PSU}(2,\mathcal{L})}$. Let $\underline{\mathcal{G}} = (\mathcal{G}, \mathbb{F}(s_1^*\mathcal{G}))$ be an object of $\mathcal{PU}(n, e; C; \widetilde{A_{n,0}})(\mathrm{Spec} k)$, then $\mathbb{F}(s_1^*\mathcal{G}) = (s_1^*\mathcal{G} = F_1(s_1^*\mathcal{G}))$. Thus $\underline{\mathcal{G}}$ can be considered as an object of $\mathcal{PU}(n, e; C; \widetilde{A_{0,0}})$ and the diagram

$$\begin{array}{ccc} \mathcal{PSU}(2, \mathcal{L}; C; A_{n,0}) & \rightarrow & \mathcal{U}(2n, 2n(g-1); C) \\ \downarrow h & & \| \\ \mathcal{PSU}(2, \mathcal{L}(-s_1); C) & \rightarrow & \mathcal{U}(2n, 2n(g-1); C) \end{array} \tag{3.5}$$

is commutative. Hence for $a = 0$ the proposition is proved. The case $a = n$ is similar. □

By a similar argument as above, we obtain the following proposition whose proof is left to the reader.

Proposition 3.1.3. *Let $n \geq a \geq b \geq 0$. Then $(SD)_{2,n}$ holds for $\mathcal{PSU}(2, \mathcal{L}; C; A_{a,b}; \Lambda_2, \ldots, \Lambda_m)$ if and only if $(SD)_{2,n}$ holds for $\mathcal{PSU}(2, \mathcal{L}; C; A_{n,n-a+b}; \Lambda_2, \ldots, \Lambda_m)$.*

3.2. Genus one case

Proposition 3.2.1. *Let C be a smooth projective irreducible curve of genus one over an algebraically closed field k of characteristic zero. Let $\mathcal{L}$ be a line bundle of degree d on C. Then $(SD)_{2,n}$ holds for $\mathcal{PSU}(2, L; C)$.*

Proof. It suffices to consider the case that e determined by the equation 2.4 is an integer. We may assume that $\mathcal{L} = \mathcal{O}_C$ or $\mathcal{L} = \mathcal{O}_C(x)$ with $x \in C$. By [B3], we have

$$\dim \mathrm{H}^0(\mathcal{PSU}(2, \mathcal{L}; C), \Xi^{(n)}) = \begin{cases} n+1 & \text{if } \mathcal{L} = \mathcal{O}_C \\ 1 & \text{if } \mathcal{L} = \mathcal{O}_C(x). \end{cases}$$

Case (1). $\mathcal{L} = \mathcal{O}_C$.

Let PSU be the coarse moduli space of $\mathcal{PSU}(2, \mathcal{L}; C)^{ss}$. We have

$$\mathrm{H}^0(\mathcal{PSU}, \Xi^{(n)}) \overset{g}{\hookrightarrow} \mathrm{H}^0(\mathcal{PSU}^{ss}, \Xi^{(n)}) \simeq \mathrm{H}^0(PSU, \Xi^{(n)}_{PSU}). \tag{3.6}$$

Let $\mathcal{M}$ be a universal bundle on $C \times \mathrm{Pic}^\circ(C)$. The 2-bundle $\mathcal{M} \oplus \mathcal{M}^{-1}$ on $C \times \mathrm{Pic}^\circ(C)$ gives rise to a morphism $f : \mathrm{Pic}^\circ(C) \to PSU$. This factors as $\mathrm{Pic}^\circ(C) \xrightarrow{f_1} \mathrm{Pic}^\circ(C)/[-1] \xrightarrow{f_2} PSU$, and f_2 is an isomorphism. Since $pr_{\mathrm{Pic}^\circ(C)*}\mathcal{M} = 0$ and $R^1 pr_{\mathrm{Pic}^\circ(C)*}\mathcal{M} = k(\mathbf{o})$, we have $f^* \Xi^{(n)}_{PSU} \simeq \mathcal{O}(2n \cdot \mathbf{o})$. Therefore $f_2^* \Xi^*_{PSU}$ is isomorphic to the line bundle $\mathcal{O}(n)$ on $\mathrm{Pic}^\circ(C)/[-1] \simeq \mathbb{P}^1$, hence $\dim \mathrm{H}^0(PSU, \Xi^{(n)}_{PSU}) = n+1$ and g is an isomorphism. If $\mathcal{G} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ with $\mathcal{M}_i \in \mathrm{Pic}^\circ(C)$, then the pull-back of the section $g(\varphi^*_{\mathcal{G}} \sigma_\Theta)$ by f is a section of $\mathcal{O}(2n \cdot \mathbf{o})$ whose divisor of zero is $\sum_{i=1}^n ([\mathcal{M}_i] + [\mathcal{M}_i^\vee])$. Thus the divisor of zero of $f_2^*(g(\varphi^*_{\mathcal{G}} \sigma_\Theta))$ is $\sum_{i=1}^n f_1([\mathcal{M}_i])$. Varying $\mathcal{M}_i$, these divisors span $\mathrm{H}^0(\mathrm{Pic}^\circ(C)/[-1], \mathcal{O}(n))$.

Case (2). $\mathcal{L} \simeq \mathcal{O}_C(x)$.

Note first that in this case n is even because we are assuming that e is an integer. Let $0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C(x) \to 0$ be a non-split exact sequence. Since $\dim \mathrm{H}^0(\mathcal{PSU}, \Xi^{(n)}) = 1$, we have only to find an n-bundle $\mathcal{G}$ of degree $-n/2$ such that $\mathrm{H}^0(C, \mathcal{E} \otimes \mathcal{G}) = 0$ because $\mathrm{H}^0(C, \mathcal{E} \otimes \mathcal{G}) = 0$ implies $\varphi^*_{\mathcal{G}} \sigma_\Theta \neq 0$ by Remark 2.1.14. Take $\mathcal{N} \in \mathrm{Pic}^\circ(C)$ such that $\mathcal{N}^{\otimes 2} \not\simeq \mathcal{O}_C$. Put $\mathcal{G} := (\mathcal{E}^\vee \otimes \mathcal{N})^{\oplus(n/2)}$. If $\mathrm{H}^0(C, \mathcal{E} \otimes \mathcal{G}) \neq 0$, then we have a non-zero homomorphism $\alpha : \mathcal{E} \to \mathcal{E} \otimes \mathcal{N}$. Since $\mathcal{E}$ is stable, α is an isomorphism so that we have $\wedge^2 \alpha : \bigwedge^2 \mathcal{E} \xrightarrow{\sim} \left(\bigwedge^2 \mathcal{E}\right) \otimes \mathcal{N}^{\otimes 2}$. This contradicts the choice of $\mathcal{N}$. Therefor $\mathrm{H}^0(C, \mathcal{E} \otimes \mathcal{G}) = 0$.

□

Proposition 3.2.2. *Let C and $\mathcal{L}$ be as in Proposition* 3.2.1. *Let s be a closed point of C and Λ be a Young diagram of type $\leq (2, n)$. Then $(SD)_{2,n}$ holds for $\mathcal{PSU}(2, \mathcal{L}; C, \Lambda)$.*

Proof. By Proposition 3.1.2, we may assume that d is odd. By Proposition 3.1.3, we may assume that $\Lambda = A_{n,a}$ $(0 \leq a \leq n)$. If $a = n$, then the proposition follows from Remark 2.1.17 and Proposition 3.2.1. If $a = 0$, the proposition follows from Proposition 3.1.2 and Remark 2.1.17 and Proposition 3.2.1. Therefore we may assume that $0 < a < n$. We may assume that $d = 1$. If a is odd, then $e = a/2$ is not an integer. Thus we may assume that a is even. In this case we have $\dim \mathrm{H}^0(\mathcal{PSU}(2,\mathcal{L};C,\Lambda),\Xi^{(n)}) = n+1-a$. Let PSU be the coarse moduli space of $\mathcal{PSU}^{ss}$. Let $\mathcal{E}$ be a stable 2-bundle on C with $\det\mathcal{E}\simeq\mathcal{L}$. For $(l\subset s^*\mathcal{E})\in\mathbb{P}(s^*\mathcal{E})$, $(\mathcal{E},s^*\mathcal{E}\supset l)$ is parabolic stable. Thus we have an isomorphism $f:\mathbb{P}(s^*\mathcal{E})\xrightarrow{\sim} PSU$ and $f^*\Xi^{(n)}_{PSU}\simeq\mathcal{O}_{\mathbb{P}^1}(n-a)$. For $1\le i\le n-a$, let $\mathcal{M}_i$ be a line bundle on C of degree zero. For $1\le j\le a/2$, let $\mathcal{F}_j$ be a stable 2-bundle on C of degree one. Put $\mathcal{G}:=\bigoplus_{i=1}^{n-a}\mathcal{M}_i\oplus\bigoplus_{j=1}^{a/2}\mathcal{F}_j$. By letting the first filter $F_1(s^*\mathcal{G})$ of $s^*\mathcal{G}$ be $s^*\mathcal{M}_1\oplus\cdots\oplus s^*\mathcal{M}_{n-a}\oplus 0\oplus\cdots\oplus 0$, we obtain the filter $\mathbb{F}(s^*\mathcal{G})$ of $s^*\mathcal{G}$ of type $\widetilde{A_{n,a}}$. Put $\underline{\mathcal{G}}:=(\mathcal{G},\mathbb{F}(s^*\mathcal{G}))$. We choose the above $\mathcal{F}_1,\dots,\mathcal{F}_{a/2}$ so general that we have $\mathrm{H}^0(C,\mathcal{E}\otimes\mathcal{F}_j(-s))=0$ for $1\le\forall j\le a/2$. If $\mathcal{M}$ is a line bundle on C of degree zero, the subset

$$\left\{(l\subset s^*\mathcal{E})\,\middle|\,\mathrm{H}^0\left(\mathrm{Ker}(\mathcal{E}\to s^*\mathcal{E}/l)\otimes\mathcal{M}\right)\neq 0\right\}\subset\mathbb{P}(s^*\mathcal{E})$$

is a one point, which we denote by $\Phi_{\mathcal{M}}$. Then the divisor of zero of the section $\varphi^*_{\underline{\mathcal{G}}}\sigma_\Theta$ is $\sum_{i=1}^{n-a}\Phi_{\mathcal{M}_i}$. Since the morphism $\Phi:\mathrm{Pic}^\circ(C)\to\mathbb{P}(s^*\mathcal{E})$ that maps $\mathcal{M}$ to $\Phi_{\mathcal{M}}$ is surjective, the divisors $\sum_{i=1}^{n-a}\Phi_{\mathcal{M}_i}$ span $\mathrm{H}^0(\mathbb{P}(s^*\mathcal{E}),\mathcal{O}(n-a))$ when we vary $\mathcal{M}_i$. □

Proposition 3.2.3. *Let C and $\mathcal{L}$ be as in Proposition 3.2.1. For $i=1,2$, let s_i be a closed point of C and Λ_i be a Young diagram of type $\le(2,n)$. Then $(SD)_{2,n}$ holds for $\mathcal{PSU}(2,\mathcal{L};C;\Lambda_1,\Lambda_2)$.*

Proof. By Proposition 3.1.3, we may assume that $\Lambda_i=A_{n,n-a_i}$ $(0\le a_i\le n)$. If $a_i=0$ or n, then using Proposition 3.1.2 and Remark 2.1.17, we can attribute the proposition to Proposition 3.2.2. Therefore we may assume that $0<a_i<n$. Moreover if d is even, we perform elementary transformation at s_1 so that we may assume that d is odd by Proposition 3.1.2. Moreover, if $a_1+a_2>n$, we perform elementary transformation at s_1 and s_2 so that we may assume that $a_1+a_2\le n$ by Proposition 3.1.2. After all, we may assume that d is odd and $\Lambda_i=A_{n,n-a_i}$ with $0<a_i<n$ and $a_1+a_2\le n$. We may assume that $d=1$ and that $e=(3n-a_1-a_2)/2$ is an integer. We have $\dim\mathrm{H}^0(\mathcal{PSU}(2,\mathcal{L};C;\Lambda_1,\Lambda_2),\Xi^{(n)})=(a_1+1)(a_2+1)$. Let PSU be the coarse moduli space of $\mathcal{PSU}(2,\mathcal{L};C;\Lambda_1,\Lambda_2)^{ss}$. Let $\mathcal{E}$ be a stable 2-bundle on C with $\det\mathcal{E}\simeq\mathcal{L}$. For $\forall(l_i\subset s_i^*\mathcal{E})\in\mathbb{P}(s_i^*\mathcal{E})$, $(\mathcal{E},s_1^*\mathcal{E}\supset l_1,s_2^*\mathcal{E}\supset l_2)$ is parabolic semistable. Thus we have the morphism $f:\mathbb{P}(s_1^*\mathcal{E})\times\mathbb{P}(s_2^*\mathcal{E})\to PSU$. We can see that f is an isomorphism and $f^*\Xi^{(n)}\simeq\mathcal{O}_{\mathbb{P}^1}(a_1)\boxtimes\mathcal{O}_{\mathbb{P}^1}(a_2)$. If $\mathcal{M}$ is a line bundle on C of degree one, the subset

$$\left\{(l_i\subset s_i^*\mathcal{E}_i)\,\middle|\,\mathrm{H}^0\left(\mathrm{Ker}(\mathcal{E}\to s_i^*\mathcal{E}/l_i)\otimes\mathcal{M}\otimes\mathcal{O}_C(-s_{3-i})\right)\neq 0\right\}\subset\mathbb{P}(s_i^*\mathcal{E})$$

is a one point, which we denote by $\Phi^{(i)}_{\mathcal{M}}$. Let $\mathcal{M}_1,\dots,\mathcal{M}_{a_1},\mathcal{N}_1,\dots,\mathcal{N}_{a_2}$ be line bundles of degree one. For $1\le j\le(n-a_1-a_2)/2$, let $\mathcal{F}_j$ be a stable 2-bundle of degree 3. Put

$$\mathcal{G}:=\mathcal{M}_1\oplus\cdots\oplus\mathcal{M}_{a_1}\oplus\mathcal{N}_1\oplus\cdots\oplus\mathcal{N}_{a_2}\oplus\bigoplus_{j=1}^{(n-a_1-a_2)/2}\mathcal{F}_j.$$

If we let $F_1(s_1^*\mathcal{G})$ and $F_1(s_2^*\mathcal{G})$ be

$$F_1(s_1^*\mathcal{G}) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_{a_1} \oplus 0 \oplus \cdots \oplus 0 \bigoplus 0 \subset s_1^*\mathcal{G}$$
$$F_2(s_2^*\mathcal{G}) = 0 \oplus \cdots \oplus 0 \oplus \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_{a_2} \bigoplus 0 \subset s_2^*\mathcal{G},$$

then $\mathbb{F}(s_1^*\mathcal{G}) = (s_1^*\mathcal{G} \supset F_1(s_1^*\mathcal{G}))$ and $\mathbb{F}(s_2^*\mathcal{G}) = (s_2^*\mathcal{G} \supset F_1(s_2^*\mathcal{G}))$ are filtrations of type $\widetilde{A_{n,n-a_1}}$ and $\widetilde{A_{n,n-a_2}}$ respectively. Put $\underline{\mathcal{G}} := (\mathcal{G}, \mathbb{F}(s_1^*\mathcal{G}), \mathbb{F}(s_2^*\mathcal{G}))$. We choose the above $\mathcal{F}_j$ generally. Then the divisor of zero of the section $\varphi_{\underline{\mathcal{G}}}^*\sigma_\Theta$ is $p_1^*(\sum_{i=1}^{a_1} \Phi_{\mathcal{M}_i}^{(1)}) + p_2^*(\sum_{i=1}^{a_2} \Phi_{\mathcal{M}_i}^{(2)})$, where p_i is the i-th projection $p_i : \mathbb{P}(s_1^*\mathcal{E}) \times \mathbb{P}(s_2^*\mathcal{E}) \to \mathbb{P}(s_i^*\mathcal{E})$. As varying $\mathcal{M}_i$ and $\mathcal{N}_j$, these $\varphi_{\underline{\mathcal{G}}}^*\sigma_\Theta$ span $\mathrm{H}^0(\mathbb{P}(s_1^*\mathcal{E}) \times \mathbb{P}(s_2^*\mathcal{E}), \mathcal{O}(a_1) \boxtimes \mathcal{O}(a_2))$. □

3.3. Base-change property

Let k be a field of characteristic zero and let R be a finitely generated reduced k-algebra. Let $\pi : \mathcal{C} \to S := \mathrm{Spec}R$ be a smooth projective morphism such that every geometric fiber is an irreducible curve of genus $g \geq 2$. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ of degree d on each fiber. Let $s : S \to \mathcal{C}$ be a section of π and Λ be a Young diagram of type $\leq(2,n)$.

Proposition 3.3.1. *If $\mathcal{PSU}$ is $\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S)$ or $\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda)$, then the property (♠) in Proposition 2.3.1 holds.*

Proof. We shall deal with the case $\mathcal{PSU} = \mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda)$, the other case is similar and simpler. We may assume that $\Lambda = A_{n,a}$ with $0 < a < n$. Let PSU be the coarse moduli space of $\mathcal{PSU}^{ss}$. Put $T = \mathrm{Spec}R'$, R' being an R-algebra. Then we have

$$\mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}_T; \mathcal{C}_T/T; \Lambda)^{ss}, \Xi^{(n)}\right) \simeq \mathrm{H}^0\left(PSU \times_S T, \Xi^{(n)}_{PSU \times_S T}\right).$$

Since $\dim \mathrm{H}^0(PSU \times_S \mathrm{Spec}k(x), \Xi^{(n)})$ is constant for any $x \in S$, we infer that the natural homomorphism $\mathrm{H}^0(PSU, \Xi^{(n)}) \otimes_R R' \to \mathrm{H}^0(PSU \times_S \mathrm{Spec}R', \Xi^{(n)})$ is an isomorphism for any R-algebra R' by the usual base-change theorem for schemes (cf. [Mum, Chapter II, §5, Crollary 2]). Therefor in order to complete the proof, it suffices to prove that the restriction map $\mathrm{H}^0(\mathcal{PSU}(2, \mathcal{L}_T; \mathcal{C}_T/T; \Lambda), \Xi^{(n)}) \to \mathrm{H}^0(\mathcal{PSU}(2, \mathcal{L}_T; \mathcal{C}_T/T; \Lambda)^{ss}, \Xi^{(n)})$ is an isomorphism. To prove this we shall apply Proposition 4.1.5. All we have to prove is the following claim.

Claim. Let C be a smooth projective irreducible curve of genus $g \geq 2$ over a field k, $\mathcal{L}$ be a line bundle of degree d on C and s be a closed point of C. Let $\tau : SH^{(N)} \to \mathcal{PSU}(2, \mathcal{L}; C; A_{n,a})$ $(0 < a < n)$ be the atlas described in the paragraph 2.1.7. Then we have

$$\mathrm{codim}_{SH^{(N)}}(SH^{(N)} \setminus SH^{ss(N)}) > g - 1.$$

Proof of Claim. Put $h(N) := d + 2(\deg \mathcal{O}(1) + 1 - g)$ as in the paragraph 2.1.7. It is easily seen that $\dim SH^{(N)} = 3g - 2 + h(N)^2$. If a 2-bundle $\mathcal{E}$ on C

with a 1-dimensional filter $l \subset s^*\mathcal{E}$ is not parabolic semistable with respect to the weights $(0,(n-a)/n)$, then we have either (i) there exists a line subbundle $\mathcal{M} \subset \mathcal{E}$ with $\deg \mathcal{M} - (n-a)/2n > d/2$, or (ii) there exists a line subbundle $\mathcal{M} \subset \mathcal{E}$ with $s^*\mathcal{M} = l$ and $\deg \mathcal{M} + (n-a)/2n > d/2$. 2-bundles $\mathcal{E}$ with a 1-dimensional filter $l \subset s^*\mathcal{E}$ satisfying (i) are parameterized by a scheme of dimension less than or equal to

$$\max\left\{\dim\left\{\bigcup_{d'}\bigcup_{\mathcal{M}} \mathbb{P}_*\mathrm{H}^1(C, \mathcal{M}^{\otimes 2} \otimes \mathcal{L}^\vee)\right\} + 1, g\right\}, \tag{3.7}$$

where $d' > (d/2) + (n-a)/2n$ and $\mathcal{M}$ is a line bundle of degree d'. $(\mathcal{E}, l \subset s^*\mathcal{E})$ satisfying (ii) are parameterized by a scheme of dimension less than or equal to

$$\max\left\{\dim\left\{\bigcup_{d'}\bigcup_{\mathcal{M}} \mathbb{P}_*\mathrm{H}^1(C, \mathcal{M}^{\otimes 2} \otimes \mathcal{L}^\vee)\right\}, g\right\}, \tag{3.8}$$

where $d' > (d/2) - (n-a)/2n$ and $\mathcal{M}$ is a line bundle of degree d'. Note that

$$\dim\left\{\bigcup_{\mathcal{M}} \mathbb{P}_*\mathrm{H}^1(C, \mathcal{M}^{\otimes 2} \otimes \mathcal{L}^\vee)\right\} \leq 2g - 2 - (2\deg \mathcal{M} - \deg \mathcal{L}).$$

Therefore we have

$$\begin{aligned}\dim\left(SH^{(N)} \setminus SH^{ss(N)}\right) &\leq \max\left\{2g - 1 - \frac{n-a}{n}, 2g - 2 + \frac{n-a}{n}, g\right\} + h(N)^2 \\ &< 2g - 1 + h(N)^2.\end{aligned}$$

Hence we have $\mathrm{codim}_{SH^{(N)}}(SH^{(N)} \setminus SH^{ss(N)}) > g - 1$. This completes the proof of the claim. □

□

3.4. $\mathcal{PSU}$ for nodal curves

Let $\pi : \mathcal{C} \to S$ be a flat projective morphism such that every geometric fiber is a connected nodal curve of arithmetic genus g. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ of degree d on each fiber. Let $s_1, \ldots, s_m$ be sections of π such that $s_i(S) \subset smooth(\pi)$ and $s_i(S) \cap s_j(S) = \emptyset$ for $i \neq j$, and $\Lambda_1, \ldots, \Lambda_m$ be Young diagrams of type $\leq (2, n)$. We shall extend the definition of the stack $\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m)$ to families of nodal curves as follows.

If T is an affine scheme and $f : T \to S$ is a morphism, an object of $\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda_1, \ldots, \Lambda_m)(T)$ is following data:

(a) T-flat coherent sheaf $\mathcal{E}$ on $\mathcal{C} \times_S T$ such that its restriction to each fiber is torsion-free and of rank 2 on every irreducible component,

(b) an isomorphism $\beta : \mathcal{E} \to \mathcal{H}om_{\mathcal{O}_{\mathcal{C}\times_S T}}(\mathcal{E}, \mathcal{L}_T)$ of $\mathcal{O}_{\mathcal{C}\times_S T}$-modules such that ${}^t\beta = -\beta$,

(c) a filtration $\mathbb{F}((s_i)_T^*\mathcal{E}) = ((s_i)_T^*\mathcal{E} \supseteq F_1((s_i)_T^*\mathcal{E}) \supset \ldots \supset F_{l(\Lambda_i)}((s_i)_T^*\mathcal{E}))$ of type Λ_i for each $1 \leq i \leq m$.

Isomorphisms of the groupoid $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)(T)$ are defined obviously.

We denote by $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)^{lf}$ the open substack of $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)$ consisting of locally free sheaves.

The line bundle $\varXi^{(n)}_{\mathcal{PSU}}$ on $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)$ is defined by (2.2).

3.4.1. Just as in the smooth case, we can describe an atlas of $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)$ as follows. Fix a π-very-ample line bundle $\mathcal{O}_{\mathcal{C}}(1)$ on $\mathcal{C}$. For a positive integer N, we define the open substack $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)^{(N)}$ of $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)$ as in the paragraph 2.1.7. Let $SH^{(N)}$ be the S-scheme parameterizing sheaves $\mathcal{E}$ on $\mathcal{C}_t$ ($t\in S$) as in (a) such that $\mathcal{E}(N)$ is globally generated and $\mathrm{H}^1(\mathcal{C}_t,\mathcal{E}(N)) = 0$, together with a basis of $\mathrm{H}^0(\mathcal{C}_t,\mathcal{E}(N))$, anti-symmetric isomorphism β as in (b) above and filtrations $\mathbb{F}(s_i^*\mathcal{E})$ as in (c) above. Then $SH^{(N)} \to \mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)^{(N)}$ is an atlas. Taking disjoint union, we obtain an atlas $\coprod_N SH^{(N)} \to \mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda_1,\dots,\Lambda_m)$. We put $SH^{lf(N)} := SH^{(N)} \times_{\mathcal{PSU}}\mathcal{PSU}^{lf}$.

3.4.2. Set-up. Now let us restrict ourselves to the following special situation that will be retained in the rest of this section. Let $S := \mathrm{Spec}k[[t]]$, k being an algebraically closed field of characteristic zero. S_o denotes the closed point of S, η denotes the generic point of S and $\bar{\eta}$ denotes the geometric point $\mathrm{Spec}\overline{k(\eta)} \to S$ of η. $\pi:\mathcal{C}\to S$ is smooth over η and $\mathcal{C}\times_S S_o$ has two irreducible component C_1 and C_2, both being smooth, and C_1 and C_2 intersects transversally at one point, which we denote by P. We assume that there is an isomorphism $\widehat{\mathcal{O}_{\mathcal{C},P}} \simeq k[[x_1,x_2,t]]/(x_1x_2-t^{\upsilon})$ of $k[[t]]$-algebra, with υ a positive integer. Let us be given only one section s of π such that $s(S_o)\in C_1\setminus\{P\}$, and let Λ be a Young diagram of type $\le(2,n)$. Let d_i be the degree of $\mathcal{L}|_{C_i}$ and g_i be the genus of C_i. We have $d=d_1+d_2$ and $g=g_1+g_2$.

Lemma 3.4.3. *Let $(\mathcal{E},\beta:\mathcal{E}\xrightarrow{\sim}\mathcal{H}om(\mathcal{E},\mathcal{L}|_{C_1\cup C_2}),\mathbb{F}((s_{S_o})^*\mathcal{E}))$ be an object of $\mathcal{PSU}(2,\mathcal{L};\mathcal{C}/S;\Lambda)(S_o)$. Then either (i) or (ii) below holds.*
(i) $\mathcal{E}$ is locally free.
(ii) $\mathcal{E}_P\simeq\mathfrak{m}_P^{\oplus 2}$, where $\mathfrak{m}_P$ is the ideal sheaf of $\{P\}$ in $\mathcal{C}_o=C_1\cup C_2$.

Proof. We know that $\mathcal{E}_P\simeq\mathfrak{m}_P^{\oplus a}\oplus\mathcal{O}_P^{\oplus(2-a)}$ with $2\ge a\ge 0$. We have only to exclude the case $a=1$. Let f_i be the degree of $(\mathcal{E}|_{C_i})$/torsion. Then the degree of $\mathcal{H}om(\mathcal{E},\mathcal{L}|_{C_1\cup C_2})|_{C_i}$/torsion is $-f_i+2d_i-a$. By β, we have $f_i=-f_i+2d_i-a$. Hence a is even. □

Local Structure. We easily see that the morphism $SH^{lf(N)}\to S$ is smooth. By [Fal, §3], we know that if $x\in SH^{(N)}$ is S_o-valued point such that the corresponding torsion-free sheaf $\mathcal{E}$ belongs to the case (ii) in Lemma 3.4.3, then we have an isomorphism $\widehat{\mathcal{O}}_{SH^{(N)},x}\simeq k[[z_1,z_2,z_3,z_4,w_1,w_2,\dots]]/(z_1z_2-z_3z_4-t^{\upsilon})$. From these we derive the following lemma.

Lemma 3.4.4. *$SH^{(N)}\to S$ is (if not empty) flat, geometrically normal and geometrically irreducible.*

Proof. By the above result cited from [Fal, §3], $SH^{(N)} \to S$ is flat and its every geometric fiber is Cohen-Macaulay and regular in codimension one, thus it is normal. To prove geometric irreducibility, it suffices to prove that every geometric fiber of $SH^{lf(N)} \to S$ is irreducible. This can be proved exactly the same argument as the non-singular case. $\square$

3.5. Induction step

We retain the notation in the paragraph 3.4.2.

Theorem 3.5.1. *If $(SD)_{2,n}$ holds for $\mathcal{PSU}(2, \mathcal{L}|_{C_1}; C_1; \Lambda, \Gamma)$ and $\mathcal{PSU}(2, \mathcal{L}|_{C_2}; C_2; \Gamma)$ for any Young diagram Γ of type $\leq (2,n)$, then $(SD)_{2,n}$ holds for $\mathcal{PSU}(2, \mathcal{L}_{\bar{\eta}}; \mathcal{C}_{\bar{\eta}}, \Lambda)$. Here the Young diagram Λ is assigned to the point $s(S_o)$ on C_1, and Γ is assigned to the point P on C_1 and C_2.*

Let e be the rational number determined by

$$2e + nd + 2n(1-g) = |\Lambda|. \tag{3.9}$$

If e is not an integer Theorem 3.5.1 holds automatically. From now on, we assume that e is an integer. Put $\mathcal{PSU}_1 := \mathcal{PSU}(2, \mathcal{L}|_{C_1}; C_1; \Lambda)$ and $\mathcal{PSU}_2 := \mathcal{PSU}(2, \mathcal{L}|_{C_2}; C_2)$. Let $(\mathcal{E}_1^{univ}, \det \mathcal{E}_1^{univ} \simeq \mathcal{L}|_{C_1}, \mathbb{F}((s_{\mathcal{PSU}_1})^* \mathcal{E}_1^{univ}))$ and $(\mathcal{E}_2^{univ}, \det \mathcal{E}_2^{univ} \simeq \mathcal{L}|_{C_2})$ be the universal objects over $\mathcal{PSU}_1$ and $\mathcal{PSU}_2$ respectively. Let $\phi_i : \mathcal{PSU}_1 \times \mathcal{PSU}_2 \to \mathcal{PSU}_i$ be the i-th projection. Since we have the isomorphisms of sheaves on $\mathcal{PSU}_1 \times \mathcal{PSU}_2$

$$\begin{aligned} \phi_1^*(P)^*_{\mathcal{PSU}_1} \det \mathcal{E}_1^{univ} &\simeq \phi_1^*(P)^*_{\mathcal{PSU}_1} (\mathcal{L}|_{C_1})_{\mathcal{PSU}_1} \\ &\simeq \phi_2^*(P)^*_{\mathcal{PSU}_2} (\mathcal{L}|_{C_2})_{\mathcal{PSU}_2} \\ &\simeq \phi_2^*(P)^*_{\mathcal{PSU}_2} \det \mathcal{E}_2^{univ}, \end{aligned}$$

we can consider the S-stack $\overline{SL}(\phi_1^*(P)^*_{\mathcal{PSU}_1} \mathcal{E}_1^{univ}, \phi_2^*(P)^*_{\mathcal{PSU}_2} \mathcal{E}_2^{univ})$ over $\mathcal{PSU}_1 \times \mathcal{PSU}_2$. (See the subsection 4.2 for the definition of $\overline{SL}(*,*)$.)

3.5.2. Explicitly, if T is an affine S_o-scheme, an object of the groupoid $\overline{SL}(\phi_1^*(P)^*_{\mathcal{PSU}_1} \mathcal{E}_1^{univ}, \phi_2^*(P)^*_{\mathcal{PSU}_2} \mathcal{E}_2^{univ})(T)$ is following data (♡):

- 2-bundles $\mathcal{E}_i$ on $C_i \times T$ $(i = 1, 2)$,
- isomorphisms $\det \mathcal{E}_i \simeq (\mathcal{L}|_{C_i})_T$ $(i = 1, 2)$, (or equivalently, anti-symmetric isomorphisms $\mathcal{E}_i \simeq \mathcal{H}om(\mathcal{E}_i, (\mathcal{L}|_{C_i})_T)$),
- a filtration $\mathbb{F}((s)^*_T \mathcal{E}_1)$ of $(s)^*_T \mathcal{E}_1$ of type Λ,
- a 2-bundle quotient $\gamma : (P)^*_T \mathcal{E}_1 \oplus (P)^*_T \mathcal{E}_2 \to \mathcal{Q} \to 0$ such that the diagram

$$\begin{array}{ccccc} (P)^*_T(\mathcal{L}|_{C_1})_T & \simeq & \det(P)^*_T \mathcal{E}_1 & \xrightarrow{\wedge^2 \gamma_1} & \det \mathcal{Q} \\ \wr\! \mid & & & & \| \\ (P)^*_T(\mathcal{L}|_{C_2})_T & \simeq & \det(P)^*_T \mathcal{E}_2 & \xrightarrow{\wedge^2 \gamma_2} & \det \mathcal{Q} \end{array} \tag{3.10}$$

commutes, where $\gamma_i := \gamma|_{(P)^*_T \mathcal{E}_i}$.

3.5.3. Let ι_i be the inclusion morphism $\iota_i : C_i \to C_1 \cup C_2$. For given data (♡) as above, the composite of $(\iota_1)_{T*}\mathcal{E}_1 \oplus (\iota_2)_{T*}\mathcal{E}_2 \to (P)_{T*}(P)^*_T\mathcal{E}_1 \oplus (P)_{T*}(P)^*_T\mathcal{E}_2$ and $(P)_{T*}((P)^*_T\mathcal{E}_1 \oplus (P)^*_T\mathcal{E}_2) \to (P)_{T*}\mathcal{Q} \to 0$ gives rise to a surjective homomorphism $(\iota_1)_{T*}\mathcal{E}_1 \oplus (\iota_2)_{T*}\mathcal{E}_2 \to (P)_{T*}\mathcal{Q} \to 0$ on $(C_1 \cup C_2) \times T$. If $\mathcal{E}$ denotes its kernel, it is a flat family of torsion-free sheaves on $C_1 \cup C_2$ parameterized by T. Besides, by taking direct sum of the alternate bi-linear forms $\mathcal{E}_i \otimes \mathcal{E}_i \to (\mathcal{L}|_{C_i})_T$ $(i = 1, 2)$, we obtain the alternate bilinear form $((\iota_1)_{T*}\mathcal{E}_1 \oplus (\iota_2)_{T*}\mathcal{E}_2) \otimes ((\iota_1)_{T*}\mathcal{E}_1 \oplus (\iota_2)_{T*}\mathcal{E}_2) \to (\mathcal{L}|_{C_1})_T \oplus (\mathcal{L}|_{C_2})_T$.

Claim 3.5.3.1. There is a unique bilinear form $\mathcal{E} \otimes \mathcal{E} \to (\mathcal{L}|_{C_1 \cup C_2})_T$ such that the diagram

$$\begin{array}{ccc} \mathcal{E} \otimes \mathcal{E} & \to & (\mathcal{L}|_{C_1 \cup C_2})_T \\ \downarrow & & \downarrow \\ ((\iota_1)_{T*}\mathcal{E}_1 \oplus (\iota_2)_{T*}\mathcal{E}_2) \otimes ((\iota_1)_{T*}\mathcal{E}_1 \oplus (\iota_2)_{T*}\mathcal{E}_2) & \to & (\mathcal{L}|_{C_1})_T \oplus (\mathcal{L}|_{C_2})_T \end{array} \tag{3.11}$$

commutes.

Proof. Since we have an exact sequence

$$0 \to (\mathcal{L}|_{C_1 \cup C_2})_T \to (\mathcal{L}|_{C_1})_T \oplus (\mathcal{L}|_{C_2})_T \to ((P)_*(\mathcal{L} \otimes k(P)))_T \to 0,$$

it suffices to prove that $\mathcal{E} \otimes \mathcal{E} \to ((P)_*(\mathcal{L} \otimes k(P)))_T$ is a zero map. Let $(P)^*_T\mathcal{E}_i \otimes (P)^*_T\mathcal{E}_i \to ((P)_*(\mathcal{L} \otimes k(P)))_T$, the pull-back of $\mathcal{E}_i \otimes \mathcal{E}_i \to (P^*\mathcal{L})_T$ by $(P)_T$, be denoted by $\wedge$. Put $\mathcal{R} := \mathrm{Ker}(\gamma : (P)^*_T\mathcal{E}_1 \oplus (P)^*_T\mathcal{E}_2 \twoheadrightarrow \mathcal{Q})$. Let $g_i : \mathcal{R} \to (P)^*_T\mathcal{E}_i$ be the composite $\mathcal{R} \to (P)^*_T\mathcal{E}_1 \oplus (P)^*_T\mathcal{E}_2 \to (P)^*_T\mathcal{E}_i$. To show that $\mathcal{E} \otimes \mathcal{E} \to ((P)_*(\mathcal{L} \otimes k(P)))_T$ is a zero map, we only need to verify that $g_1(r) \wedge g_1(r') - g_2(r) \wedge g_2(r')$ is zero for $\forall r, r' \in \mathcal{R}$. This follows from Remark 4.2.2 (2). □

One can easily see that this alternate bi-linear form $\mathcal{E} \otimes \mathcal{E} \to (\mathcal{L}|_{C_1 \cup C_2})_T$ gives the isomorphism $\mathcal{E} \simeq \mathcal{H}om(\mathcal{E}, (\mathcal{L}|_{C_1 \cup C_2})_T)$. The filtration $\mathbb{F}((s)^*_T\mathcal{E}_1)$ induces a filtration of $(s)^*_T\mathcal{E}$ because $(s)^*_T\mathcal{E}_1 \simeq (s)^*_T\mathcal{E}$.

Summing up, from the data (♡), we obtained an object of $\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda)(T)$. For short, let us put $\overline{SL} := \overline{SL}(\phi_1(P)^*_{\mathcal{PSU}_1}\mathcal{E}_1^{univ}, \phi_2(P)^*_{\mathcal{PSU}_2}\mathcal{E}_2^{univ})$ and $SL := SL(\phi_1(P)^*_{\mathcal{PSU}_1}\mathcal{E}_1^{univ}, \phi_2(P)^*_{\mathcal{PSU}_2}\mathcal{E}_2^{univ})$. Then the above procedure induces the following commutative diagram.

$$\begin{array}{ccc} \overline{SL} & \xrightarrow{\bar{f}} & \mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda) \\ \cup & & \cup \\ SL & \xrightarrow{f} & \mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda)^{lf} \end{array} \tag{3.12}$$

Here f is an isomorphism.

Lemma 3.5.4. *If $\mathcal{M}$ is a line bundle on $\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda)$, then we have the following commutative diagram in which all homomorphisms are isomorphisms.*

$$\begin{array}{ccc} \mathrm{H}^0\left(\overline{SL}, \bar{f}^*\mathcal{M}\right) & \xleftarrow[\bar{f}^*]{\sim} & \mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda), \mathcal{M}\right) \\ \wr\downarrow (a) & & \wr\downarrow (b) \\ \mathrm{H}^0\left(SL, f^*\mathcal{M}\right) & \xleftarrow[f^*]{\sim} & \mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda)^{lf}, \mathcal{M}\right) \end{array} \tag{3.13}$$

Proof. The commutativity is clear. By Lemma 3.4.4 and Proposition 4.1.5, (b) is an isomorphism. Since f is an isomorphism, f^* is an isomorphism. Since (a) is injective, we know that (a) and $\bar{f}^*$ are isomorphisms. $\square$

Lemma 3.5.5. *Let N be a positive integer such that $2(N-2)\deg(\mathcal{O}(1)|_{C_i}) \geq 6g_i - 1 - d_i$ for $i = 1, 2$. Then codimension of $\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda) \setminus \mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda)^{(N)}$ in $\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda)$ is greater than or equal to 2.*

Proof. Since $\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2; \Lambda) \to \mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2)$ is a $Flag_\Lambda$-bundle, in order to prove the lemma we may ignore the filtration. Since the codimension of $\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2) \setminus \mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2)^{lf}$ in $\mathcal{PSU}(2, \mathcal{L}|_{C_1 \cup C_2}; C_1 \cup C_2)$ is greater than or equal to 2, we only need to prove that the codimension of $\mathcal{PSU}^{lf} \setminus \mathcal{PSU}^{lf(N)}$ in $\mathcal{PSU}^{lf}$ is greater than or equal to 2. Let $\mathcal{F}$ be an object of $\mathcal{PSU}^{lf} \setminus \mathcal{PSU}^{lf(N)}(\mathrm{Spec}k)$, then either $\mathrm{H}^1(C_1 \cup C_2, \mathcal{F} \otimes \mathcal{O}(N)) \neq 0$ or $\mathcal{F} \otimes \mathcal{O}(N)$ is not globally generated. In any case we have $\mathrm{H}^1(C_1 \cup C_2, \mathcal{F} \otimes \mathcal{O}(N-1)) \neq 0$. Put $\mathcal{F}_i := \mathcal{F}|_{C_i}$.

Claim. We have either $\mathrm{H}^1(C_1, \mathcal{F}_1 \otimes (\mathcal{O}(N-2)|_{C_1})) \neq 0$ or $\mathrm{H}^1(C_2, \mathcal{F}_2 \otimes (\mathcal{O}(N-2)|_{C_2})) \neq 0$.

Proof of Claim. Assume that both are zero. Then for $i = 1, 2$, $\mathrm{H}^1(C_i, \mathcal{F}_i \otimes (\mathcal{O}(N-1)|_{C_i})) = 0$ and $\mathcal{F}_i \otimes (\mathcal{O}(N-1)|_{C_i})$ is globally generated. By the long exact sequence of

$$0 \to \mathcal{F} \otimes \mathcal{O}(N-1) \to \bigoplus_{i=1}^{2} \mathcal{F}_i \otimes (\mathcal{O}(N-1)|_{C_i}) \to \mathcal{F} \otimes k(P) \to 0,$$

we have $\mathrm{H}^1(C_1 \cup C_2, \mathcal{F} \otimes \mathcal{O}(N-1)) = 0$. This is a contradiction. $\square$

For $j = 1, 2$, let $\mathcal{PSU}_j^\dagger$ be the open substack of $\mathcal{PSU}(2, \mathcal{L}|_{C_j}; C_j)$ such that $\mathcal{F} \in \mathcal{PSU}(2, \mathcal{L}|_{C_j}; C_j)$ is in $\mathcal{PSU}_j^\dagger$ if and only if $\mathrm{H}^1(C_j, \mathcal{F} \otimes (\mathcal{O}(N-2)|_{C_j})) = 0$. Then by the above claim, we have only to prove that $\mathrm{codim}(\mathcal{PSU}(2, \mathcal{L}|_{C_j}; C_j) \setminus \mathcal{PSU}_j^\dagger, \mathcal{PSU}(2, \mathcal{L}|_{C_j}; C_j)) \geq 2$. If $\mathcal{F} \in \mathcal{PSU}(2, \mathcal{L}|_{C_j}; C_j) \setminus \mathcal{PSU}_j^\dagger(\mathrm{Spec}k)$, we have a non-zero homomorphism $\alpha : \mathcal{F} \otimes (\mathcal{O}(N-2)|_{C_j}) \to K_{C_j}$ by Serre duality. Put $A := \mathrm{Im}\alpha$. Then $\mathrm{Ker}(\alpha) \simeq (\mathcal{L}|_{C_j}) \otimes A^\vee \otimes (\mathcal{O}(2N-4)|_{C_j})$. We have

$$\begin{aligned}
\deg \mathrm{Ker}(\alpha) - \deg A &= d_j + 2(N-2)\deg(\mathcal{O}(1)|_{C_j}) - 2\deg A \\
&\geq d_j + 2(N-2)\deg(\mathcal{O}(1)|_{C_j}) - 2(2g_j - 2) \\
&\geq 2g_j + 3 \\
&> 2g_j - 2.
\end{aligned}$$

Therefore we have $\mathrm{Ext}^1(A, \mathrm{Ker}(\alpha)) = 0$. Hence we have $\mathcal{F} \otimes (\mathcal{O}(N-2)|_{C_j}) \simeq A \oplus \mathrm{Ker}(\alpha)$. We have

$$\begin{aligned}
\dim \mathrm{Aut}(A \oplus \mathrm{Ker}(\alpha)) &= 2 + h^0(\mathrm{Ker}(\alpha) \otimes A^\vee) \\
&\geq 2 + 2g_j + 3 + 1 - g_j \\
&= g_j + 6.
\end{aligned}$$

Thus if $\mathrm{Aut}'(A \oplus \mathrm{Ker}(\alpha)) := \{f \in \mathrm{Aut}(A \oplus \mathrm{Ker}(\alpha)) | \det(f) = \mathrm{id}\}$, then $\dim \mathrm{Aut}'(A \oplus \mathrm{Ker}(\alpha)) \geq g_j + 5$. For $N' \geq N$, let $\tau : SH_j^{(N')} \to \mathcal{PSU}(2; \mathcal{L}|_{C_j}; C_j)^{(N')}$ be the atlas described in the paragraph 2.1.7. We have $\dim SH_j^{(N')} = h(N')^2 + 3g_j - 3$, where $h(N') := d_j + 2(N' \deg(\mathcal{O}(1)|_{C_j}) + 1 - g_j)$. We have

$$\begin{aligned}\dim(SH_j^{(N')} \setminus \tau^{-1}(\mathcal{PSU}_j^{\dagger})) &\leq \dim \mathrm{Pic}(C_j) + h(N')^2 - \dim \mathrm{Aut}'(A \oplus \mathrm{Ker}(\alpha)) \\ &\leq g_j + h(N')^2 - (g_j + 5) = h(N')^2 - 5 \\ &\leq h(N')^2 + 3g_j - 5 \leq \dim SH_j^{(N')} - 2.\end{aligned}$$

This completes the proof. □

Corollary 3.5.6. *Ler R' be a flat $k[[t]]$-algebra and put $T := \mathrm{Spec}R'$. If $\mathcal{M}$ is a line bundle on $\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda)$, then the natural homomorphism*

$$\mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda), \mathcal{M}\right) \otimes_{k[[t]]} R' \to \mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}_T; \mathcal{C}_T/T; \Lambda), \mathcal{M}_T\right)$$

is an isomorphism.

Proof. By Lemma 3.5.5 and Proposition 4.1.5, for a large N, we have the isomorphisms

$$\mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda), \mathcal{M}\right) \xrightarrow{\sim} \mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda)^{(N)}, \mathcal{M}\right)$$

and

$$\mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}_T; \mathcal{C}_T/T; \Lambda), \mathcal{M}_T\right) \xrightarrow{\sim} \mathrm{H}^0\left(\mathcal{PSU}(2, \mathcal{L}_T; \mathcal{C}_T/T; \Lambda)^{(N)}, \mathcal{M}_T\right).$$

Now the corollary follows from Proposition 4.1.1. □

Let $\zeta : \overline{SL}(\phi_1^*(P)^*_{\mathcal{PSU}_1}\mathcal{E}_1^{univ}, \phi_2^*(P)^*_{\mathcal{PSU}_2}\mathcal{E}_2^{univ}) \to \mathcal{PSU}_1 \times \mathcal{PSU}_2$ be the projection.

Lemma 3.5.7. *We have an isomorphism*

$$\bar{f}^* \Xi^{(n)} \simeq \zeta^*\left(\phi_1^* \Xi^{(n)}_{\mathcal{PSU}_1} \otimes \phi_2^* \Xi^{(n)}_{\mathcal{PSU}_2}\right) \otimes \mathcal{O}_{\overline{SL}}(n\mathbb{B}), \tag{3.14}$$

where $\mathbb{B} = \mathbb{B}(\phi_1^*(P)^*_{\mathcal{PSU}_1}\mathcal{E}_1^{univ}, \phi_2^*(P)^*_{\mathcal{PSU}_2}\mathcal{E}_2^{univ})$, which is a divisor of $\overline{SL}(\phi_1^*(P)^*_{\mathcal{PSU}_1}\mathcal{E}_1^{univ}, \phi_2^*(P)^*_{\mathcal{PSU}_2}\mathcal{E}_2^{univ})$. (See the subsection 4.2 for the definition of $\mathbb{B}(*, *)$.)

Proof. Let us be given the data (♡) in the paragraph 3.5.2, and let e be as in the equation (3.9). Then we have

$$\begin{aligned}(\det \mathbb{R}pr_{T*}\mathcal{E})^{\otimes(-n)} &\otimes \bigotimes_{j=0}^{l(\Lambda)} \det\left(F_j((s)^*_T\mathcal{E})/F_{j+1}((s)^*_T\mathcal{E})\right)^{\otimes\gamma_j(\Lambda)} \\ &\simeq (\det \mathbb{R}pr_{T*}\mathcal{E}_1)^{\otimes(-n)} \otimes (\det \mathbb{R}pr_{T*}\mathcal{E}_2)^{\otimes(-n)} \otimes \mathcal{Q}^{\otimes n} \\ &\otimes \bigotimes_{j=0}^{l(\Lambda)} \det\big(F_j((s)^*_T\mathcal{E}_1)/F_{j+1}((s)^*_T\mathcal{E}_1)\big)^{\otimes\gamma_j(\Lambda)}.\end{aligned}$$

This proves the lemma. $\square$

Let e be as in the equation (3.9) (and we are assuming that e is an integer). Let $\mathcal{G}$ be an S-flat coherent sheaf on $\mathcal{C}$ that is torsion-free of rank n and $\chi = e + n(1-g)$ on every fiber. Let $\mathbb{F}(s^*\mathcal{G}) = \{s^*\mathcal{G} \supseteq F_1(s^*\mathcal{G}) \supset \cdots \supset F_{l(\Lambda)}(s^*\mathcal{G})\}$ be a filtration of type $\widetilde{\Lambda}$. Put $\underline{\mathcal{G}} := (\mathcal{G}, \mathbb{F}(s^*\mathcal{G}))$. Then as in the paragraph 2.1.15, we have the morphism of S-stacks

$$\varphi_{\underline{\mathcal{G}}} : \mathcal{PSU}(2, \mathcal{L}; \mathcal{C}/S; \Lambda)^{lf} \to \mathcal{U}(2n, 2n(g-1); \mathcal{C}/S).$$

For a positive integer l, put $S^{(1/l)} := \mathrm{Spec}k[[t^{1/l}]]$ and $\mathcal{C}^{(1/l)} := \mathcal{C} \times_S S^{(1/l)}$. By the base-change, we have

$$(\varphi_{\underline{\mathcal{G}}})_{S^{(1/l)}} : \mathcal{PSU}(2, \mathcal{L}; \mathcal{C}^{(1/l)}/S^{(1/l)}; \Lambda)^{lf} \to \mathcal{U}(2n, 2n(g-1); \mathcal{C}^{(1/l)}/S^{(1/l)}).$$

Lemma 3.5.8. *There exists a positive integer l such that we have an isomorphism $(\varphi_{\underline{\mathcal{G}}})^*_{S^{(1/l)}}\Theta_{\mathcal{U}} \simeq \Xi^{(n)}_{\mathcal{PSU}}$.*

Proof. Let t_1, t_2 be sections $S \to \mathcal{C}$ such that $t_i(S) \subset smooth(\pi)$, $t_1(S) \cap t_2(S) = \emptyset$ and $t_i \cap C_i \neq \emptyset$. $\mathcal{O}_{\mathcal{C}}(t_1 + t_2)$ is a π-ample line bundle. One can find a positive integer b and a short exact sequence of coherent $\mathcal{O}_{\mathcal{C}}$-modules

$$0 \to \mathcal{G} \to \mathcal{O}_{\mathcal{C}}(b(t_1+t_2))^{\oplus n} \to \mathcal{T} \to 0 \tag{3.15}$$

such that $\mathrm{Supp}(\mathcal{T}) \to S$ is a finite morphism. Since $\mathcal{G}|_{C_1 \cup C_2}$ is torsion-free, $\mathcal{T}$ is S-flat. Then we can find a positive integer l such that $\mathcal{T}_{S^{(1/l)}}$, the pull-back of $\mathcal{T}$ to $\mathcal{C}^{(1/l)}$, has a filtration $\mathcal{T}_{S^{(1/l)}} = \mathcal{T}_{S^{(1/l)},0} \supset \mathcal{T}_{S^{(1/l)},1} \supset \cdots \supset \mathcal{T}_{S^{(1/l)},\xi} = 0$ with the property that for $1 \le j \le \xi$, $\mathcal{T}_{S^{(1/l)},j-1}/\mathcal{T}_{S^{(1/l)},j}$ is a vector bundle on $u_j(S^{(1/l)})$, where $u_j : S^{(1/l)} \to \mathcal{C}^{(1/l)}$ is a section of $\mathcal{C}^{(1/l)} \to S^{(1/l)}$. Let us fix isomorphisms

$$\begin{aligned} &\det\left(F_j(s^*\mathcal{G})/F_{j+1}(s^*\mathcal{G})\right) \simeq \mathcal{O}_S, t_j^*\mathcal{L} \simeq \mathcal{O}_S, u_j^*\mathcal{L}_{S^{(1/l)}} \simeq \mathcal{O}_{S^{(1/l)}}, \\ &t_j^*\mathcal{O}_{\mathcal{C}}(ct_j) \simeq \mathcal{O}_S, \det\left(\mathcal{T}_{S^{(1/l)},j-1}/\mathcal{T}_{S^{(1/l)},j}\right) \simeq \mathcal{O}_{S^{(1/l)}}. \end{aligned} \tag{3.16}$$

Let $(\mathcal{B}, \det\mathcal{B} \simeq \mathcal{L}_{S^{(1/l)}}, \mathbb{F}((s)^*_T\mathcal{B}))$ be an object of $\mathcal{PSU}(2, \mathcal{L}_{S^{(1/l)}}; \mathcal{C}^{(1/l)}/S^{(1/l)}/S^{(1/l)}; \Lambda)^{lf}(T)$, T being an affine $S^{(1/l)}$-scheme. For this object, by the construction of $\varphi_{\underline{\mathcal{G}}}$ explained in the paragraph 2.1.15 and Lemma 2.1.5, $(\varphi_{\underline{\mathcal{G}}})^*_{S^{(1/l)}}\Theta_{\mathcal{U}}$ corresponds to a line bundle on T

$$\begin{aligned} \det\left(\mathbb{R}pr_{T*}\mathcal{B} \otimes \mathcal{G}_T\right)^{\vee} \otimes \left(\bigotimes_{j=0}^{l(\Lambda)} \det\left(F_j(s^*\mathcal{G})/F_{j+1}(s^*\mathcal{G})\right)^{\otimes \gamma_j(\Lambda)} \otimes \mathcal{O}_T \right) \\ \otimes \bigotimes_{j=0}^{l(\Lambda)} \det\left(F_j(s^*\mathcal{B})/F_{j+1}(s^*\mathcal{B})\right)^{\otimes \gamma_j(\widetilde{\Lambda})}. \end{aligned}$$

Using the above short exact sequence (3.15), the filtrations of $\mathcal{T}_{S^{(1/l)}}$ and the isomorphisms (3.16), we have

$$\begin{aligned}\det \mathbb{R}pr_{T*}(\mathcal{B}\otimes\mathcal{G}_T) &\simeq \det \mathbb{R}pr_{T*}(\mathcal{B}\otimes\mathcal{O}_C(b(t_1+t_2))_T)^{\otimes n}\otimes\det \mathbb{R}pr_{T*}(\mathcal{B}\otimes\mathcal{T}_T)\\ &\simeq (\det \mathbb{R}pr_{T*}\mathcal{B})^{\otimes n}.\end{aligned}$$

This proves the lemma. $\square$

Let ε denote the closed immersion $\mathbb{B}(\phi_1^*(P)_{\mathcal{PSU}_1}\mathcal{E}_1^{univ}, \phi_2^*(P)_{\mathcal{PSU}_2}\mathcal{E}_2^{univ}) \hookrightarrow \overline{SL}$, and we put $\mathbb{B} := \mathbb{B}(\phi_1^*(P)_{\mathcal{PSU}_1}\mathcal{E}_1^{univ}, \phi_2^*(P)_{\mathcal{PSU}_2}\mathcal{E}_2^{univ})$. (See the subsection 4.2 for the definition of $\mathbb{B}(*,*)$.) Put $V_j := \mathrm{H}^0(\overline{SL}, \zeta^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2})\otimes \mathcal{O}_{\overline{SL}}(j\mathbb{B}))$. We have the exact sequence of k-vector spaces

$$0\to V_{j-1}\to V_j\to \mathrm{H}^0\left(\mathbb{B}, (\zeta|_{\mathbb{B}})^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2})\otimes\varepsilon^*\mathcal{O}_{\overline{SL}}(j\mathbb{B})\right), \tag{3.17}$$

and let $\overline{V}_j \subset \mathrm{H}^0\left(\mathbb{B}, (\zeta|_{\mathbb{B}})^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2})\otimes\varepsilon^*\mathcal{O}_{\overline{SL}}(j\mathbb{B})\right)$ be the image of V_j. If the data ($\heartsuit$) in the paragraph 3.5.2 is a T-valued point of $\mathbb{B}$, then $\mathcal{Q} = \mathcal{Q}_1\oplus\mathcal{Q}_2$ with $\mathcal{Q}_i$ a line bundle and γ is the direct sum of $(P)_T^*\mathcal{E}_1\to\mathcal{Q}_1\to 0$ $(P)_T^*\mathcal{E}_2\to\mathcal{Q}_2\to 0$. (See Proposition 4.2.3.) This gives a T-valued point of $\mathcal{PSU}(2,\mathcal{L}|_{C_1};C_1;\Lambda,A_{n,n-j})\times\mathcal{PSU}(2,\mathcal{L}|_{C_2};C_2;A_{j,0})$. Therefore for $1\le j\le n$, $\mathbb{B}\simeq\mathcal{PSU}(2,\mathcal{L}|_{C_1};C_1;\Lambda,A_{n,n-j})\times\mathcal{PSU}(2,\mathcal{L}|_{C_2};C_2;A_{j,0})$, and we have an isomorphism

$$\begin{aligned}&(\zeta|_{\mathbb{B}})^*\left(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2}\right)\otimes\varepsilon^*\mathcal{O}_{\overline{SL}}(j\mathbb{B})\\ &\qquad\simeq \Xi^{(n)}_{\mathcal{PSU}(2,\mathcal{L}|_{C_1};C_1;\Lambda,A_{n,n-j})}\boxtimes\Xi^{(n)}_{\mathcal{PSU}(2,\mathcal{L}|_{C_2};C_2;A_{j,0})}.\end{aligned}$$

For $j=0$, we have

$$\mathrm{H}^0\left(\mathbb{B}, (\zeta|_{\mathbb{B}})^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2})\right)\simeq\mathrm{H}^0\left(\mathcal{PSU}_1\times\mathcal{PSU}_2, \Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2},\right)$$

and we have isomorphisms

$$\begin{aligned}&\mathcal{PSU}_1\times\mathcal{PSU}_2\simeq\mathcal{PSU}(2,\mathcal{L}|_{C_1};C_1;\Lambda,A_{n,n})\times\mathcal{PSU}(2,\mathcal{L}|_{C_2};C_2,A_{0,0})\\ &\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2}\simeq\Xi^{(n)}_{\mathcal{PSU}(2,\mathcal{L}|_{C_1};C_1;\Lambda,A_{n,n})}\boxtimes\Xi^{(n)}_{\mathcal{PSU}(2,\mathcal{L}|_{C_2};C_2;A_{0,0})}.\end{aligned}$$

Put $\mathcal{PSU}_1^\sharp := \mathcal{PSU}(2,\mathcal{L}|_{C_1};C_1;\Lambda,A_{n,n-j})$ and $\mathcal{PSU}_2^\sharp := \mathcal{PSU}(2,\mathcal{L}|_{C_2};C_2;A_{j,0})$. Then in any case for $0\le j\le n$, we have an isomorphism of k-vector spaces

(3.18)
$$\mathrm{H}^0(\mathbb{B},(\zeta|_{\mathbb{B}})^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2})\otimes\varepsilon^*\mathcal{O}_{\overline{SL}}(j\mathbb{B}))\simeq\mathrm{H}^0(\mathcal{PSU}_1^\sharp\times\mathcal{PSU}_2^\sharp,\Xi^{(n)}_{\mathcal{PSU}_1^\sharp}\boxtimes\Xi^{(n)}_{\mathcal{PSU}_2^\sharp}),$$

which is determined up to $k^\times$.

Let $e_{1,j}$ and $e_{2,j}$ be the rational numbers satisfying

$$\begin{aligned} 2e_{1,j} + nd_1 + 2n(1-g_1) &= |\Lambda| + 2n - j \\ 2e_{2,j} + nd_2 + 2n(1-g_2) &= j. \end{aligned} \tag{3.19}$$

Note that $e_{1,j}$ is an integer if and only if so is $e_{2,j}$, for $e = e_{1,j} + e_{2,j}$ and we are assuming that e is an integer. By Proposition 4.1.8, the isomorphism (3.18) and Remark 2.1.12, we have

$$\mathrm{H}^0\left(\mathbb{B}, (\zeta|_{\mathbb{B}})^*(\varXi^{(n)}_{\mathcal{PSU}_1} \boxtimes \varXi^{(n)}_{\mathcal{PSU}_2}) \otimes \varepsilon^* \mathcal{O}_{\overline{SL}}(j\mathbb{B})\right) = 0$$

if $e_{1,j}$ and $e_{2,j}$ are not integers. Therefore we have

(3.20)

$$\begin{aligned} &\dim \mathrm{H}^0\left(\mathcal{PSU}(2, C_1 \cup C_2; \mathcal{L}|_{C_1 \cup C_2}; \Lambda), \varXi^{(n)}_{\mathcal{PSU}}\right) \\ &\quad = \dim V_n = \sum_{j=0}^{n} \dim \overline{V}_j = \sum_{0 \le j \le n,\ e_{1,j}:\text{integer}} \dim \overline{V}_j \\ &\quad \le \sum_{0 \le j \le n,\ e_{1,j}:\text{integer}} \dim \mathrm{H}^0\left(\mathbb{B}, (\zeta|_{\mathbb{B}})^*(\varXi^{(n)}_{\mathcal{PSU}_1} \boxtimes \varXi^{(n)}_{\mathcal{PSU}_2}) \otimes \varepsilon^* \mathcal{O}_{\overline{SL}}(j\mathbb{B})\right). \end{aligned}$$

Now fix j so that $e_{1,j}$ and $e_{2,j}$ are integers. Let $\mathcal{G}_i$ be a vector bundle on C_i of rank n and of degree $e_{i,j}$ and let $\mathbb{F}((s)^*_{S_o}\mathcal{G}_1)$ be a filtration of $(s)^*_{S_0}\mathcal{G}_1$ of type $\widetilde{\Lambda}$. Let $q : \mathcal{G}_1|_P \oplus \mathcal{G}_2|_P \to U$ be an n-dimensional quotient of k-vector spaces such that $q|_{(\mathcal{G}_1|_P)}$ is an isomorphism and $q|_{(\mathcal{G}_2|_P)}$ is of rank j. Put $\underline{\mathcal{G}_1} := (\mathcal{G}_1, \mathbb{F}((s)^*_{S_o}\mathcal{G}_1))$. For the data $(\underline{\mathcal{G}_1}, \mathcal{G}_2, q)$, we shall define the morphism of stacks $\psi_{(\underline{\mathcal{G}_1}, \mathcal{G}_2, q)} : \overline{SL} \to \mathcal{U}(2n, 2n(g-1); C_1 \cup C_2)$ as follows. Let T be an affine k-scheme and let us be given the data (♡) in the paragraph 3.5.2. Let $\rho : \overline{SL}\left((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2\right) \to T$ be the projection, and $0 \to \mathcal{V}^{univ} \to \rho^*\left((P)^*_T\mathcal{E}_1 \oplus (P)^*_T\mathcal{E}_2\right) \xrightarrow{\gamma^{univ}} \mathcal{Q}^{univ} \to 0$ be the universal quotient. Let ε' denote the closed immersion $\mathbb{B}\left((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2\right) \hookrightarrow \overline{SL}\left((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2\right)$. Put $\mathbb{B}' := \mathbb{B}\left((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2\right)$. For $i = 1, 2$ we can find a line subbundle $\mathcal{M}_i$ of $(\rho|_{\mathbb{B}'})^*((P)^*_T\mathcal{E}_i)$ and an isomorphism of the short exact sequences

$$\begin{array}{ccccccccc} 0 \to & \varepsilon'^*\mathcal{V}^{univ} & \to & \varepsilon'^*\rho^*(\bigoplus_{i=1}^2 (P)^*_T\mathcal{E}_i) & \to & \varepsilon'\mathcal{Q}^{univ} & \to 0 \\ & \wr| & & \| & & \wr| & \\ 0 \to & \mathcal{M}_1 \oplus \mathcal{M}_2 & \to & (\rho|_{\mathbb{B}'})^*(\bigoplus_{i=1}^2 (P)^*_T\mathcal{E}_i) & \to & \bigoplus_{i=1}^2 \frac{(\rho|_{\mathbb{B}'})^*(P)^*_T\mathcal{E}_i}{\mathcal{M}_i} & \to 0. \end{array} \tag{3.21}$$

Let $\alpha' : \rho^*\left(((P)^*_T\mathcal{E}_1 \otimes_k (\mathcal{G}_1|_P)) \oplus ((P)^*_T\mathcal{E}_2 \otimes_k (\mathcal{G}_2|_P))\right) \to \mathcal{Q}^{univ} \otimes_k U$ be given by $(x_1 \otimes y_1, x_2 \otimes y_2) \mapsto \gamma^{univ}(x_1) \otimes q(y_1) - \gamma^{univ}(x_2) \otimes q(y_2)$. Let $\mathcal{K}$ be the kernel of the composite of morphisms

$$\begin{aligned} &\mathcal{Q}^{univ} \otimes_k U \twoheadrightarrow (\mathcal{Q}|_{\mathbb{B}'}) \otimes_k U \simeq (\oplus_{i=1}^2((\rho|_{\mathbb{B}'})^*(P)^*_T\mathcal{E}_i/\mathcal{M}_i)) \otimes_k U \\ &\to ((\rho|_{\mathbb{B}'})^*(P)^*_T\mathcal{E}_2/\mathcal{M}_2) \otimes_k U \twoheadrightarrow ((\rho|_{\mathbb{B}'})^*(P)^*_T\mathcal{E}_2/\mathcal{M}_2) \otimes (U/q(\mathcal{G}_2|_P)). \end{aligned} \tag{3.22}$$

Lemma 3.5.9. *(1) $\mathcal{K}$ is a vector bundle of rank $2n$ on $\overline{SL}((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2)$.*

(2) α' factors as

$$\rho^*\Big(((P)^*_T\mathcal{E}_1 \otimes_k (\mathcal{G}_1|_P)) \oplus ((P)^*_T\mathcal{E}_2 \otimes_k (\mathcal{G}_2|_P))\Big) \xrightarrow{\beta'} \mathcal{K} \subset \mathcal{Q}^{univ} \otimes_k U,$$

and β' is surjective.

(3) The subsheaf $\varepsilon'^*\mathrm{Ker}(\beta') \subset \bigoplus_{i=1}^2((\rho|_{\mathbb{B}'})^*(P)^*_T\mathcal{E}_i \otimes_k (\mathcal{G}_i|_P))$ is $\mathcal{M}_1 \otimes (q|_{(\mathcal{G}_1|_P)})^{-1}(\mathrm{Im}(q|_{(\mathcal{G}_2|_P)})) \oplus \big(\mathcal{M}_2 \otimes_k (\mathcal{G}_2|_P) + (\rho|_{\mathbb{B}'})^*(P)^*_T\mathcal{E}_2 \otimes_k \mathrm{Ker}(q|_{(\mathcal{G}_2|_P)})\big)$.

(4) The subsheaf $\bigwedge^{2n}\mathcal{K} \subset \bigwedge^{2n}(\mathcal{Q}^{univ} \otimes_k U) \simeq (\bigwedge^2 \mathcal{Q}^{univ})^{\otimes n} \otimes (\bigwedge^n U)^{\otimes 2}$ is the image of the injective homomorphism $(\bigwedge^2 \mathcal{Q}^{univ})^{\otimes j} \otimes (\bigwedge^n U)^{\otimes 2} \otimes (\mathcal{L}|_P)^{\otimes(n-j)} \hookrightarrow (\bigwedge^2 \mathcal{Q}^{univ})^{\otimes n} \otimes (\bigwedge^n U)^{\otimes 2}$ given the the $(n-j)$-th tensor of the canonical section of $(\bigwedge^2 \mathcal{Q}^{univ}) \otimes (\mathcal{L}|_P)^\vee$.

Proof. We fix an isomorphism $\mathcal{L}|_P \simeq k$. We choose bases $\{\mathbf{g}_{1,1}, \dots \mathbf{g}_{1,n}\}$, $\{\mathbf{g}_{2,1}, \dots \mathbf{g}_{2,n}\}$, and $\{\mathbf{u}_1, \dots \mathbf{u}_n\}$ of the k-vector spaces $\mathcal{G}_1|_P$, $\mathcal{G}_2|_P$ and U respectively so that $q(\mathbf{g}_{1,i}) = \mathbf{u}_i$ $(1 \le i \le n)$, $q(\mathbf{g}_{2,i}) = \mathbf{u}_i$ $(1 \le i \le j)$ and $q(\mathbf{g}_{2,i}) = 0$ $(j < i)$.

We can check this lemma locally on T, so making T small if necessary, we may assume that we have an isomorphism $(P)^*_T\mathcal{E}_i \simeq \mathcal{O}_T\mathbf{e}_{i,1} \oplus \mathcal{O}_T\mathbf{e}_{i,2}$ so that $\det(P)^*_T\mathcal{E}_i \xrightarrow{\sim} (P)^*_T(\mathcal{L}|_{C_i})_T = \mathcal{O}_T$, the pull-back by $(P)_T$ of the isomorphism in the data $(\heartsuit)$, is given by $\mathbf{e}_{i,1} \wedge \mathbf{e}_{i,2} \mapsto 1$. Let $T = \mathrm{Spec}R$.

$\overline{SL}((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2)$ is covered by five affine open subschemes $SL((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2)$ and W_{λ_1,λ_2} $(\lambda_i = 1, 2)$, where W_{λ_1,λ_2} is the open subscheme of $\overline{SL}((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2)$ defined by $\gamma^{univ}(\rho^*\mathbf{e}_{i,\lambda_i}) \ne 0 (i = 1, 2)$. (1), (2) and (4) are clear over $SL((P)^*_T\mathcal{E}_1, (P)^*_T\mathcal{E}_2)$. Now we shall check them over $W_{1,2}$. For other W_{λ_1,λ_2}, the proofs are quite similar. We can find an isomorphism $W_{1,2} \simeq \mathrm{Spec}R[z_1, z_2, z_3]$ so that over $W_{1,2}$, we have an isomorphism $\mathcal{Q}^{univ}|_{W_{1,2}} \simeq \mathcal{O}\mathbf{q}_1 \oplus \mathcal{O}\mathbf{q}_2$ such that $\gamma^{univ}|_{\rho^*(P)^*_T\mathcal{E}_1}$ and $\gamma^{univ}|_{\rho^*(P)^*_T\mathcal{E}_2}$ are given respectively by the matrices

$$\begin{pmatrix} 1 & z_1 \\ 0 & z_2 \end{pmatrix} \text{ and } \begin{pmatrix} z_2 & 0 \\ z_3 & 1 \end{pmatrix}.$$

$\mathbb{B}'$ is defined by $z_2 = 0$. The composite of morphisms (3.22) is the projection

$$\left(\bigoplus_{i=1}^n \mathcal{O}_{W_{1,2}}\mathbf{q}_1 \otimes \mathbf{u}_i\right) \oplus \left(\bigoplus_{i=1}^n \mathcal{O}_{W_{1,2}}\mathbf{q}_2 \otimes \mathbf{u}_i\right) \to \bigoplus_{i=j+1}^n (\mathcal{O}_{W_{1,2}}/(z_2))\mathbf{q}_2 \otimes \mathbf{u}_i,$$

so $\mathcal{K}$ is the subsheaf

$$\left(\bigoplus_{i=1}^n \mathcal{O}_{W_{1,2}}\mathbf{q}_1 \otimes \mathbf{u}_i\right) \oplus \left(\bigoplus_{i=1}^j \mathcal{O}_{W_{1,2}}\mathbf{q}_2 \otimes \mathbf{u}_i \oplus \bigoplus_{i=j+1}^n \mathcal{O}_{W_{1,2}} z_2\mathbf{q}_2 \otimes \mathbf{u}_i\right).$$

This proves (1) and (4).

Using the above bases, over $W_{1,2}$, the morphism α' is expressed as

$$\begin{aligned}
&\mathbf{e}_{1,1}\otimes\mathbf{g}_{1,i}\mapsto\mathbf{q}_1\otimes\mathbf{u}_i,\\
&\mathbf{e}_{1,2}\otimes\mathbf{g}_{1,i}\mapsto(z_1\mathbf{q}_1+z_2\mathbf{q}_2)\otimes\mathbf{u}_i,\\
&\mathbf{e}_{2,1}\otimes\mathbf{g}_{2,i}\mapsto-(z_2\mathbf{q}_1+z_3\mathbf{q}_2)\otimes\mathbf{u}_i\quad\text{if } 1\le i\le j,\\
&\mathbf{e}_{2,2}\otimes\mathbf{g}_{2,i}\mapsto-\mathbf{q}_2\otimes\mathbf{u}_i\quad\text{if } 1\le i\le j,\\
&\mathbf{e}_{2,*}\otimes\mathbf{g}_{2,i}\mapsto 0\quad\text{if } j<i.
\end{aligned}$$

From this, we know that $\mathrm{Im}\alpha'=\mathcal{K}$. This proves (2).

Now let us proves (3). Note that $\mathbb{B}'$ is covered by four affine open subschemes $W_{\lambda_1,\lambda_2}\cap\mathbb{B}'$. As above, we check (3) only for $W_{1,2}$. The restriction of the sections $\mathbf{q}_1\otimes\mathbf{u}_i$ $(1\le i\le n)$, $\mathbf{q}_2\otimes\mathbf{u}_i$ $(1\le i\le j)$, $z_2\mathbf{q}_2\otimes\mathbf{u}_i$ $(j<i\le n)$ of $\mathcal{K}$ to $\mathbb{B}'$ forms a basis of $\mathcal{K}|_{\mathcal{B}'}$. Using this basis, over $W_{1,2}\cap\mathbb{B}'=\mathrm{Spec}R[z_1,z_3]$, the surjection $\beta'|_{\mathbb{B}'}$ is expressed as

$$\begin{aligned}
&\mathbf{e}_{1,1}\otimes\mathbf{g}_{1,i}\mapsto\mathbf{q}_1\otimes\mathbf{u}_i,\\
&\mathbf{e}_{1,2}\otimes\mathbf{g}_{1,i}\mapsto z_1\mathbf{q}_1\otimes\mathbf{u}_i\quad\text{if } 1\le i\le j,\\
&\mathbf{e}_{1,2}\otimes\mathbf{g}_{1,i}\mapsto z_1\mathbf{q}_1\otimes\mathbf{u}_i+(z_2\mathbf{q}_2\otimes\mathbf{u}_i)\quad\text{if } j<i,\\
&\mathbf{e}_{2,1}\otimes\mathbf{g}_{2,i}\mapsto-z_3\mathbf{q}_2\otimes\mathbf{u}_i\quad\text{if } 1\le i\le j,\\
&\mathbf{e}_{2,2}\otimes\mathbf{g}_{2,i}\mapsto-\mathbf{q}_2\otimes\mathbf{u}_i\quad\text{if } 1\le i\le j,\\
&\mathbf{e}_{2,*}\otimes\mathbf{g}_{2,i}\mapsto 0\quad\text{if } j<i.
\end{aligned}$$

From this, we know that the subsheaf $\mathrm{Ker}(\beta'|_{\mathbb{B}'})(=\varepsilon'^*\mathrm{Ker}(\beta'))$ of $\oplus_{i=1}^2((\rho|_{\mathbb{B}'})^*(P)_T^*\mathcal{E}_i\otimes_k(\mathcal{G}_i|_P))$ is a direct sum of the subsheaf generated by $(z_1\mathbf{e}_{1,1}-\mathbf{e}_{1,2})\otimes\mathbf{g}_{1,i}$ $(1\le i\le j)$ and the subsheaf generated by $(\mathbf{e}_{2,1}-z_3\mathbf{e}_{2,2})\otimes\mathbf{g}_{2,i}$ $(1\le i\le j)$ and $\mathbf{e}_{2,*}\otimes\mathbf{g}_{2,i}$ $(j<i)$. Since $\mathcal{M}_1=\mathcal{O}\cdot(z_1\mathbf{e}_{1,1}-\mathbf{e}_{1,2})$ and $\mathcal{M}_2=\mathcal{O}\cdot(\mathbf{e}_{2,1}-z_3\mathbf{e}_{2,2})$, (3) is proved. □

$\gamma:(P)_T^*\mathcal{E}_1\oplus(P)_T^*\mathcal{E}_2\twoheadrightarrow\mathcal{Q}$ in the data (♡) corresponds to a section $\sigma: T\to\overline{SL}((P)_T^*\mathcal{E}_1,(P)_T^*\mathcal{E}_2)$. $\psi_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}$ is defined to associate to the data (♡) the kernel of the composite of morphisms

$$\begin{aligned}
&\iota_{1*}\big(\mathcal{E}_1\otimes(\mathcal{G}_1)_T\big)\oplus\iota_{2*}\big(\mathcal{E}_2\otimes(\mathcal{G}_2)_T\big)\\
&\twoheadrightarrow(s)_{T*}\big((s)_T^*\mathcal{E}_1\otimes_k(\mathcal{G}_1|_P)\big)\oplus(P)_{T*}\big((P)_T^*\mathcal{E}_1\otimes_k(\mathcal{G}_1|_P)\oplus(P)_T^*\mathcal{E}_2\otimes_k(\mathcal{G}_2|_P)\big)\\
&\xrightarrow{(s)_{T*}(\mu_\Lambda)\oplus(P)_{T*}(\sigma^*(\beta'))}(s)_{T*}\mathcal{Q}\left(\mathbb{F}((s)_T^*\mathcal{E}_1),\mathbb{F}((\mathcal{G}_1|_P)_T)\right)\oplus(P)_{T*}(\sigma^*\mathcal{K}),
\end{aligned}$$

where $\mu_\Lambda:(s)_T^*\mathcal{E}_1\otimes_k(\mathcal{G}_1|_P)\to\mathcal{Q}\left(\mathbb{F}((s)_T^*\mathcal{E}_1),\mathbb{F}((\mathcal{G}_1|_P)_T)\right)$ is the surjective morphism defined in Definition 2.1.4.

3.5.10. Let $\mathcal{G}$ be the kernel of the composite of morphisms

$$\iota_{1*}\mathcal{G}_1\oplus\iota_{2*}\mathcal{G}_2\twoheadrightarrow(P)_*\big((\mathcal{G}_1|_P)\oplus(\mathcal{G}_2|_T)\big)\xrightarrow{(P)_*q}(P)_*U.$$

Since $s^*\mathcal{G}=s^*\mathcal{G}_1$, we can define a filtration of $s^*\mathcal{G}$ of type $\widetilde{\Lambda}$ by $\mathbb{F}(s^*\mathcal{G}_1)$. Put $\underline{\mathcal{G}}:=(\mathcal{G},\mathbb{F}(s^*\mathcal{G}))$. Then we have the following diagram.

$$
\begin{tikzcd}
\overline{SL} \arrow[rr, "\psi_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}"] \arrow[d, equal] & & \mathcal{U}(2n, 2n(g-1), C_1 \cup C_2) \arrow[dd, equal] \\
\overline{SL} \arrow[r, "\overline{f}"] & \mathcal{PSU}(2, \mathcal{L}|_{C_1\cup C_2}; C_1 \cup C_2; \Lambda) & \\
SL \arrow[u, phantom, "\bigcup"] \arrow[r, "f"] & \mathcal{PSU}(2, \mathcal{L}|_{C_1\cup C_2}; C_1 \cup C_2; \Lambda)^{lf} \arrow[u, phantom, "\bigcup"] \arrow[r, "\varphi_{\underline{\mathcal{G}}}"] & \mathcal{U}(2n, 2n(g-1), C_1 \cup C_2)
\end{tikzcd}
$$

Lemma 3.5.11. *The above diagram is commutative.*

Proof. Let $\mathcal{E}$ be the sheaf on $(C_1 \cup C_2) \times T$ constructed in the paragraph 3.5.3. The commutativity of the above diagram is equivalent to the fact that if $\gamma|_{(P)_T^*\mathcal{E}_i} : (P)_T^*\mathcal{E}_i \to \mathcal{Q}$ in the data ($\heartsuit$) is an isomorphism for $i = 1, 2$, then we have $\varphi_{\underline{\mathcal{G}}}(\mathcal{E}) \simeq \psi_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}(\mathcal{E})$. This is easily checked. □

Lemma 3.5.12. *We have an isomorphism of line bundles on $\overline{SL}$:*

$$
\psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\Theta_{\mathcal{U}} \simeq \zeta^*\left(\Xi^{(n)}_{\mathcal{PSU}_1} \boxtimes \Xi^{(n)}_{\mathcal{PSU}_2}\right) \otimes \mathcal{O}_{\overline{SL}}(j\mathbb{B}). \tag{3.23}
$$

Proof. We check this isomorphism for the data ($\heartsuit$) in the paragraph 3.5.2. We have

$$
\begin{aligned}
\psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\Theta_{\mathcal{U}} \simeq{}& (\det \mathbb{R}pr_{T*}\mathcal{E}_1)^{\otimes(-n)} \otimes \bigotimes_{j=0}^{l(\Lambda)} \det\left(F_j((s)_T^*\mathcal{E}_1)/F_{j+1}((s)_T^*\mathcal{E}_1)\right) \\
&\otimes (\det \mathbb{R}pr_{T*}\mathcal{E}_2)^{\otimes(-n)} \otimes \sigma^* \det \mathcal{K}.
\end{aligned}
$$

By Lemma 3.5.9 (4), we have $\det \mathcal{K} \simeq \left(\wedge^2\mathcal{Q}^{univ}\right)^{\otimes j}$. This proves the lemma. □

By the above lemma, the pull-back of the canonical section $\sigma_{\mathcal{U}}$ of $\Theta_{\mathcal{U}}$ by $\psi_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}$ gives a section (up to $k^\times$) of $\zeta^*(\Xi^{(n)}_{\mathcal{PSU}_1} \boxtimes \Xi^{(n)}_{\mathcal{PSU}_2}) \otimes \mathcal{O}_{\overline{SL}}(j\mathbb{B})$, that is, an element of V_j, which we denote by $\psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\sigma_{\mathcal{U}} (\in V_j)$. By inclusion $V_j \hookrightarrow V_n$, we can regard $\psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\sigma_{\mathcal{U}}$ as a global section of $\zeta^*(\Xi^{(n)}_{\mathcal{PSU}_1} \boxtimes \Xi^{(n)}_{\mathcal{PSU}_2}) \otimes \mathcal{O}_{\overline{SL}}(n\mathbb{B})$. On the other hand, by Lemma 3.5.4, the global section $\varphi^*_{\underline{\mathcal{G}}}(\sigma_{\mathcal{U}})$ of $\Xi^{(n)}_{\mathcal{PSU}}|_{\mathcal{PSU}^{lf}}$ extends uniquely to a global section of $\Xi^{(n)}_{\mathcal{PSU}}$ over $\mathcal{PSU} = \mathcal{PSU}(2, \mathcal{L}|_{C_1\cup C_2}; C_1 \cup C_2; \Lambda)$, which we denote again by $\varphi^*_{\underline{\mathcal{G}}}(\sigma_{\mathcal{U}})$. $\bar{f}^*\varphi^*_{\underline{\mathcal{G}}}(\sigma_{\mathcal{U}})$ gives a global section of $\zeta^*(\Xi^{(n)}_{\mathcal{PSU}_1} \boxtimes \Xi^{(n)}_{\mathcal{PSU}_2}) \otimes \mathcal{O}_{\overline{SL}}(n\mathbb{B})$ by Lemma 3.5.7.

Lemma 3.5.13. *Up to $k^\times$-multiple, we have $\bar{f}^*\varphi^*_{\underline{\mathcal{G}}}(\sigma_{\mathcal{U}}) = \psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\sigma_{\mathcal{U}}$ in V_n.*

Proof. By Lemma 3.5.4, it suffices to prove that $\bar{f}^*\varphi^*_{\underline{\mathcal{G}}}(\sigma_{\mathcal{U}}) = \psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\sigma_{\mathcal{U}}$ over SL. By Lemma 3.5.11, over SL, we have an isomorphism $(\varphi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\Theta_{\mathcal{U}})|_{SL}$

$\simeq f^*\varphi^*_{\underline{\mathcal{G}}}\Theta_{\mathcal{U}}$ under which the pull-backs of $\sigma_{\mathcal{U}}$ on the both sides correspond. Over SL, we have isomorphisms

$$\begin{aligned}(\varphi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\Theta_{\mathcal{U}})|_{SL} &\simeq \zeta^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes \Xi^{(n)}_{\mathcal{PSU}_2})\otimes \mathcal{O}_{\overline{SL}}(j\mathbb{B})|_{SL}\\ &\xrightarrow{\sim} \zeta^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes \Xi^{(n)}_{\mathcal{PSU}_2})\otimes \mathcal{O}_{\overline{SL}}(n\mathbb{B})|_{SL}\end{aligned}$$

and

$$f^*\varphi^*_{\underline{\mathcal{G}}}\Theta_{\mathcal{U}} \simeq \zeta^*(\Xi^{(n)}_{\mathcal{PSU}_1}\boxtimes \Xi^{(n)}_{\mathcal{PSU}_2})\otimes \mathcal{O}_{\overline{SL}}(n\mathbb{B})|_{SL}.$$

Thus over SL, $\bar{f}^*\varphi^*_{\underline{\mathcal{G}}}(\sigma_{\mathcal{U}})$ and $\psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\sigma_{\mathcal{U}}$ differ up to $\mathrm{H}^0(SL,\mathcal{O})^\times$-multiple. By Lemma 3.5.4, we have $\mathrm{H}^0(SL,\mathcal{O})\simeq k$, which completes the proof of the lemma. □

Since $\dim q|^{-1}_{(\mathcal{G}_1|_P)}\big(\mathrm{Im}(q|_{(\mathcal{G}_2|_P)})\big)=j$, $q|^{-1}_{(\mathcal{G}_1|_P)}\big(\mathrm{Im}(q|_{(\mathcal{G}_2|_P)})\big)$ gives a filtration $\mathbb{F}(\mathcal{G}_1|_P)$ of $\mathcal{G}_1|_P$ of type $\widetilde{A_{n,n-j}}$. Similarly $\mathrm{Ker}(q|_{(\mathcal{G}_2|_P)})$ gives a filtration $\mathbb{F}(\mathcal{G}_2|_P)$ of $\mathcal{G}_2|_P$ of type $\widetilde{A_{j,0}}$. Put $\underline{\mathcal{G}_1}^\sharp := (\mathcal{G}_1,\mathbb{F}(s^*\mathcal{G}_1),\mathbb{F}(\mathcal{G}_1|_P))$ and $\underline{\mathcal{G}_2}^\sharp := (\mathcal{G}_2,\mathbb{F}(\mathcal{G}_2|_P))$. Let

$$\begin{aligned}h:\mathcal{U}(2n,2n(g_1-1);C_1)\times\mathcal{U}(2n,2n(g_2-1);C_2)\\ \to\mathcal{U}(2n,2n(g-1),C_1\cup C_2)\end{aligned}$$

be given by $(\mathcal{F}_1,\mathcal{F}_2)\mapsto \iota_{1*}\mathcal{F}_1\oplus\iota_{2*}\mathcal{F}_2$.

Lemma 3.5.14. *For short, put $\mathcal{U}_i:=\mathcal{U}(2n,2n(g_i-1);C_i)$ and $\mathcal{U}(C_1\cup C_2):=\mathcal{U}(2n,2n(g-1);C_1\cup C_2)$.*

For $0\le j\le n$, the following diagram is commutative:

$$\begin{array}{ccc}\mathcal{PSU}_1^\sharp\times\mathcal{PSU}_2^\sharp & \xrightarrow{\varphi_{\underline{\mathcal{G}_1}^\sharp}\times\varphi_{\underline{\mathcal{G}_2}^\sharp}} & \mathcal{U}_1\times\mathcal{U}_2\\ \uparrow & & \downarrow h\\ \mathbb{B} & & \mathcal{U}(C_1\cup C_2)\\ \varepsilon\downarrow & & \|\\ \overline{SL} & \xrightarrow{\psi_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}} & \mathcal{U}(C_1\cup C_2),\end{array}\tag{3.24}$$

where the morphism $\mathbb{B}\to\mathcal{PSU}_1^\sharp\times\mathcal{PSU}_2^\sharp$ is an isomorphism for $1\le j\le n$, and $\mathbb{P}^1\times\mathbb{P}^1$-bundle for $j=0$.

Proof. This follows from (3) of Lemma 3.5.9. □

Corollary 3.5.15. *The restriction of the section $\psi^*_{(\underline{\mathcal{G}_1},\mathcal{G}_2,q)}\sigma_{\mathcal{U}}(\in V_j)$ to $\mathbb{B}$ corresponds to the section $\varphi^*_{\underline{\mathcal{G}_1}^\sharp}\sigma_{\mathcal{U}_1}\boxtimes\varphi^*_{\underline{\mathcal{G}_2}^\sharp}\sigma_{\mathcal{U}_2}$ (up to $k^\times$-multiple) by the isomorphism* (3.18).

Proof. This is a direct consequence of Lemma 3.5.14. □

3.5.16. Completion of the proof of Theorem 3.5.1.

Claim 3.5.16.1. There exist a finite number of rank n torsion-free sheaves $\mathcal{G}_b$ $(1 \leq b \leq B)$ on $C_1 \cup C_2$ of degree e with a filtration $\mathbb{F}(s^*\mathcal{G}_b)$ of $s^*\mathcal{G}_b$ of type $\widetilde{\Lambda}$ such that the set $\{\varphi^*_{\underline{\mathcal{G}_b}}\sigma_{\mathcal{U}}\}_{b=1}^B$ spans the vector space $\mathrm{H}^0\big(\mathcal{PSU}(2,\mathcal{L}|_{C_1\cup C_2}; C_1 \cup C_2; \Lambda)^{lf}, \Xi^{(n)}\big)$.

Proof of Claim 3.5.16.1. By assumption, for $0 \leq j \leq n$ with $e_{1,j}$ and $e_{2,j}$ integers, we can find a finite number of n-bundles $\mathcal{G}_{j,i,b_i}$ $(i = 1, 2$ and $1 \leq b_i \leq B_i)$ with filtrations $\mathbb{F}(s^*\mathcal{G}_{j,1,b_1})$ of type $\widetilde{\Lambda}$, $\mathbb{F}(\mathcal{G}_{j,1,b_1}|_P)$ of type $\widetilde{A_{n,n-j}}$ and $\mathbb{F}(\mathcal{G}_{j,2,b_2}|_P)$ of type $\widetilde{A_{j,0}}$ such that the set $\Big\{\varphi^*_{\mathcal{G}_{j,1,b_1}}\sigma_{\mathcal{U}_1} \boxtimes \varphi^*_{\mathcal{G}_{j,2,b_2}}\sigma_{\mathcal{U}_2}\,|1 \leq b_i \leq B_i\Big\}$ span the vector space $\mathrm{H}^0(\mathcal{PSU}^\sharp_1, \Xi^{(n)}) \otimes \mathrm{H}^0(\mathcal{PSU}^\sharp_2, \Xi^{(n)})$. Put $\underline{\mathcal{G}_{j,1,b_1}} := (\mathcal{G}_{j,1,b_1}, \mathbb{F}(s^*\mathcal{G}_{j,1,b_1}))$, $\underline{\mathcal{G}_{j,1,b_1}}^\sharp := (\mathcal{G}_{j,1,b_1}, \mathbb{F}(s^*\mathcal{G}_{j,1,b_1}), \mathbb{F}(\mathcal{G}_{j,1,b_1}|_P))$ and $\underline{\mathcal{G}_{j,2,b_2}}^\sharp := (\mathcal{G}_{j,2,b_2}, \mathbb{F}(\mathcal{G}_{j,2,b_2}|_P))$. Take a surjective homomorphism $q_{j,(b_1,b_2)} : (\mathcal{G}_{j,1,b_1}|_P) \oplus (\mathcal{G}_{j,2,b_2}|_P) \to k^n$ such that $q_{j,(b_1,b_2)}|_{(\mathcal{G}_{j,1,b_1}|_P)}$ is an isomorphism, $\mathrm{Ker}(q_{j,(b_1,b_2)}|_{(\mathcal{G}_{j,2,b_2}|_P)}) = F_2(\mathcal{G}_{j,2,b_2}|_P)$ and $(q_{j,(b_1,b_2)}|_{(\mathcal{G}_{j,1,b_1}|_P)})^{-1}(\mathrm{Im}(q_{j,(b_1,b_2)}|_{(\mathcal{G}_{j,2,b_2}|_P)})) = F_1(\mathcal{G}_{j,1,b_1}|_P)$. As $\mathcal{G}$ is made from $\mathcal{G}_1$ and $\mathcal{G}_2$ in the paragraph 3.5.10, we let $\mathcal{G}_{j,(b_1,b_2)}$ be the kernel of the composite of morphisms $\iota_{1*}\mathcal{G}_{j,1,b_1} \oplus \iota_{2*}\mathcal{G}_{j,2,b_2} \to P_*(\mathcal{G}_{j,1,b_1}|_P \oplus \mathcal{G}_{j,2,b_2}|_P) \xrightarrow{(P)_*q_{j,(b_1,b_2)}} P_*k^n$. The filtration $\mathbb{F}(s^*\mathcal{G}_{j,1,b_1})$ induces a filtration $\mathbb{F}(s^*\mathcal{G}_{j,(b_1,b_2)})$. Put $\underline{\mathcal{G}_{j,(b_1,b_2)}} := (\mathcal{G}_{j,(b_1,b_2)}, \mathbb{F}(s^*\mathcal{G}_{j,(b_1,b_2)}))$. By Corollary 3.5.15, the image of $\varphi^*_{(\underline{\mathcal{G}_{j,1,b_1}}, \mathcal{G}_{j,2,b_2}, q_{j,(b_1,b_2)})}\sigma_{\mathcal{U}} \in V_j$ to $\overline{V_j} \subset \mathrm{H}^0(\mathbb{B}, (\zeta|_{\mathbb{B}})^*(\Xi^{(n)}_{\mathcal{PSU}_1} \boxtimes \Xi^{(n)}_{\mathcal{PSU}_2}) \otimes \varepsilon^*\mathcal{O}_{\overline{SL}}(j\mathbb{B})) \simeq \mathrm{H}^0(\mathcal{PSU}^\sharp_1, \Xi^{(n)}_{\mathcal{PSU}^\sharp_1}) \otimes \mathrm{H}^0(\mathcal{PSU}^\sharp_2, \Xi^{(n)}_{\mathcal{PSU}^\sharp_2})$ is $\varphi^*_{\underline{\mathcal{G}_{j,1,b_1}}^\sharp}\sigma_{\mathcal{U}} \boxtimes \varphi^*_{\underline{\mathcal{G}_{j,2,b_2}}^\sharp}\sigma_{\mathcal{U}}$. From this we know

- $\overline{V_j} = \mathrm{H}^0(\mathbb{B}, (\zeta|_{\mathbb{B}})^*(\Xi^{(n)}_{\mathcal{PSU}_1} \boxtimes \Xi^{(n)}_{\mathcal{PSU}_2}) \otimes \varepsilon^*\mathcal{O}_{\overline{SL}}(j\mathbb{B}))$,
- $\Big\{\varphi^*_{(\underline{\mathcal{G}_{j,1,b_1}}, \mathcal{G}_{j,2,b_2}, q_{j,(b_1,b_2)})}\sigma_{\mathcal{U}}\,|0 \leq j \leq n,\ 1 \leq b_i \leq B_i\Big\}$ spans the vector space V_n,
- $\dim V_n = \sum_{j=0}^n \dim \mathrm{H}^0(\mathcal{PSU}^\sharp_1, \Xi^{(n)}_{\mathcal{PSU}^\sharp_1}) \cdot \dim \mathrm{H}^0(\mathcal{PSU}^\sharp_2, \Xi^{(n)}_{\mathcal{PSU}^\sharp_2})$.

Now Claim 3.5.16.1 follows from Lemma 3.5.13. □

As in the above claim, we choose $\underline{\mathcal{G}_b} = (\mathcal{G}_b, \mathbb{F}(s^*\mathcal{G}_b))$ $(1 \leq b \leq B)$ so that the set $\{\varphi^*_{\underline{\mathcal{G}_b}}\sigma_{\mathcal{U}}\}$ spans $\mathrm{H}^0(\mathcal{PSU}(2, \mathcal{L}|_{C_1\cup C_2}; C_1 \cup C_2; \Lambda)^{lf}, \Xi^{(n)})$. There exists a positive integer M such that for each b, there exists $\underline{\widetilde{\mathcal{G}_b}} = (\widetilde{\mathcal{G}_b}, \mathbb{F}((s)^*_{S^{(1/M)}}\widetilde{\mathcal{G}_b}))$, where $\widetilde{\mathcal{G}_b}$ is a $S^{(1/M)}$-flat coherent sheaf on $\mathcal{C}^{(1/M)}$ and $\mathbb{F}((s)^*_{S^{(1/M)}}\widetilde{\mathcal{G}_b})$ is a filtration of type $\widetilde{\Lambda}$ such that the restriction of $\underline{\widetilde{\mathcal{G}_b}}$ to the central fiber $C_1 \cup C_2$ is $\underline{\mathcal{G}_b}$. By Lemma 3.5.8, if necessary, we can replace M so that $\varphi^*_{\underline{\widetilde{\mathcal{G}_b}}}\Theta_{\mathcal{U}} \simeq \Xi^{(n)}_{\mathcal{PSU}}$ $(1 \leq b \leq B)$, where $\mathcal{PSU} = \mathcal{PSU}(2, \mathcal{L}_{S^{(1/M)}}; \mathcal{C}^{(1/M)}/S^{(1/M)}; \Lambda)^{lf}$. If we restrict the sections $\varphi^*_{\underline{\widetilde{\mathcal{G}_b}}}\sigma_{\mathcal{U}} \in \mathrm{H}^0(\mathcal{PSU}(2, \mathcal{L}_{S^{(1/M)}}; \mathcal{C}^{(1/M)}/S^{(1/M)}; \Lambda)^{lf}, \Xi^{(n)})$ $(1 \leq b \leq B)$ to the central fiber, they span the vector space $\mathrm{H}^0(\mathcal{PSU}(2, \mathcal{L}|_{C_1\cup C_2}; C_1 \cup$

$C_2, \Lambda)^{lf}, \Xi^{(n)})$. We also have

$$\dim \mathrm{H}^0\left(\mathcal{PSU}(2,(\mathcal{L})_{\mathrm{Spec}k((t^{1/M}))};\mathcal{C}\times_S \mathrm{Spec}k((t^{1/M}))/\mathrm{Spec}k((t^{1/M}));\Lambda),\Xi^{(n)}\right)$$

$$\stackrel{(*)}{=}\sum_{j=0}^{n}\dim \mathrm{H}^0\left(\mathcal{PSU}(2,L_1;C_1;\Lambda,A_{n,n-j}),\Xi^{(n)}\right)\cdot\dim \mathrm{H}^0\left(\mathcal{PSU}(2,L_2;C_2;A_{j,0}),\Xi^{(n)}\right)$$

$$\stackrel{(**)}{=}\dim \mathrm{H}^0\left(\mathcal{PSU}(2,\mathcal{L}|_{C_1\cup C_2};C_1\cup C_2,\Lambda)^{lf},\Xi^{(n)}\right),$$

where $(*)$ follows from Remark 2.1.11 and $(**)$ was proved in the proof of Claim 3.5.16.1.

Then applying Proposition 4.1.6 as $\mathcal{X} = \mathcal{PSU}(2,\mathcal{L}_{S^{(1/M)}};\mathcal{C}^{(1/M)}/S^{(1/M)};\Lambda)^{lf(N)}$ with $N \gg 0$ and noting that $\mathrm{H}^0(\mathcal{PSU}(2,\mathcal{L}_{S^{(1/M)}};\mathcal{C}^{(1/M)}/S^{(1/M)};\Lambda)^{lf},\Xi^{(n)}) \simeq \mathrm{H}^0(\mathcal{PSU}(2,\mathcal{L}_{S^{(1/M)}};\mathcal{C}^{(1/M)}/S^{(1/M)};\Lambda)^{lf(N)},\Xi^{(n)})$ for $N \gg 0$, we know that $\mathrm{H}^0\left(\mathcal{PSU}(2,\mathcal{L}_{S^{(1/M)}};\mathcal{C}^{(1/M)}/S^{(1/M)};\Lambda)^{lf},\Xi^{(n)}\right)$ is a finite free $k[[t]]$-module and that $\mathrm{H}^0\left(\mathcal{PSU}(2,\mathcal{L}_{S^{(1/M)}};\mathcal{C}^{(1/M)}/S^{(1/M)};\Lambda)^{lf},\Xi^{(n)}\right) \otimes k \simeq \mathrm{H}^0\left(\mathcal{PSU}(2,\mathcal{L}|_{C_1\cup C_2};C_1\cup C_2;\Lambda)^{lf},\Xi^{(n)}\right)$. By Nakayama's Lemma, $\{\varphi^*_{\underline{\widetilde{\mathcal{G}_b}}}\sigma_{\mathcal{U}}\}$ generate $\mathrm{H}^0\left(\mathcal{PSU}(2,\mathcal{L}_{S^{(1/M)}};\mathcal{C}^{(1/M)}/S^{(1/M)};\Lambda)^{lf},\Xi^{(n)}\right)$ as a $k[[t^{1/M}]]$-module. Hence they generate the $k((t^{1/M}))$-vector space

$$\mathrm{H}^0\left(\mathcal{PSU}(2,\mathcal{L}_{S^{(1/M)}};\mathcal{C}^{(1/M)}/S^{(1/M)};\Lambda)^{lf},\Xi^{(n)}\right)\otimes_{k[[t^{1/M}]]}k((t^{1/M})),$$

which is isomorphic to

$$\mathrm{H}^0\left(\mathcal{PSU}(2,(\mathcal{L})_{\mathrm{Spec}k((t^{1/M}))};\mathcal{C}\times_S \mathrm{Spec}k((t^{1/M}))/\mathrm{Spec}k((t^{1/M}));\Lambda),\Xi^{(n)}\right)$$

again by Proposition 4.1.6. This complete the proof of Theorem 3.5.1.

3.6. $SL(2)-GL(n)$ strange duality conjecture

Theorem 3.6.1. *Let C be an irreducible smooth projective curve of genus $g \geq 1$ over an algebraically closed field k of characteristic zero. Let s be a closed point of C. Fix a positive integer n. Let Λ be a Young diagram of type $\leq (2,n)$ and let $\mathcal{L}$ be a line bundle of degree d on C. Assume that the 1-pointed curve (C,s) is general. Then $(SD)_{2,n}$ holds for $\mathcal{PSU}(2,\mathcal{L};C;\Lambda)$.*

Proof. We proceed by induction on g. If $g=1$, then the theorem follows from Proposition 3.2.2. Assume that the theorem is true for $g-1 \geq 1$. We can find a family of nodal curves $\pi : \mathcal{C} \to S = \mathrm{Spec}k[[t]]$ with a section s and a line bundle $\mathcal{L}$ on $\mathcal{C}$ described in the paragraph 3.4.2 such that $g_1 = 1$, $g_2 = g-1$, and $(SD)_{2,n}$ holds for $\mathcal{PSU}(2,\mathcal{L}|_{C_2};C_2;\Gamma)$ for any Young diagram Γ of type $\leq (2,n)$, where Γ is assigned to the point P. By Proposition 3.2.3, $(SD)_{2,n}$ holds for $\mathcal{PSU}(2,\mathcal{L}|_{C_1};C_1;\Lambda,\Gamma)$ for any Young diagram Γ of type $\leq (2,n)$. Theorem 3.5.1 implies that $(SD)_{2,n}$ holds for $\mathcal{PSU}(2,\mathcal{L}_{\bar{\eta}};\mathcal{C}_{\bar{\eta}};\Lambda)$. By Proposition 3.3.1 and Proposition 2.3.1, we know that $(SD)_{2,n}$ holds for $\mathcal{PSU}(2,\mathcal{L};C;\Lambda)$ for general (C,s). ∎

Corollary 3.6.2. *Let C be an irreducible smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic zero, and let $\mathcal{L}$ be a line bundle of degree d on C. Let n be a positive integer such that the rational number $e = n(g-1) - nd/2$ is an integer. Assume that C is general, then Conjecture* 1.0.4 *holds for $r = 2$.*

Proof. For a rank n bundle $\mathcal{G}$ of degree e on C, put $\Theta^{\mathcal{G}} := \{\mathcal{E} \in SU(2,\mathcal{L}) | \mathrm{H}^0(C, \mathcal{G} \otimes \mathcal{E}) \neq 0\}$. Then by Theorem 3.6.1, we can find a finite number of $\mathcal{G}_b$'s such that the effective divisors $\Theta^{\mathcal{G}_b}$'s span the vector space $\mathrm{H}^0(SU(2,\mathcal{L}), \Xi^{\otimes n})$. We can deform $\mathcal{G}_b$'s a little bit so that they become semistable. The details are left to the reader. □

4. Appendix

4.1. Some propositions on stacks

Proposition 4.1.1. *Let $\mathcal{X}$ be a quasi-compact algebraic stack over $S = \mathrm{Spec}R$ and $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Let R' be a flat R-algebra. Then the natural homomorphism of R'-modules*

$$\mathrm{H}^0(\mathcal{X}, \mathcal{F}) \otimes_R R' \to \mathrm{H}^0(\mathcal{X}_{R'}, \mathcal{F}_{R'})$$

is an isomorphism.

Proof. We can find an affine atlas $\phi : X \to \mathcal{X}$. Since $X \times_{\mathcal{X}} X$ is quasi-compact, it is covered by finite affine open subschemes Y_j. Put $Y := \coprod Y_j$. For $i = 1, 2$, let $\psi_i : Y \to X$ be the composite $Y \to X \times_{\mathcal{X}} X \xrightarrow{i\text{-th proj.}} X$. Put $M^0 := \mathrm{H}^0(X, \phi^*\mathcal{F})$ and $M^1 := \mathrm{H}^0(Y, (\phi \circ \psi_1)^*\mathcal{F})$. There is a natural isomorphism $(\phi \circ \psi_1)^*\mathcal{F} \simeq (\phi \circ \psi_2)^*\mathcal{F}$, and it induces an isomorphism $\alpha : \mathrm{H}^0(Y, (\phi \circ \psi_2)^*\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^0(Y, (\phi \circ \psi_1)^*\mathcal{F})$. Put $\theta := \psi_1^* - \alpha \circ \psi_2^*$. Then for any R-algebra R', we have

$$\mathrm{H}^0(\mathcal{X}_{R'}, \mathcal{F}_{R'}) \simeq \mathrm{Ker}(M^0 \otimes_R R' \xrightarrow{\theta \otimes \mathrm{id}_{R'}} M^1 \otimes_R R').$$

If R' is R-flat, we have

$$\mathrm{Ker}(M^0 \otimes_R R' \xrightarrow{\theta \otimes \mathrm{id}_{R'}} M^1 \otimes_R R') \simeq \mathrm{Ker}(M^0 \xrightarrow{\theta} M^1) \otimes_R R'.$$

This proves the proposition. □

Proposition 4.1.2. *Let $\mathcal{X}$ be an algebraic stack over* $\mathrm{Spec}R$. *Let $(\mathcal{X}_n)_{n=0}^{\infty}$ be an increasing sequence of non-empty open substack of $\mathcal{X}$. Let $\mathcal{L}$ be a quasi-coherent sheaf on $\mathcal{X}$, and let $\mathcal{L}_n$ be $\mathcal{L}|_{\mathcal{X}_n}$. Assume that $\mathcal{X} = \cup_{n=0}^{\infty} \mathcal{X}_n$ (i.e, for any affine R-scheme T and any object $a \in \mathcal{X}(T)$, there exists n such that $a \in \mathcal{X}_n(T)$).*

(1) *The natural morphism of R-modules*

$$\mathrm{H}^0(\mathcal{X}, \mathcal{L}) \to \varprojlim \mathrm{H}^0(\mathcal{X}_n, \mathcal{L}_n)$$

is an isomorphism.

(2) *Assume moreover that R is a field, all $\mathcal{X}_n$ are quasi-compact and the restriction map* $\mathrm{H}^0(\mathcal{X}_n, \mathcal{L}_n) \to \mathrm{H}^0(\mathcal{X}_{n'}, \mathcal{L}_{n'})$ *is injective for any $n' \leq n$. Then for any R-algebra R', the natural map*

$$\mathrm{H}^0(\mathcal{X}, \mathcal{L}) \otimes_R R' \to \mathrm{H}^0(\mathcal{X}_{R'}, \mathcal{L}_{R'})$$

is an isomorphism.

Proof. (1) This directly follows from the definition of $\mathrm{H}^0(\mathcal{X}, -)$.

(2) This follows from (1) and Proposition 4.1.1 together with Lemma 4.1.3. □

Lemma 4.1.3. *Let R be a commutative ring and N be a free R-module. Let $\{f_{n'n} : M_n \to M_{n'}\}_{n \geq n'}$ be a projective system of R-modules indexed by non-negative integers. Assume that $f_{n'n} : M_n \to M_{n'}$ is injective for any $n \geq n'$. Then the natural morphism*

$$\left(\varprojlim M_n\right) \otimes_R N \to \varprojlim (M_n \otimes_R N)$$

is an isomorphism.

Proof. Easy. □

Remark 4.1.4. The injectivity of $\mathrm{H}^0(\mathcal{X}_n, \mathcal{L}_n) \to \mathrm{H}^0(\mathcal{X}_{n'}, \mathcal{L}_{n'})$ in Proposition 4.1.2 (2) is satisfied if each $\mathcal{X}_n$ has an irreducible and reduced atlas, and $\mathcal{L}$ is a line bundle.

Proposition 4.1.5. *Let $\mathcal{X}$ be an algebraic stack over $S = \mathrm{Spec}R$. Assume that we have an atlas $\tau : \coprod_{n \geq 0} H_n \to \mathcal{X}$ such that H_n is flat and of finite type over S, and $H_n \times_S \mathrm{Spec}k(x)$ is irreducible and normal for any $x \in S$. Let $\mathcal{X}^\circ$ be an open substack of $\mathcal{X}$. Put $H_n^\circ := H_n \times_{\mathcal{X}} \mathcal{X}^\circ$. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. If the codimension of $(H_n \times_S \mathrm{Spec}k(x)) \setminus (H_n^\circ \times_S \mathrm{Spec}k(x))$ in $H_n \times_S \mathrm{Spec}k(x)$ is greater than or equal to two, then the restriction map $\mathrm{H}^0(\mathcal{X}, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}^\circ, \mathcal{L}|_{\mathcal{X}^\circ})$ is an isomorphism.*

Proof. It suffices to prove that the restriction map

$$\mathrm{H}^0(H_n, \tau^*\mathcal{L}) \to \mathrm{H}^0(H_n^\circ, \tau^*\mathcal{L}|_{H_n^\circ})$$

is an isomorphism, for the gluing condition is satisfied because it is satisfied generically. To prove this, we have only to prove that, for $\forall y \in H_n \setminus H_n^\circ$, $\mathrm{depth}\mathcal{O}_{H_n,y} \geq 2$ by [EGAIV, (5.10.5)]. By [Mat, Theorem 50], we can check this fiberwisely. □

Proposition 4.1.6. *Let $(R, \mathfrak{m})$ be a discrete valuation ring, $\mathcal{X}$ a quasi-compact Artin-stack over* $\mathrm{Spec}R$ *and $\mathcal{L}$ a line bundle on $\mathcal{X}$. Assume that $\mathcal{X}$ is flat over* $\mathrm{Spec}R$. *Put $k := R/\mathfrak{m}$ and $K := \mathrm{Frac}R$.*

(1) *Assume that* $\mathrm{H}^0(\mathcal{X},\mathcal{L})\otimes_R k \to \mathrm{H}^0(\mathcal{X}\times_{\mathrm{Spec}R}\mathrm{Spec}k,\mathcal{L}\otimes_R k)$ *is surjective. Then for any R-algebra S*

$$\mathrm{H}^0(\mathcal{X},\mathcal{L})\otimes_R S \to \mathrm{H}^0(\mathcal{X}\times_{\mathrm{Spec}R}\mathrm{Spec}S,\mathcal{L}\otimes_R S) \tag{4.1}$$

is an isomorphism.

(2) *Assume moreover that* $\dim_K \mathrm{H}^0(\mathcal{X}\times\mathrm{Spec}K,\mathcal{L}\otimes_R K) = \dim_k \mathrm{H}^0(\mathcal{X}\times\mathrm{Spec}k,\mathcal{L}\otimes_R k) < \infty$*. Then* $\mathrm{H}^0(\mathcal{X},\mathcal{L})$ *is a free R-module of finite rank.*

Proof. Since $\mathcal{X}$ is quasi-compact and flat over R, we can find a 2-term complex $M^0 \xrightarrow{\alpha} M^1$ of flat R-modules such that, for any R-algebra S, we have an isomorphism

$$\mathrm{H}^0(\mathcal{X}\times\mathrm{Spec}S,\mathcal{L}\otimes_R S)\simeq \mathrm{Ker}(M^0\otimes_R S \xrightarrow{\alpha\otimes\mathrm{id}} M^1\otimes_R S),$$

which is functorial in S.

If S is an R-algebra, we have the exact sequences

$$0\to\mathrm{Ker}(\alpha)\otimes_R S\to M^0\otimes_R S\to\mathrm{Im}(\alpha)\otimes_R S\to 0$$

and

$$0\to\mathrm{Tor}_1^R(S,\mathrm{Coker}(\alpha))\to\mathrm{Im}(\alpha)\otimes_R S\to M^1\otimes_R S\to\mathrm{Coker}(\alpha)\otimes_R S\to 0.$$

Since (4.1) is an isomorphism, we have $\mathrm{Tor}_1^R(k,\mathrm{Coker}(\alpha))=0$. One can easily check that this implies $\mathrm{Tor}_1^R(N,\mathrm{Coker}(\alpha))=0$ for any R-module N. This completes the proof of (1).

Let us go on to the proof of (2). Put $d:=\dim_K \mathrm{H}^0(\mathcal{X}\times\mathrm{Spec}K,\mathcal{L}\otimes_R K)=\dim_k \mathrm{H}^0(\mathcal{X}\times\mathrm{Spec}k,\mathcal{L}\otimes_R k)$. Let $\gamma: R^{\oplus d}\to\mathrm{Ker}(\alpha)$ be an R-module homomorphism such that $\gamma\otimes_R k:(R^{\oplus d})\otimes_R k\to\mathrm{Ker}(\alpha)\otimes_R k$ is an isomorphism. Then we have $\mathrm{Ker}(\gamma)\otimes_R k=0$, $\mathrm{Coker}(\gamma)\otimes_R k=0$ and $\mathrm{Tor}_1^R(\mathrm{Coker}(\gamma),k)=0$. Since $\mathrm{Ker}(\gamma)$ is a finitely generated R-module, we have $\mathrm{Ker}(\gamma)=0$ by Nakayama's Lemma. By assumption, we have $\mathrm{Coker}(\gamma)\otimes_R K=0$. Then by Lemma 4.1.7, we have $\mathrm{Coker}(\gamma)=0$ and complete the proof of (2). □

Lemma 4.1.7. *Let* $(R,\mathfrak{m})$*,*k *and* K *be as in Proposition* 4.1.6*. Let* L *be an R-module. If* $\mathrm{Tor}_1^R(L,k)=0$ *and* $L\otimes_R K=0$*, then* $L=0$*.*

Proof. Let t be a generator of $\mathfrak{m}$. $L\otimes_R K=0$ implies that for any $x\in L$ there exists $n>0$ such that $t^n\cdot x=0$. $\mathrm{Tor}_1^R(L,k)=0$ implies that $\mathrm{Tor}_1^R(L,R/\mathfrak{m}^n)=0$ for any $n>0$. This implies that $L\xrightarrow{t^n\cdot}L$ is injective. Hence $L=0$. □

Proposition 4.1.8. *Let* $\mathcal{X}^{(1)}$*,* $\mathcal{X}^{(2)}$ *be Artin stacks over a field k, and let* $\mathcal{F}^{(1)}$*,* $\mathcal{F}^{(2)}$ *be quasi-coherent sheaves on* $\mathcal{X}^{(1)}$ *and* $\mathcal{X}^{(2)}$ *respectively. Let* $\phi^{(i)}:\mathcal{X}^{(1)}\times\mathcal{X}^{(2)}\to\mathcal{X}^{(i)}$ *be the projection. Assume that for* $i=1,2$ *there exists a quasi-compact open substack* $\mathcal{X}^{(i)\circ}$ *of* $\mathcal{X}^{(i)}$ *such that the restriction morphisms*

$$\mathrm{H}^0(\mathcal{X}^{(i)},\mathcal{F}^{(i)})\to\mathrm{H}^0(\mathcal{X}^{(i)\circ},\mathcal{F}^{(i)})$$

$$\mathrm{H}^0(\mathcal{X}^{(1)}\times\mathcal{X}^{(2)},\phi^{(1)*}\mathcal{F}^{(1)}\otimes\phi^{(2)*}\mathcal{F}^{(2)})$$
$$\to\mathrm{H}^0(\mathcal{X}^{(1)\circ}\times\mathcal{X}^{(2)\circ},\phi^{(1)*}\mathcal{F}^{(1)}\otimes\phi^{(2)*}\mathcal{F}^{(2)})$$

are isomorphisms. Then the natural k-linear map

$$\mathrm{H}^0(\mathcal{X}^{(1)},\mathcal{F}^{(1)})\otimes\mathrm{H}^0(\mathcal{X}^{(2)},\mathcal{F}^{(2)})\to\mathrm{H}^0(\mathcal{X}^{(1)}\times\mathcal{X}^{(2)},\phi^{(1)*}\mathcal{F}^{(1)}\otimes\phi^{(2)*}\mathcal{F}^{(2)})$$

is an isomorphism.

Proof. We may assume that $\mathcal{X}^{(1)}$ and $\mathcal{X}^{(2)}$ are quasi-compact. Let $\tau^{(i)}: X^{(i)}\to\mathcal{X}^{(i)}$ be an affine atlas, and $\tau_j^{(i)}: X^{(i)}\times_{\mathcal{X}^{(i)}}X^{(i)}\to X^{(i)}$ be the j-th projection. Note that $X^{(i)}\times_{\mathcal{X}^{(i)}}X^{(i)}$ is quasi-compact and separated by the definition of artin stacks [L-M]. Put $\varphi_j^{(i)}:=\tau^{(i)}\circ\tau_j^{(i)}$. Since $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ are canonically isomorphic, the two vector spaces $\mathrm{H}^0(X^{(i)}\times_{\mathcal{X}^{(i)}}X^{(i)},\varphi_1^{(i)*}\mathcal{F}^{(i)})$ and $\mathrm{H}^0(X^{(i)}\times_{\mathcal{X}^{(i)}}X^{(i)},\varphi_2^{(i)*}\mathcal{F}^{(i)})$ are canonically isomorphic. We denote this vector space by $V_1^{(i)}$. Put $V_0^{(i)}:=\mathrm{H}^0(X^{(i)},\tau^{(i)*}\mathcal{F}^{(i)})$. Put $H_0^{(i)}:=\left\{v\in V_0^{(i)}\,\middle|\,\tau_1^{(i)*}(v)=\tau_2^{(i)*}(v)\text{ in }V_1^{(i)}\right\}$. and $H_0:=\left\{v\in V_0^{(1)}\otimes V_0^{(2)}\,\middle|\,(\tau_1^{(1)*}\otimes\tau_1^{(2)*})(v)=(\tau_2^{(1)*}\otimes\tau_2^{(2)*})(v)\right\}$. By Künneth theorem for quasi-coherent sheaves on quasi-compact and separated schemes over a field, we have isomorphisms

$$\begin{aligned}H_0^{(i)}&\simeq\mathrm{H}^0\left(\mathcal{X}^{(i)},\mathcal{F}^{(i)}\right)\\ H_0&\simeq\mathrm{H}^0\left(\mathcal{X}^{(1)}\times\mathcal{X}^{(2)},\phi^{(1)*}\mathcal{F}^{(1)}\otimes\phi^{(2)*}\mathcal{F}^{(2)}\right).\end{aligned}$$

Thus we need to prove that $H_0^{(1)}\otimes H_0^{(2)}\xrightarrow{\sim}H_0$. It is clear that $H_0^{(1)}\otimes H_0^{(2)}\subset H_0$. We choose bases $\{\boldsymbol{e}_\lambda^{(1)}\}_\lambda$ and $\{\boldsymbol{e}_\mu^{(2)}\}_\mu$ of $V_0^{(1)}$ and $V_0^{(2)}$ so that part $\{\boldsymbol{e}_\lambda^{(1)}\}_\lambda$ and $\{\boldsymbol{e}_\mu^{(2)}\}_\mu$ forms bases of $H_0^{(1)}$ and $H_0^{(2)}$ respectively. Let $v:=\sum c_{\lambda\mu}\boldsymbol{e}_\lambda^{(1)}\otimes\boldsymbol{e}_\mu^{(2)}$ be in H_0. We have

$$\sum c_{\lambda\mu}\tau_1^{(1)*}(\boldsymbol{e}_\lambda^{(1)})\otimes\tau_1^{(2)*}(\boldsymbol{e}_\mu^{(2)})=\sum c_{\lambda\mu}\tau_2^{(1)*}(\boldsymbol{e}_\lambda^{(1)})\otimes\tau_2^{(2)*}(\boldsymbol{e}_\mu^{(2)}).\tag{4.2}$$

Let $\delta^{(i)}: X^{(i)}\to X^{(i)}\times_{\mathcal{X}^{(i)}}X^{(i)}$ be the diagonal morphism. Applying $\delta^{(1)*}\otimes\mathrm{id}$ to the equation (4.2), we obtain

$$\sum_\lambda\boldsymbol{e}_\lambda^{(1)}\otimes\left\{\sum_\mu c_{\lambda\mu}\tau_1^{(2)*}\boldsymbol{e}_\mu^{(2)}\right\}=\sum_\lambda\boldsymbol{e}_\lambda^{(1)}\otimes\left\{\sum_\mu c_{\lambda\mu}\tau_2^{(2)*}\boldsymbol{e}_\mu^{(2)}\right\}.$$

Therefore for $\forall\lambda$, we have $\sum_\mu c_{\lambda\mu}\boldsymbol{e}_\mu^{(2)}\in H_0^{(2)}$, in other words, $c_{\lambda\mu}=0$ if $\boldsymbol{e}_\mu^{(2)}\notin H_0^{(2)}$. Interchanging the role of $H_0^{(2)}$ and $H_0^{(1)}$, we have $c_{\lambda\mu}=0$ if $\boldsymbol{e}_\lambda^{(1)}\notin H_0^{(1)}$. Hence $v\in H_0^{(1)}\otimes H_0^{(2)}$. □

4.2. A Compactification of SL_2

Let S be a scheme, $\mathcal{V}_1$ and $\mathcal{V}_2$ be locally free $\mathcal{O}_S$-modules of rank 2. Let us be given an isomorphism $\delta:\wedge^2\mathcal{V}_1\xrightarrow{\sim}\wedge^2\mathcal{V}_2$.

Definition 4.2.1. We shall define three functors $\mathcal{SL}(\mathcal{V}_1,\mathcal{V}_2)$, $\overline{\mathcal{SL}}(\mathcal{V}_1,\mathcal{V}_2)$ and $\mathcal{B}(\mathcal{V}_1,\mathcal{V}_2)$ from the category of S-schemes to the category of sets as follows.

(1) For an S-scheme $f : T \to S$, $\mathcal{SL}(\mathcal{V}_1, \mathcal{V}_2)(T)$ is the set of isomorphims $\alpha : f^*\mathcal{V}_1 \xrightarrow{\sim} f^*\mathcal{V}_2$ such that $\wedge^2\alpha = f^*(\delta)$.

(2) For an S-scheme $f : T \to S$, $\overline{\mathcal{SL}}(\mathcal{V}_1, \mathcal{V}_2)(T)$ is the set of equivalence classes of rank-2 bundle quotients $\beta : f^*\mathcal{V}_1 \oplus f^*\mathcal{V}_2 \to \mathcal{W} \to 0$ such that the diagram

$$\begin{array}{ccc} \bigwedge^2 f^*\mathcal{V}_1 & \xrightarrow{f^*(\delta)} & \bigwedge^2 f^*\mathcal{V}_2 \\ {\scriptstyle \wedge^2\beta_1}\downarrow & & \downarrow{\scriptstyle \wedge^2\beta_2} \\ \bigwedge^2 \mathcal{W} & = & \bigwedge^2 \mathcal{W} \end{array} \tag{4.3}$$

commutes, where $\beta_i := \beta|_{f^*\mathcal{V}_i}$.

(3) For an S-scheme $f : T \to S$, $\mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)(T)$ is the set of equivalence classes of rank-2 bundle quotients $\beta : f^*\mathcal{V}_1 \oplus f^*\mathcal{V}_2 \to \mathcal{W} \to 0$ such that the diagram (4.3) commutes and $\wedge^2\beta_i = 0$.

Remark 4.2.2. (1) If $\alpha : f^*\mathcal{V}_1 \xrightarrow{\sim} f^*\mathcal{V}_2$ is a T-valued point of $\mathcal{SL}(\mathcal{V}_1, \mathcal{V}_2)$, then $f^*\mathcal{V}_1 \oplus f^*\mathcal{V}_2 \xrightarrow{\alpha\oplus\mathrm{id}} f^*\mathcal{V}_2$ is a T-valued point of $\overline{\mathcal{SL}}(\mathcal{V}_1, \mathcal{V}_2)$.

(2) In Definition 4.2.1 (2), if we let $\mathcal{K} := \mathrm{Ker}(\beta)$, then the commutativity of the diagram (4.3) is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} \bigwedge^2 \mathcal{K} & = & \bigwedge^2 \mathcal{K} \\ \downarrow & & \downarrow \\ \bigwedge^2 f^*\mathcal{V}_1 & \xrightarrow{f^*(\delta)} & \bigwedge^2 f^*\mathcal{V}_2. \end{array}$$

(3) We denote the schemes that represent the functors $\mathcal{SL}(\mathcal{V}_1, \mathcal{V}_2)$, $\overline{\mathcal{SL}}(\mathcal{V}_1, \mathcal{V}_2)$ and $\mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ by $SL(\mathcal{V}_1, \mathcal{V}_2)$, $\overline{SL}(\mathcal{V}_1, \mathcal{V}_2)$ and $\mathbb{B}(\mathcal{V}_1, \mathcal{V}_2)$ respectively. Then $SL(\mathcal{V}_1, \mathcal{V}_2)$ is an open subscheme of $\overline{SL}(\mathcal{V}_1, \mathcal{V}_2)$, and $\mathbb{B}(\mathcal{V}_1, \mathcal{V}_2)$ is a closed subscheme of $\overline{SL}(\mathcal{V}_1, \mathcal{V}_2)$. Set-theoretically, $\overline{SL}(\mathcal{V}_1, \mathcal{V}_2)$ is a disjoint union of $SL(\mathcal{V}_1, \mathcal{V}_2)$ and $\mathbb{B}(\mathcal{V}_1, \mathcal{V}_2)$.

Proposition 4.2.3. *We have an isomorphism* $\mathbb{P}(\mathcal{V}_1) \times_S \mathbb{P}(\mathcal{V}_2) \xrightarrow{\sim} \mathbb{B}(\mathcal{V}_1, \mathcal{V}_2)$ *of S-schemes.*

Proof. For an S-scheme $f : T \to S$, if $(f^*\mathcal{V}_1 \twoheadrightarrow \mathcal{L}_1, f^*\mathcal{V}_2 \twoheadrightarrow \mathcal{L}_2)$ is a T-valued point of $\mathbb{P}(\mathcal{V}_1) \times_S \mathbb{P}(\mathcal{V}_2)$, then $f^*\mathcal{V}_1 \oplus f^*\mathcal{V}_2 \twoheadrightarrow \mathcal{L}_1 \oplus \mathcal{L}_2$ is a T-valued point of $\mathbb{B}(\mathcal{V}_1, \mathcal{V}_2)$. One can check that this gives a bijective correspondence between the sets of T-valued points of $\mathbb{P}(\mathcal{V}_1) \times_S \mathbb{P}(\mathcal{V}_2)$ and $\mathbb{B}(\mathcal{V}_1, \mathcal{V}_2)$. $\square$

Acknowledgements. The author would like to thank Professor Hiromichi Takagi for warm encouragement. He would like to thank Professor Minoru Itoh for useful discussion. He would also like to thank Dr. Masao Aoki for discussion on algebraic stacks.

Note added in proof. In [M-O], Marian and Oprea proved Conjecture 1.0.3. Their method also proves the strange duality for parabolic bundles formulated in this paper.

Research Institute for Mathematical Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto, 606-8502, Japan
e-mail: abeken@kurims.kyoto-u.ac.jp

References

[B1] A. Beauville, *Fibrés de rang* 2 *sur les courbes, fibré déterminant et fonctions thêta* II, Bull. Soc. Math. France **119** (1991), 259–291.

[B2] ———, *Vector bundles on curves and generalized theta functions: recent results and open problems*, Complex Algebraic Geometry, MSRI Publications **28** (1995), 17–33.

[B3] ———, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993),75–96, Israel Math. Conf. Proc. 9, Bar-Ilan Univ., Ramat Gan (1996).

[B-L] A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), 385–419.

[BNR] A. Beauville, M. S. Narasimhan and S. Ramanan, *Spectral curves and the generalized theta divisor*, J. Reine Agnew. Math. **398** (1989), 169–179.

[Bel] P. Belkale, *The strange duality conjecture for generic curves*, math. AG/0602018.

[D-T] R. Donagi and L. Tu, *Theta functions for $SL(n)$ versus $GL(n)$*, Math. Res. Lett. **1** (1994), 345–357.

[Fal] G. Faltings, *Moduli-stacks for bundles on semistable curves*, Math. Ann. **304** (1996), 489–515.

[Ful] W. Fulton, *Young tableaux*, London Math. Soc., Student Text **35** (1997).

[EGAIV] A. Grothendieck and J. Dieudonné, *Éléments de géometrie algébrique* IV, Publications Mathémathiques IHES, N. 20, 24, 28, 32 (1964, 1965, 1966, 1967).

[L] Y. Laszlo, *A propos de l'espace des modules de fibrés de rang* 2 *sur une courbe*, Math. Ann. **299** (1994), 597–608.

[L-M] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 2000.

[Le] J. LePotier, *Lectures on Vector bundles*, Cambridge Stud. Adv. Math. **54**, Cambridge University Press, 1997.

[M-O] A. Marian and D. Oprea, *The rank-level duality for non-abelian theta functions*, math.AG/0605097

[Mat] H. Matsumura, *Commutative algebra*, Bebjamin, 1970.

[M-S] V. M. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. **248** (1980), 205–239.

[Mum] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, Oxford University Press, 1970.

[P] C. Pauly, *Espaces de modules de fibrés paraboliques et blocs conformes*, Duke Math. J. **84** (1996), 217–235.

[vG-P] B. van Geemen and E. Previato, *Prym varieties and the Verlinde formula*, Math. Ann. **294** (1992), 741–754.