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Time frequency analysis and multipliers of the spaces $M(p,q)(R^d)$ and $S(p,q)(R^d)$

By

A. Turan Gürkanli

Abstract

In the second section of this paper, in analogy to modulation spaces, we define the space $M(p,q)\left(R^d\right)$ to be the subspace of tempered distributions $f \in S'\left(R^d\right)$ such that the Gabor transform $V_g\left(f\right)$ of f is in the Lorentz space $L\left(p,q\right)\left(R^{2d}\right)$, where the window function g is a rapidly decreasing function. We endow this space with a suitable norm and show that the $M(p,q)\left(R^d\right)$ becomes a Banach space and is invariant under time-frequency shifts for $1 \leq p,q \leq \infty$. We also discuss the dual space of $M(p,q)\left(R^d\right)$ and the multipliers from $L^1\left(R^d\right)$ into $M(p,q)\left(R^d\right)$. In the third section we intend to study the intersection space $S\left(p,q\right)\left(R^d\right) = L^1\left(R^d\right) \cap M\left(p,q\right)\left(R^d\right)$ for $1 < p < \infty$, $1 \leq q \leq \infty$. We endow it with the sum norm and show that $S\left(p,q\right)\left(R^d\right)$ becomes a Banach convolution algebra. Further we prove that it is also a Segal algebra. In the last section we discuss the multipliers of $S\left(p,q\right)\left(R^d\right)$ and $M\left(p,q\right)\left(R^d\right)$.

1. Introduction

Through out this paper $C_0\left(R^d\right)$ and $S\left(R^d\right)$ denote the space of complex-valued continuous function on R^d that vanish at infinity, and the space of complex-valued continuous functions on R^d rapidly decreasing at infinity, respectively. In this paper we will work on R^d with Lebesgue measure dx. Let f be a measurable complex valued function on R^d. The translation and modulation operators are defined as $T_x f\left(t\right) = f\left(t-x\right)$ and $M_w f\left(t\right) = e^{2\pi i w t} f\left(t\right)$ for $x, w \in R^d$, respectively. It is easy to see that $T_x M_t = e^{-2\pi i x t} M_t T_x$. For $1 \leq p \leq \infty$ we write $\left(L^p\left(R^d\right), \|\cdot\|_p\right)$ for the Lebesgue spaces. It is also easy to show that $\|T_x M_t f\|_p = \|f\|_p$, [13].

Let $\langle x,t\rangle = \sum\limits_{i=1}^{d} x_i t_i$ be the usual scalar product on R^d. For $f \in L^1\left(R^d\right)$

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the Fourier transform $\hat{f}$ (or Ff) is given by

$$\hat{f}\,(t) = \int\limits_{R^d} f\,(x)\, e^{-2\pi i \langle x,t\rangle} dx.$$

It is known that $\hat{f} \in C_0(R^d)$.

The subject of Fourier analysis is one of the oldest subjects in mathematical analysis and in engineering. When f is thought of as an analog signal, then its domain R is called time-domain. In this case the Fourier transform $\hat{f}$ of f describes the spectral behavior of the signal f. Then the domain of $\hat{f}$ is called frequency domain. The Fourier transform provides only non-localized frequency information. For any $f \in L^1\left(R^d\right)$ its Fourier transform $\hat{f}\,(t)$ alone is not very useful for extracting information of the information of the spectrum $\hat{f}$ from local observation of the signal f. Thus the idea of Short-Time Fourier transform (STFT) or Gabor transform comes up. This transform maps the time domain signal into the joint time and frequency domain. Given any fix function $g \neq 0$ (called the window function) the Short-Time Fourier transform (STFT) or Gabor transform of a function f with respect to g is given by

$$V_g f\,(x,w) = \int\limits_{R^d} f\,(t)\, \overline{g\,(t-x)} e^{-2\pi i t w} dt,$$

for $x, w \in R^d$. It is known that if $f, g \in L^2\left(R^d\right)$ then $V_g f \in L^2\left(R^d \times R^d\right)$ and $V_g f$ is uniformly continuous. Moreover

$$V_g\left(T_u M_\eta f\right)(x,w) = e^{-2\pi i u w} V_g f\,(x-u, w-\eta)$$

for all $x, w, u, \eta \in R^d$, [13].

Let f be measurable function defined on a measure space (X, μ). For $y > 0$ we define

$$\lambda_f(y) = \mu\left(\{\, x \in X \,:\, |\, f(x)\,| > y \,\}\right).$$

The function $\lambda_f(y)$ is called the distribution function of f. The rearrangement of f is defined by

$$f^*(t) = \inf\{\, y > 0 \,:\, \lambda_f(y) \leq t \,\} = \sup\{\, y > 0 \,:\, \lambda_f(y) > t \,\}, \qquad t > 0.$$

Also, the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(x)\, dx.$$

It is easy to see that λ_f, f^*, f^{**} are nonincreasing and right continuous functions on $(0,\infty)$. The Lorentz space $L(p,\,q)(X,\,\mu)$ (shortly $L(p,\,q)$) is defined to be the vector space of all (equivalence classes) of measurable functions f such that $\| f \|^{*}_{(p,q)} < \infty$ where

$$\| f \|^{*}_{(p,q)} = \left(\frac{q}{p} \int_{0}^{\infty} \left[t^{\frac{1}{p}} f^{*}(t) \right]^{q} \frac{dt}{t} \right)^{\frac{1}{q}}, \quad 0 < p < \infty, 0 < q < \infty,$$

$$\| f \|^{*}_{(p,q)} = \sup_{t>0} t^{\frac{1}{p}} f^{*}(t), \quad 0 < p \le \infty, q = \infty.$$

It is known that $\| f \|^{*}_{(p,p)} = \| f \|_{p}$ and so $L(p,\, p) = L^{p}$. If $0 < q_1 \le q_2 \le \infty$, $0 < p < \infty$ then $\| f \|^{*}_{(p,q_2)} \le \| f \|^{*}_{(p,q_1)}$ holds and hence $L(p,\, q_1) \subset L(p,\, q_2)$, [14]. Also $L(p,\, q)(X,\, \mu)$ is a normed space with the norm

$$\| f \|_{(p,q)} = \left(\frac{q}{p} \int_{0}^{\infty} \left[t^{\frac{1}{p}} f^{**}(t) \right]^{q} \frac{dt}{t} \right)^{\frac{1}{q}}, \quad 0 < p < \infty, 0 < q < \infty,$$

$$\| f \|_{(p,q)} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), \quad 0 < p \le \infty, q = \infty\,.$$

It is also known that if $1 < p < \infty$, $1 \le q \le \infty$ we have

$$\|\cdot\|^{*}_{(p,\,q)} \le \|\cdot\|_{(p,\,q)} \le \frac{p}{p-1} \|\cdot\|^{*}_{(p,\,q)}.$$

(see O'Neil [18] and Yap [25])

For two Banach modules B_1 and B_2 over a Banach algebra A we write $M_A(B_1, B_2)$ or $Hom_A(B_1, B_2)$ for the space of all bounded linear operators satisfying $T(ab) = aT(b)$ for all $a \in A, b \in B_1$. This operators are called multiplier (right) or module homomorphism from B_1 into B_2 ([20], [21], [15]). It is known that

$$Hom_A(B_1, B_2^{*}) \cong (B_1 \otimes_A B_2)^{*},$$

where B_2^{*} is dual of B_2 and $B_1 \otimes_A B_2$ is the A-module tensor product of B_1 and B_2, (See Theorem 1.4 in [21]).

Let G be a locally compact Abelian group. A subalgebra $S^1(G)$ of $L^1(G)$ is called a Segal algebra if:

1) $S^1(G)$ is dense in $L^1(G)$ and if $f \in S^1(G)$ then $T_a f \in S^1(G)$, where $T_a f(x) = f(a^{-1}x)$;

2) $S^1(G)$ is a Banach algebra under some norm $\|.\|_{S^1}$ which also satisfies $\|f\|_{S^1} = \|T_a f\|_{S^1}$ for all $f \in S^1(G), a \in G$;

3) if $f \in S^1(G)$ then for every $\varepsilon > 0$ there exists a neighbourhood U of the identity element of G such that $\|T_y f - f\|_{S^1} < \varepsilon$ for all $y \in U$.

Let G be a locally compact Abelian group with dual group $\hat{G}$ and Haar measures dx and $d\hat{x}$, respectively. For $1 < p < \infty, 1 \le q \le \infty$, $A(p,q)(G)$ denotes the vector space of all functions $f \in L^1(G)$ whose Fourier transforms $\hat{f}$ belong to Lorentz space $L(p,q)\left(\hat{G}\right)$. For every $f \in A(p,q)(G)$ we supply a norm in $A(p,q)(G)$ by

$$\|f\|_{A(p,q)} = \|f\|_{L^1} + \left\|\hat{f}\right\|_{(p,q)},$$

where $\left\|\hat{f}\right\|_{(p,q)}$ is the norm of $\hat{f}$ in the Lorentz space $L(p,q)\left(\hat{G}\right)$. L. Y. H. Yap showed that $A(p,q)(G)$ is a Segal algebra [25]. Later a number of authors such as e.g Y.K. Chen and H.C. Lai [1], T.S. Quek and L.Y.H. Yap [19] worked on these spaces.

The main purpose of this paper is to define the spaces $M(p,q)\left(R^d\right)$ and $S(p,q)\left(R^d\right)$ like $A(p,q)\left(R^d\right)$ using the Gabor transform instead of Fourier transform and study the properties of these spaces. Also to show that some of the results for $A(p,q)\left(R^d\right)$ are true for $S(p,q)\left(R^d\right)$, and the spaces $M(p,q)\left(R^d\right)$ and $S(p,q)\left(R^d\right)$ are a kind of generalization of modulation space $M^{p,q}\left(R^d\right)$, (see, [13]).

2. The space $M(p,q)\left(R^d\right)$

Using the Gabor transform with respect to a rapidly decreasing function define a space $M(p,q)\left(R^d\right)$ of tempered distributions as follows.

Definition 2.1. Fix a non zero window $g \in S\left(R^d\right)$ and $1 \leq p,q \leq \infty$. We let $M(p,q)\left(R^d\right)$ denote the subspace of tempered distributions $S'\left(R^d\right)$ consisting of $f \in S'\left(R^d\right)$ such that the Gabor transform $V_g(f)$ of f is in the Lorentz space $L(p,q)\left(R^{2d}\right)$. We endow the vector space $M(p,q)\left(R^d\right)$ with the norm

$$\|f\|_{M(p,q)} = \|V_g(f)\|_{(p,q)}, \tag{2.1}$$

where $\|\cdot\|_{(p,q)}$ is the norm of the Lorentz space [14]. Since if $p=q$, $L(p,q)\left(R^{2d}\right) = L^p\left(R^{2d}\right)$, we denote $M(p,p)\left(R^d\right) = M(p)\left(R^d\right)$.

Before we begin to study the structure of $M(p,q)\left(R^d\right)$ we recall the adjoint operator of V_g. Given a non-zero window γ and a function F on R^{2d} we define

$$\left\langle V_\gamma^* F, f\right\rangle = \left\langle F, V_\gamma f\right\rangle. \tag{2.2}$$

Lemma 2.1. *Let $1 \leq q \leq p < \infty$. If $f \in L^1\left(R^d\right)$ is bounded and continuous then $f \in L(p,q)\left(R^d\right)$.*

Proof. If $p \geq q$ we have

$$\int_0^\infty x^{\frac{q}{p}-1}\left[f^*(x)\right]^q dx = \int_0^1 x^{\frac{q}{p}-1}\left[f^*(x)\right]^q dx + \int_1^\infty x^{\frac{q}{p}-1}\left[f^*(x)\right]^q dx.$$

Since f^* is continuous on [0,1] we write

$$\begin{aligned}\int_0^1 x^{\frac{q}{p}-1}\left[f^*(x)\right]^q dx &\le \left(\operatorname*{supp}_{x\in[0,1]} f^*(x)\right)^q \int_0^1 x^{\frac{q}{p}-1}dx \\ &= \frac{p}{q}\left(\operatorname*{supp}_{x\in[0,1]} f^*(x)\right)^q < \infty.\end{aligned} \tag{2.3}$$

Also, since $f \in L^1\left(R^d\right)$ and bounded, we write

$$\begin{aligned}\int_1^\infty x^{\frac{q}{p}-1}\left[f^*(x)\right]^q dx &\le \int_1^\infty \left[f^*(x)\right]^q dx \le \int_0^\infty \left[f^*(x)\right]^q dx \\ &= \int_{R^d} |f(t)|^q dt \le \|f\|_\infty^{q-1}\|f\|_1 < \infty.\end{aligned} \tag{2.4}$$

Finally using (2.3) and (2.4) we have

$$\int_0^\infty x^{\frac{q}{p}-1} f^{*^q}(x)\,dx < \infty.$$

That means, $f \in L(p,q)\left(R^d\right)$. □

Proposition 2.1. *If $1 \le p,q < \infty$ and $g \in S\left(R^d\right)$ then $S\left(R^d\right) \subset M(p,q)\left(R^d\right)$ is dense in $M(p,q)\left(R^d\right)$.*

Proof. Let $f \in S\left(R^d\right)$. If $p \le q$ we write

$$\begin{aligned}\|f\|_{M(p,q)} = \|V_g f\|_{(p,q)} &\le \left\{\sup_{z\in R^{2d}} (1+|z|)^n V_g f(z)\right\}\left\|(1+|z|)^{-n}\right\|_{(p,q)} \\ &\le \left\{\sup_{z\in R^{2d}} (1+|z|)^n V_g f(z)\right\}\left\|(1+|z|)^{-n}\right\|_{p}.\end{aligned} \tag{2.5}$$

Then the right hand side of this expression is finite for sufficiently large n. If $p \ge q$ the right hand side of

$$\|f\|_{M(p,q)} = \|V_g f\|_{(p,q)} \le \left\{\sup_{z\in R^{2d}} (1+|z|)^n V_g f(z)\right\}\left\|(1+|z|)^{-n}\right\|_{(p,q)}$$

is also finite for sufficiently large n by Lemma 2.1. Hence $S\left(R^d\right) \subset M(p,q)\left(R^d\right)$. If one uses the techniques in the proof of Proposition 11.3.4, [13], one obtains that $S\left(R^d\right)$ is dense in $M(p,q)\left(R^d\right)$. □

Theorem 2.1. *Assume that $g,\gamma \in S\left(R^d\right)$ are non-zero windows and $1 \le p,q < \infty$. Then*

1. *V_γ^* maps $L(p,q)\left(R^{2d}\right)$ into $M(p,q)\left(R^d\right)$ and satisfies*

$$\left\|V_\gamma^* F\right\|_{M(p,q)} \le \|V_g\gamma\|_1 \|F\|_{(p,q)}.$$

2. *The inversion formula*

$$f = \frac{1}{\langle \gamma, g \rangle} \iint\limits_{R^{2d}} V_g (f) (x, w) M_w T_x \gamma dx dw$$

holds in $M(p,q) \left(R^d\right)$.

Proof.

1. We prove first that $V_\gamma^* F$ is a tempered distribution. Let $f \in S\left(R^d\right)$. Then $V_\gamma (f) \in L\left(p^{'}, q^{'}\right)\left(R^{2d}\right)$ by Proposition 2.1 and Definition of $M\left(p^{'}, q^{'}\right)\left(R^{2d}\right)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. If $p \geq q \left(\text{ hence } p^{'} \leq q^{'}\right)$, by Holder's inequality for Lorentz space we write

$$\begin{aligned} \left|\left\langle V_\gamma^* F, f\right\rangle\right| &= \left|\left\langle F, V_\gamma f\right\rangle\right| = \left|\iint\limits_{R^{2d}} F(x,w) \overline{V_\gamma (f)(x,w)} dx dw\right| \\ &\leq \|F\|_{(p,q)} \|V_\gamma f\|_{\left(p', q'\right)} \leq \|F\|_{(p,q)} \|V_\gamma f\|_{p'}. \end{aligned} \tag{2.6}$$

for all $f \in S\left(R^d\right)$. Thus from (2.6) we obtain

$$\begin{aligned} \left|\left\langle V_\gamma^* F, f\right\rangle\right| &\leq \|F\|_{(p,q)} \|V_\gamma f\|_{p'} \\ &\leq \|F\|_{(p,q)} \left\{\sup_{z \in R^{2d}} (1 + |z|)^n V_\gamma f(z)\right\} \left\|(1+|z|)^{-n}\right\|_{p'}. \end{aligned} \tag{2.7}$$

This expression is finite for sufficiently large n. Using the equivalence of seminorms ([13], Corollary 11.2.6) it follows that $V_\gamma^* F \in S'\left(R^d\right)$. If $p \leq q \left(\text{hence } p^{'} \geq q^{'}\right)$ then

$$\left\|(1+|z|)^{-n}\right\|_{\left(p', q'\right)}$$

is finite for sufficienly large n by Lemma 2.1. Hence

$$\begin{aligned} \left|\left\langle V_\gamma^* F, f\right\rangle\right| &\leq \|F\|_{(p,q)} \|V_\gamma f\|_{p'} \\ &\leq \|F\|_{(p,q)} \left\{\sup_{z \in R^{2d}} (1 + |z|)^n V_\gamma f(z)\right\} \left\|(1+|z|)^{-n}\right\|_{\left(p', q'\right)} \end{aligned} \tag{2.8}$$

is also finite. This implies that $V_\gamma^* F \in S'\left(R^d\right)$. Since $V_\gamma^* F \in S'\left(R^d\right)$, it has Gabor transform and we have

$$\begin{aligned} V_g V_\gamma^* F(u, \eta) &= \left\langle V_\gamma^* F, M_\eta T_u g\right\rangle = \left\langle F, V_\gamma \left(M_\eta T_u g\right)\right\rangle \\ &= \iint\limits_{R^{2d}} F(x,w) V_g \gamma (u - x, \eta - w) e^{-2\pi i x (\eta - w)} dx dw. \end{aligned}$$

Then

$$\left|V_g V_\gamma^* F\,(u,\eta)\right| \leq (|F| * |V_g\gamma|)\,(u,\eta). \tag{2.9}$$

Since $V_g\gamma \in S\left(R^{2d}\right) \subset L^1\left(R^{2d}\right)$ and $L\,(p,q)\left(R^{2d}\right)$ is $L^1\left(R^{2d}\right)$-module we obtain

$$\begin{aligned}\left\|V_\gamma^* F\right\|_{M(p,q)} &= \left\|V_g\left(V_\gamma^* F\right)\right\|_{(p,q)} \\ &\leq \||F| * |V_g\gamma|\|_{(p,q)} \leq \|F\|_{(p,q)}\,\|V_g\gamma\|_1.\end{aligned} \tag{2.10}$$

2. Since every element of $M\,(p,q)\left(R^d\right)$ is a tempered distribution, we complete the proof of this part by Theorem 11.2.3 and Corollary 11.2.7 in [13]. □

Since $L\,(p,q)\left(R^{2d}\right)$ is a solid translation invariant Banach Function space then $M\,(p,q)\left(R^d\right)$ is a Coorbit space. Hence it is a Banach space for $1 \leq q \leq \infty$ and the definition of $M\,(p,q)\left(R^d\right)$ is independed of the choice of the window function $g \in S\left(R^d\right)$. Also $M\,(p,q)\left(R^d\right)$ is invariant under time-frequency shifts and $\|T_x M_w f\|_{M(p,q)} = \|f\|_{M(p,q)}$. Different windows yield equivalent norms (see Theorem 4.2 in [5]).

Proposition 2.2. *Let $1 < p < \infty$, $1 \leq q < \infty$, and $g \in S\left(R^d\right)$. Then the mapping $z \to T_z f$ of R^d into $M\,(p,q)\left(R^d\right)$ is continuous for every $f \in M\,(p,q)\left(R^d\right)$.*

Proof. Let $f \in M\,(p,q)\left(R^d\right)$ and $z \in R^d$. We write

$$\begin{aligned}\|T_z f - f\|_{M(p,q)} &= \|V_g(T_z f - f)\|_{(p,q)} = \|V_g\,(T_z f) - V_g f\|_{(p,q)} \\ &= \left\|e^{-2\pi wzi} T_{(z,0)}(V_g f) - V_g f\right\|_{(p,q)} \\ &\leq \left\|e^{-2\pi wzi} T_{(z,0)}(V_g f) - e^{-2\pi wzi} V_g f\right\|_{(p,q)} \\ &\quad + \left\|e^{-2\pi wzi} V_g f - V_g f\right\|_{(p,q)} \\ &= \left\|e^{-2\pi wzi} (T_{(z,0)}(V_g f) - V_g f)\right\|_{(p,q)} \\ &\quad + \left\|(e^{-2\pi wzi} - 1) V_g f\right\|_{(p,q)}.\end{aligned} \tag{2.11}$$

Since

$$\left|e^{-2\pi wzi} (T_{(z,0)}(V_g f) - V_g f)\,(x,w)\right| = \left|(T_{(z,0)}(V_g f) - V_g f)\,(x,w)\right|$$

then

$$\left\|e^{-2\pi wzi} (T_{(z,0)}(V_g f) - V_g f)\right\|_{(p,q)} = \left\|(T_{(z,0)}(V_g f) - V_g f)\right\|_{(p,q)}. \tag{2.12}$$

It is known by Proposition 2.3 in [25] that the mapping $(z,t) \to T_{(z,t)} F$ is continuous for every $F \in L(p,q)(R^{2d})$. If we apply the argument to $V_g f$, the mapping $(z,t) \to T_{(z,t)}(V_g f)$ is continuous. Hence the right side of (2.12) tends to zero as z tends to zero. This implies the first term on the right side in (2.11) tends to zero as z tends to zero. Now let $h_z(x,w) = \left|e^{-2\pi zwi} - 1\right| |V_g f(x,w)|$. It is easy to see that $h_z(x,w) \to 0$ as $z \to 0$ for all $(x,w) \in R^{2d}$. This implies that the rearrangements of $(e^{-2\pi zwi} - 1)(V_g f(x,w))$ tends to zero as z tends to zero. Since

$$h_z(x,w) = \left|e^{-2\pi zwi} - 1\right| |V_g f(x,w)| \le 2 |V_g f(x,w)|$$

and $V_g f \in L(p,q)\left(R^{2d}\right)$ we write $(h_z(x,w))^* \le (2|V_g f(x,w)|)^*$. Thus by the Lebesgue dominated convergence theorem

$$\left\| \left|e^{-2\pi zwi} - 1\right| |V_g f(x,w)| \right\|_{(p,q)} = \left\| (e^{-2\pi zwi} - 1) V_g f \right\|_{(p,q)}$$

tends to zero as z tends to zero. This implies the second term on the right side in (2.11) tends to zero as z tends to zero. This completes the proof. $\square$

Theorem 2.2. *$M(p,q)\left(R^d\right)$ is an essential Banach convolution module over $L^1\left(R^d\right)$.*

Proof. Let $f \in M(p,q)\left(R^d\right)$ and $h \in L^1\left(R^d\right)$. It is known by Lemma 3.1.1 in [13] that

$$V_g(f*h)(x,w) = e^{-2\pi ixw}(f*h)*M_w g^{\sim} \tag{2.13}$$

where $M_w g^{\sim}(x) = e^{2\pi ixw} g^{\sim}(x) = e^{2\pi ixw}\overline{g}(-x)$. Hence

(2.14)

$$\begin{aligned}
\|f*h\|_{M(p,q)} &= \|V_g(f*h)\|_{(p,q)} = \|(f*h)*M_w g^{\sim}\|_{(p,q)} \\
&= \|f*(h*M_w g^{\sim})\|_{(p,q)} = \left\| \int_{R^d} f(u)(h*M_w g^{\sim})(x-u)\,du \right\|_{(p,q)} \\
&\le \int_{R^d} \|f(u)(h*M_w g^{\sim})(x-u)\|_{(p,q)}\,du \\
&= \int_{R^d} |f(u)| \, \|(h*M_w g^{\sim})(x-u)\|_{(p,q)}\,du \\
&= \int_{R^d} |f(u)| \left\| T_{(0,u)}(h*M_w g^{\sim})(x) \right\|_{(p,q)}\,du.
\end{aligned}$$

Since $L(p,q)\left(R^{2d}\right)$ is strongly translation invariant, by Lemma 3.1 in [1] and

(2.15) we write

$$\begin{aligned}
\|f*h\|_{M(p,q)} &\leq \int\limits_{R^d} |f(u)| \left\|T_{(0,u)}\left(h*M_w g^{\sim}\right)(x)\right\|_{(p,q)} du \\
&= \int\limits_{R^d} |f(u)| \left\|\left(h*M_w g^{\sim}\right)\right\|_{(p,q)} du = \left\|\left(h*M_w g^{\sim}\right)\right\|_{(p,q)} \int\limits_{R^d} |f(u)| du \\
&= \left\|\left(h*M_w g^{\sim}\right)\right\|_{(p,q)} \|f\|_1 = \|h\|_{M(p,q)} \|f\|_1 .
\end{aligned}$$

Now let $f \in M(p,q)\left(R^d\right)$. Since the mapping $z \rightarrow T_z f$ of R^d into $M(p,q)\left(R^d\right)$ is continuous by Proposition 2.2, then given $\epsilon > 0$ there exists a compact neighbourhood U of $0 \in R^d$ such that

$$\|T_z f - f\|_{M(p,q)} < \varepsilon$$

for all $z \in U$. Assume that $g \in L^1\left(R^d\right)$ is non-negative continuous function with compact support $suppg \subset U$ and $\int\limits_{R^d} g = 1$. Then

$$\begin{aligned}
\|g*f-f\|_{M(p,q)} &= \left\| \int\limits_{R^d} g(z) f(y-z)\, dz - \int\limits_{R^d} g(z) f(y)\, dz \right\|_{M(p,q)} \\
&\leq \int\limits_{R^d} \|g(z)(f(y-z)-f(y))\|_{M(p,q)} dz \\
&\leq \int\limits_{R^d} |g(z)| \|T_z f - f\|_{M(p,q)}\, dz \leq \varepsilon \int\limits_{R^d} g(z)\, dz = \varepsilon .
\end{aligned}$$

Since $M(p,q)\left(R^d\right)$ is Banach module over $L^1\left(R^d\right)$ and $g*f \in L^1\left(R^d\right) * M(p,q)\left(R^d\right)$ then $L^1\left(R^d\right) * M(p,q)\left(R^d\right)$ is dense in $M(p,q)\left(R^d\right)$. Hence $M(p,q)\left(R^d\right) = L^1\left(R^d\right) * M(p,q)\left(R^d\right)$ by Module Factorization Theorem in [24]. That means $M(p,q)\left(R^d\right)$ is an essential module over $L^1\left(R^d\right)$. □

Theorem 2.3. *The dual space of $M(p,q)\left(R^d\right)$, $1 < p,q < \infty$ is $M(p',q')\left(R^d\right)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and hence these spaces are reflexive. Also, the dual pair is of the form*

$$\langle f,h\rangle = \langle V_g f, V_g h\rangle \int\limits_{R^{2d}} V_g f(z)\, V_g h(z)\, dz$$

for all $f \in M(p,q)\left(R^d\right)$, $h \in M(p',q')\left(R^d\right)$.

Proof. Let $u \in M\left(p^{'},q^{'}\right)\left(R^d\right)$. It is known that the dual space of $L(p,q)\left(R^{2d}\right)$ is $L\left(p^{'},q^{'}\right)\left(R^{2d}\right)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and they are reflexive. Also the dual form is of the form

$$\langle s,t\rangle = \int\limits_{R^{2d}} s(z)\,t(z)\,dz, \quad s\in L(p,q)\left(R^{2d}\right), t\in L\left(p^{'},q^{'}\right)(R^{2d}), \tag{2.15}$$

(see [17], [3]). Thus

$$l_u(f) = \int\limits_{R^{2d}} V_g f(z)\,V_g u(z)\,dz \tag{2.16}$$

defines a linear functional on $M(p,q)\left(R^d\right)$ and, by Hölder's inequality for the Lorentz space, we have

$$|l_u(f)| \le \|V_g u\|_{\left(p^{'},q^{'}\right)} \|V_g f\|_{(p,q)} \tag{2.17}$$

for all $f\in M(p,q)\left(R^d\right)$. That means l_u is a bounded linear functional on $M(p,q)\left(R^d\right)$.

Conversely assume that $l\in\left(M(p,q)\left(R^d\right)\right)^*$. It is easy to see that $M(p,q)\left(R^d\right)$ is isometrically isomorphic to the closed subspace

$$N = \left\{V_g f\in L(p,q)\left(R^{2d}\right) : f\in M(p,q)\left(R^d\right)\right\} \tag{2.18}$$

of $L(p,q)\left(R^{2d}\right)$. Hence $\widetilde{l}(V_g f) := l(f)$ defines a continuous linear functional on N and by the Theorem of Hahn-Banach, $\widetilde{l}$ extends continuously to $L(p,q)\left(R^{2d}\right)$. Thus by (2.7) in [14] or [1], there exists $K\in L\left(p^{'},q^{'}\right)\left(R^{2d}\right)$, such that

$$\widetilde{l}(V_g f) = \int\limits_{R^{2d}} V_g f(z)\,K(z)\,dz = l(f). \tag{2.19}$$

Also, since $K\in L\left(p^{'},q^{'}\right)\left(R^{2d}\right)$, from Theorem 2.1 there exists $k\in M\left(p^{'},q^{'}\right)\left(R^d\right)$ such that $k=V_g^* K$. Thus every continuous linear functional on $M(p,q)\left(R^d\right)$ is of the form (2.20) and $\left(M(p,q)\left(R^d\right)\right)^* = M\left(p^{'},q^{'}\right)\left(R^d\right)$. □

3. The Space $S(p,q)\left(R^d\right)$.

Let p,q be real numbers such that $1<p<\infty, 1\le q<\infty$ and $g\in S\left(R^d\right), g\neq 0$. Write $L^1\left(R^d\right)\cap M(p,q)\left(R^d\right)$ as $S(p,q)\left(R^d\right)$ and for $f\in S(p,q)\left(R^d\right)$ define

$$\|f\|_S = \|f\|_1 + \|f\|_{M(p,q)} = \|f\|_1 + \|V_g f\|_{(p,q)}. \tag{3.1}$$

It is easy to verify that

$$S(p,q)\left(R^d\right) = \left\{f\in L^1\left(R^d\right) : V_g f\in L(p,q)\left(R^{2d}\right)\right\}. \tag{3.2}$$

In this section we will discuss some properties of this space.

Theorem 3.1. *For $1 < p < \infty, 1 \leq q < \infty$ the space $S(p,q)$ is a Banach convolution algebra with the norm*

$$\|f\|_S = \|f\|_1 + \|V_g f\|_{(p,q)}.$$

Proof. Let (f_n) be a Cauchy sequence in $S(p,q)$. Then (f_n) is a Cauchy sequence in $L^1(R^d)$ and $(V_g f_n)$ is a Cauchy sequence in $L(p,q)(R^{2d})$. Since $L^1(R^d)$ and $L(p,q)(R^{2d})$ are Banach spaces (f_n) converges to a function f in $L^1(R^d)$ and $(V_g f_n)$ converges to a function h in $L(p,q)(R^{2d})$. This implies that $(V_g f_n)$ has a subsequence $(V_g f_{n_k})$ which converges pointwise to h almost everywhere. Let $\varepsilon > 0$ be given. Since (f_n) converges to f in $L^1(R^d)$, there exists $n_0 \in N$ such that

$$\|f_n - f\|_1 < \frac{\varepsilon}{\|\hat{g}\|_1} \tag{3.3}$$

for all $n \geq n_0$. If we apply the Lemma 3.1.1 in [13] and the Hölder's inequality, we have

$$\begin{aligned} |V_g f_n(x,w) - V_g f(x,w)| &= |V_g(f_n - f)(x,w)| \\ &= \left|\left\langle (f_n - f)^\wedge, T_x M_{-w}\hat{g}\right\rangle\right| \\ &\leq \left\|(f_n - f)^\wedge\right\|_\infty \|T_x M_{-w}\hat{g}\|_1 \\ &= \left\|(f_n - f)^\wedge\right\|_\infty \|\hat{g}\|_1 \leq \|f_n - f\|_1 \|\hat{g}\|_1. \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$|V_g f_n(x,w) - V_g f(x,w)| < \frac{\varepsilon}{\|\hat{g}\|_1}. \|\hat{g}\|_1 = \varepsilon. \tag{3.5}$$

That means, $(V_g f_n)$ converges pointwise to $V_g f$.

If one uses the inequality

$$\begin{aligned} &|V_g f_{n_k}(x,w) - V_g f(x,w)| \\ &\quad = |V_g f_{n_k}(x,w) - V_g f(x,w) + V_g(f_n)(x,w) - V_g(f_n)(x,w)| \\ &\quad \leq |V_g f_{n_k}(x,w) - V_g f_n(x,w)| + |V_g f_n(x,w) - V_g f(x,w)| \\ &\quad \leq \|f_{n_k} - f_n\|_1 \|\hat{g}\|_1 + \|f_n - f\|_1 \|\hat{g}\|_1 \end{aligned}$$

and (3.5), one obtains that $(V_g f_{n_k})$ also converges pointwise to $V_g f$. Finally, using the inequality

$$|V_g f(x,w) - h(x,w)| \leq |V_g f_{n_k}(x,w) - V_g f(x,w)| + |V_g f_{n_k}(x,w) - h(x,w)|$$

we have $V_g f(x,w) = h(x,w)$ a.e. Then, given any $\varepsilon > 0$, there exist $n_1, n_2 \in N$ such that

$$\|f_n - f\|_1 < \frac{\varepsilon}{2} \quad \textit{and} \quad \|V_g(f_n - f)\|_{(p,q)} = \|V_g f_n - V_g f\|_{(p,q)} < \frac{\varepsilon}{2}$$

for all $n > n_1$ and $n > n_2$. Hence for all $n > n_0 = \max\{n_1, n_2\}$ we have

$$\|f_n - f\|_S = \|f_n - f\|_1 + \|V_g(f_n - f)\|_{(p,q)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.6}$$

Thus $S(p,q)(R^d)$ is a Banach space.

Now let $f, h \in S(p,q)(R^d)$. We write

$$\begin{aligned} \|V_g(f*h)\|_{(p,q)} &= \|(f*h)*M_w g^{\sim}\|_{(p,q)} = \|f*(h*M_w g^{\sim})\|_{(p,q)} \\ &= \left\| \int_{R^d} f(u)(h*M_w g^{\sim})(x-u)\,du \right\|_{(p,q)} \\ &\leq \int_{R^d} \|f(u)(h*M_w g^{\sim})(x-u)\|_{(p,q)}\,du \\ &= \int_{R^d} |f(u)|\,\|(h*M_w g^{\sim})\|_{(p,q)}\,du = \|f\|_1 \cdot \|V_g h\|_{(p,q)}. \end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned} \|f*h\|_S &= \|f*h\|_1 + \|V_g(f*h)\|_{p,q} \\ &= \|f*h\|_1 + \|f*V_g h\|_{p,q} \\ &\leq \|f\|_1 \cdot \|h\|_1 + \|f\|_1 \cdot \|V_g h\|_{p,q} \\ &= \|f\|_1 \cdot \left(\|h\|_1 + \|V_g h\|_{p,q}\right) \\ &= \|f\|_1 \cdot \|h\|_S \leq \|f\|_S \cdot \|h\|_S. \end{aligned} \tag{3.8}$$

It is easy to prove the other conditions for $S(p,q)(R^d)$ to be a Banach algebra. This completes the proof. □

Proposition 3.1. *Let $1 < p < \infty$, $1 \leq q < \infty$ and $g \in S(R^d)$. Then*
a) $S(p,q)(R^d)$ is strongly translation invariant.
b) The mapping $z \to T_z f$ of R^d into $S(p,q)(R^d)$ is continuous.

Proof. a) Let $f \in S(p,q)(R^d)$ and $z \in R^d$. It is known that $\|T_z f\|_1 = \|f\|_1$. It is also known by Lemma 3.1 in [1] and Lemma 3.1.3 in [13] that

$$\left\|T_{(x,w)} V_g f\right\|_{(p,q)} = \|V_g f\|_{(p,q)}$$

and

$$|V_g(T_x M_w f)(u,v)| = \left|e^{-2\pi i x v} V_g f(u-x, v-w)\right| = \left|T_{(x,w)} V_g f(u,v)\right|.$$

Thus we obtain

$$\|V_g(T_x M_w f)\|_{(p,q)} = \left\|T_{(x,w)} V_g f\right\|_{(p,q)} = \|V_g f\|_{(p,q)} = \|f\|_{M(p,q)}. \tag{3.9}$$

From (3.9) we write

$$\begin{aligned}\|T_z f\|_{M(p,q)} &= \|V_g (T_z f)\|_{(p,q)} = \left\|T_{(z,o)} V_g f\right\|_{(p,q)} \\ &= \|V_g f\|_{(p,q)} = \|f\|_{M(p,q)}.\end{aligned}$$

This implies

$$\begin{aligned}\|T_z f\|_S &= \|T_z f\|_1 + \|T_z f\|_{M(p,q)} \\ &= \|f\|_1 + \|f\|_{M(p,q)} = \|f\|_S.\end{aligned} \tag{3.10}$$

b) It is known that the function $z \to T_z f$ of R^d into $L^1\left(R^d\right)$ is continuous. We proved in Proposition 2.2 that $z \to T_z f$ is continuous from R^d into $M(p,q)\left(R^d\right)$. Then it is easily proved that $z \to T_z f$ is continuous from R^d into $S(p,q)\left(R^d\right)$. □

The following important Theorem follows immediately from Theorem 3.1 and Proposition 3.1.

Theorem 3.2. *For $1 < p < \infty$, $1 \leq q < \infty$ and $g \in S\left(R^d\right)$, the space $S(p,q)\left(R^d\right)$ is a Segal algebra.*

Proof. We have already proved in Theorem 3.1 and Proposition 3.1 some of the necessary conditions for Segal algebras. To complete the proof it is enough to show that $S(p,q)\left(R^d\right)$ is dense in $L^1\left(R^d\right)$. It is known that $S\left(R^d\right) \subset L^1\left(R^d\right)$ is dense in $L^1\left(R^d\right)$. It is also proved in Proposition 2.1 that $S\left(R^d\right) \subset M(p,q)\left(R^d\right)$. Hence $S\left(R^d\right) \subset S(p,q)\left(R^d\right)$. Since $S\left(R^d\right)$ is dense in $L^1\left(R^d\right)$ then $S(p,q)\left(R^d\right)$ is dense in $L^1\left(R^d\right)$. □

Theorem 3.3. *$S(p,q)\left(R^d\right)$ is an essential Banach ideal in $L^1\left(R^d\right)$.*

Proof. Let $f \in S(p,q)\left(R^d\right)$ and $h \in L^1\left(R^d\right)$. Since, by Theorem 2.2, $M(p,q)\left(R^d\right)$ is an essential Banach convolution module over $L^1\left(R^d\right)$, we have $f * h \in M(p,q)\left(R^d\right)$ and

$$\|f * h\|_{M(p,q)} \leq \|f\|_{M(p,q)} \cdot \|h\|_1 . \tag{3.11}$$

Also $f * h \in L^1\left(R^d\right)$ and

$$\|f * h\|_1 \leq \|f\|_1 \cdot \|h\|_1 . \tag{3.12}$$

This implies $f * h \in S(p,q)\left(R^d\right)$ and

$$\begin{aligned}\|f * h\|_{S(p,q)} &= \|f * h\|_1 + \|f * h\|_{M(p,q)} \\ &\leq \|f\|_1 \|h\|_1 + \|f\|_{M(p,q)} \|h\|_1 \\ &\leq \|h\|_1 \left(\|f\|_1 + \|f\|_{M(p,q)}\right) = \|h\|_1 \|f\|_{S(p,q)} .\end{aligned} \tag{3.13}$$

In order to see that $L^1\left(R^d\right) * S(p,q)\left(R^d\right)$ is dense in $S(p,q)\left(R^d\right)$, take any $h \in S(p,q)\left(R^d\right)$. Hence $h \in L^1\left(R^d\right)$ and $h \in M(p,q)\left(R^d\right)$. Since the map $z \to T_z h$ of R^d into $S(p,q)\left(R^d\right)$ is continuous, the maps $z \to T_z h$ of R^d into $L^1\left(R^d\right)$ and $z \to T_z h$ of R^d into $M\left(R^d\right)$ are continuous. Thus given $\varepsilon > 0$ there exists a compact neighbourhood U of $0 \in R^d$ such that

$$\|T_z h - h\|_1 < \frac{\varepsilon}{2} \tag{3.14}$$

and

$$\|T_z h - h\|_{M(p,q)} < \frac{\varepsilon}{2} \tag{3.15}$$

for all $z \in U$. Assume that $f \in L^1\left(R^d\right)$ is non-negative continuous function with compact support $suppf \subset U$ and $\int\limits_{R^d} f(t)\,dt = 1$. Then

$$\|f * h - h\|_{M(p,q)} < \frac{\varepsilon}{2} \tag{3.16}$$

by Theorem 2.2. Also,

$$\begin{aligned} \|f * h - h\|_1 &= \left\| \int\limits_{R^d} f(z)\, h(y-z)\,dz - \int\limits_{R^d} f(z)\, h(y)\,dz \right\|_1 \\ &\le \left\| \int\limits_{R^d} f(z)\,(h(y-z) - h(y))dz \right\|_1 \\ &\le \int\limits_{R^d} |f(z)|\, \|T_z h - h\|_1\,dz \le \frac{\varepsilon}{2} \int\limits_{R^d} f(z)\,dz = \frac{\varepsilon}{2}. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) we see that

$$\|f * h - h\|_{S(p,q)} \le \|f * h - h\|_1 + \|f * h - h\|_{M(p,q)} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $f * h \in L^1\left(R^d\right)$ and $f * h \in M(p,q)\left(R^d\right)$ then $f * h \in S(p,q)\left(R^d\right)$. This shows that $L^1\left(R^d\right) * S(p,q)\left(R^d\right)$ is dense in $S(p,q)\left(R^d\right)$. Hence $L^1\left(R^d\right) * S(p,q)\left(R^d\right) = S(p,q)\left(R^d\right)$ by Module Factorization Theorem (see [24]). This completes the proof. □

Consider for each $p, q\,(1 \le p, q < \infty)$ the mapping Φ from $S(p,q)\left(R^d\right)$ into $L^1\left(R^d\right) \times L(p,q)\left(R^d\right)$ defined by $\Phi(f) = (f, V_g f)$. This is a linear isometry of $S(p,q)\left(R^d\right)$ into $L^1\left(R^d\right) \times L(p,q)\left(R^d\right)$ with the norm

$$|||f||| = \|f\|_1 + \|V_g f\|_{(p,q)}\,, (f \in S(p,q)(R^d)). \tag{3.18}$$

Hence we consider $S(p,q)\left(R^d\right)$ as a closed subspace of the Banach space $L^1\left(R^d\right) \times L(p,q)\left(R^d\right)$. Let

$$H = \left\{(f, V_g f) : f \in S(p,q)\left(R^d\right)\right\}$$

and

$$(3.19)\quad K=\left\{\begin{array}{c}(\varphi,\psi):(\varphi,\psi)\in L^{\infty}\left(R^d\right)\times L\left(p^{'},q^{'}\right)\left(R^{2d}\right),\\ \int\limits_{R^d} f(y)\,\varphi(y)\,dy+\iint\limits_{R^{2d}} V_g f(x,w)\,\psi(x,w)\,dxdw=0,\ \text{for}\\ \text{all }\ (f,V_g f)\in H\end{array}\right\},$$

where $\frac{1}{p}+\frac{1}{p^{'}}=1$ and $\frac{1}{q}+\frac{1}{q^{'}}=1$.

The following Proposition is easily proved by Duality Theorem 1.7 in [17].

Proposition 3.2. *For each $p,q\,(1\le p,q<\infty)$ the dual space $S(p,q)\left(R^d\right)^{*}$ of $S(p,q)\left(R^d\right)$ is isomorphic to $L^{\infty}\left(R^d\right)\times L(p',q')\left(R^d\right)/K$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$.*

4. Multipliers of $S(p,q)\left(R^d\right)$ and $M(p,q)\left(R^d\right)$

Proposition 4.1. *If $g\in S\left(R^d\right)$ and $1\le p,q<\infty$ then the multiplier space $M\left(L^1\left(R^d\right),M(p',q')\left(R^d\right)\right)$ is isomorphic to $M(p',q')\left(R^d\right)$.*

Proof. By Theorem 2.2 and Corollary 2 in [21] we write

$$\begin{aligned}M\left(L^1\left(R^d\right),M(p',q')\left(R^d\right)\right)&=\left(L^1\left(R^d\right)*M(p,q)\left(R^d\right)\right)^{*}\\&=\left(M(p,q)\left(R^d\right)\right)^{*}=M(p',q')\,(R^d).\end{aligned}$$

□

Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity with compactly supported Fourier transforms (band limited functions) in $L^1\left(R^d\right)$. Define the vector space

$$M_S\left(R^d\right)=\left\{\mu\in M\left(R^d\right):\|\mu*e_\alpha\|_S<C(\mu)\ \text{ for all } \alpha\in I\right\},$$

where $M\left(R^d\right)$ is the space of bounded regular Borel measure on R^d and $C(\mu)$ is a constant depending on the measure μ. Since $S(p,q)\left(R^d\right)$ is a Segal algebra then it is an essential ideal in $L^1\left(R^d\right)$ and hence $M_S\left(R^d\right)$ is uniquely defined as independent of approximate identity by Proposition 3, in [3].

Let $1<p<\infty$, $1\le q<\infty$ and $g\in S\left(R^d\right)$. The space $S(p,q)\left(R^d\right)$ is a Segal algebra by Theorem 3.2, and the following Proposition is proved by Theorem 4 in [3].

Proposition 4.2. *The following are equivalent*:

1. *$T\in M\left(L^1\left(R^d\right),S(p,q)\left(R^d\right)\right)$.*

2. *There exists a unique $\mu\in M_S\left(R^d\right)$ such that $Tf=\mu*f$ for all $f\in L^1\left(R^d\right)$. Moreover, the spaces $M\left(L^1\left(R^d\right),S(p,q)\left(R^d\right)\right)$ and $M_S\left(R^d\right)$ are homeomorphic.*

Proposition 4.3. *If $g\in S\left(R^d\right)$ and $1\le p,q<\infty$ then the multiplier space $M\left(L^1\left(R^d\right),S^{*}(p,q)\left(R^d\right)\right)$ is isomorphic to $L^{\infty}\left(R^d\right)\times L(p',q')\left(R^d\right)/K$ where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$ and $S^{*}(p,q)\left(R^d\right)$ is the dual of $S\left(R^d\right)$.*

Proof. By Theorem 3.3 we write

$$L^1(R^d) * S(p,q)(R^d) = S(p,q)(R^d). \tag{4.1}$$

Hence by Corollary 2.13 in [21] and Proposition 3.2 we have

(4.2)
$$\begin{aligned} M\left(L^1\left(R^d\right), S^*\left(p,q\right)\left(R^d\right)\right) &= M(L^1\left(R^d\right), L^\infty\left(R^d\right) \times L\left(p',q'\right)\left(R^d\right)/K) \\ &= L^\infty\left(R^d\right) \times L\left(p',q'\right)\left(R^d\right)/K. \end{aligned}$$

□

Lemma 4.1. *Let G be a non-compact locally compact abelian group and $1 \leq p < \infty, 1 < q < \infty$. If $f \in L\left(p,q\right)(G)$ then*

$$\lim_{s\to\infty} \left\|f + T_s f\right\|_{(p,q)} = 2^{\frac{1}{p}} \left\|f\right\|_{(p,q)}.$$

Proof. Suppose that g is a simple function, that is

$$g = \sum\nolimits_{j=1}^{n} c_j^{'} \chi_{E_j}$$

where each E_j is measurable and compact with $\mu\left(E_j\right) > 0$ and $E_j \cap E_k = \phi$ for $j \neq k$. Let $d_j = \mu\left(E_1\right) + \mu\left(E_2\right) + ... + \mu\left(E_j\right)$, $1 \leq j \leq n$ and $d_0 = 0$. Then if we set $c_j = \left|c_j^{'}\right|$, for $1 \leq j \leq n$ and $c_1 \geq c_2 \geq ... \geq c_n \geq 0$ then

$$g^*(t) = \begin{cases} c_1, \text{ if } 0 \leq t < d_1 \\ c_j, \text{ if } d_{j-1} \leq t < d_j \\ 0, \text{ if } d_n \leq t \end{cases}$$

for $1 \leq j \leq n$, [14]. Also we write

$$\begin{aligned} \left(\left\|g\right\|_{(p,q)}^*\right)^q &= \frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} g^*\left(t\right)\right)^q \frac{dt}{t} \\ &= \left(c_1^q - c_2^q\right) d_1^{\frac{q}{p}} + \left(c_2^q - c_3^q\right) d_2^{\frac{q}{p}} + \cdots + \left(c_{n-1}^q - c_n^q\right) d_{n-1}^{\frac{q}{p}} + c_n^q d_n^{\frac{q}{p}}. \end{aligned}$$

If $s \notin \cup_{j,k=1}^n E_j E_k^{-1}$ then the supports of g and $T_s g$ are disjoint and that means

$$g + T_s g = \sum\nolimits_{j=1}^{n} c_j^{'} \chi_{E_j \cup sE_j} \text{ and } \left(E_j \cup sE_j\right) \cap \left(E_k \cup sE_k\right) = \phi \text{ for } k \neq j.$$

Also we obtain

$$\begin{aligned} \widetilde{d_j} &= \mu\left(E_1 \cup sE_j\right) + \mu\left(E_2 \cup sE_j\right) + \cdots + \mu\left(E_j \cup sE_j\right) \\ &= 2\left(\mu\left(E_1\right) + \mu\left(E_2\right) + \cdots + \mu\left(E_j\right)\right) = 2d_j. \end{aligned}$$

Then

$$(g + T_s g)^*(t) = \begin{cases} c_1, \text{ if } 0 \leq t < \widetilde{d_1} \\ c_j, \text{ if } \widetilde{d_{j-1}} \leq t < \widetilde{d_j} \\ 0, \text{ if } \widetilde{d_n} \leq t \end{cases}$$

where $c_j = \left|c_j^{'}\right|$, for $1 \leq j \leq n$ and $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. Hence

$$\begin{aligned}
&\left(\|g + T_s g\|_{(p,q)}^*\right)^q \\
&\quad = \left(c_1^q - c_2^q\right) \tilde{d}_1^{\frac{q}{p}} + \left(c_2^q - c_3^q\right) \tilde{d}_2^{\frac{q}{p}} + \cdots + \left(c_{n-1}^q - c_n^q\right) \tilde{d}_{n-1}^{\frac{q}{p}} + c_n^q \tilde{d}_n^{\frac{q}{p}} \\
&\quad = 2^{\frac{q}{p}} \left(\left(c_1^q - c_2^q\right) d_1^{\frac{q}{p}} + \left(c_2^q - c_3^q\right) d_2^{\frac{q}{p}} + \cdots + \left(c_{n-1}^q - c_n^q\right) d_{n-1}^{\frac{q}{p}} + c_n^q d_n^{\frac{q}{p}}\right) \\
&\quad = 2^{\frac{q}{p}} \left(\|g\|_{(p,q)}^*\right)^q
\end{aligned}$$

This implies

$$\|g + T_s g\|_{(p,q)}^* = 2^{\frac{1}{p}} \|g\|_{(p,q)}^*. \tag{4.3}$$

Now let $f \in L(p,q)(G)$ and $\varepsilon > 0$ be given. By the density of simple function in $L(p,q)(G)$, [14] we can choose a simple function $g = \sum_{j=1}^n c_j \chi_{E_j}$ such that

$$\|f - g\|_{(p,q)}^* < \frac{\varepsilon}{4}. \tag{4.4}$$

Let the support of g be $\cup_{j=1}^n E_j$. Then if $s \notin \cup_{j,k=1}^n E_j E_k^{-1}$ by using (4.3) and (4.4) we have

$$\begin{aligned}
&\left|\|f + T_s f\|_{(p,q)}^* - 2^{\frac{1}{p}} \|f\|_{(p,q)}^*\right| \\
&\quad \leq \left|\|f + T_s f\|_{(p,q)}^* - \|g + T_s g\|_{(p,q)}^*\right| \\
&\qquad + \left|\|g + T_s g\|_{(p,q)}^* - 2^{\frac{1}{p}} \|g\|_{(p,q)}^*\right| + \left|2^{\frac{1}{p}} \|f\|_{(p,q)}^* - 2^{\frac{1}{p}} \|g\|_{(p,q)}^*\right| \\
&\quad \leq \|f - g\|_{(p,q)}^* + \|T_s f - T_s g\|_{(p,q)}^* + 2^{\frac{1}{p}} \|f - g\|_{(p,q)}^* \\
&\quad \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2^{\frac{1}{p}} \varepsilon \leq \varepsilon.
\end{aligned}$$

This completes the proof. □

Lemma 4.2. *Let G be a non-compact locally compact abelian group and $1 \leq p < \infty, 1 < q < \infty$. For any continuous complex valued function $s \to c(s)$ on G with $|c(s)| = 1$ and $f \in L(p,q)(G)$ we have*

$$\lim_{s \to \infty} \|f + c(s) T_s f\|_{(p,q)}^* = \lim_{s \to \infty} \|f + T_s f\|_{(p,q)}^*.$$

Proof. Let $g \in C_c(G)$. Assume that supp$g$ $=K$. If $s \in KK^{-1}$ then the supports of g and $T_s g$ are disjoint. This implies that the supports of g and $c(s) T_s g$ are disjoint. Thus the distribution function of $g + T_s g$ is

$$\begin{aligned}
\lambda_{g + T_s g}(y) &= \mu\left\{x \in G : |g + T_s g|(t) > y\right\} \\
&= \mu\left\{x \in G : |g|(t) + |T_s g|(t) > y\right\}, \quad y > 0.
\end{aligned} \tag{4.5}$$

Also since the supports of g and $c(s)\,T_s g$ are disjoint and $|c(s)| = 1$ then from (4.5) we have

$$\begin{aligned}
\lambda_{g+c(s)T_s g}(y) &= \mu\left\{x \in G : |g + c(s)\,T_s g|(t) > y\right\} \\
&= \mu\left\{x \in G : |g|(t) + |c(s)\,T_s g|(t) > y\right\} \\
&= \mu\left\{x \in G : |g|(t) + |c(s)|\,|T_s g|(t) > y\right\} \\
&= \mu\left\{x \in G : |g|(t) + |T_s g|(t) > y\right\} = \lambda_{g+T_s g}(t), \quad y > 0.
\end{aligned}$$

This implies $(g + c(s)\,T_s g)^* = (g + T_s g)^*$ and hence

$$\|g + c(s)\,T_s g\|^*_{(p,q)} = \|g + T_s g\|^*_{(p,q)}. \tag{4.6}$$

Now let $f \in L(p,q)(G)$ and $\varepsilon > 0$ be given. Since $C_c(G)$ is dense in $L(p,q)(G)$, [26], there exists $g \in L(p,q)(G)$ such that

$$\|f - g\|^*_{(p,q)} < \frac{\varepsilon}{4}. \tag{4.7}$$

By using (4.6) and (4.7) we obtain

$$\begin{aligned}
&\left|\|f + c(s)\,T_s f\|^*_{(p,q)} - \|f + T_s f\|^*_{(p,q)}\right| \\
&\quad\leq \left|\|f + c(s)\,T_s f\|^*_{(p,q)} - \|g + c(s)\,T_s g\|^*_{(p,q)}\right| \\
&\qquad + \left|\|g + c(s)\,T_s g\|^*_{(p,q)} - \|g + T_s g\|^*_{(p,q)}\right| \\
&\qquad + \left|\|f + T_s f\|^*_{(p,q)} - \|g + T_s g\|^*_{(p,q)}\right| \\
&\quad\leq \|(f - g) + c(s)\,(T_s f - T_s g)\|^*_{(p,q)} \\
&\qquad + \left|\|f + T_s f\|^*_{(p,q)} - \|g + T_s g\|^*_{(p,q)}\right| \\
&\quad\leq \|f - g\|^*_{(p,q)} + |c(s)|\,\|T_s(f - g)\|^*_{(p,q)} \\
&\qquad + \|f - g\|^*_{(p,q)} + \|T_s(f - g)\|^*_{(p,q)} \\
&\quad= 4\,\|f - g\|^*_{(p,q)}.
\end{aligned} \tag{4.8}$$

Then combining (4.7) and (4.8) we have

$$\left|\|f + c(s)\,T_s f\|^*_{(p,q)} - \|f + T_s f\|^*_{(p,q)}\right| \leq 4\,\|f - g\|^*_{(p,q)} < 4\frac{\varepsilon}{4} = \varepsilon.$$

This completes the proof. $\square$

Theorem 4.1. *Let $T : S(p,q)\left(R^d\right) \to L^1\left(R^d\right)$ be a linear transformation and $1 < p, q < \infty$, $g \in S\left(R^d\right)$. Then the following are equivalent:*

1. $T \in M\left(S(p,q)\left(R^d\right), L^1\left(R^d\right)\right)$.

2. *There exists a unique $\mu \in M\left(R^d\right)$ such that $Tf = \mu * f$ for each $f \in S(p,q)\left(R^d\right)$, where $M\left(R^d\right)$ is the space of bounded regular Borel measures on R^d.*

Proof. 1. Let $\mu \in M\left(R^d\right)$ and $f \in S\left(p,q\right)\left(R^d\right)$. Then

$$\left\|Tf\right\|_1 = \left\|\mu * f\right\|_1 \leq \left\|\mu\right\| \left\|f\right\|_1 \leq \left\|\mu\right\| \left\|f\right\|_S.$$

It is easy to prove the other conditions to be multiplier from $S\left(p,q\right)\left(R^d\right)$ into $L^1\left(R^d\right)$. Hence $T \in M\left(S\left(p,q\right)\left(R^d\right), L^1\left(R^d\right)\right)$.

Conversely assume that $T \in M\left(S\left(p,q\right)\left(R^d\right), L^1\left(R^d\right)\right)$. Then

$$\left\|Tf\right\|_1 \leq \left\|T\right\| \left\|f\right\|_S = \left\|T\right\| \left(\left\|f\right\|_1 + \left\|V_g f\right\|_{(p,q)}\right). \tag{4.9}$$

By using Lemma 3.5.1 in [15], Lemma 4.1 , Lemma 4.2 and (4.9) we deduce that

(4.10)

$$\begin{aligned}
2\left\|Tf\right\|_1 &= \lim_{s\to\infty} \left\|Tf + T_s Tf\right\|_1 \leq \lim_{s\to\infty} \left\|T\right\| \left(\left\|f + T_s f\right\|_1 + \left\|f + T_s f\right\|_{M(p,q)}\right)\\
&\leq \lim_{s\to\infty} \left\|T\right\| \left(2\left\|f\right\|_1 + \left\|V_g\left(f + T_s f\right)\right\|_{(p,q)}\right)\\
&= \lim_{s\to\infty} \left\|T\right\| \left(2\left\|f\right\|_1 + \left\|V_g f + V_g(T_s f)\right\|_{(p,q)}\right)\\
&= \lim_{s\to\infty} \left\|T\right\| \left(2\left\|f\right\|_1 + \left\|V_g f + e^{-2\pi swi} T_{(s,0)} V_g f\right\|_{(p,q)}\right)\\
&= \lim_{s\to\infty} \left\|T\right\| \left(2\left\|f\right\|_1 + \left\|V_g f + T_{(s,0)} V_g f\right\|_{(p,q)}\right)\\
&= \left\|T\right\| \left(2\left\|f\right\|_1 + \lim_{s\to\infty} \left\|V_g f + (T_{(s,0)} V_g f)\right\|_{(p,q)}\right)\\
&= \left\|T\right\| (2\left\|f\right\|_1 + 2^{\frac{1}{p}} \left\|V_g f\right\|_{(p,q)})
\end{aligned}$$

for all $f \in S\left(p,q\right)\left(R^d\right)$. This implies

$$\left\|Tf\right\|_1 \leq \left\|T\right\| \left(\left\|f\right\|_1 + 2^{\frac{1}{p}-1} \left\|V_g f\right\|_{(p,q)}\right).$$

Repeating this process n times we see that

$$\left\|Tf\right\|_1 \leq \left\|T\right\| \left(\left\|f\right\|_1 + 2^{n(\frac{1}{p}-1)} \left\|V_g f\right\|_{(p,q)}\right).$$

Since $p > 1$ then we have $\lim_{n\to\infty} 2^{n(\frac{1}{p}-1)} = 0$ and so we conclude that

$$\left\|Tf\right\|_1 \leq \left\|T\right\| \left\|f\right\|_1. \tag{4.11}$$

Also since $S\left(p,q\right)\left(R^d\right)$ is dense in $L^1\left(R^d\right)$ then $T \in M\left(L^1\left(R^d\right)\right)$. Hence by Theorem 0.1 in [10], there exists unique $\mu \in M\left(R^d\right)$ such that $Tf = \mu * f$ for all $f \in S\left(p,q\right)\left(R^d\right)$. □

Theorem 4.2. *Let $1 < p,q < \infty$ and $g \in S\left(R^d\right)$. Then the multipliers*

$$M\left(S\left(p,q\right)\left(\left(R^d\right)\right), S\left(p,q\right)\left(\left(R^d\right)\right)\right)$$

is isometrically isomorphic to $M\left(R^d\right)$.

Proof. By Theorem 4.1 we write

$$M\left(S\left(p,q\right)\left(\left(R^{d}\right)\right),S\left(p,q\right)\left(\left(R^{d}\right)\right)\right)\subset M(S\left(p,q\right)\left(R^{d}\right),L^{1}\left(R^{d}\right)) \\ =M\left(R^{d}\right). \tag{4.12}$$

Conversely let $\mu \in M\left(R^{d}\right)$. It is known by Theorem 3.3 that $S\left(p,q\right)\left(R^{d}\right)$ is an essential Banach ideal in $L^{1}\left(R^{d}\right)$. Also for each $\mu \in M\left(R^{d}\right)$ and $f \in S\left(p,q\right)\left(R^{d}\right)$ we write $\mu * f \in S\left(p,q\right)\left(R^{d}\right)$ and there exists a constant $C>0$ such that

$$\left\|Tf\right\|_{S}=\left\|\mu * f\right\|_{S}\leq C.\left\|\mu\right\|\left\|f\right\|_{S},$$

(see Lemma 2 in [3], and Proposition 2.1 in [11]). Then we conclude that $T\in M\left(S\left(p,q\right)\left(\left(R^{d}\right)\right),S\left(p,q\right)\left(\left(R^{d}\right)\right)\right)$. Thus

$$M\left(S\left(p,q\right)\left(R^{d}\right),L^{1}\left(R^{d}\right)\right)=M\left(R^{d}\right)\subset M(S\left(p,q\right)\left(R^{d}\right),S\left(p,q\right)\left(R^{d}\right)). \tag{4.13}$$

Hence combining (4.12) and (4.13) we obtain

$$M(S\left(p,q\right)\left(R^{d}\right),S\left(p,q\right)\left(R^{d}\right))=M\left(R^{d}\right).$$

□

For the case $p=q=1$ the Theorem 4.1. and Theorem 4.2. are not true. As an example if $p=q=1$ then $S\left(p,q\right)\left(R^{d}\right)=S_{0}\left(R^{d}\right)$, where the Segal algebra $S_{0}\left(R^{d}\right)$ is known Feichtinger algebra [13]. It is also known that the multiplier space of $S_{0}\left(R^{d}\right)$ is bigger than $M(R^{d})$.

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Ondokuz Mayis University
Faculty of Arts and Sciences
Department of Mathematics
55139, Kurupelit, Samsum
Turkey
e-mail: gurkanli@omu.edu.tr

References

[1] Y.-K. Chen and H. C. Lai, *Multipliers of Lorentz spaces*, Hokkaido Math. J. **4** (1975), 247–260.

[2] R. S. Doran and J. Wichmann, Approximate Identities and Factorization in Banach Modules, Lect. Notes in Math. 768, Springer-Verlag, 1979.

[3] M. Dogan and A. T. Gürkanlı, *Multipliers of the space $S_\omega(G)$*, Mathematica Balkanica, New Series **15**, Fasc.3-4 (2001), 199–212.

[4] C. Duyar and A. T. Gürkanlı, *Multipliers and relative completion in weighted Lorentz space*, Acta Mathematica Scientia, **23B**-4 (2003), 467–476.

[5] H. G. Feichtinger, *Banach spaces related to integrable group representations and their atomic decompositions1*, J.Funct. Anal. **86**-2 (1989), 307–339.

[6] H. G. Feichtinger and A. T. Gürkanlı, *On a family of weighted convolution algebras*, Internat. J. Math. Math. Sci. **13**-3 (1990), 517–526.

[7] H. G. Feichtinger and K. Gröchenig, *Gabor frames and Time-Frequency analysis of distributions*, J. Funct. Anal. **146**-2 (1997), 464–495.

[8] ———, Gabor wavelets and the Heisenberg Group: Gabor expansions and short time Fourier transform from the group theoretical point view. *Wavelets*, 359–397, Wavelet Anal. Appl. **2**, Academic Press, Boston, MA, 1992.

[9] R. H. Fischer, A. T. Gürkanlı and T. S. Liu, *On a family of weighted space*, Math. Slovaca **46**-1 (1996), 71–82.

[10] G. I. Gaudry, *Multipliers of weighted Lebesgue and measure spaces*, Proc. London Math. Soc. (3) **19** (1969), 327–340.

[11] A. T. Gürkanlı, *Multipliers of some Banach ideals and Wiener-Ditkin sets*, Math. Slovaca **55**-2 (2005), 237–248.

[12] ———, *On the inclusion of some Lorentz spaces*, J. Math. Kyoto Univ. **44**-2 (2004), 441–450.

[13] K. Gröchenig, Foundation of Time-Frequency Analysis, Birkhäuser, Boston, 2001.

[14] R. A. Hunt, *On $L(p,q)$spaces*, L'enseignement Mathematique T, XII, fasc. 4 (1966), 249–276.

[15] R. Larsen, Introduction to the Theory of Multipliers, Springer Verlag, Berlin-Heidelberg-New York, 1971.

[16] E. H. Lieb, *Integral bounds for radar ambiguity and Wigner distributions*, J. Math. Phys. **31** (3) (1990), 594–599.

[17] T. S. Liu and A. Van Rooij, *Sums and intersections of normed linear spaces*, Mathematische Nachrichten **42** (1969), 29–42.

[18] R. O'Neil, *Convolution operators and $L(p,q)$ spaces*, Duke Math. J. **30** (1963), 129–142.

[19] T. S. Quek and L. Y. H. Yap, *A test for membership in Lorentz spaces and some applications*, Hokkaido Math. J. **17** (1988), 279–288.

[20] M. A. Rieffel, *Induced Banach representation of Banach algebras and locally compact groups*, J. Funct. Anal. **1** (1967), 443–491.

[21] ______, *Multipliers and tensor products of L^p -spaces of locally compact groups*, Studia Math. **33** (1969), 71–82.

[22] S. Saeki and E. L. Thome, *Lorentz spaces as L_1- modules and multipliers*, Hokkaido Math. J. **23** (1994), 55–92.

[23] A. F. Talamanca and G. I. Gaudry, *Multipliers and sets of uniqueness of L^p*, Mich. Math. J. **17** (1970), 179–191.

[24] H. G. Wang, Homogeneous Banach Algebras, Marcel Dekker, Inc., New York-Basel, 1972.

[25] L. H. Y. Yap, *On two classes of Subalgebras of $L^1(G)$*, Proc. Japan Acad. **48** (1972), 315–319.

[26] ______, *Some remarks on convolution operators and $L(p,q)$ spaces*, Duke Math. J. **36** (1969), 647–658.