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On the diffeomorphism groups of rational and ruled 4-manifolds*

By

Bang-He Li and Tian-Jun Li

Abstract

Let $A(M)$ be the automorphism group of the middle homology of a smooth 4-manifold M and $D(M)$ be the subgroup induced by diffeomorphisms of M. In this paper we give explicit generators of $D(M)$ for rational and ruled 4-manifolds. We also prove the uniqueness of reduced forms for classes with minimal genus 0 and non-negative square.

1. Introduction

On a smooth 4-manifold M, each diffeomorphism induces an automorphism of the lattice of the second integral cohomology. Hence there is a natural map from the group of diffeomorphisms Diff(M) to the automorphism group of the lattice $A(M)$. Let D(M) be the image of this natural map. In other words, an automorphism is in D(M) if it is realized by an orientation-preserving diffeomorphism.

Let $M = CP^2 \# n\overline{CP^2}$. If $n \leq 9$, there is the classical result of Wall ([10]) that $D(M)$ coincides with $A(M)$. While for $n > 9$, Friedman and Morgan ([1]) showed that $D(M)$ is a subgroup of $A(M)$ with infinite index via the Donaldson theory. To characterize $D(M)$, Friedman and Morgan introduced the complex concepts of P-cell and super P-cell. In [7], another characterization of $D(M)$ was given via K-symplectic cones. However it is still rather abstract. In particular, there is neither description of a generating set of $D(M)$, nor the coset space when $n > 9$. Thus the structure of $D(M)$ with $n > 9$ still remains mysterious.

Based on results on the minimal genus of a class with square -1 or -2 in [7], in this note we use a very simple argument to present an explicit and finite generating set of $D(M)$ for any n. The knowledge of minimal genus is also used to explicitly write down infinitely many cosets of $D(M)$ in $A(M)$ when $n > 9$. We are also able to determine the uniqueness of reduced representatives of spherical classes of nonnegative squares.

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For irrational ruled 4-manifolds, we give a similar presentation of $D(M)$.

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2. Spherical reflections, reduced classes and minimal genus

For any oriented, closed 4-manifold M and $\gamma \in H_2(M, \mathbb{Z})$ with $\gamma^2 = \gamma \cdot \gamma = \pm 1$ or ± 2, there is an automorphism $R(\gamma)$ of the lattice called the reflection along γ,

$$R(\gamma)\beta = \beta - \frac{2(\gamma \cdot \beta)}{\gamma \cdot \gamma}\gamma.$$

Notice that $R(\gamma) = R(-\gamma)$. The following is Lemma 2.5 in [7].

Lemma 2.1. *Suppose $\psi \in A(M)$ and $\gamma = \psi(\gamma_0)$. Then $R(\gamma) = \psi \circ R(\gamma_0) \circ \psi^{-1}$.*

Now suppose that M is smooth and γ is represented by a smoothly embedded sphere. Proposition 2.4 in Chapter III in [1] then says that $R(\gamma) \in D(M)$ when $\gamma \cdot \gamma = -1$ or -2. Notice that, simply by looking at the opposite orientation of M, we still have $R(\gamma) \in D(M)$ when $\gamma \cdot \gamma = 1$ or 2.

Let $M = CP^2 \# n\overline{CP^2}$. Let H be a generator of $H_2(CP^2; \mathbb{Z})$ and $E_i, 1 \leq i \leq n$, be a generator of the H_2 of each of the $\overline{CP}^2$. $H, E_1, \ldots, E_n$ are naturally considered as classes in $H^2(CP^2 \# n\overline{CP}^2; \mathbb{Z})$ and form a basis. We will call such a basis a standard basis and always assume such a basis is chosen.

Given such a basis, we introduce the following notations:

$$\begin{aligned} R_0 &= R(H), \\ R_i &= R(E_i), \quad 1 \leq i \leq n, \\ R_{ij} &= R(E_i - E_j), \quad 1 \leq i < j \leq n, \quad R_{ii} = \mathrm{id}, \\ R_{123} &= R(H - E_1 - E_2 - E_3). \end{aligned}$$

According to Wall ([10]), an automorphism is called trivial if it is of the form $R_i, 1 \leq i \leq n$, or R_{ij}. It was shown in [10] that trivial automorphisms are in D(M). In fact, R_i, R_{ij}, R_{123} are all in $D(M)$ as $H, E_i, E_i - E_j, H - E_1 - E_2 - E_j$ are all represented by embedded spheres.

Definition 2.2. For $CP^2 \# n\overline{CP^2}$, a class $\xi = aH - \sum_{i=1}^n b_i E_i$ is called reduced if

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq 0, \quad , a \geq b_1 + b_2 + b_3.$$

If $n \leq 2$, it is understood that $b_i = 0$ for $i \geq n$ in the last condition.

Definition 2.3. For $M = CP^2 \# n\overline{CP^2}$, let $D_1(M)$ be the group generated by

$$R_0, \quad R_{123}, \quad R_i, 1 \leq i \leq n, \quad R_{1j}, 2 \leq j \leq n.$$

Since $R_{1i}(E_i - E_j) = E_1 - E_j$, by Lemma 2.1, we have

Lemma 2.4. *R_{ij} is generated by R_{1i} and R_{1j}.*

Consequently, $D_1(M)$ is actually the subgroup of $D(M)$ generated by trivial automorphisms, R_0 and R_{123}.

Definition 2.5. We say a class has a reduced form if it is equivalent to a reduced class under the action of $D_1(CP^2\#n\overline{CP^2})$.

It is proved in [6] that for any n, each class with nonnegative square has a reduced form. The following result in [7] concerning the existence of a reduced form for a class with square -1 or -2 is crucial.

Proposition 2.6. *Let $M = CP^2\#n\overline{CP^2}$. Let C be a class with square -1 or -2.*

1. *If C has a reduced form, then C has positive minimal genus.*
2. *If C does not have a reduced form, then C has minimal genus 0.*

2(a). *If C has square -1, C is equivalent to E_1 if $b^-(M) \neq 2$; when $b^-(M) = 2$, another possibility is that it is characteristic, and equivalent to $H - E_1 - E_2$.*

2(b). *If C has square -2, C is equivalent to $E_1 - E_2$ if $b^-(M) \neq 3$; when $b^-(M) = 3$, another possibility is that it is characteristic, and equivalent to $H - E_1 - E_2 - E_3$.*

3. Rational manifolds

We call a smooth, closed, orientable 4-manifold M a rational manifold if it is diffeomorphic to $S^2 \times S^2$ or $CP^2\#n\overline{CP^2}$.

It is well known that $D(CP^2\#2\overline{CP^2})$ is generated by $R(H - E_1 - E_2), R_0, R_1, R_2$ and R_{12}; $D(CP^2\#\overline{CP^2})$ is generated by R_0 and R_1; and $D(CP^2)$ is generated by R_0. For $S^2 \times S^2$, let $x = [S^2 \times pt]$ and $y = [pt \times S^2]$ be the generators of $H_2(S^2 \times S^2, \mathbb{Z})$. Then $A(S^2 \times S^2) = D(S^2 \times S^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by the reflections along $x + y$ and $x - y$. Our first main result says that the same phenomenon is valid for all rational manifolds.

Theorem 3.1. *For $M = CP^2\#n\overline{CP^2}$ with $n \geq 3$, we have $D(M) = D_1(M)$. Consequently, for any rational 4-manifold $M, D(M)$ is finitely generated by reflections along minimal genus 0 classes with squares ± 1 or ± 2.*

Proof. For any $\sigma \in D(CP^2\#n\overline{CP^2})$, by Proposition 2.6, there exists an element σ_1 in $D_1(CP^2\#n\overline{CP^2})$ such that $\sigma_1\sigma E_n = E_n$ or a reduced class. If the latter would happen, then the minimal genus of $\sigma_1\sigma E_n$ is positive by Proposition 2.6, a contradiction to the fact that it should be zero. Hence $\sigma_1\sigma E_n = E_n$, and $E_{n-1} \cdot E_n = 0$ implies $\sigma_1\sigma E_{n-1} = aH - b_1E_1 - \cdots - b_{n-1}E_{n-1}$. The same argument shows that there is an element $\sigma_2 \in D_1(CP^2\#(n-1)\overline{CP^2})$ such that $\sigma_2\sigma_1\sigma E_{n-1} = E_{n-1}$. Continuing in this way, we will have $\sigma_1, \sigma_2, \ldots, \sigma_{n-3}$ such that $\sigma_i \in D_1(CP^2\#(n-i+1)\overline{CP^2})$, and

$$\sigma_i\sigma_{i-1}\cdots\sigma_1\sigma E_{n-i+1} = E_{n-i+1}$$

Obviously

$$\sigma_{n-3}\sigma_{n-4}\cdots\sigma_1\sigma E_i = E_i, \text{ for } i = 4, 5, \ldots, n$$

Hence, $\sigma_{n-3}\sigma_{n-4}\cdots\sigma_1\sigma$ is in $A(CP^2\#3\overline{CP^2})$. By [11], $A(CP^2\#3\overline{CP^2}) = D_1(CP^2\#3\overline{CP^2})$. Thus $\sigma \in D_1(CP^2\#n\overline{CP^2})$ and the proof is complete. □

Remark 3.2. Notice that the proof of Theorem 3.1 does not rely on the previous characterizations of $D(M)$ in [1] and [7].

Remark 3.3. For $n \leq 9$, Wall [11] showed that the full automorphism group $A(M)$ has the same description and that enabled him to identify $D(M)$ with $A(M)$ in this case.

For the infiniteness of index of $D(CP^2\#n\overline{CP^2})$ in $A(CP^2\#n\overline{CP^2})$ with $n > 9$, the following result tell us how easy it is to find elements in $A(CP^2\#n\overline{CP^2})$ but not in $D(CP^2\#n\overline{CP^2})$, and how "big" the index is.

Theorem 3.4. *Let $M = CP^2\#n\overline{CP^2}$ and γ be a reduced class with square -1 or -2. Then the reflection $R(\gamma)$ is not in $D(CP^2\#n\overline{CP^2})$. Moreover, For a a positive integer, let*

$$\xi_a = a\left(3H - \sum_{i=1}^{9} E_i\right) - E_{10}, \quad \eta_a = a\left(3H - \sum_{i=1}^{9} E_i\right) - E_{10} - E_{11}.$$

Then, if $a \neq a'$,

1. *ξ_a and $\xi_{a'}$ are in different cosets if $n \geq 10$;*
2. *$\eta_a, \eta_{a'}$, ξ_a and $\xi_{a'}$ are in different cosets if $n \geq 11$.*

Proof. Observe that if an automorphism takes $\pm E_i$ to a reduced class, then it cannot be in $D(M)$ by Proposition 2.6.

If γ is reduced and has square -1 or -2, and $b_l > 0$ and $b_{l+1} = 0$, then

$$R(\gamma)E_l = E_l + \frac{2b_l}{|\gamma^2|}\gamma$$

is still reduced. Therefore $R(\gamma)$ is not in $D(M)$.

Notice that this applies to the classes ξ_a which have square -1 and η_a which have square -2. Now we further show the corresponding reflections are in different cosets. This follows from

$$\begin{aligned}
R(\xi_{a'})R(\xi_a)(-E_{10}) &= \xi_{2(a'-a)}, \\
R(\eta_{a'})R(\eta_a)(-E_{11}) &= (a'-a)\left(3H - \sum_{i=1}^{9} E_i\right) - E_{11}, \\
R(\eta_{a'})R(\xi_a)(E_{11}) &= \xi_{a'}.
\end{aligned}$$

□

For classes with square 1 or 2, we have,

Proposition 3.5. *Let $M = CP^2 \# n\overline{CP^2}$ with $n \geq 9$ and γ be a reduced class with square 1 or 2. Then $R(\gamma)$ is not in $D(M)$ if $b_3 > 0$.*

Proof. We use the same observation as above.

Since b_3 is assumed to be positive, so $b_1 > 0$. If $a > b_1 + b_2 + b_3$, then $R(\gamma)(-E_1)$ is reduced.

Now assume that $a = b_1 + b_2 + b_3$. First of all, it is impossible to have $b_1 = b_2 = \cdots = b_n$ as n is assumed to be as least 9.

Suppose there is an k with $3 \leq k < n$ such that $b_k > b_{k+1} > 0$. Then $R(\gamma)(-E_{k+1})$ is reduced.

Suppose there is an k with $3 \leq k < n$ such that $b_1 \geq b_2 > b_3 = \cdots = b_k$ and $b_{k+1} = 0$, then $R(\gamma)(-E_3)$ can be written as

$$a'H - b_1'E_1 - b_2'E_2 - (b_3' + 1)E_3 - b_3' \sum_{i=4}^{k} E_i,$$

where $b_1' \geq b_2' > b_3'$ and $a' = b_1' + b_2' + b_3'$. Hence

$$R(H - E_1 - E_2 - E_3)R(\gamma)(-E_3) = a^{"}H - b_1^{"}E_1 - b_2^{"}E_2 - b_3^{"}E_3 - b_3' \sum_{i=4}^{k} E_i$$

satisfies:

$$\begin{aligned} a^{"} &= b_1' + b_2' + b_3' - 1, \\ b_1^{"} &= b_1' - 1, \\ b_2^{"} &= b_2' - 1, \\ b_3^{"} &= b_3'. \end{aligned}$$

Therefore

$$R(H - E_1 - E_2 - E_3)R(\gamma)(-E_3)$$

is reduced.

Similarly, if there is an k with $3 \leq k < n$ such that $b_1 \geq b_2 > b_3 = \cdots = b_k$ and $b_{k+1} = 0$, then

$$R(H - E_1 - E_2 - E_3)R(\gamma)(-E_2)$$

is reduced. □

The uniqueness of reduced form is an interesting and difficult question[*1]. The only known result is Theorem 2.3 in [11]: any class with positive square in $H_2(CP^2 \# 2\overline{CP^2}; \mathbb{Z})$ has a unique reduced form.

[*1] This has recently been answered affirmatively in [4].

The result of Wall can be generalized to include the classes with square zero by the following argument. If $aH-b_1E_1-b_2E_2$ is reduced with $a^2=b_1^2+b_2^2$, then there are integers $t\ge s\ge 0$ and $k>0$ such that

$$a=k(t^2+s^2),\quad b_1=max\{k(t^2-s^2),2kts\},\quad b_2=min\{k(t^2-s^2),2kts\}$$

$a\ge b_1+b_2$ implies $2s^2\ge 2ts$. This, together with $t\ge s\ge 0$, implies $s=0$ or $s=t$. Therefore, reducedness implies $b_2=0$ and $a=b_1$.

We prove the following general result concerning uniqueness.

Theorem 3.6. *For $M=CP^2\#n\overline{CP^2}$, a minimal genus 0 class with nonnegative square has a unique reduced form in the following list*:

1. $2H$,
2. $(k+1)H-kE_1,\qquad k\ge 0$,
3. $(k+1)H-kE_1-E_2,\qquad k\ge 1$,
4. $kH-kE_1,\qquad k\ge 0$.

Proof. Let $\xi=aH-b_1E_1-\cdots-b_nE_n$ be a reduced minimal genus 0 class with $\xi^2\ge 0$ and $b_n>0$. Let $\xi'=aH-b_1E_1-\cdots-b_nE_n-E_{n+1}-\cdots-E_{n+s}$ with $s=\xi^2+2$. Then ξ' is a minimal genus 0 class with square -2. In particular, $R(\xi')$ is defined and lies in $D(CP^2\#(n+s)\overline{CP^2})$. Now we calculate that

$$R(\xi')E_{n+s}=\xi'+E_{n+s}.$$

If $n>2$, $\xi'+E_{n+s}$ would be reduced with square -1. Then, by Proposition 2.6, its minimal genus would be positive. This contradiction leads to the conclusion that $n\le 2$.

By [3], [5] and [6], the reduced minimal genus 0 classes in $CP^2\#2\overline{CP^2}$ with nonzero squares are just the following classes: 1). $2H$, 2). $(k+1)H-kE_1, k\ge 0$, 3). $(k+1)H-kE_1-E_2, k\ge 1$, 4). $kH-kE_1, k\ge 0$. By just looking at the squares and divisibility, we see that all the classes listed in 1)–4) are in different orbits under the action of $D(CP^2\#2\overline{CP^2})$. So, for minimal genus 0 classes with nonnegative square, its reduced from is unique. □

Remark 3.7. Theorem 3.6 immediately implies many things about minimal genus 0 classes, e.g. it implies Theorem 4.2 and part of Theorem D in [7].

4. Irrational ruled manifolds

Let Σ be a Riemann surface of positive genus, and N be the nontrivial S^2-bundle over Σ. Then, up to diffeomorphisms, $\Sigma\times S^2$ and N are the only minimal irrational 4-manifold. Any non-minimal irrational ruled 4-manifold is diffeomorphic to $N\#n\overline{CP^2}$ with $n\ge 1$. Let $U,T,E_1,\ldots,E_n$ be the standard basis of $H_2(N\#n\overline{CP^2};\mathbb{Z})$, where U is represented by a section with $U^2=1$, and T is represented by a fiber of N as an S^2-bundle.

Notice that for all irrational ruled 4-manifolds: $\Sigma \times S^2$ and $N\#n\overline{CP^2}, n = 0, 1, \ldots$, there is, up to sign, a unique primitive minimal genus 0 class f with square zero, i.e. the class $[pt \times S^2]$ or T respectively. Let $A^f(M) \subset A(M)$ be the subgroup preserving f up to sign. Then $D(M) \subset A^f(M)$, as minimal genus are preserved under diffeomorphisms[*2].

It is easy to see that, for $M = \Sigma \times S^2$ or N, $A(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $D(M) = A^f(M) \cong \mathbb{Z}_2$ is generated by -id (see e.g. [8]). Since the only spherical classs for $S \times S^2$ and N have square zero, -id cannot be a reflection along spherical class with squares equal to ± 1 or ± 2.

For $M = N\#n\overline{CP^2}$, let -Id$_N \in A(M)$ such that -Id$_N(U) = -U$, -Id$_N(T) = -T$ but -Id$_N(E_i) = E_i$, $i = 1, \ldots, n$. Then, since -id$\in D(N)$, it follows from Lemma 2 in [10] that -Id$_N \in D(M)$.

Proposition 4.1. *For $n \geq 2$, the group $D(N\#n\overline{CP^2})$ is generated by*

$$\text{-}Id_N, \quad R(T - E_1 - E_2), \quad R_i, 1 \leq i \leq n, \quad R_{1j}, 1 < j \leq n.$$

While, $D(N\#\overline{CP^2}) \cong \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2)$ is generated by -Id$_N, R_1$ and $R(T - E_1)$, where $\times$ and $*$ means direct and free product respectively.

Proof. First observe that the only minimal genus 0 classes with square -1 are of the form $bT \pm E_i$ for some i. To further proceed we willneed two lemmas.

Lemma 4.2. *Let $M = N\#n\overline{CP^2}$. Let $D'(M)$ be the subgroup of $D(M)$ generated by $R(bT - E_1), R_1$ if $n = 1$, and*

$$R(bT - E_j), R_i, R(T + E_i - E_j), R_{1j}, i \geq 1, j > 1,$$

if $n \geq 2$. Then for any $\sigma \in D(M)$, there exists a $\sigma' \in D'(M)$ such that $\sigma'\sigma$ preserves each E_i.

Proof. We first look at the case $n = 1$. In this case we have $\sigma E_1 = bT \pm E_1$. Since E_1 is not characteristic, and $bT \pm E_1$ is characteristic if and only if b is odd, we have $b = 2b_1$ for some $b_1 \in \mathbb{Z}$; let $\sigma_1 = id$, if $c_1 = 1$, and $\sigma_1 = R(E_1)$, if $c_1 = -1$, then $R(b_1T - E)\sigma_1$ maps $bT - c_1E_1$ to E_1.

For $n \geq 2$, $\sigma E_n = bT - c_iE_i$, there are two possibilities for b: odd and even. Let σ_1 =id, if $c_i = 1$, and $\sigma_1 = R(E_i)$, if $c_i = -1; \sigma_2 = id$, if $i = 1$, and $\sigma_2 = R_{1i}$, if $i > 1$. Then if b is even, $R_{1n}R((b/2)T - E_1)\sigma_2\sigma_1$ maps σE_n to E_n. If b is odd, $R(T + E_1 - E_n)R([b/2]T - E_1)\sigma_2\sigma_1$ maps σE_n to E_n. Therefore, we can find an automorphism σ_3 in the group generated by $R(bT - E_1), R_i, R(T + E_1 - E_j), R_{1j}, i \geq 1, j > 1$, such that

$$\sigma_3\sigma E_i = E_i, \qquad \text{for } i \geq 2$$
$$\sigma_3\sigma E_1 = cT - E_1$$

[*2]This can be also seen if we observe that f corresponds to a generator of the image of the cup product on H^1 in H^2.

Since $E_1+E_2+\cdots+E_n$ is not characteristic, and $\sigma_3\sigma(E_1+E_2+\cdots+E_n)=cT-E_1+E_2+\cdots+E_n$ is characteristic if and only if c is odd, we see that c must be even. Thus

$$R((c/2)T-E_1)\sigma_3\sigma E_i=E_i, \text{ for } i=1,\dots,n.$$

□

Lemma 4.3. *For $n\geq 1$, if $\sigma\in A^f(N\#n\overline{CP^2})$ takes E_i to E_i, for $i=1,\dots,n$. Then $\sigma=id$ or -Id_N.*

Proof. Write $\sigma(U)=cU+dT$ and $\sigma(T)=\epsilon T$ with $\epsilon=\pm 1$. We have $d=0$ and $c=\epsilon$, since

$$1=U\cdot T=\sigma(U)\cdot\sigma(T)=cb, \quad 1=\sigma(U)\cdot\sigma(U)=c^2+2cd.$$

Hence $\sigma=$id or -Id$_N$. □

By direct calculations, we find that $R(bT-E_1)$ sends (U,T,E_1) to

$$(U+2b^2T-2bE_1, \quad T, \quad 2bT-E_1),$$

and $R(aT-E_1)R_1R(bT-E_1)$ sends (U,T,E_1) to

$$(U+2(a+b)^2T-2(a+b)E_1, \quad T, \quad 2(a+b)T-E_1).$$

Since they both preserve $E_i, i\geq 2$, we have

$$R(aT-E_1)R_1R(bT-E_1)=R((a+b)T-E_1).$$

From this, we have, for $m\geq 2$,

$$R(mT-E_1)=R(T-E_1)R_1R(T-E_1)R_1\cdots R(T-E_1)R_1R(T-E_1),$$

with $R(T-E_1)$ appearing on the right side m times. And, for $m\leq -1$, we have

$$R(mT-E_1)=R_1R(T-E_1)R_1R(T-E_1)\cdots R_1R(T-E_1)R_1,$$

with $R(T-E_1)$ appearing on the right side $-m$ times.

Therefore $R(bT-E_1)$ is generated by $R(T-E_1)$ and R_1. Since $a\neq b$ implies $R(aT-E_1)\neq R(bT-E_1)$, the group generated by $R(T-E_1)$ and R_1 is the free product of $\mathbb{Z}_2$ and $\mathbb{Z}_2$. Since -Id$_N$ is in the center of $D(N\#\overline{CP^2})$, we have

$$D(N\#\overline{CP^2})\cong\mathbb{Z}_2\times(\mathbb{Z}_2*\mathbb{Z}_2).$$

For $n\geq 2$, we have

$$R(T-E_1-E_2)(T-E_1)=E_2.$$

Thus by Lemma 2.1, $R(T-E_1)$ is generated by $R(T-E_1-E_2)$ and R_2. Thus $R(bT-E_1)$ is generated by R_1, R_2 and $R(T-E_1-E_2)$.

Now, look at $R(T+E_1-E_j)$ with $j>1$. Observe that

$$R_1 R_{2j}(T+E_1-E_j) = T-E_1-E_2.$$

Thus by Lemma 2.1, $R(T+E_1-E_j)$ is generated by $R(T-E_1-E_2)$ and $R_{2j}, j>1$. The proof of the proposition is now complete by Lemma 2.2. □

Notice that the proof is rather elementary. It does not use the deep result Proposition 2.6.

It is remarked in [2] that $D(M) = A^f(M)$. This result was not used in the proof of Proposition 4.1. In fact we can also directly derive this.

Proposition 4.4 ([2]). *For $M = N\#n\overline{CP^2}$, $\sigma \in A(M)$ is in $D(M)$ if and only if $\sigma T = \pm T$, i.e. $D(M) = A^f(M)$.*

Proof. We have remarked that $D(M) \subset A^f(M)$. Now if $\sigma \in A(M)$ and $\sigma T = \pm T$. By Lemma 4.2, there is an element σ' in $D(N\#n\overline{CP^2})$ such that $\sigma'\sigma E_i = E_i, 1 \le i \le n$. Then. since $\sigma'\sigma(T) = \sigma'(\pm T) = \pm T$, we have that $\sigma'\sigma$ is either id or -Id$_N$ by Lemma 4.3. □

Remark 4.5. It is interesting to compare $D(N\#\overline{CP^2})$ with $D(CP^2\#2\overline{CP^2})$. In [11], it is proved that the following automorphisms:

$$\begin{array}{cccccc}
 & & \alpha & \beta & \gamma & \delta \\
E_1 & \to & E_2 & -2E_1-E_2-2H & E_2 & -E_1 \\
E_2 & \to & -E_1 & -E_1-2E_2-2H & E_1 & -E_2 \\
H & \to & H & 2E_1+2E_2+3H & H & -H
\end{array}$$

generate $D(CP^2\#2\overline{CP^2})$ with defining relations:

$$\begin{array}{ccc}
\alpha^4 = \beta^2 = \gamma^2 = \delta^2 = 1, & \gamma\alpha\gamma = \alpha^{-1}, & \gamma\beta\gamma = \beta, \\
\delta\alpha\delta = \alpha, & \delta\beta\delta = \beta, & \delta\gamma\delta = \gamma.
\end{array}$$

We can restate this result of Wall in the following form:

$$D(CP^2\#2\overline{CP^2}) \cong \mathbb{Z}_2 \times (\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 * \mathbb{Z}_4)),$$

where the generators from left to right are $\delta, \gamma, \beta, \alpha$, and the semidirect product $\ltimes$ is given by the homomorphism $h : \mathbb{Z}_2 \to \mathrm{Auto}(\mathbb{Z}_2 * \mathbb{Z}_4)$ with

$$\gamma^h(\beta) = \beta, \quad \gamma^h(\alpha) = \alpha^{-1}$$

Results similar to Theorem 3.4 for $N\#n\overline{CP^2}$ are easy to obtain. We are satisfied with the following statement.

Theorem 4.6. *For $M = N\#n\overline{CP^2}, n \ge 1$, $D(M) = A^f(M)$ is a subgroup of $A(M)$ with infinite index.*

Proof. Let $H' = U, E'_0 = U - T$. Then in the basis $H', E'_0, E_1, \dots, E_n$, for $\xi = aH' - bE'_0 - b_1E_1 - \cdots - b_nE_n$, $\xi^2 = a^2 - b^2 - b_1^2 - \cdots - b_n^2$. So there is a natural isomorphism of $A(N\#n\overline{CP^2})$ and $A(CP^2\#(n+1)\overline{CP^2})$ in virtue of this basis and the standard basis of $CP^2\#(n+1)\overline{CP^2}$. Thus, by the uniqueness of the reduced class in an orbit of classes with square zero under the action of $A(CP^2\#2\overline{CP^2})$ which we proved in the introduction, any primitive class $\xi = aH' - bE'_0 - b_1E_1$, with $a^2 = b^2 + b_1^2$ is equivalent to $H' - E'_0 = T$ via the action of $A(N\#\overline{CP^2})$. There are infinite coprime pairs $(s,t) \in \mathbb{Z}$ with $s \geq t > 0$. For any such a pair, let $\xi(s,t) = (s^2+t^2)H' - (s^2-t^2)E'_0 - 2stE_1$. Then there is an element $A(s,t)$ in $A(N\#\overline{CP^2})$ which sends T to $\xi(s,t)$. Suppose $A(s_1,t_1)$ and $A(s_2,t_2)$ are in the same right coset of $D(N\#n\overline{CP^2})$ in $A(N\#n\overline{CP^2})$, i. e. there are $B_1, B_2 \in D(N\#n\overline{CP^2})$ such that

$$A(s_1,t_1)B_1 = A(s_2,t_2)B_2$$

Since $B_iT = \pm T$, we have then $A(s_1,t_1)T = \pm A(s_2,t_2)T$, i.e.

$$\xi(s_1,t_1) = \pm\xi(s_2,t_2)$$

Therefore, $(s_1,t_1) = (s_2,t_2)$. This shows that $(s_1,t_1) \neq (s_2,t_2)$ implies $A(s_1,t_1)$ and $A(s_2,t_2)$ are in the different right cosets. ☐

Lab. of Math. Mechanization
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100080, China
e-mail: Libh@amss.ac.cn

School of Mathematics
University of Minnesota
Minneapolis, MN 55455, USA
e-mail: tjli@math.umn.edu

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