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Another proof of theorems of De Concini and Procesi

By

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Abstract

We give a new proof of some characteristic-free fundamental theorems in invariant theory first proved in C. De Concini and C. Procesi, A characteristic free approach to invariant theory, *Adv. Math.* **21** (1976), 330–354. We treat the action of the general linear group and the symplectic group. Our approach is geometric, and utilizes the fact that the categorical quotients are principal fiber bundles off codimension two or more.

1. Introduction

Let k be an algebraically closed field, and $n \geq m \geq t \geq 2$. Set $r = t-1$, $E = k^r$, $V = k^n$, and $W = k^m$. Let $M = \mathrm{Hom}(E, W) \times \mathrm{Hom}(V, E)$, and $G = GL(E)$. G acts on M by $g(\varphi, \psi) = (\varphi g^{-1}, g\psi)$. Let $Y_t := \{f \in \mathrm{Hom}(V, W) \mid \mathrm{rank}\, f < t\}$. It is easy to see that $\pi\colon M \to Y_t$ given by $\pi(\varphi, \psi) = \varphi \circ \psi$ is well-defined and G-invariant.

De Concini and Procesi [4] proved that π is a categorical quotient. The case of characteristic zero was proved in [13]. In other words, π induces an isomorphism $k[Y_t] \cong k[M]^G$. Yet another interpretation is as follows. M is isomorphic to $E^{\oplus n} \oplus (E^*)^{\oplus m}$ as a G-module, where G acts via

$$g(e_1, \ldots, e_n, e_1^*, \ldots, e_m^*) = (ge_1, \ldots, ge_n, e_1^* g^{-1}, \ldots, e_m^* g^{-1}).$$

It is easy to see that $\xi_{ij} = e_i^*(e_j)$ is a G-invariant polynomial function, i.e., an element in $k[M]^G$. The De Concini–Procesi theorem says that $k[M]^G$ is generated by ξ_{ij}, and the kernel of the surjective map $k[x_{ij}] \to k[M]^G$ given by $x_{ij} \mapsto \xi_{ij}$ is generated by the determinantal ideal $I_t(x_{ij})$ generated by the all t-minors of the matrix (x_{ij}).

The purpose of this article is to give a short and geometric proof to the theorem. Here we give a sketch of the proof.

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We need to assume that Y_t is Cohen–Macaulay. This was first proved by Hochster and Eagon [8]. See also [3] and [2]. The Cohen–Macaulay property of Y_t is highly non-trivial, so our proof is not completely self-contained. But the rest of the argument is easy and self-contained.

It is easy to see that Y_t satisfies the (R_1) condition, so Y_t is normal. As the codimension of Y_{t-1} in Y_t is at least two, when we set $U = Y_t \setminus Y_{t-1}$, we have $k[Y_t] = \Gamma(U, \mathcal{O}_U)$. On the other hand, $\pi|_{\pi^{-1}(U)} \colon \pi^{-1}(U) \to U$ is a principal G-bundle. Hence $\Gamma(U, \mathcal{O}_U) \to \Gamma(\pi^{-1}(U), \mathcal{O}_M)^G$ is an isomorphism. This is enough to prove the theorem.

In Section 2, we prepare necessary results to prove our main theorem. In Section 3, we prove the main theorem as described above. In Section 4, we extend the result to arbitrary base ring, rather than an algebraically closed base field. The key is good filtrations of representations of algebraic groups. In Section 5, we show that a similar argument is effective for the symplectic group action.

The results proved in Section 4 and 5 are also already proved in [4], and there is no new result in this paper. What is new here is a simple geometric proof of their results based on the Cohen–Macaulay property of the candidates of the invariant subrings.

2. Preliminaries

Let k be an algebraically closed field.

In the sequel, we will apply set theoretic argument to k-varieties. For example, given a k-morphism $f : X \to Y$, we will say that f is an isomorphism, since the induced map of the k-valued points $f(k) : X(k) \to Y(k)$ is bijective. This argument itself is obviously false (for example, consider $\operatorname{Spec} k[t] \to \operatorname{Spec} k[t^2, t^3]$), but the reader should interpret it into the correct argument: as for any finitely generated k-algebra A, the induced map of the set of A-valued points $f(A) : X(A) \to Y(A)$ is bijective, f is an isomorphism. We will abuse such a "k-valued points only" argument for brevity, only in the case where the interpretation to the correct argument is straightforward (but annoying).

Let m, n, t be integers such that $2 \leq t \leq m \leq n$. Set $V := k^n$, $W := k^m$, and $E := k^r$, where $r := t - 1$. Set $M := \operatorname{Hom}(E, W) \times \operatorname{Hom}(V, E)$, and $Y_t := Y_t(V, W) = \{f \in \operatorname{Hom}(V, W) \mid \operatorname{rank} f < t\} \subset \operatorname{Hom}(V, W)$. Let $S = k[x_{ij}]_{1 \leq i \leq m,\, 1 \leq j \leq n}$ be the polynomial ring in mn variables over k, so that S is the coordinate ring of $\operatorname{Hom}(V, W)$. Then Y_t is a closed subscheme of $\operatorname{Hom}(V, W)$ defined by I_t, where $I_t = I_t(x_{ij})$ is the ideal of S generated by t-minors of the matrix (x_{ij}). Let $k[M] := k[w_{il}, v_{lj} \mid 1 \leq i \leq m,\ 1 \leq l < t,\ 1 \leq j \leq n]$. Then $k[M]$ is the coordinate ring of M.

We define $\pi : M \to Y_t$ by $\pi(\varphi, \psi) = \varphi \circ \psi$ for $\varphi \in \operatorname{Hom}(E, W)$ and $\psi \in \operatorname{Hom}(V, E)$. This is a well-defined morphism, since a linear map which factors through the r-dimensional space E has rank less than t. The associated map of the coordinate rings is given by $x_{ij} \mapsto \sum_{l=1}^{t-1} w_{il} v_{lj}$.

The following argument is taken from [3, pp.4–5] for convenience of readers.

Obviously, π is surjective. As M is irreducible, Y_t is also irreducible.

Let $V = U \oplus \tilde{U}$ be a direct sum decomposition with $\dim U = r$. If $f \in Y_t$ and $f|_U$ is injective, then there exist unique linear maps $g\colon \tilde{U} \to U$ and $h\colon U \to W$ such that $f(u \oplus \tilde{u}) = h(u) + h(g(\tilde{u}))$ for all $u \in U$ and $\tilde{u} \in \tilde{U}$. So if we set

$$N := \{f \in Y_t \mid f|_U \text{ injective}\},$$

then there is an isomorphism of k-schemes

$$\operatorname{Hom}(\tilde{U}, U) \times (\operatorname{Hom}(U, W) \setminus Y_{r-1}(U, W)) \to N.$$

Since the variety on the left is an open subvariety of $\operatorname{Hom}(\tilde{U}, U) \times \operatorname{Hom}(U, W)$, we have that

$$\begin{aligned}\dim Y_t(V, W) = \dim N = \dim \operatorname{Hom}(\tilde{U}, U) \times \operatorname{Hom}(U, W)&\\ = (m - r)r + rn = mr + nr - r^2.&\end{aligned}$$

Moreover, N is non-singular. Varying U, we have that $Y_t(V, W) \setminus Y_{t-1}(V, W)$ is non-singular. So we have the following.

Proposition 2.1 ([3, (1.1)])**.**
1. $Y_t(V, W)$ *is irreducible.*
2. $\dim Y_t(V, W) = mr + nr - r^2$.
3. Y_t *is non-singular off* Y_{t-1}.

Hence we have

Lemma 2.1. $\dim Y_t - \dim Y_{t-1} = m + n - 2r + 1 \geq 3$. *In particular,* Y_t *satisfies the* (R_2) *condition.*

We need the following theorem, which was first proved by Hochster and Eagon [8]. See also [3].

Theorem 2.1. Y_t *is Cohen–Macaulay.*

Since Y_t is irreducible and satisfies the (R_1) and the (S_2) conditions, we have the following immediately.

Corollary 2.1. Y_t *is normal and integral.*

Definition 2.1. Let G be an affine algebraic group over k, X a k-scheme of finite type on which G acts, and $f\colon X \to Y$ a k-morphism. We say that f is G-invariant if $f(gx) = f(x)$ holds for $x \in X$ and $g \in G$. We say that f is a principal G-bundle if f is faithfully flat, G-invariant, and the map $\Phi\colon G \times X \to X \times_Y X$ given by $\Phi(g, x) = (gx, x)$ for $g \in G$ and $x \in X$ is an isomorphism.

It is not so difficult to show that a principal G-bundle is a universally submersive geometric quotient in the sense of [12]. We do not prove this because we will not use it later. What we need is the following.

Lemma 2.2. *Let G be an affine algebraic group over k, X a k-scheme of finite type on which G-acts, and $\pi\colon X \to Y$ a principal G-bundle. Then the canonical map $\mathcal{O}_Y \to (\pi_*\mathcal{O}_X)^G$ is an isomorphism.*

Proof. First, consider the case $X = G \times Y$ and π is the second projection. The question is local on Y, so we may assume that $Y = \operatorname{Spec} A$ is affine. Then the assertion reads $A \to (k[G] \otimes A)^G$ is an isomorphism, and this is trivial.

Applying this observation to the second projection $G \times X' \to X'$, and considering the X'-isomorphism $\Phi\colon G \times X' \cong X \times_Y X'$ which is also a G-isomorphism, we have $\mathcal{O}_{X'} \to ((p_2)_*\mathcal{O}_{X\times_Y X'})^G$ is an isomorphism, where X' is the scheme X with the trivial G-action, and $p_2\colon X \times_Y X' \to X'$ is the second projection.

Now apply $(\pi')^*$ to $\mathcal{O}_Y \to (\pi_*\mathcal{O}_X)^G$, where $\pi'\colon X' \to Y$ is π (remember that $X' = X$). Since the G-invariance is compatible with the flat base change, the result is

$$\mathcal{O}_{X'} \to ((\pi')^*\pi_*\mathcal{O}_X)^G \cong ((p_2)_*\mathcal{O}_{X\times_Y X'})^G.$$

We know that this is an isomorphism. As π' is faithfully flat, we have that $\mathcal{O}_Y \to (\pi_*\mathcal{O}_X)^G$ is also an isomorphism. □

3. Main Theorem

Let k be an algebraically closed field, and $m, n, t \in \mathbb{Z}$ such that $2 \le t \le m \le n$. Set $V := k^n$, $W := k^m$, and $E := k^r$, where $r = t - 1$. As in Section 2, set $M := \operatorname{Hom}(E, W) \times \operatorname{Hom}(V, E)$, and $Y_t := Y_t(V, W) = \{f \in \operatorname{Hom}(V, W) \mid \operatorname{rank} f < t\}$. Consider the morphism $\pi\colon M \to Y_t$ given by $\pi(\varphi, \psi) = \varphi \circ \psi$. Denote the associated k-algebra map $S/I_t \to k[M]$ by $\pi^\#$, where S/I_t is as in Section 2.

Let $G := GL(E)$. Then G acts on M via $g \cdot (\varphi, \psi) = (\varphi g^{-1}, g\psi)$ for $g \in G$, $\varphi \in \operatorname{Hom}(E, W)$, and $\psi \in \operatorname{Hom}(V, E)$. Then obviously, π is G-invariant.

The objective of this section is to give a new proof to the following theorem.

Theorem 3.1 (De Concini–Procesi [4, section 3])**.** *Let the notation be as above. Then the associated map $\pi^\#$ is injective, and the image of $\pi^\#$ is identified with $k[M]^G$.*

Proof. Since $\pi\colon M \to Y_t$ is dominating and Y_t is integral, $\pi^\#$ is injective. As π is G-invariant, an injective map $\pi^\#\colon k[Y_t] = S/I_t \to k[M]^G$ is induced. It suffices to prove that $\pi^\#$ is surjective.

Set $U := Y_t \setminus Y_{t-1}$. Then since Y_t is normal, we have that $\Gamma(U, \mathcal{O}_U) = k[Y_t]$ by Lemma 2.1.

On the other hand, we claim that $\Gamma(U, \mathcal{O}_U) = \Gamma(\pi^{-1}(U), \mathcal{O}_M)^G$. If the claim is true, $\pi^\#$ is surjective and the proof is complete, since $\Gamma(\pi^{-1}(U), \mathcal{O}_M)^G \supset k[M]^G$. Set $\tilde{U} = \pi^{-1}(U)$. To prove the claim, it suffices to show that

$$\pi|_{\tilde{U}}\colon \tilde{U} \to U$$

is a principal G-bundle by Lemma 2.2.

Note that $\tilde{U} = \{(\varphi, \psi) \in M \mid \psi \text{ surjective and } \varphi \text{ injective}\}$.

It is obvious that $\pi|_{\tilde{U}}$ is G-invariant, since π is.

We prove that $\Phi\colon G \times \tilde{U} \to \tilde{U} \times_U \tilde{U}$ given by $\Phi(g, \tilde{u}) = (g\tilde{u}, \tilde{u})$ is an isomorphism. Let $((\varphi, \psi), (\varphi', \psi')) \in \tilde{U} \times_U \tilde{U}$. Since $\varphi\psi = \varphi'\psi'$ and φ and φ' are injective, we have that

$$\operatorname{Ker}\psi = \operatorname{Ker}\varphi\psi = \operatorname{Ker}\varphi'\psi' = \operatorname{Ker}\psi'.$$

By the homomorphism theorem, it is easy to see that there exists a unique $g \in G(k)$ such that $\psi = g\psi'$. Since

$$\varphi'\psi' = \varphi\psi = \varphi g\psi'$$

and ψ' is surjective, we have that $\varphi' = \varphi g$. So

$$((\varphi, \psi), (\varphi', \psi')) = ((\varphi' g^{-1}, g\psi'), (\varphi', \psi')) = \Phi(g, (\varphi', \psi')).$$

Hence Φ is surjective.

Next, assume that $\Phi(g, (\varphi, \psi)) = \Phi(g', (\varphi', \psi'))$ for $g, g' \in G$, $\varphi, \varphi' \in \operatorname{Hom}(E, W)$, and $\psi, \psi' \in \operatorname{Hom}(V, E)$. Then obviously $(\varphi, \psi) = (\varphi', \psi')$. Since $g\psi = g'\psi$ and ψ is surjective, $g = g'$. Hence Φ is injective.

Since Φ is bijective, Φ is an isomorphism.

Next, we prove that $\pi|_{\tilde{U}}$ is faithfully flat. Since π is surjective, $\pi|_{\tilde{U}}$ is surjective. We only need to prove that $\pi|_{\tilde{U}}$ is flat.

Since Φ is an isomorphism, for each closed point $x \in \tilde{U}$, the morphism $G \to \pi^{-1}(\pi(x))$ given by $g \mapsto gx$ is an isomorphism. So each fiber is integral, and the dimension of fibers is constant. Since both U and $\tilde{U}$ are non-singular, the flatness of $\pi|_{\tilde{U}}$ now follows easily from [10, Corollary to Theorem 23.1]. $\square$

4. Arbitrary base ring

In this section, we extend Theorem 3.1 to an arbitrary base ring R, rather than an algebraically closed base field k.

First consider the case where the base ring R is an arbitrary field. The map $S/I_t \to R[M]^{G_R}$ of graded R-algebras is an isomorphism if (and only if) it is an isomorphism after taking a faithfully flat base change of the base ring R. As a field extension is faithfully flat, the map in problem is an isomorphism by Theorem 3.1.

Next consider the base ring $R = \mathbb{Z}$. Since $\operatorname{Hom}_{\mathbb{Z}}(E_{\mathbb{Z}}, W_{\mathbb{Z}}) \times \operatorname{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, E_{\mathbb{Z}})$ is isomorphic to $E_{\mathbb{Z}}^{\oplus n} \oplus (E_{\mathbb{Z}}^*)^{\oplus m}$ as a $G_{\mathbb{Z}}$-module, we have an isomorphism

$$\mathbb{Z}[M_{\mathbb{Z}}] \cong (\operatorname{Sym} E_{\mathbb{Z}}^*)^{\otimes n} \otimes (\operatorname{Sym} E_{\mathbb{Z}})^{\otimes m}.$$

Note that $\operatorname{Sym} E_{\mathbb{Z}}$ and $\operatorname{Sym} E_{\mathbb{Z}}^*$ have good filtrations as $GL(E_{\mathbb{Z}})$-modules. To verify this, it suffices to prove the assertion over an algebraically closed field rather than $\mathbb{Z}$, since $\operatorname{Sym} E_{\mathbb{Z}}$ and $\operatorname{Sym} E_{\mathbb{Z}}^*$ are direct sums of $GL(E_{\mathbb{Z}})$-modules which are $\mathbb{Z}$-finite free, see [6, Corollary III.4.1.8]. The proof for the case that the base is an algebraically closed field can be found in [1, (4.3)].

By Mathieu's tensor product theorem [11] and [6, Corollary III.4.1.8], the tensor product $\mathbb{Z}[M_{\mathbb{Z}}]$ of $\operatorname{Sym} E_{\mathbb{Z}}$ and $\operatorname{Sym} E_{\mathbb{Z}}^*$ has a good filtration. In particular, $H^i(G_{\mathbb{Z}}, \mathbb{Z}[M_{\mathbb{Z}}]) = 0$ for $i > 0$, see [6, Proposition III.2.3.8] (note that the trivial module is a Weyl module). By the universal coefficient theorem, the canonical map $\mathbb{Z}[M_{\mathbb{Z}}]^{G_{\mathbb{Z}}} \otimes k \to k[M]^G$ is an isomorphism for any field k. It follows that $S/I_t \otimes k \to \mathbb{Z}[M_{\mathbb{Z}}]^{G_{\mathbb{Z}}} \otimes k$ is an isomorphism for any k. This shows that $S/I_t \to \mathbb{Z}[M_{\mathbb{Z}}]^{G_{\mathbb{Z}}}$ is an isomorphism, since each homogeneous component of S/I_t and $\mathbb{Z}[M_{\mathbb{Z}}]^{G_{\mathbb{Z}}}$ are finitely generated $\mathbb{Z}$-modules with $\mathbb{Z}[M_{\mathbb{Z}}]^{G_{\mathbb{Z}}}$ torsion-free, and a $\mathbb{Z}$-linear map $P \to Q$ between finitely generated $\mathbb{Z}$-modules P and Q with Q torsion-free is an isomorphism if and only if its base change to an arbitrary field $P \otimes k \to Q \otimes k$ is so.

Now consider arbitrary base ring R. By the universal coefficient theorem again, $\mathbb{Z}[M_{\mathbb{Z}}]^{G_{\mathbb{Z}}} \otimes R \to R[M_R]^{G_R}$ is an isomorphism. Hence

Theorem 4.1 (De Concini–Procesi [4, section 3]). *Let R be an arbitrary commutative ring. The canonical map $S/I_t \to R[M_R]^{G_R}$ is an isomorphism.*

5. Symplectic group action

We apply the strategy above to another example.

Let k be an algebraically closed field, t and n integers such that $4 \leq 2t \leq n$, $V := k^n$ and $E := k^{2t-2}$.

Let $A = (a_{ij})$ be an alternating $2t \times 2t$ matrix over a commutative ring R. Namely, $a_{ji} = -a_{ij}$ and $a_{ii} = 0$. We define the Pfaffian of A to be

$$\operatorname{Pfaff}(A) = \sum_{\sigma \in \Gamma} (-1)^{\sigma} a_{\sigma 1\, \sigma 2} a_{\sigma 3\, \sigma 4} \cdots a_{\sigma(2t-1)\, \sigma(2t)} \in R,$$

where

$$\Gamma = \{\sigma \in \mathfrak{S}_{2t} \mid \sigma 1 < \sigma 3 < \cdots < \sigma(2t-1),\ \sigma(2i-1) < \sigma(2i)\ (1 \leq i \leq t)\}.$$

Note that $(\operatorname{Pfaff}(A))^2 = \det A$ (on the other hand, if A is an alternating matrix of odd size, $\det A = 0$).

Let $\langle\ ,\ \rangle : E \times E \to k$ be the bilinear form given by $(\langle e_i, e_j \rangle) = \tilde{J}$, where

$$J = J_{t-1} = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix} \text{ and } \tilde{J} = \tilde{J}_{t-1} = \begin{pmatrix} O & J \\ -J & O \end{pmatrix},$$

and $e_1, \ldots, e_{2t-2}$ is the standard basis of E. Note that the bilinear form $\langle\ ,\ \rangle$ induces a k-linear map $\rho : \bigwedge^2 E \to k$ given by $\rho(v \wedge v') = \langle v, v' \rangle$. We define G to be the symplectic group $\{\varphi \in \operatorname{End}(E) \mid \rho \circ \bigwedge^2 \varphi = \rho\}$. Note that $\dim G = 2(t-1)^2 + t - 1$, see [7, p. 3].

Define M to be the affine space $\operatorname{Hom}(V, E)$. Note that G acts on M in a natural way. Define Y_t to be the variety of $n \times n$ alternating matrices such that the Pfaffians of the all main $2t \times 2t$ submatrices vanish, where a main $2t \times 2t$ submatrix of an alternating $n \times n$ matrix $A = (a_{ij})$ is an alternating $2t \times 2t$ matrix of the form $(a_{i_u i_v})_{1 \le u,v \le 2t}$, where $1 \le i_1 < \cdots < i_{2t} \le n$.

Thus Y_t is a closed subscheme of the affine space $(\bigwedge^2 V)^*$. Define $\pi\colon M \to (\bigwedge^2 V)^*$ by $\pi(\varphi) = \rho \circ \bigwedge^2 \varphi$.

Note that there is a well-defined Pfaffian map $\mathrm{Pf}_{2t}^V\colon \bigwedge^{2t} V \to \mathrm{Sym} \bigwedge^2 V = k[(\bigwedge^2 V)^*]$ given by $\mathrm{Pf}_{2t}^V(v_1 \wedge \cdots \wedge v_{2t}) = \mathrm{Pfaff}(v_i \wedge v_j)$. Note that $A \in (\bigwedge^2 V)^*$ lies in Y_t if and only if the composite

$$\bigwedge^{2t} V \xrightarrow{\mathrm{Pf}_{2t}} \mathrm{Sym} \bigwedge^2 V = k[(\bigwedge^2 V)^*] \xrightarrow{A} k$$

is zero.

Let $\varphi \in \mathrm{Hom}(V, E)$. As the diagram

$$\begin{array}{ccccccc} & & \bigwedge^{2t} V & \xrightarrow{\mathrm{Pf}_{2t}^V} & \mathrm{Sym} \bigwedge^2 V & \xrightarrow{\pi(\varphi)} & k \\ & & \downarrow \bigwedge^{2t} \varphi & & \downarrow \mathrm{Sym} \bigwedge^2 \varphi & & \downarrow \mathrm{id}_k \\ 0 & = & \bigwedge^{2t} E & \xrightarrow{\mathrm{Pf}_{2t}^E} & \mathrm{Sym} \bigwedge^2 E & \xrightarrow{\rho} & k \end{array}$$

commutes, π factors through Y_t. So we have the morphism $\pi\colon M \to Y_t$. By the definition of the symplectic group G, π is G-invariant.

Fix the standard basis $f_1, \ldots, f_n$ of V. For $f \in (\bigwedge^2 V)^*$, the alternating matrix $A(f) = (f(f_i \wedge f_j))$ corresponds. We say that $\varphi \in \mathrm{Hom}(V, V^*)$ is alternating if the representation matrix (a_{ij}) given by $\varphi(f_j) = \sum_i a_{ij} f_i^*$ is alternating, where $f_1^*, \ldots, f_n^*$ is the dual basis of $f_1, \ldots, f_n$. This notion is independent of the choice of basis of V. Denote the space of alternating maps by $\mathrm{Alt}(V) \subset \mathrm{Hom}(V, V^*)$. Corresponding to $f \in (\bigwedge^2 V)^*$, an alternating map $a(f) \in \mathrm{Alt}(V)$ is given by $a(f)(v) = \sum_i f(f_i \wedge v) f_i^*$. The representation matrix of $a(f)$ is $A(f)$.

Similarly, we fix the basis $e_1, \ldots, e_{2t-2}$ of E, and corresponding to $h \in (\bigwedge^2 E)^*$, the $(2t-2) \times (2t-2)$ alternating matrix $A(h)$ corresponds, and the alternating map $a(h)$ corresponds. Note that $A(\rho) = \tilde{J}$. We denote by $\tilde{\rho} \in \mathrm{Alt}(E)$ the corresponding map $a(\rho)$.

It is easy to see that $a(\pi(\varphi)) = \varphi^* \circ \tilde{\rho} \circ \varphi$. In other words, $A(\pi(\varphi)) = {}^t X \tilde{J} X$, where X is the representation matrix of φ.

Lemma 5.1. *Let A be an $n \times n$ alternating matrix of rank $2r$. Then there exists some $X \in GL_n(k)$ such that*

$${}^t X A X = \begin{pmatrix} \tilde{J}_r & O \\ O & O \end{pmatrix}.$$

Proof. Induction on n. If $A = O$, then there is nothing to be proved. Assume that $A \ne O$. Since $A = (a_{ij})$ is alternating, there exists some $1 \le i < j \le n$ such that $a_{ij} \ne 0$. Exchanging the first row and the ith row and then exchanging the first column and the ith column, we may assume that $a_{1j} \ne 0$. Exchanging the jth column and the nth column, and then exchanging the jth row and the nth row, we may assume that $a_{1n} \neq 0$. Multiplying appropriate X from the right and changing A except for the first column, we may assume that $a_{1n} = 1$, and $a_{1j} = 0$ for $j < n$, and then multiplying ${}^tX$ from the left, A is still alternating. Multiplying appropriate Y from the left and changing A except for the first row, we may further assume that $a_{in} = 0$ for $i > 1$. Multiplying ${}^tY$ from the right, A is still alternating. Thus we may assume that A is of the form

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & A_1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where A_1 is an $(n-2) \times (n-2)$ alternating matrix. By induction assumption, the rest is easy. □

Lemma 5.2. *$\pi\colon M \to Y_t$ is surjective.*

Proof. Let $\psi \in Y_t$. Let $\operatorname{rank}\psi = 2l$, where $l \leq t-1$. It is easy to find $Z \in \operatorname{Mat}(2t-2, k)$ such that

$${}^tZ\tilde{J}_{t-1}Z = \begin{pmatrix} \tilde{J}_l & O \\ O & O \end{pmatrix}.$$

By Lemma 5.1, we may write

$${}^tXA(\psi)X = \begin{pmatrix} \tilde{J}_l & O \\ O & O \end{pmatrix}$$

with $X \in GL_n(k)$. Let Y be the $2l \times n$ matrix consisting of the first $2l$ rows of X^{-1}. Then we have ${}^tY\tilde{J}_lY = A(\psi)$. If $T = ZY'$, where Y' is the $(2t-2) \times n$ matrix whose first $2l$ rows are Y, and the rest is zero, then ${}^tT\tilde{J}_{t-1}T = A(\psi)$. If $\varphi \in \operatorname{Hom}(V, E)$ whose representation matrix is T, then $\pi(\varphi) = \psi$. □

Lemma 5.3. *Y_t is Cohen–Macaulay and integral.*

Proof. It is well-known that $k[Y_t]$ is a graded ASL on a distributive lattice, see [5, section 12] and [9]. Hence $k[Y_t]$ is Cohen–Macaulay by [5, Theorem 8.1]. It is also reduced by [3, (5.7)]. Since M is irreducible and π is surjective, Y_t is irreducible. Hence Y_t is integral. □

Set $U := Y_t \setminus Y_{t-1} = \{\varphi \in \operatorname{Alt}(V) \mid \operatorname{rank}\varphi = 2(t-1)\}$.

Lemma 5.4. *U is non-singular.*

Proof. By Lemma 5.3, U is integral. On the other hand, $GL(V)$ acts on U by $(g, u) \mapsto g^*ug$. By Lemma 5.1, this action is transitive. Hence U is non-singular. □

Proposition 5.1. *The morphism $\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U$ is a principal G-bundle.*

Proof. Note that π is G-invariant. Consider that $U \subset \mathrm{Alt}(V)$. Note that $\pi^{-1}(U) = \{\varphi \in \mathrm{Hom}(V, E) \mid \varphi \text{ surjective}\}$.

Consider the morphism $\Phi\colon G \times \pi^{-1}(U) \to \pi^{-1}(U) \times_U \pi^{-1}(U)$. Let $(\varphi, \psi) \in \pi^{-1}(U) \times_U \pi^{-1}(U)$. Then $\varphi^* \tilde{\rho} \varphi = \psi^* \tilde{\rho} \psi$. Hence

$$\operatorname{Ker}\varphi = \operatorname{Ker}(\varphi^* \tilde{\rho}\varphi) = \operatorname{Ker}(\psi^* \tilde{\rho}\psi) = \operatorname{Ker}\psi.$$

By the homomorphism theorem, there exists some $g \in GL(E)$ such that $\varphi = g\psi$. Since $\psi^* g^* \tilde{\rho} g \psi = \psi^* \tilde{\rho}\psi$, ψ is surjective, and ψ^* is injective, we have that $g^* \tilde{\rho} g = \tilde{\rho}$. This shows that $g \in G$. Since $(\varphi, \psi) = \Phi(g, \psi)$, Φ is surjective. The injectivity of Φ is easy. So Φ is an isomorphism.

Since π is surjective by Lemma 5.2, $\pi|_{\pi^{-1}(U)}$ is also surjective. Since U and $\pi^{-1}(U)$ are non-singular by Lemma 5.4, $\pi|_{\pi^{-1}(U)}$ is faithfully flat as in the proof of Theorem 3.1. □

Corollary 5.1. $\dim Y_t = (2n - 2t + 1)(t - 1)$.

Proof. By the proposition,

$$\begin{aligned}\dim Y_t = \dim U = \dim \pi^{-1}(U) - \dim G \\ = 2n(t-1) - (2(t-1)^2 + (t-1)) = (2n - 2t + 1)(t-1).\end{aligned}$$

□

Corollary 5.2. $\operatorname{codim}_{Y_t} Y_{t-1} \geq 5$.

Proof. By Corollary 5.1,

$$\operatorname{codim}_{Y_t} Y_{t-1} = 2(n - 2t) + 5 \geq 5.$$

□

Corollary 5.3. *Y_t is normal.*

Proof. By Corollary 5.2 and Lemma 5.4, we have that Y_t satisfies the (R_4) condition. Since Y_t is Cohen–Macaulay by Lemma 5.3, Y_t is normal. □

Theorem 5.1 ([4, (6.6), (6.7)])**.** *Let R be a commutative ring, and consider the morphism $\pi\colon M_R \to (Y_t)_R$ over R. Then the associated ring homomorphism $R[(Y_t)_R] \to R[M_R]^{G_R}$ is an isomorphism.*

Proof. As a G-module, $\mathrm{Hom}(V, E) \cong E^{\oplus n}$. Note that $E \cong E^*$ as a G-module.

By [1, section 4] and [6, Corollary III.4.1.8], $\operatorname{Sym} E$ has a good filtration as a G-module. By Mathieu's theorem [11], $\mathbb{Z}[M_{\mathbb{Z}}]$ has a good filtration. As in Section 4, we may assume that the base ring R is an algebraically closed field k.

Since Y_t is integral and π is surjective, the associated map $k[Y_t] \to k[M]^G$ is injective. We want to prove that this is surjective. Since Y_t is normal by

Corollary 5.3 and $\operatorname{codim}_{Y_t} Y_{t-1} \geq 2$ by Corollary 5.2, we have that $k[Y_t] \cong \Gamma(U, \mathcal{O}_U)$, where $U = Y_t \setminus U_{t-1}$. Since $\pi|_{\pi^{-1}(U)} \colon \pi^{-1}(U) \to U$ is a principal G-bundle by Proposition 5.1, $\Gamma(U, \mathcal{O}_U) \to \Gamma(\pi^{-1}(U), \mathcal{O}_M)^G$ is surjective by Lemma 2.2. Hence $k[Y_t] \to k[M]^G$ is surjective, as desired. $\square$

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