Minimal norm interpolation in harmonic Hilbert spaces and Wiener amalgam spaces on locally compact abelian groups

By

H. G. Feichtinger, S. S. Pandey and T. Werther

Abstract

The family of harmonic Hilbert spaces is a natural enlargement of those classical L^2-Sobolev spaces on $\mathbb{R}^d$ which consist of continuous functions. In the present paper we demonstrate that the use of basic results from the theory of Wiener amalgam spaces allows to establish fundamental properties of harmonic Hilbert spaces even if they are defined over an arbitrary locally compact abelian group $\mathcal{G}$. Even for $\mathcal{G} = \mathbb{R}^d$ this new approach improves previously known results. In this paper we present results on minimal norm interpolators over lattices and show that the infinite minimal norm interpolations are the limits of finite minimal norm interpolations. In addition, the new approach paves the way for the study of stability problems and error analysis for norm interpolations in harmonic Hilbert and Banach spaces on locally compact abelian groups.

1. Introduction

Babuska [3] introduced the concept of periodic Hilbert spaces in order to study universally optimal quadrature formulas. Subsequently Prager [16] has studied in detail the relationship between optimal approximation of linear functionals on periodic Hilbert spaces and minimal norm interpolation. In a more recent paper Delvos [6] has introduced the notion of harmonic Hilbert spaces for the real line $\mathbb{R}$ and discussed the interpolation problems over an infinite uniform lattice for ℓ^2-data in $\mathbb{R}$.

In the present paper we introduce the concept of harmonic Hilbert spaces over a locally compact abelian group $\mathcal{G}$, establish the properties of these spaces, and provide a characterization of minimal norm interpolators. The use of Wiener amalgam spaces provides a clear understanding of the subject. This paper opens new viewpoints for the study of optimal norm interpolation in various types of harmonic Banach spaces over locally compact abelian groups.

The article is organized as follows. In Section 2 we introduce the notation and state basic facts of harmonic analysis on compact abelian groups that we

essentially need for proving the main statements. The next section gives a short overview on Wiener amalgam spaces and contains a summary of some of their important properties. In Section 4 we introduce weight functions and derive the reproducing kernel of harmonic Hilbert spaces. Then we show in Section 5 how the concept of Riesz-basis leads to certain principal shift invariant subspaces. In particular, it follows that the bi-orthogonal system of the Riesz sequence of uniform translates of the kernel is generated by the same family of translates of the so-called Lagrange interpolator. Section 6 contains the core results of the paper, namely the explicit derivation of minimal norm interpolations. The simple construction of the Lagrange interpolator allows for efficient computational schemes of minimal norm interpolators. In the final section we discuss the approximation of the minimal norm interpolation by optimal interpolations of finite samples.

2.   Preliminaries and notation

Let $\mathcal{G}$ be a locally compact abelian (LCA) group and $\hat{\mathcal{G}}$ its dual group with the normalized Haar measures dx and $d\gamma$, respectively. The dual group $\hat{\mathcal{G}}$ is defined as the set of all characters $\gamma : \mathcal{G} \to \mathrm{T}$, i.e., continuous homomorphisms from $\mathcal{G}$ into the torus T with pointwise multiplication as group operation, and the standard (compact-open) topology.

For $1 \leq p \leq \infty$, L^p-spaces are denoted by their usual symbols. For $f \in L^1(\mathcal{G})$, the Fourier transform $\mathcal{F}$ is defined as

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{\mathcal{G}} f(x)\overline{\gamma(x)}dx \,.$$

There are many excellent textbooks on abstract harmonic analysis such as [17] and [13] that comprise a rigorous treatise of Fourier transform on LCA groups. Here, we only state those facts that are most useful for the development of the presented results.

Using the Pontryagin duality theorem which allows to write $\langle \gamma, x \rangle$ or $\langle x, \gamma \rangle$ unambiguously for $\gamma(x)$, the Fourier inversion formula for $h = \hat{f}$ in $L^1(\hat{\mathcal{G}})$ writes as

$$f(x) = \mathcal{F}^{-1}h(x) = \int_{\hat{\mathcal{G}}} \hat{f}(\gamma)\langle \gamma, x \rangle d\gamma \,.$$

The range of the Fourier transform, which is an injective and bounded linear mapping from L^1 to C^0, the space of continuous functions that vanish at infinity, is denoted by $\mathcal{F}L^1$. The convolution theorem states that

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g), \qquad f, g \in L^1 \,,$$

where "$*$" denotes the usual convolution. By transport of the norm $\|\mathcal{F}f\|_{\mathcal{F}L^1} = \|f\|_{L^1}$, the normed space $(\mathcal{F}L^1, \|\cdot\|_{\mathcal{F}L^1})$ turns into a Banach algebra with respect to pointwise multiplication. Due to Plancherel's theorem one has $\mathcal{F}L^1 = L^2 * L^2$, which ensures that $\mathcal{F}L^1$ is a dense subspace of C^0 endowed with the sup-norm $\|\cdot\|_\infty$.

Whenever a Banach space $(X, \|\cdot\|_X)$ is continuously embedded into another Banach space $(Y, \|\cdot\|_Y)$, i.e., there exists a constant $c > 0$ such that $\|x\|_Y \leq c\|x\|_X$ for all $x \in X$, we write $X \hookrightarrow Y$.

Throughout this paper we use the symbol T_x for the translation operator

$$T_x f(y) = f(y - x), \quad x, y \in \mathcal{G}.$$

For convenience, all index sets, sums, and lattices on $\mathcal{G}$ and $\hat{\mathcal{G}}$ used in the sequel are assumed to be countable.

Let Λ be a subgroup of $\mathcal{G}$. According to [14], we call Λ a lattice if the quotient group $\mathcal{G}/\Lambda$ is compact. The lattice size $s(\Lambda)$ is defined as the measure of a fundamental domain of Λ in $\mathcal{G}$, i.e., we choose a measurable set $U \subset \mathcal{G}$ such that every $x \in \mathcal{G}$ can be uniquely written as $x = \lambda + u$ for some $\lambda \in \Lambda$ and $u \in U$, cf. [14]. Then the lattice size $s(\Lambda)$ is the measure of U. For an equivalence class in $\mathcal{G}/\Lambda$ we write $\dot{x} = \{x + \lambda\}$. Let $d\dot{x}$ denote the normalized Haar measure on $\mathcal{G}/\Lambda$. For $f \in L^1(\mathcal{G})$, we then have Weil's formula [17]

$$(2.1) \qquad \int_{\mathcal{G}} f(x)dx = s(\Lambda) \int_{\mathcal{G}/\Lambda} \Big(\sum_{\lambda \in \Lambda} f(x + \lambda) \Big) d\dot{x}.$$

The annihilator of Λ is the subgroup

$$\Lambda^\perp = \big\{ \chi \in \hat{\mathcal{G}} \,\big|\, \chi(\lambda) = 1, \ \lambda \in \Lambda \big\}$$

and Weil's formula becomes in this context the identity

$$(2.2) \qquad \int_{\hat{\mathcal{G}}} \hat{f}(\gamma)d\gamma = s(\Lambda^\perp) \int_{\hat{\mathcal{G}}/\Lambda^\perp} \Big(\sum_{\chi \in \Lambda^\perp} \hat{f}(\gamma + \chi) \Big) d\dot{\gamma}.$$

Note that the dual group $\hat{\Lambda}$ of Λ is naturally isomorphic to $\hat{\mathcal{G}}/\Lambda^\perp$, [17]. For $(c_\lambda) \in \ell^2(\Lambda)$, the sum $\sum_{\lambda \in \Lambda} c_\lambda \langle \gamma, \lambda \rangle$ is a Fourier series on $\hat{\mathcal{G}}/\Lambda^\perp$ and Plancherel's theorem [17] yields

$$(2.3) \qquad \int_{\hat{\mathcal{G}}/\Lambda^\perp} \Big| \sum_{\lambda \in \Lambda} c_\lambda \langle \gamma, \lambda \rangle \Big|^2 d\dot{\gamma} = \sum_{\lambda \in \Lambda} |c_\lambda|^2.$$

For $\mathcal{G} = \mathbb{R}$ and $\Lambda = \alpha\mathbb{Z}$, $\alpha > 0$, this is just the classical Fourier series expansion of periodic functions. In this case we simply have $\Lambda^\perp = \alpha^{-1}\mathbb{Z}$.

We make use of these fundamental results in the proof of Theorem 5.1.

3. Wiener amalgam spaces

Wiener amalgam spaces on LCA groups and their properties under Fourier transform have been studied in a series of papers starting with [8]. We refer to [11], [15] for a compact survey on their properties. Here, we only provide a short overview over the facts needed in the present context.

First we introduce the continuous version of Wiener amalgam spaces. Let ψ be a non-zero, non-negative function on $\mathcal{G}$ with compact support, i.e.,

$\text{supp}(\psi) \subset \Omega$ where $\Omega \subseteq G$ is compact. Given any Banach space $(B, \|\cdot\|_B)$ of functions on $\mathcal{G}$ on which $\mathcal{F}L^1$ acts by pointwise multiplication, i.e., $\mathcal{F}L^1 \cdot B \subseteq B$, the Wiener amalgam space $W(B, L^q)$ is defined by

$$(3.1) \quad W(B, L^q) = \left\{ f \in B_{loc} \,\big|\, \|f\|_{W(B,L^q)} := \left(\int_{\mathcal{G}} \|f \cdot T_x \psi\|_B^q \, dx \right)^{1/q} < \infty \right\}$$

for $1 \leq q < \infty$ (with obvious modifications for $q = \infty$). The expression $f \in B_{loc}$ means that the function f can at least locally be measured by the B-norm hence the integrand in the definition of $\|\cdot\|_{W(B,L^q)}$ is a well-defined non-negative function. As stated in [11], these spaces are Banach spaces and do not depend on ψ (up to equivalence of norms).

An equivalent, but discrete, definition of Wiener amalgam spaces uses a so called $\mathcal{F}L^1$-bounded, uniform partition of unity (for short BUPU), that is a sequence of non-negative functions $\{\psi_i\}$ corresponding to a sequence $\{g_i\}$ in $\mathcal{G}$ such that

    1. $\text{supp}(\psi_i) \subset g_i + \Omega$,
    2. $\sup_i \sharp\{ j \,|\, (g_i + \Omega) \cap (g_j + \Omega) \} < \infty$,
    3. $\sum_i \psi_i(g) \equiv 1$,
    4. ψ_i are bounded in $\mathcal{F}L^1$, i.e., $\|\psi_i\|_{\mathcal{F}L^1} \leq C < \infty$ for all i.

With the help of such a BUPU we define the discrete Wiener amalgam space

$$(3.2) \qquad W(B, \ell^q) = \left\{ f \in B_{loc} \,\big|\, \|f\|_{W(B,\ell^q)} := \left(\sum_i \|f\psi_i\|_B^q \right)^{1/q} < \infty \right\}$$

for $1 \leq q < \infty$. Note that again, $W(B, \ell^q)$ is a Banach space independent from the partition of unity (up to equivalence of norms). In all cases of this paper we will use a lattice Λ in $\mathcal{G}$ instead of an arbitrary sequence $\{g_i\}$.

In contrast to the Lebesque-spaces L^p on $\mathbb{R}^d$, where inclusion fails for different p, Wiener amalgam spaces enjoy the so-called *coordinate-wise inclusion*, i.e., if $B_{1,loc} \hookrightarrow B_{2,loc}$ and $q_1 \leq q_2$, then $W(B_1, \ell^{q_1}) \hookrightarrow W(B_2, \ell^{q_2})$.

Wiener amalgam spaces can also be defined with additional weight functions that are described in the following section, cf. [11].

4. Reproducing kernel Hilbert spaces

We first state sufficient conditions on weight functions that are used to define harmonic Hilbert spaces. It is beyond the scope of this work to elaborate on these conditions.

A strictly positive and continuous function w is called a *submultiplicative* (or *Beurling*) weight function on $\hat{\mathcal{G}}$ if

$$(4.1) \qquad w(\gamma_1 + \gamma_2) \leq w(\gamma_1)\, w(\gamma_2) \quad \text{for all} \quad \gamma_1, \gamma_2 \in \hat{\mathcal{G}},$$

It satisfies the *Beurling-Domar non-quasianalyticity condition* [17, VI, sect. 3] if

$$(4.2) \qquad \text{(BD)} \qquad \sum_{n=1}^{\infty} n^{-2} \log w(n\gamma) < \infty \quad \text{for all} \quad \gamma \in \hat{\mathcal{G}}.$$

The standard example of such weight functions are weights of polynomial type on $\mathbb{R}^d$ such as

$$w_s(\gamma) = (1 + |\gamma|)^s \simeq (1 + |\gamma|^2)^{s/2}\,, \qquad \gamma \in \mathbb{R}^d,$$

for $s \geq 0$, or subexponential weights such as

$$w(\gamma) = e^{\alpha|\gamma|^\delta}(1 + |\gamma|)^s\,, \qquad \gamma \in \mathbb{R}^d,$$

for $\alpha > 0$, $0 < \delta < 1$, and $s \in \mathbb{R}$. For detailed studies on such weight functions on the Euclidian space $\mathbb{R}^d$ we refer to [7], [9], [10].

Throughout this paper, we assume that $w^{-1} \in L^2(\hat{\mathcal{G}})$ which implies $w^{-2} \in W(C^0, \ell^1)$. Obviously, this is satisfied for the above example w_s whenever $s > d/2$. In contrast to [6] where the author included the box function as a possible weight leading to band-limited functions that we discuss separately in Example 2, we restrict our discussion to submultiplicative weight functions satisfying (BD) in order to obtain stronger results and general statements valid for arbitrary lattices. From our point of view the band-limited case should be seen as a limiting case, requiring sometimes separate arguments.

Among others, the submultiplicativity of w in conjunction with the assumption $w^{-2} \in L^1(\hat{\mathcal{G}})$, allows to apply the convolution theorem for Wiener amalgam spaces [8], [15], in order to derive the following crucial property: for any lattice Λ there exist positive constants a and b such that

$$(4.3) \qquad a \leq \sum_{\chi \in \Lambda^\perp} w^{-2}(\gamma + \chi) \leq b\,, \qquad \gamma \in \hat{\mathcal{G}}\,.$$

The upper bound follows from the fact that the Haar measure for $\Lambda^\perp$, i.e. $\mu = \sum_{\chi \in \Lambda^\perp} \delta_\chi$ belongs to $W(M, \ell^\infty)$, where M denotes the space of bounded measures, while on the other hand the submultiplicativity implies that $w^{-2} \in W(C^0, \ell^1)$. Since the $\Lambda^\perp$ periodization of w^{-2} equals $\mu * w^{-2} \subseteq W(M, \ell^\infty) * W(C^0, \ell^1) \subseteq W(C, \ell^\infty)$ and $W(C, \ell^\infty)$ coincides with the space of bounded continuous functions on $\mathcal{G}$, the upper bound is valid, see [11]. Since w^{-2} is strictly positive the periodicity implies that there is a strictly positive lower bound.

We define the *harmonic Hilbert* space corresponding to the weight w as

$$(4.4) \qquad \mathcal{H}_w(\mathcal{G}) = \mathcal{F}^{-1}L_w^2(\hat{\mathcal{G}})$$

with inner product

$$\langle f, g \rangle_w = \int_{\hat{\mathcal{G}}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)}w^2(\gamma)d\gamma\,.$$

Lemma 4.1. *$\mathcal{H}_w(\mathcal{G})$ is a Hilbert space.*

Proof. Left to the reader. □

In the case of the weight functions w_s, $\mathcal{H}_{w_s}$ is known as the Sobolev space of fractional order s [1].

Remark 1. It has been shown in [7] that if $w^{-1} \in L^2$ is *subadditive*, i.e.,

$$w(\gamma_1 + \gamma_2) \leq C(w(\gamma_1) + w(\gamma_2)), \qquad \gamma_1, \gamma_2 \in \hat{\mathcal{G}},$$

then L_w^2 is a Banach convolution algebra. As a consequence, $\mathcal{H}_w$ turns into a Banach algebra with respect to pointwise multiplication. This is the case for w_s, $s > d/2$, and subexponential weights.

We now state a result which is fundamental for sampling functions in $\mathcal{H}_w$ along some lattice Λ.

Theorem 4.1. *If $w^{-1} \in L^2(\hat{\mathcal{G}})$, then*

$$\mathcal{H}_w(\mathcal{G}) = W(\mathcal{F}^{-1}L_w^2, \ell^2) \hookrightarrow W(C^0, \ell^2)(\mathcal{G}).$$

Proof. By virtue of Corollary 7 of [11], we have $\mathcal{F}^{-1}L_w^2 = W(\mathcal{F}^{-1}L_w^2, \ell^2)$. We emphasize that this result is a consequence of both the submultiplicativity and (BD). Now, since $w^{-1} \in L^2$, an easy application of the Cauchy-Schwartz inequality implies $L_w^2 \hookrightarrow L^1$ which in turn results in the following inclusions

$$\mathcal{F}^{-1}L_w^2 \hookrightarrow \mathcal{F}^{-1}L^1 \hookrightarrow C^0,$$

(by the Riemann-Lebesgue lemma). Hence, by the coordinate-wise inclusion properties of Wiener amalgam spaces we obtain

$$\mathcal{H}_w = W(\mathcal{H}_w, \ell^2) \hookrightarrow W(C^0, \ell^2).$$

□

As a consequence, $\mathcal{H}_w(\mathcal{G})$ is a *reproducing kernel Hilbert space* (RKHS). This is a Hilbert space consisting of continuous functions in which the point evaluation functionals are continuous. Therefore, by virtue of the Riesz Representation theorem, for each $x \in \mathcal{G}$ there exists a unique function, say k_x, in the RKHS, such that the point evaluation at x of any function f in the RKHS can be performed by means of the inner product with k_x. In the case of $\mathcal{H}_w$, that is

$$f(x) = \langle f, k_x \rangle_w, \qquad f \in \mathcal{H}_w(\mathcal{G}), x \in \mathcal{G}.$$

The kernel $k(x, y) = k_x(y)$ defines a continuous function on $\mathcal{G} \times \mathcal{G}$ containing all the information about the scalar product. A first detailed survey on RKHS goes back to [2].

In the present situation the kernel of $\mathcal{H}_w$ turns out to consist of translations of a single function.

Proposition 4.1. *$\mathcal{H}_w(\mathcal{G})$ is a reproducing kernel Hilbert space with kernel $k(x, y) = \phi(x - y)$, where*

$$\phi = \mathcal{F}^{-1}w^{-2}.$$

That is, $f(x) = \langle f, T_x\phi \rangle_w$ for all $f \in \mathcal{H}_w$ and $x \in \mathcal{G}$.

Proof. Define ϕ by $\hat{\phi} = w^{-2}$. Since $w^{-2} \in L^1$, $\phi \in \mathcal{H}_w$. Next we compute

$$
\begin{aligned}
\langle f, T_x\phi \rangle_w &= \int_{\hat{\mathcal{G}}} \hat{f}(\gamma)\overline{\widehat{T_x\phi}(\gamma)}w^2(\gamma)d\gamma \\
&= \int_{\hat{\mathcal{G}}} \hat{f}(\gamma)\overline{\overline{\langle x,\gamma \rangle}w^{-2}(\gamma)}w^2(\gamma)d\gamma \\
&= \int_{\hat{\mathcal{G}}} \hat{f}(\gamma)\langle x,\gamma \rangle d\gamma \; = \; f(x)\,,
\end{aligned}
$$
(4.5)

the last step following from the Fourier inversion theorem (note that $\hat{f} \in L^1$). The uniqueness of the kernel completes our proof of the fact that $k_x = T_x\phi$. $\square$

Remark 2. The fact that $\mathcal{H}_w \hookrightarrow C^0$ can also be derived immediately from (4.5). The use of Wiener amalgam spaces, however, reveals the important property that the sequence of samples of any function in $\mathcal{H}_w$ with respect to any lattice Λ is square summable. This property has led to the definition of so-called ℓ^2- *puzzles* in [19].

Corollary 4.1. *For any lattice Λ the mapping*

$$
Q : f \mapsto (f(\lambda))_{\lambda \in \Lambda}
$$
(4.6)

is bounded from $\mathcal{H}_w(\mathcal{G})$ to $\ell^2(\Lambda)$.

Proof. Choose a fixed lattice Λ' of $\mathcal{G}$ with fundamental domain Ω'. Then the family $\{T_{\lambda'}\chi'_{\Omega}\}_{\lambda' \in \Lambda'}$ where χ denotes the characteristic function, forms a BUPU. We now consider the norm of $W(C,\ell^2)$ with respect to this BUPU. For any $\lambda' \in \Lambda'$, there is at most a finite number of points of Λ in $\lambda' + \Omega'$, say $n_{\lambda'}$. These numbers are uniformly bounded. Therefore, $\|(f(\lambda))\|_{\ell^2} \leq \sup(n_{\lambda'})\|f\|_{W(C,\ell^2)}$. Hence, the result follows from Theorem 4.1. $\square$

We will later see that the mapping Q is surjective.

5. Riesz basis

Riesz bases are a well-established concept in Hilbert space theory [5].

Definition 5.1. A family of vectors $\{h_n\}$ in a Hilbert space $\mathcal{H}$ is called a Riesz sequence if there exist bounds $0 < a \leq b < \infty$ such that

$$
a\|c\|^2_{\ell^2} \leq \big\| \sum_n c_n h_n \big\|^2 \leq b\|c\|^2_{\ell^2}
$$
(5.1)

for all sequences $c = (c_n) \in \ell^2$.

Riesz sequences generalize the concept of orthogonal sequences as one can see from the following properties. Let V denote the closed linear span of $\{h_n\}$.

For every Riesz sequence $\{h_n\}$ there exists a unique *dual* or *bi-orthogonal sequence* $\{\tilde{h}_n\}$ in V such that

$$\langle h_n, \tilde{h}_m \rangle = \delta_{nm}, \qquad n, m \in \mathbb{N},$$

and the orthogonal projection P_V from $\mathcal{H}$ onto V is given by

$$(5.2) \qquad P_V h = \sum_n \langle h, \tilde{h}_n \rangle h_n = \sum_n \langle h, h_n \rangle \tilde{h}_n, \quad \text{for all} \quad h \in \mathcal{H},$$

cf. [21]. A Riesz sequence $\{h_n\}$ obviously constitutes a (Riesz) basis for V.

The following statement is a standard result in Fourier analysis, cf. [18]. For the sake of completeness we include the prove.

Theorem 5.1. *For any lattice Λ of $\mathcal{G}$, the sequence $\{T_\lambda \phi\}_{\lambda \in \Lambda}$ is a Riesz sequence in $L^2(\mathcal{G})$ if and only if there exist positive constants a, b such that*

$$(5.3) \qquad a \leq \sum_{\chi \in \Lambda^\perp} |\hat{\phi}(\gamma + \chi)|^2 \leq b \qquad a. \ e.$$

Proof. We compute

$$\big\| \sum_{\lambda \in \Lambda} c_\lambda T_\lambda \phi \big\|_2^2 \overset{\text{Plancherel}}{=} \big\| \sum_{\lambda \in \Lambda} c_\lambda \gamma_\lambda \hat{\phi} \big\|_2^2$$

$$= \int_{\hat{\mathcal{G}}} |\hat{\phi}(\gamma)|^2 \big| \sum_{\lambda \in \Lambda} c_\lambda \langle \lambda, \gamma \rangle \big|^2 d\gamma$$

$$\overset{(2.2)}{=} s(\Lambda^\perp) \int_{\hat{\mathcal{G}}/\Lambda^\perp} \sum_{\chi \in \Lambda^\perp} \Big[|\hat{\phi}(\gamma + \chi)|^2 \big| \sum_{\lambda \in \Lambda} c_\lambda \langle \lambda, \gamma + \chi \rangle \big|^2 \Big] d\dot{\gamma}$$

$$= s(\Lambda^\perp) \int_{\hat{\mathcal{G}}/\Lambda^\perp} \sum_{\chi \in \Lambda^\perp} \Big[|\hat{\phi}(\gamma + \chi)|^2 \big| \sum_{\lambda \in \Lambda} c_\lambda \langle \lambda, \gamma \rangle \langle \chi, \lambda \rangle \big|^2 \Big] d\dot{\gamma}$$

$$= s(\Lambda^\perp) \int_{\hat{\mathcal{G}}/\Lambda^\perp} \sum_{\chi \in \Lambda^\perp} \Big[|\hat{\phi}(\gamma + \chi)|^2 \big| \sum_{\lambda \in \Lambda} c_\lambda \langle \lambda, \gamma \rangle \big|^2 \Big] d\dot{\gamma}$$

$$= s(\Lambda^\perp) \int_{\hat{\mathcal{G}}/\Lambda^\perp} \sum_{\chi \in \Lambda^\perp} |\hat{\phi}(\gamma + \chi)|^2 \big| \sum_{\lambda \in \Lambda} c_\lambda \langle \lambda, \gamma \rangle \big|^2 d\dot{\gamma}.$$

Since this holds for all $(c)_\lambda \in \ell^2(\Lambda)$, the statement follows from (2.3). $\qquad \square$

A similar result holds for $\mathcal{H}_w(\mathcal{G})$ instead of $L^2(\mathcal{G})$.

Theorem 5.2. *For any lattice Λ of $\mathcal{G}$, the sequence $\{T_\lambda \phi\}_{\lambda \in \Lambda}$ is a Riesz sequence in $\mathcal{H}_w(\mathcal{G})$ if and only if there exist positive constants a, b such that*

$$(5.4) \qquad a \leq \sum_{\chi \in \Lambda^\perp} \hat{\phi}(\gamma + \chi) \leq b \qquad a. \ e.$$

Proof.   Analogue to the proof of Theorem 5.1.   □

Recalling condition (4.3) on the weight, we finally obtain that $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ forms a Riesz basis for its closed linear span, the so-called *spline-type space*

$$V_\Lambda(\phi) = \mathrm{span}\big\{\, T_\lambda\phi \,\big|\, \lambda\in\Lambda \,\big\} = \left\{ f = \sum_{\lambda\in\Lambda} c_\lambda T_\lambda\phi \,\big|\, (c_\lambda)\in\ell^2(\Lambda) \right\}$$

which is a closed subspace of $\mathcal{H}_w(\mathcal{G})$.

We now look for the dual basis of $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ in $\mathcal{H}_w$. Let us define

(5.5)
$$\hat{\psi}(\gamma) = \frac{\hat{\phi}(\gamma)}{\sum_{\chi\in\Lambda^\perp} \hat{\phi}(\gamma+\chi)}\,.$$

Note that the denominator is a $\Lambda^\perp$-periodic square integrable function. Since any such function has a unique Fourier series presentation with ℓ^2-coefficients, we can easily see that ψ belongs to $V_\Lambda(\phi)$. Similar to the proof of Theorem 5.1 we compute

(5.6)
$$\langle T_\lambda\phi, T_{\lambda'}\psi\rangle_w = \delta_{\lambda,\lambda'}\,, \qquad \lambda,\lambda'\in\Lambda\,.$$

Hence, $\{T_\lambda\psi\}_{\lambda\in\Lambda}$ is the dual basis of $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ in $V_\Lambda(\phi)$. It follows from (5.2) that every function in $V_\Lambda(\phi)$ can be written as

(5.7)
$$f = \sum_{\lambda\in\Lambda} \langle f, T_\lambda\phi\rangle \, T_\lambda\psi = \sum_{\lambda\in\Lambda} f(\lambda)\, T_\lambda\psi\,.$$

As a consequence, every function in $V_\Lambda(\phi)$ is completely determined by its samples on Λ.

Because of the reproducing kernel property of ϕ, (5.6) is equivalent to

$$\psi(\lambda-\lambda') = \delta_{\lambda,\lambda'}\,, \qquad \lambda,\lambda'\in\Lambda\,.$$

Therefore, ψ is also called the *Lagrange interpolator* for Λ.

It is important to note that the bi-orthogonal system of $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is again generated by the same translates of a single function, namely the Lagrange interpolator.

Remark 3.   Following Theorem 5.1, $V_\Lambda(\phi)$ is also a closed subspace of L^2. Similar to above, the dual basis of $\{T_\lambda\psi\}_{\lambda\in\Lambda}$ in L^2 is generated by the dual atom ψ_2 given by

$$\hat{\psi_2}(\gamma) = \frac{\hat{\phi}(\gamma)}{\sum_{\chi\in\Lambda^\perp} |\hat{\phi}(\gamma+\chi)|^2}\,.$$

Example 5.1.   In [12], the authors give a detailed study of interpolation and stable reconstruction of functions in the Sobolev space $\mathcal{H}_{w_s}(\mathbb{R}^d)$ from samples taken over a lattice of varying lattice size.

Example 5.2. A well-known example is the space of band-limited functions

$$B = \left\{ f \in L^2(\mathbb{R}) \,\middle|\, \mathrm{supp}(\hat{f}) \subset [-1/2, 1/2] \right\}$$

endowed with the L^2-inner product. It might be seen as a harmonic Hilbert space for the weight function

$$w(x) = \begin{cases} 1 & x \in [-1/2, 1/2], \\ \infty & x \notin [-1/2, 1/2] \end{cases}$$

although the weight function does not satisfy the conditions (4.1) and (4.2). Nevertheless, results similar to those above can be obtained. For instance, B is a RKHS with the so-called sinc-kernel

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

which is the inverse Fourier transform of the box function

$$\xi(x) = \begin{cases} 1 & x \in [-1/2, 1/2], \\ 0 & x \notin [-1/2, 1/2]. \end{cases}$$

Note that $\xi = w^{-2}$. The set of integer shifts of the sinc-kernel constitutes an orthonormal sequence. It is even an orthonormal basis of B. Consider the lattice $\Lambda = \alpha\mathbb{Z}$ with $\alpha < 1$. The corresponding annihilator is $\Lambda^{\perp} = \alpha^{-1}\mathbb{Z}$, and it can easily be seen that

$$\sum_{k \in \mathbb{Z}} w^{-2}(x - \alpha^{-1}k)$$

is not bounded away from zero. In particular $\{T_{\alpha k}\mathrm{sinc}\}_{k \in \mathbb{Z}}$ is an overcomplete basis system, a so-called *frame* [5] and not a Riesz basis, in accordance to Theorem 5.1. The fact that the periodization that is not bounded away from zero almost everywhere leads to a frame, is a general property and provides a characterization of such frames as shown in [4].

6. Minimal norm interpolation

The problem of minimal norm interpolation in a harmonic Hilbert space on the real line has been discussed by Delvos, [6]. In the present section we study the corresponding problem in a more general setting by integrating results stated above.

Theorem 6.1. *Let Λ be a lattice in $\mathcal{G}$ and $(c_\lambda) \in \ell^2(\Lambda)$. The interpolation problem $f(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$ has a unique minimal norm solution in $\mathcal{H}_w(\mathcal{G})$. It coincides with the interpolating element in $V_\Lambda(\phi)$ with $\hat{\phi} = w^{-2}$.*

Proof. Since Theorem 4.1 shows that $\mathcal{H}_w(\mathcal{G}) \hookrightarrow W(C, \ell^2)(\mathcal{G})$, the mapping

$$f \in \mathcal{H}_w(\mathcal{G}) \to (f(\lambda)) \in \ell^2(\Lambda)$$

is well-defined and bounded, cf. Corollary 4.1. Due to Theorem 5.2, $\{T_\lambda\phi\}$ is a Riesz basis for $V_\Lambda(\phi)$ which, by duality, consists of all functions of the form

$$f = \sum_{\lambda \in \Lambda} c_\lambda T_\lambda \psi$$

for some $(c_\lambda) \in \ell^2$ and the Lagrange interpolator ψ which is the dual Riesz atom for $\{T_\lambda\phi\}$. Due to the Lagrange property of ψ it follows that

$$f(\lambda) = c_\lambda, \qquad \lambda \in \Lambda,$$

for any such $f \in V_\Lambda(\phi)$. Hence the mapping

$$f \in V_\Lambda(\phi) \to (f(\lambda)) \in \ell^2(\Lambda)$$

is bijective. In particular, there exists a unique element $f_c \in V_\Lambda(\phi)$ with $f(\lambda) = c_\lambda$. Since $V_\Lambda(\phi)$ is closed, we can split $\mathcal{H}_w(\mathcal{G})$ into the direct sum

$$(6.1) \qquad\qquad \mathcal{H}_w(\mathcal{G}) = V_\Lambda(\phi) \oplus V_\Lambda^\perp(\phi),$$

where $V_\Lambda^\perp(\phi) = \left\{ f \in \mathcal{H}_w(\mathcal{G}) \,\middle|\, f(\lambda) = 0, \ \lambda \in \Lambda \right\}$. Assume that some $g \in \mathcal{H}_w(\mathcal{G})$ interpolates c on Λ. Then $g - f_c \in V_\Lambda^\perp(\phi)$ and we obtain

$$\|g\|_w^2 = \|(g - f_c) + f_c\|_w^2 = \|g - f_c\|_w^2 + \|f_c\|_w^2 \geq \|f_c\|_w^2.$$

Hence, f_c is the unique minimal norm element in $\mathcal{H}_w(\mathcal{G})$. $\qquad\square$

Corollary 6.1. *The minimal norm interpolation of the sequence $(f(\lambda))$ for some $f \in \mathcal{H}_w(\mathcal{G})$ sampled on a lattice Λ in $\mathcal{G}$ coincides with the orthogonal projection of f onto $V_\Lambda(\phi)$, say Pf.*

Proof. This follows immediately from

$$Pf(\lambda) = \langle T_\lambda\phi, Pf \rangle = \langle PT_\lambda\phi, f \rangle = \langle T_\lambda\phi, f \rangle = f(\lambda).$$

$\qquad\square$

From a computational perspective it is important to note that in order to solve the presented optimization problem subject to interpolation it suffices to compute or approximate the Lagrange interpolator via (5.5). Then, the minimal norm interpolation simply results from the quasi-interpolation (5.7). Therefore, the complexity for computing the minimal norm interpolation basically reduces to the computation of the Lagrange interpolator.

7. Orthogonal projection

In the final section we show that the minimal norm interpolation for a finite number of lattice elements converges to the minimal norm interpolation for Λ when increasing the number of lattice elements.

Let Λ be a lattice of $\mathcal{G}$. We denote by $\{\Lambda_F\}$ a nested sequence of finite subsets of Λ with $\bigcup_F \Lambda_F = \Lambda$. We define

$$V_F(\phi) = \text{span}\big\{ T_\lambda \phi \,\big|\, \lambda \in \Lambda_F \big\}.$$

We recall that $\big\{ T_\lambda \phi \,\big|\, \lambda \in \Lambda_F \big\}$ is obviously a Riesz basis for $V_F(\phi)$ whose dual basis is given by $\big\{ P_F T_\lambda \psi \,\big|\, \lambda \in \Lambda_F \big\}$ where P_F denotes the orthogonal projection onto $V_F(\phi)$. It is obvious that $P_F T_\lambda \psi = 0$ for all $\lambda \notin F$.

For a fixed element $c = (c_\lambda) \in \ell^2(\Lambda)$ we have seen that

$$g = \sum_{\lambda \in \Lambda} c_\lambda T_\lambda \psi$$

is the minimal norm interpolation on Λ. Set $c_F = (c_\lambda)_{\lambda \in \Lambda_F}$. By the same arguments used in Theorem 6.1, we easily deduce that the minimal norm element for c_F in $\mathcal{H}_w(\mathcal{G})$ is given by

$$g_F = \sum_{\lambda \in \Lambda_F} c_\lambda P_F T_\lambda \psi = P_F\Big(\sum_{\lambda \in \Lambda_F} c_\lambda T_\lambda \psi \Big) = P_F\Big(\sum_{\lambda \in \Lambda} c_\lambda T_\lambda \psi \Big) = P_F g.$$

In other words, the minimal norm interpolation for c_F is just the orthogonal projection of the minimal norm interpolation g for c.

Exploiting the nested structure of the subspaces, a standard argument in wavelet theory, e.g., [20], immediately implies

(7.1) $$\|g - g_F\|_w \to 0 \quad \text{for } F \text{ increasing}.$$

Since $\mathcal{H}_w(\mathcal{G})$ is continuously embedded in $C(\mathcal{G})$, the convergence in (7.1) holds true also for the sup-norm.

Faculty of Mathematics
University of Vienna
Nordbergstrasse 15
A-1090 Vienna
Austria
e-mail: hans.feichtinger@univie.ac.at

Department of Mathematics
R. D. University
Jabalpur
India
e-mail: sheelpandey@hotmail.com

Faculty of Mathematics
University of Vienna
Nordbergstrasse 15
A-1090 Vienna
Austria
e-mail: tobias.werther@univie.ac.at

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