Illinois Journal of Mathematics
Volume 61, Numbers 3–4, Fall and Winter 2017, Pages 415–441
S 0019-2082

DISTINGUISHING $\Bbbk$-CONFIGURATIONS

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Dedicated to the memory of A. V. Geramita

ABSTRACT. A $\Bbbk$-configuration is a set of points $\mathbb{X}$ in $\mathbb{P}^2$ that satisfies a number of geometric conditions. Associated to a $\Bbbk$-configuration is a sequence $(d_1, \dots, d_s)$ of positive integers, called its type, which encodes many of its homological invariants. We distinguish $\Bbbk$-configurations by counting the number of lines that contain d_s points of $\mathbb{X}$. In particular, we show that for all integers $m \gg 0$, the number of such lines is precisely the value of $\Delta\mathbf{H}_{m\mathbb{X}}(md_s - 1)$. Here, $\Delta\mathbf{H}_{m\mathbb{X}}(-)$ is the first difference of the Hilbert function of the fat points of multiplicity m supported on $\mathbb{X}$.

1. Introduction

In the late 1980's, Roberts and Roitman [13] introduced special configurations of points in $\mathbb{P}^2$ which they named $\Bbbk$-configurations. We recall this definition:

DEFINITION 1.1. A $\Bbbk$-configuration of points in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions: there exist integers $1 \le d_1 < \cdots < d_s$, subsets $\mathbb{X}_1, \dots, \mathbb{X}_s$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \dots, \mathbb{L}_s \subseteq \mathbb{P}^2$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^{s} \mathbb{X}_i$;
(2) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subseteq \mathbb{L}_i$ for each $i = 1, \dots, s$, and;
(3) $\mathbb{L}_i$ $(1 < i \le s)$ does not contain any points of $\mathbb{X}_j$ for all $1 \le j < i$.

Received May 25, 2017; received in final form February 15, 2018.

Y.S. Shin is the corresponding author.

Shin's research was supported by the Basic Science Research Program of the NRF (Korea) under grant (No. 2016R1D1A1B03931683). Van Tuyl's research was supported in part by NSERC Discovery Grant 2014-03898.

2010 *Mathematics Subject Classification.* 13D40, 14M05.

In this case, the $\mathbb{k}$-configuration is said to be of type $(d_1, \ldots, d_s)$.

This definition was first extended to $\mathbb{P}^3$ by Harima [12], and later to all $\mathbb{P}^n$ by Geramita, Harima, and Shin (see [8], [9]). As shown by Roberts and Roitman [13, Theorem 1.2], all $\mathbb{k}$-configurations of type $(d_1, \ldots, d_s)$ have the same Hilbert function (which can be computed from the type). This result was later generalized by Geramita, Harima, and Shin [7, Corollary 3.7] to show that all the graded Betti numbers of the associated graded ideal $I_{\mathbb{X}}$ only depend upon the type.

Interestingly, $\mathbb{k}$-configurations of the same type can have very different geometric properties. Figure 1 shows various examples of $\mathbb{k}$-configurations of type $(1, 2, 3)$. Note that the different shapes correspond to different sets of $\mathbb{X}_i$, i.e., the star is the point of $\mathbb{X}_1$, the squares are the two points of $\mathbb{X}_2$, and the circles are the three points $\mathbb{X}_3$. From a geometric point-of-view, these configurations are all qualitatively different in that the number of lines containing 3 points in each configuration is different (e.g., there are four lines that contain 3 points of $\mathbb{X}$ in the first configuration, but only one such line in the last configuration). However, from an algebraic point-of-view, because these sets of points are all $\mathbb{k}$-configurations of type $(1, 2, 3)$, the graded resolutions (and consequently, the Hilbert functions) of these sets of points are all the same.

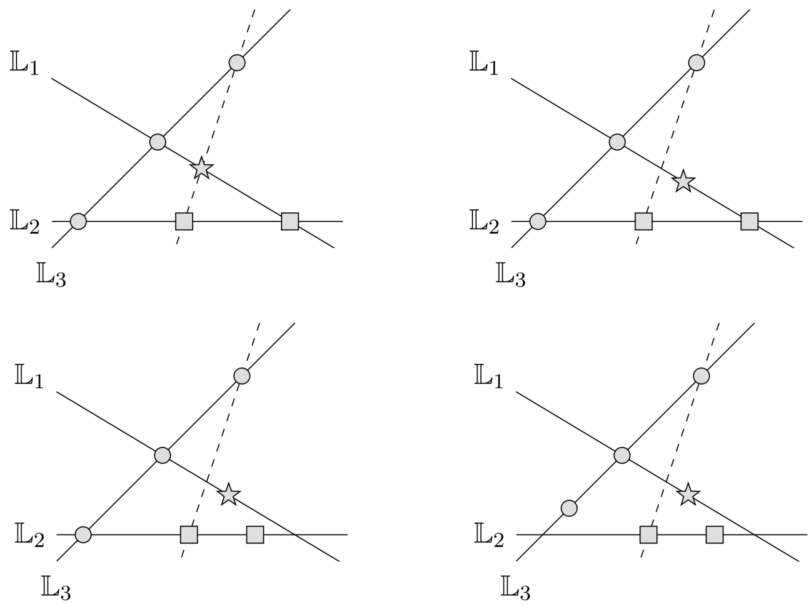

Figure 1. Four different $\mathbb{k}$-configurations of type $(1, 2, 3)$.

So the algebra does not "see" these lines, and so we cannot distinguish these $\Bbbk$-configurations.

Our goal in this paper is to determine how one can distinguish these $\Bbbk$-configurations from an algebraic point-of-view. In particular, we wish to distinguish $\Bbbk$-configurations by the number of lines that contain d_s points of $\mathbb{X}$. It can be shown (see Remark 2.12) that the first difference function of the Hilbert function of $\mathbb{X}$ produces only an upper bound on the number of lines. We show that one can obtain an exact value if one instead considers the Hilbert function of the set of fat points supported on the $\Bbbk$-configuration. Precisely, we prove the following theorem.

THEOREM 1.2. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a $\Bbbk$-configuration of type $d = (d_1, \ldots, d_s) \neq (1)$. Then there exists an integer m_0 such that for all $m \geq m_0$,*

$$\Delta\mathbf{H}_{m\mathbb{X}}(md_s - 1) = \text{number of lines containing exactly } d_s \text{ points of } \mathbb{X},$$

where $\Delta\mathbf{H}_{m\mathbb{X}}(-)$ is the first difference function of the Hilbert function of fat points of multiplicity m supported on $\mathbb{X}$. Furthermore, if $d_s > s$, then $m_0 = 2$, and if $d_s = s$, then $m_0 = s + 1$.

In other words, the number of lines that contain d_s points of $\mathbb{X}$ is encoded in the Hilbert function of fat points supported on $\mathbb{X}$. This provides us with an algebraic method to differentiate $\Bbbk$-configurations. Note we exclude the $\Bbbk$-configuration of type $d = (1)$ since $\mathbb{X}$ is a single point, and there are an infinite number of lines through this point. Thematically, this paper is similar to works of Bigatti, Geramita, and Migliore [2], and Chiantini and Migliore [4] which derived geometric consequences about points from the Hilbert function.

We now give an outline of the paper. In Section 2, we define all the relevant terminology involving $\Bbbk$-configurations, and some properties of $\Bbbk$-configurations. We also recall a procedure to bound values of the Hilbert function of a set of fat points due to Cooper, Harbourne, and Teitler [5], which will be our main tool. In Section 3, we focus on the case $d_s > s$ and prove Theorem 1.2 in this case. In Section 4, we focus on the case that $d_s = s$. A more subtle argument is needed to prove Theorem 1.2 since an extra line may come into play. We will also require a result of Catalisano, Trung, and Valla [3] to complete this case. In the final section, we give a reformulation of Theorem 1.2, and make a connection to a question of Geramita, Migliore, and Sabourin [10] on the number of Hilbert functions of fat points whose support has a fixed Hilbert function.

2. Background results

This section collects the necessary background results. We first review Hilbert functions and ideals of (fat) points in $\mathbb{P}^2$. We then introduce a number of lemmas describing $\Bbbk$-configurations. Throughout the remainder of this paper, $R = \Bbbk[x_0, x_1, x_2]$ is a polynomial ring over an algebraically closed field $\Bbbk$.

2.1. Points in $\mathbb{P}^2$ and Hilbert functions. We recall some general facts about (fat) points in $\mathbb{P}^2$ and their Hilbert functions. These results will be used later in our study of $\Bbbk$-configurations.

Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of distinct points in $\mathbb{P}^2$. If I_{P_i} is the ideal associated to P_i in $R = \Bbbk[x_0, x_1, x_2]$, then the homogeneous ideal associated to $\mathbb{X}$ is the ideal $I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s}$. Given s positive integers $m_1, \ldots, m_s$ (not necessarily distinct), the scheme defined by the ideal $I_{\mathbb{Z}} = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$ is called a set of *fat points*. We say that m_i is the *multiplicity* of the point P_i. If $m_1 = \cdots = m_s = m$, then we say $\mathbb{Z}$ is a *homogeneous set of fat points* of multiplicity m. In this case, we normally write $m\mathbb{X}$ for $\mathbb{Z}$, and $I_{m\mathbb{X}}$ for $I_{\mathbb{Z}}$.

Note that it can be shown that $I_{m\mathbb{X}} = I_{\mathbb{X}}^{(m)}$, the mth symbolic power of the ideal $I_{\mathbb{X}}$. If $\mathbb{X} = \{P\}$, then we sometimes write I_{mP} for $I_{m\mathbb{X}}$. As well, since I_P is a complete intersection, it follows that $I_{mP} = I_P^{(m)} = I_P^m$ (see Zariski–Samuel [14, Appendix 6, Lemma 5]).

An ongoing problem at the intersection of commutative algebra and algebraic geometry is to study and classify the Hilbert functions arising from the homogeneous ideals of sets of fat points. Recall that if $I \subseteq R = \Bbbk[x_0, x_1, x_2]$ is any homogeneous ideal, then the *Hilbert function* of R/I, denoted $\mathbf{H}_{R/I}$, is the numerical function $\mathbf{H}_{R/I} : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathbf{H}_{R/I}(t) := \dim_{\Bbbk} R_t - \dim_{\Bbbk} I_t,$$

where R_t, respectively I_t, denotes the tth graded component of R, respectively I. If $I = I_{\mathbb{Z}}$ is the defining ideal of a set of (fat) points $\mathbb{Z}$, then we usually write $\mathbf{H}_{\mathbb{Z}}$ for $\mathbf{H}_{R/I_{\mathbb{Z}}}$. The *first difference* of the Hilbert function $\mathbf{H}_{R/I}$, is the function

$$\Delta\mathbf{H}_{R/I}(t) := \mathbf{H}_{R/I}(t) - \mathbf{H}_{R/I}(t-1)$$
$$\text{for all } t \geq 0 \text{ where } \mathbf{H}_{R/I}(t) = 0 \text{ for all } t < 0.$$

Given a set of points $\mathbb{Z} \subseteq \mathbb{P}^2$, Cooper, Harbourne, and Teitler [5] described a procedure by which one can find both upper and lower bounds on $\mathbf{H}_{\mathbb{Z}}(t)$ for all $t \geq 0$. This procedure, which we describe below, will be instrumental in the proof of our main results.

Let $\mathbb{Z} = \mathbb{Z}_0$ be a fat point subscheme of $\mathbb{P}^2$. Choose a sequence of lines $\mathbb{L}_1, \ldots, \mathbb{L}_r$ and define $\mathbb{Z}_i$ to be the residual of $\mathbb{Z}_{i-1}$ with respect to the line $\mathbb{L}_i$ (i.e., the subscheme of $\mathbb{P}^2$ defined by the ideal $I_{\mathbb{Z}_i} : I_{\mathbb{L}_i}$). Define the associated *reduction vector* $\mathbf{v} = (v_1, \ldots, v_r)$ by taking $v_i = \deg(\mathbb{L}_i \cap \mathbb{Z}_{i-1})$. In particular, v_i is the sum of the multiplicities of the points in $\mathbb{L}_i \cap \mathbb{Z}_{i-1}$. Given $\mathbf{v} = (v_1, \ldots, v_r)$, we define functions

$$f_{\mathbf{v}}(t) = \sum_{i=0}^{r-1} \min(t - i + 1, v_{i+1}) \tag{2.1}$$

and

$$(2.2)\qquad F_{\mathbf{v}}(t) = \min_{0\le i\le r}\left(\binom{t+2}{2} - \binom{t-i+2}{2} + \sum_{j=i+1}^{r} v_j\right).$$

THEOREM 2.1 (Cooper–Harbourne–Teitler [5, Theorem 1.1]). *Let $\mathbb{Z} = \mathbb{Z}_0$ be a fat point scheme in $\mathbb{P}^2$ with reduction vector $\mathbf{v} = (v_1, \dots, v_r)$ such that $\mathbb{Z}_{r+1} = \varnothing$. Then the Hilbert function $\mathbf{H}_{\mathbb{Z}}(t)$ of $\mathbb{Z}$ is bounded by $f_{\mathbf{v}}(t) \le \mathbf{H}_{\mathbb{Z}}(t) \le F_{\mathbf{v}}(t)$.*

EXAMPLE 2.2. Let $\mathbb{X}$ be the $\Bbbk$-configuration of type $(1,3,4,5)$ in Figure 2. We illustrate how to use Theorem 2.1 to compute $\mathbf{H}_{2\mathbb{X}}(8)$. We take $\mathbb{Z} = 2\mathbb{X}$, that is, we assume that each point has multiplicity two; this is indicated by the 2 by each point.

We apply Theorem 2.1 using the sequence of lines $\mathbb{H}_1, \dots, \mathbb{H}_8$, where $\mathbb{H}_1 = \mathbb{L}_4$, $\mathbb{H}_2 = \mathbb{L}_3$, $\mathbb{H}_3 = \mathbb{L}_2$, $\mathbb{H}_4 = \mathbb{L}_1$, $\mathbb{H}_5 = \mathbb{L}_4$, $\mathbb{H}_6 = \mathbb{L}_3$, $\mathbb{H}_7 = \mathbb{L}_2$, and $\mathbb{H}_8 = \mathbb{L}_1$. The reduction vector is $\mathbf{v} = (10, 9, 8, 3, 3, 3, 2, 1)$. To see this, note that $\mathbb{Z}_0 = 2\mathbb{X}$, so $\mathbb{H}_1 \cap \mathbb{Z}_0$ consists of the five double points on $\mathbb{L}_4$, so $\deg(\mathbb{H}_1 \cap \mathbb{Z}_0) = 10$. We form $\mathbb{Z}_1$ from $\mathbb{Z}_0$ by reducing the multiplicity of each point on $\mathbb{H}_1 = \mathbb{L}_4$ by one. Then $\mathbb{H}_2 \cap \mathbb{Z}_1 = \mathbb{L}_3 \cap \mathbb{Z}_1$ consists of 4 double points and one reduced point, so $\deg(\mathbb{H}_2 \cap \mathbb{Z}_1) = 2 \cdot 4 + 1 = 9$. Figure 2 illustrates the first two steps of this procedure. Continuing in this fashion allows us to compute $\mathbf{v}$, ending when we reach $\mathbb{Z}_9 = \varnothing$.

To compute the lower bound $f_{\mathbf{v}}(8)$ using Equation (2.1), we compare the values of $8 - i + 1$ and v_{i+1} in the table below (the minimum is in bold).

i	0	1	2	3	4	5	6	7
$8-i+1$	**9**	**8**	**7**	6	5	4	3	2
v_{i+1}	10	9	8	**3**	**3**	**3**	**2**	**1**

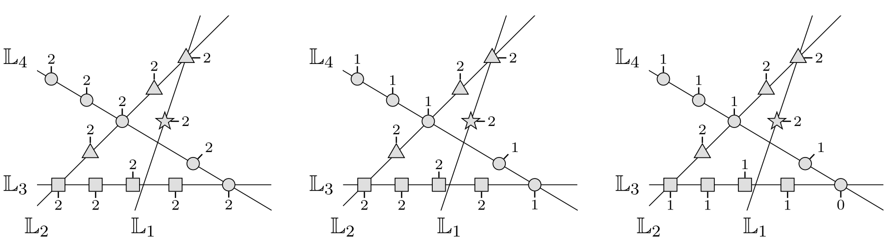

FIGURE 2. The Cooper–Harbourne–Teitler bound computation.

Adding up the minimum values gives $f_{\mathbf{v}}(8) = 36$. To compute an upper bound, we take $i = 3$ in Equation (2.2); then we have

$$F_{\mathbf{v}}(8) \le \binom{8+2}{2} - \binom{5+2}{2} + \sum_{j=4}^{8} v_j = 36.$$

This implies that $f_{\mathbf{v}}(8) = F_{\mathbf{v}}(8) = \mathbf{H}_{2\mathbb{X}}(8) = 36$.

REMARK 2.3. In Example 2.2, we have used the procedure of [5] to find actual values of the Hilbert function. In general, however, one can only expect to find bounds.

2.2. Properties of $\Bbbk$-configurations. In this section, we record a number of useful facts about $\Bbbk$-configurations.

LEMMA 2.4. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^2$ is a $\Bbbk$-configuration of type $(d_1, \dots, d_s)$. Then*

(i) $d_j \le d_s - s + j$ *for* $j = 1, \dots, s$;
(ii) *if* $d_s = s$, *then* $(d_1, \dots, d_s) = (1, \dots, s)$;
(iii) *for any line* $\mathbb{L}$ *in* $\mathbb{P}^2$, $|\mathbb{L} \cap \mathbb{X}| \le d_s$.

Proof. Statements (i) and (ii) follow directly from the definition of $\Bbbk$-configurations since $1 \le d_1 < d_2 < \cdots < d_s$. Statement (iii) is [13, Lemma 1.3]. □

By definition, there is at least one line $\mathbb{L}$ in $\mathbb{P}^2$ that meets a $\Bbbk$-configuration $\mathbb{X}$ of type $(d_1, \dots, d_s)$ at d_s points, namely, the line $\mathbb{L}_s$. As mentioned in the Introduction, our goal is to enumerate the lines that meet $\mathbb{X}$ at exactly d_s points. We begin with some useful necessary conditions for a line $\mathbb{L}$ to contain d_s points.

LEMMA 2.5. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^2$ is a $\Bbbk$-configuration of type $(d_1, \dots, d_s)$, and $\mathbb{L}_1, \dots, \mathbb{L}_s$ are the lines used to define $\mathbb{X}$. Let $\mathbb{L}$ be any line in $\mathbb{P}^2$ such that $|\mathbb{L} \cap \mathbb{X}| = d_s$.*

(i) *If* $d_s > s$, *then* $\mathbb{L} \in \{\mathbb{L}_1, \dots, \mathbb{L}_s\}$.
(ii) *If* $\mathbb{L} = \mathbb{L}_i$, *then* $d_j = d_s - s + j$ *for* $j = i, \dots, s$.

Proof. (i) If $\mathbb{L} \notin \{\mathbb{L}_1, \dots, \mathbb{L}_s\}$, then $\mathbb{L} \cap \mathbb{X} \subseteq \bigcup_{i=1}^{s} (\mathbb{L} \cap \mathbb{L}_i)$. So if $s < d_s$, $|\mathbb{L} \cap \mathbb{X}| \le \sum_{i=1}^{s} |\mathbb{L} \cap \mathbb{L}_i| = s < d_s$. In other words, if $|\mathbb{L} \cap \mathbb{X}| = d_s$, then $\mathbb{L}$ must be in $\{\mathbb{L}_1, \dots, \mathbb{L}_s\}$.

(ii) Suppose $\mathbb{L} = \mathbb{L}_i$ contains d_s points of $\mathbb{X}$. By definition, $\mathbb{L}_i$ contains the d_i points of $\mathbb{X}_i \subseteq \mathbb{X}$. Furthermore, this line cannot contain any of the points in $\mathbb{X}_1, \dots, \mathbb{X}_{i-1}$. In addition, $\mathbb{L}_i$ can contain at most one point of $\mathbb{X}_{i+1}, \dots, \mathbb{X}_s$. So $d_s = |\mathbb{L}_i \cap \mathbb{X}| \le d_i + (s - i)$. But by Lemma 2.4, we have $d_i + (s - i) \le d_s$, so $d_s = d_i + (s - i)$. To complete the proof, note that $d_i < d_{i+1} < \cdots < d_s$ is a set of $s - i + 1$ strictly increasing integers with $d_i = d_s - (s - i)$. This forces $d_j = d_s - (s - j)$ for all $j = i, \dots, s$. □

REMARK 2.6. If $\mathbb{X}$ is a $\Bbbk$-configuration of type $(d_1,\dots,d_s)$ with $d_{s-1} < d_s - 1$, the above lemma implies that there is exactly one line containing d_s points of $\mathbb{X}$, namely $\mathbb{L}_s$.

If $d_s > s$, Lemma 2.5 implies that the lines we want to count are among the $\mathbb{L}_i$'s, and consequently, there are at most s such lines. The next result shows that when $d_s = s$ (or equivalently, the type is $(1,2,\dots,s)$) the situation is more subtle. In particular, if there is a line $\mathbb{L}$ that contains s points that is not among the $\mathbb{L}_i$'s, then it must be one of two lines.

LEMMA 2.7. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^2$ is a $\Bbbk$-configuration of type $(1,2,\dots,s)$ with $s \geq 2$. Let $\mathbb{X}_1,\dots,\mathbb{X}_s$ be the subsets of $\mathbb{X}$; let $\mathbb{L}_1,\dots,\mathbb{L}_s$ be the lines used to define $\mathbb{X}$; let $\mathbb{X}_1 = \{P\}$ be the point on $\mathbb{L}_1$; and let $\mathbb{X}_2 = \{Q_1,Q_2\}$ be the two points on $\mathbb{L}_2$. If $\mathbb{L}$ is a line in $\mathbb{P}^2$ such that $|\mathbb{L}\cap\mathbb{X}| = d_s = s$, and if $\mathbb{L} \notin \{\mathbb{L}_1,\dots,\mathbb{L}_s\}$, then $\mathbb{L}$ must either be the line through P and Q_1, or the line through P and Q_2.*

Proof. Suppose $|\mathbb{L}\cap\mathbb{X}| = d_s = s$. Since $\mathbb{L} \notin \{\mathbb{L}_1,\dots,\mathbb{L}_s\}$, we have $s = |\mathbb{L}\cap\mathbb{X}| \leq |\mathbb{L}\cap\mathbb{L}_1| + \cdots + |\mathbb{L}\cap\mathbb{L}_s| = s$. In other words, $\mathbb{L}\cap\mathbb{L}_i$ is a point of $\mathbb{X}_i \subseteq \mathbb{X}$ for $i = 1,\dots,s$. So $\mathbb{L}$ must pass through P and either Q_1 or Q_2. $\square$

COROLLARY 2.8. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^2$ is a $\Bbbk$-configuration of type $(1,2,\dots,s)$ with $s \geq 2$. Then there are at most $s+1$ lines that contain s points of $\mathbb{X}$.*

Proof. The only candidates for the lines that contain s points are the s lines $\mathbb{L}_1,\dots,\mathbb{L}_s$ that define the $\Bbbk$-configuration, and by Lemma 2.7, the two lines $\mathbb{L}_{PQ_1}$ and $\mathbb{L}_{PQ_2}$, i.e., the lines that go through the point of $\mathbb{X}_1 = \{P\}$ and one of the two points of $\mathbb{X}_2 = \{Q_1,Q_2\}$. This gives us $s+2$ lines. However, if the lines $\mathbb{L}_{PQ_1}$ and $\mathbb{L}_{PQ_2}$ both contain s points, then either $\mathbb{L}_1$ is one of these two lines, or does not contain s points. Indeed if $\mathbb{L}_1$ contains s points, then $s = |\mathbb{L}_1\cap\mathbb{X}| = |\mathbb{X}_1| + |\mathbb{L}_1\cap\mathbb{L}_2| + \cdots + |\mathbb{L}_1\cap\mathbb{L}_s| = s$. In particular, $|\mathbb{L}_1\cap\mathbb{L}_2| = 1$, that is, $\mathbb{L}_1$ must contain one of the two points of $\mathbb{X}_2$, and so $\mathbb{L}_1 = \mathbb{L}_{PQ_1}$ or $\mathbb{L}_{PQ_2}$. So, there are at most $s+1$ lines that contain s points $\mathbb{X}$. $\square$

We finish this section with a useful lemma for relabelling a $\Bbbk$-configuration. This lemma exploits the fact that the lines and subsets defining a $\Bbbk$-configuration need not be unique.

LEMMA 2.9. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^2$ is a $\Bbbk$-configuration of type $(d_1,d_2,\dots,d_s)$ with $s \geq 2$. Let $\mathbb{X}_1,\dots,\mathbb{X}_s$ be the subsets of $\mathbb{X}$, and $\mathbb{L}_1,\dots,\mathbb{L}_s$ the lines used to define $\mathbb{X}$. Suppose that*

- $|\mathbb{L}_{s-k}\cap\mathbb{X}| = d_s$ *for* $k = 0,\dots,j$,
- $|\mathbb{L}_{s-k}\cap\mathbb{X}| < d_s$ *for* $k = j+1,\dots,i-1$, *and*
- $|\mathbb{L}_{s-i}\cap\mathbb{X}| = d_s$.

Set $\mathbb{T} = \mathbb{L}_{s-i}\cap(\mathbb{X}_{s-j-1}\cup\mathbb{X}_{s-j-2}\cup\cdots\cup\mathbb{X}_{s-i+1})$.

Then the $\Bbbk$*-configuration* $\mathbb{X}$ *can also be defined using the subsets* $\mathbb{X}'_1, \ldots, \mathbb{X}'_s$ *and lines* $\mathbb{L}'_1, \ldots, \mathbb{L}'_s$ *where*

- $\mathbb{X}'_k = \mathbb{X}_k$ *and* $\mathbb{L}'_k = \mathbb{L}_k$ *for* $k = 1, \ldots, s-i-1$,
- $\mathbb{X}'_k = \mathbb{X}_{k+1} \setminus \mathbb{T}$ *and* $\mathbb{L}'_k = \mathbb{L}_{k+1}$ *for* $k = s-i, \ldots, s-j-2$,
- $\mathbb{X}'_{s-j-1} = \mathbb{X}_{s-i} \cup \mathbb{T}$ *and* $\mathbb{L}'_{s-j-1} = \mathbb{L}_{s-i}$, *and*
- $\mathbb{X}'_k = \mathbb{X}_k$ *and* $\mathbb{L}'_k = \mathbb{L}_k$ *for* $k = s-j, \ldots, s$.

Proof. We need to verify that the subsets $\mathbb{X}'_i$ and lines $\mathbb{L}'_i$ define the same $\Bbbk$-configuration, that is, we need to see if they satisfy the conditions (1), (2), and (3) of Definition 1.1.

We first note that condition (1) holds since

$$\begin{aligned}\bigcup_{i=1}^{s} \mathbb{X}'_k &= (\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_{s-i-1}) \cup \left(\bigcup_{k=s-i}^{s-j-2} (\mathbb{X}_{k+1} \setminus \mathbb{T})\right) \\ &\quad \cup (\mathbb{X}_{s-i} \cup \mathbb{T}) \cup (\mathbb{X}_{s-j} \cup \cdots \cup \mathbb{X}_s) \\ &= \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_s = \mathbb{X}.\end{aligned}$$

For condition (2), it is clear that $\mathbb{X}'_k \subseteq \mathbb{L}'_k$ for all k. We now verify that $|\mathbb{X}'_k| = d_k$ for all k. For $k = 1, \ldots, s-i-1$ and $k = s-j, \ldots, s$ this is immediate since $\mathbb{X}'_k = \mathbb{X}_k$. Because $|\mathbb{L}_{s-i} \cap \mathbb{X}| = d_s$, it follows by Lemma 2.5 that $d_{s-k} = d_s - s + (s-k) = d_s - k$ for $k = i, \ldots, j+1$. Moreover, as in the proof of Lemma 2.5, $\mathbb{L}_{s-i} \cap \mathbb{L}_{s-k} \in \mathbb{X}_{s-k}$ for all $k = j+1, \ldots, i-1$. So $|\mathbb{T}| = i-1-j$, and thus

$$\left|\mathbb{X}'_{s-j+1}\right| = |\mathbb{X}_{s-i} \cup \mathbb{T}| = d_s - i + i - 1 - j = d_s - (j+1) = d_{s-j+1}.$$

Also, again since $\mathbb{L}_{s-i} \cap \mathbb{L}_{s-k} \in \mathbb{X}_{s-k}$ for $k = j+1, \ldots, i-1$, we have

$$\left|\mathbb{X}'_k\right| = |\mathbb{X}_{k+1} \setminus \mathbb{T}| = d_{k+1} - 1 = d_s - s + k + 1 - 1 = d_s - s + k = d_k$$

for $k = s-i, \ldots, s-j-2$.

Finally, for condition (3), we only need to check the line $\mathbb{L}'_{s-j-1}$ since the result is true for the other lines by the construction of $\mathbb{X}$ using the lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$. Now $\mathbb{L}'_{s-j-1} = \mathbb{L}_{s-i}$, and we know that it does not intersect with the points $\mathbb{X}'_k = \mathbb{X}_k$ with $k < s-i$. Also, by construction, $\mathbb{L}'_{s-j-1}$ does not intersect with the points of $\mathbb{X}'_{s-i}, \ldots, \mathbb{X}'_{s-j-2}$. So condition (3) holds. □

Example 2.10. Figure 3 gives an example of the relabelling. As before, the shapes denote which points belong to the subsets $\mathbb{X}_i$.

Corollary 2.11. *Suppose that* $\mathbb{X} \subseteq \mathbb{P}^2$ *is a* $\Bbbk$*-configuration of type* $(d_1, d_2, \ldots, d_s)$ *with* $s \geq 2$. *Let* $\mathbb{L}_1, \ldots, \mathbb{L}_s$ *be the lines used to define* $\mathbb{X}$. *After relabelling, we can assume that there is an* r *such that* $|\mathbb{L}_{s-j} \cap \mathbb{X}| = d_s$ *for all* $0 \leq j \leq r-1$, *but* $|\mathbb{L}_{s-j} \cap \mathbb{X}| < d_s$ *for all* $r \leq j \leq s-1$.

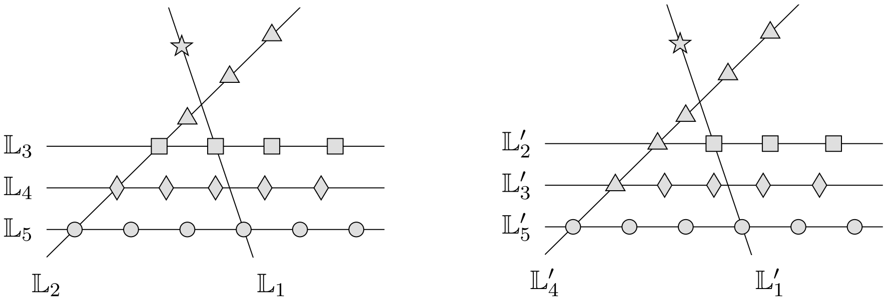

FIGURE 3. Relabelling lines of a $\mathbb{k}$-configuration of type $(1,3,4,5,6)$.

Proof. In the assumptions of Lemma 2.9, we are assuming that $\mathbb{L}_s,\ldots,\mathbb{L}_{s-j}$ all meet $\mathbb{X}$ at d_s points, but $\mathbb{L}_{s-j-1}$ does not. After applying the relabelling of Lemma 2.9, the lines $\mathbb{L}'_s,\ldots,\mathbb{L}'_{s-j-1}$ now meet $\mathbb{X}$ at d_s points. By reiterating Lemma 2.9, we arrive at the conclusion. $\square$

REMARK 2.12. In the Introduction, we mentioned that the number of lines that contain d_s points is bounded by a value of the first difference of the Hilbert function. Specifically, the number of lines that contain d_s points is bounded above by $\Delta\mathbf{H}_{\mathbb{X}}(d_s-1)+1$.

We sketch out how to prove this result. Roberts and Roitman [13, Theorem 1.2] give a formula for the Hilbert function $\mathbf{H}_{\mathbb{X}}$ of a $\mathbb{k}$-configuration in terms of the type $(d_1,d_2,\ldots,d_s)$. It follows from this formula that $\mathbf{H}_{\mathbb{X}}(d_s-1)=\sum_{i=1}^{s} d_i$, and $\mathbf{H}_{\mathbb{X}}(d_s-2)=(\sum_{i=1}^{s} d_i)-t$ where t is the number of consecutive integers at the end of $(d_1,d_2,\ldots,d_s)$. So,

$$\Delta\mathbf{H}_{\mathbb{X}}(d_s-1)=\mathbf{H}_{\mathbb{X}}(d_s-1)-\mathbf{H}_{\mathbb{X}}(d_s-2)=t.$$

Note that if $d_s=s$, then $t=s$. It follows by Lemma 2.5 that if $d_s>s$, then t is an upper bound on the number of lines that contain d_s points, and if $d_s=s$, then by Corollary 2.8, $t+1=s+1$ is an upper bound. We can combine this information in the statement that number of lines that contain d_s points is bounded above by $\Delta\mathbf{H}_{\mathbb{X}}(d_s-1)+1$.

3. The case $d_s>s$

In this section, we prove Theorem 1.2 in the case the $\mathbb{k}$-configuration $\mathbb{X}$ has type $(d_1,\ldots,d_s)$ with $d_s>s$.

THEOREM 3.1. *Let $\mathbb{X}\subseteq\mathbb{P}^2$ be a $\mathbb{k}$-configuration of type $d=(d_1,\ldots,d_s)$, and assume that there are r lines containing exactly d_s points of $\mathbb{X}$. If $d_s>s$, then $r=\Delta\mathbf{H}_{m\mathbb{X}}(md_s-1)$ for all $m\geq 2$.*

Proof. By Lemma 2.5(i), the lines containing d_s points of $\mathbb{X}$ fall among the lines $\mathbb{L}_1, \dots, \mathbb{L}_s$ defining the $\Bbbk$-configuration. By Corollary 2.11, we may assume that the lines containing exactly d_s points of $\mathbb{X}$ are $\mathbb{L}_s, \dots, \mathbb{L}_{s-r+1}$, while the lines $\mathbb{L}_{s-r}, \dots, \mathbb{L}_1$ contain less than d_s points of $\mathbb{X}$.

We will apply Theorem 2.1 to compute certain values of $\mathbf{H}_{m\mathbb{X}}$. Towards this goal, we obtain the reduction vector $\mathbf{v}$ of $\mathbb{X}$ using the sequence of lines

$$\mathbb{L}_s, \dots, \mathbb{L}_1, \mathbb{L}_s, \dots, \mathbb{L}_1, \dots, \mathbb{L}_s, \dots, \mathbb{L}_1,$$

where the subsequence $\mathbb{L}_s, \dots, \mathbb{L}_1$ is repeated m times.

We claim that, for $i = 1, \dots, r$, we have

$$v_i = md_s - i + 1.$$

Let $\mathbb{Z}_0 = m\mathbb{X}$. Since $|\mathbb{L}_s \cap \mathbb{X}| = d_s$, we have

$$v_1 = \deg(\mathbb{L}_s \cap \mathbb{Z}_0) = md_s.$$

Now let $\mathbb{Z}_1$ be the residual of $\mathbb{Z}_0$ with respect to the line $\mathbb{L}_s$. The line $\mathbb{L}_{s-1}$ contains the d_{s-1} points of $\mathbb{X}_{s-1}$, and the point $\mathbb{X}_s \cap \mathbb{L}_{s-1}$. The multiplicity of the point $\mathbb{X}_s \cap \mathbb{L}_{s-1}$ in $\mathbb{Z}_1$ is $m-1$, while the points of $\mathbb{X}_{s-1}$ have multiplicity m in $\mathbb{Z}_1$. Thus, we get

$$v_2 = \deg(\mathbb{L}_{s-1} \cap \mathbb{Z}_1) = md_{s-1} + m - 1 = m(d_s - 1) + m - 1 = md_s - 1,$$

where $d_{s-1} = d_s - 1$ by Lemma 2.5(ii). Continuing in this fashion, for $i = 3, \dots, r$, we have a scheme $\mathbb{Z}_{i-1}$. The line $\mathbb{L}_{s-i+1}$ contains the d_{s-i+1} points of $\mathbb{X}_{s-i+1}$, and the points $\mathbb{X}_s \cap \mathbb{L}_{s-i+1}$, $\mathbb{X}_{s-1} \cap \mathbb{L}_{s-i+1}$, $\dots$, $\mathbb{X}_{s-i+2} \cap \mathbb{L}_{s-i+1}$. The former have multiplicity m in $\mathbb{Z}_{i-1}$, while the latter have multiplicity $m-1$ in $\mathbb{Z}_{i-1}$. Thus, for $i = 1, \dots, r$, we get

$$\begin{aligned} v_i &= \deg(\mathbb{L}_{s-i+1} \cap \mathbb{Z}_{i-1}) = md_{s-i+1} + (m-1)(i-1) \\ &= m(d_s - i + 1) + (m-1)(i-1) = md_s - i + 1. \end{aligned}$$

Next, we claim that, for $i = r+1, \dots, s$, we have

$$v_i \leq md_s - i.$$

The line $\mathbb{L}_{s-i+1}$ contains the d_{s-i+1} points of $\mathbb{X}_{s-i+1}$, and e points at the intersections $\mathbb{X}_t \cap \mathbb{L}_{s-i+1}$ for $t > s - i + 1$. Note that $d_{s-i+1} + e < d_s$ because we assumed that, for $i = r+1, \dots, s$, the line $\mathbb{L}_{s-i+1}$ contains less than d_s points of $\mathbb{X}$. The points of $\mathbb{X}_{s-i+1}$ have multiplicity m in $\mathbb{Z}_{i-1}$, while each point $\mathbb{X}_t \cap \mathbb{L}_{s-i+1}$ has multiplicity $m-1$ in $\mathbb{Z}_{i-1}$. Thus, for $i = r+1, \dots, s$, we get

$$\begin{aligned} v_i &= \deg(\mathbb{L}_{s-i+1} \cap \mathbb{Z}_{i-1}) = md_{s-i+1} + (m-1)e \\ &= d_{s-i+1} + (m-1)(d_{s-i+1} + e) \\ &< d_s - i + 1 + (m-1)d_s = md_s - i + 1, \end{aligned}$$

where the inequality uses Lemma 2.4(i). This proves our claim.

This concludes our first round of removing the lines $\mathbb{L}_s, \dots, \mathbb{L}_1$, corresponding to the entries $v_1, \dots, v_s$ of the reduction vector $\mathbf{v}$. Now we focus on later passes. We can index later entries of the reduction vector by v_{js+i}, where $j = 1, \dots, m-1$ keeps track of the current pass (the first pass corresponding to $j = 0$), and $i = 1, \dots, s$ indicates that we are going to remove the line $\mathbb{L}_{s-i+1}$. We claim that

$$v_{js+i} \leq md_s - (js + i).$$

We proceed to estimate the multiplicity of points in $\mathbb{L}_{s-i+1} \cap \mathbb{Z}_{js+i-1}$. The line $\mathbb{L}_{s-i+1}$ contains the d_{s-i+1} points of $\mathbb{X}_{s-i+1}$; these have multiplicity at most $m-j$ in $\mathbb{Z}_{js+i-1}$, because the line $\mathbb{L}_{s-i+1}$ was removed j times in previous passes. In addition, the line $\mathbb{L}_{s-i+1}$ contains e points at the intersections $\mathbb{X}_t \cap \mathbb{L}_{s-i+1}$ for $t > s-i+1$, where $d_{s-i+1} + e \leq d_s$ as before. Each of these points has been removed j times in previous passes and once in the current pass, and therefore, it has multiplicity at most $m-j-1$ in $\mathbb{Z}_{js+i-1}$. Altogether, for $j = 1, \dots, m-1$ and $i = 1, \dots, s$, we obtain the following estimate:

$$\begin{aligned} v_{js+i} &= \deg(\mathbb{L}_{s-i+1} \cap \mathbb{Z}_{js+i-1}) \leq (m-j)d_{s-i+1} + (m-j-1)e \\ &= d_{s-i+1} + (m-j-1)(d_{s-i+1} + e) \\ &\leq d_s - i + 1 + (m-j-1)d_s = md_s - jd_s - i + 1 \\ &< md_s - js - i + 1, \end{aligned}$$

using Lemma 2.4(i) and the hypothesis $d_s > s$. This proves our claim.

Observe that after removing the lines $\mathbb{L}_s, \dots, \mathbb{L}_1$ m times, we have $\mathbb{Z}_{ms+1} = \varnothing$. In other words, $\mathbf{v} = (v_1, \dots, v_{ms})$ is a complete reduction vector for $m\mathbb{X}$. We can summarize our findings about $\mathbf{v}$ as follows:

$$\begin{aligned} v_i &= md_s - i + 1 \quad \text{for } i = 1, \dots, r, \\ v_i &\leq md_s - i \quad \text{for } i = r+1, \dots, ms. \end{aligned}$$

Now we compute the value $\mathbf{H}_{m\mathbb{X}}(md_s - 2)$ using Theorem 2.1. Recall that a lower bound is given by

$$f_{\mathbf{v}}(md_s - 2) = \sum_{i=0}^{ms-1} \min(md_s - 1 - i, v_{i+1}).$$

Based on our previous estimates, we have

$$\begin{aligned} \min(md_s - 1 - i, v_{i+1}) &= md_s - 1 - i \quad \text{for } i = 0, \dots, r-1, \\ \min(md_s - 1 - i, v_{i+1}) &= v_{i+1} \quad \text{for } i = r, \dots, ms-1. \end{aligned}$$

Hence, we get

$$f_{\mathbf{v}}(md_s - 2) = \sum_{i=0}^{r-1}(md_s - 1 - i) + \sum_{i=r}^{ms-1} v_{i+1}.$$

As for the upper bound, it is given by

$$F_{\mathbf{v}}(md_s-2)=\min_{0\le i\le ms}\left(\binom{md_s}{2}-\binom{md_s-i}{2}+\sum_{j=i+1}^{ms}v_j\right).$$

Evaluating the right-hand side for $i=r$, we get

$$\begin{aligned}F_{\mathbf{v}}(md_s-2)&\le\binom{md_s}{2}-\binom{md_s-r}{2}+\sum_{j=r+1}^{ms}v_j\\&=\sum_{h=md_s-r}^{md_s-1}h+\sum_{j=r+1}^{ms}v_j\\&=\sum_{i=0}^{r-1}(md_s-1-i)+\sum_{i=r}^{ms-1}v_{i+1}.\end{aligned}$$

Combining these bounds, we obtain

$$\mathbf{H}_{m\mathbb{X}}(md_s-2)=\sum_{i=0}^{r-1}(md_s-1-i)+\sum_{i=r}^{ms-1}v_{i+1}.$$

Similarly, we can use Theorem 2.1 to compute $\mathbf{H}_{m\mathbb{X}}(md_s-1)$. In this case, the lower bound is given by

$$\begin{aligned}f_{\mathbf{v}}(md_s-1)&=\sum_{i=0}^{ms-1}\min(md_s-i,v_{i+1})\\&=\sum_{i=0}^{r-1}(md_s-i)+\sum_{i=r}^{ms-1}v_{i+1}=\sum_{i=0}^{ms-1}v_{i+1}.\end{aligned}$$

Note that since $f_{\mathbf{v}}(md_s-1)$ is the sum of all the entries of the reduction vector, $f_{\mathbf{v}}(md_s-1)=\deg(m\mathbb{X})$ by [5, Remark 1.2.6]. On the other hand, it is well known that for any zero-dimensional scheme $\mathbb{Z}$, $\mathbf{H}_{\mathbb{Z}}(t)\le\deg(\mathbb{Z})$ for all t (see, e.g., [3]). We thus have

$$\mathbf{H}_{m\mathbb{X}}(md_s-1)=\sum_{i=0}^{r-1}(md_s-i)+\sum_{i=r}^{ms-1}v_{i+1}=\deg(m\mathbb{X}).$$

Finally, computing the first difference of the Hilbert function gives the desired result:

$$\begin{aligned}\Delta\mathbf{H}_{m\mathbb{X}}(md_s-1)&=\mathbf{H}_{m\mathbb{X}}(md_s-1)-\mathbf{H}_{m\mathbb{X}}(md_s-2)\\&=\sum_{i=0}^{r-1}(md_s-i)-\sum_{i=0}^{r-1}(md_s-1-i)=\sum_{i=0}^{r-1}1=r.\end{aligned}$$

□

EXAMPLE 3.2. Consider the $\Bbbk$-configuration $\mathbb{X}$ of type $(1,3,4,5)$ of Example 2.2. There are three lines containing $d_4 = 5$ points, namely $\mathbb{L}_2$, $\mathbb{L}_3$, and $\mathbb{L}_4$.

Our computation in Example 2.2 shows that $\mathbf{H}_{2\mathbb{X}}(2d_4 - 2) = \mathbf{H}_{2\mathbb{X}}(8) = 36$. In fact, this is an instance of the general computation carried out in the proof of Theorem 3.1. A similar computation yields $\mathbf{H}_{2\mathbb{X}}(9) = 39$. Therefore, we have

$$\Delta\mathbf{H}_{2\mathbb{X}}(9) = \mathbf{H}_{2\mathbb{X}}(9) - \mathbf{H}_{2\mathbb{X}}(8) = 39 - 36 = 3,$$

as desired.

4. The case $d_s = s$

In this section, we focus on $\Bbbk$-configurations of type $d = (d_1, \ldots, d_s)$ with $d_s = s \geq 2$ (as mentioned in the Introduction, the case $d = (1)$ is a single point). As noted in Lemma 2.4, the $\Bbbk$-configuration $\mathbb{X}$ must have type $(1, 2, \ldots, s)$. Unlike the case $d_s > s$, the value of $\Delta\mathbf{H}_{2\mathbb{X}}(2d_s - 1)$ need not equal the number of lines that contain $d_s = s$ points of $\mathbb{X}$. As a simple example, consider the $\Bbbk$-configuration of type $(1,2,3)$ given in Figure 4. This $\Bbbk$-configuration has exactly one line containing exactly three points (namely, the line $\mathbb{L}_3$). However, when we compute the Hilbert function of $2\mathbb{X}$, we get

$$\mathbf{H}_{2\mathbb{X}} : 1\ 3\ 6\ 10\ 15\ 18\ 18 \rightarrow,$$

and consequently, $\Delta\mathbf{H}_{2\mathbb{X}}(2 \cdot 3 - 1) = \mathbf{H}_{2\mathbb{X}}(5) - \mathbf{H}_{2\mathbb{X}}(4) = 18 - 15 = 3$. So, the hypothesis that $d_s > s$ in Theorem 3.1 is necessary.

In this section, we will derive a result similar to Theorem 3.1. However, in order to find the number of lines that contain s points of $\mathbb{X}$, we need to consider the Hilbert function of $m\mathbb{X}$ with $m \geq s+1$ instead of $m \geq 2$. We need a more subtle argument, in part, because of Lemma 2.7. That is, unlike the case of $d_s > s$, there may be up to two extra lines $\mathbb{L}$ that contains s points of $\mathbb{X}$, where $\mathbb{L}$ is not among the lines that defines the $\Bbbk$-configuration.

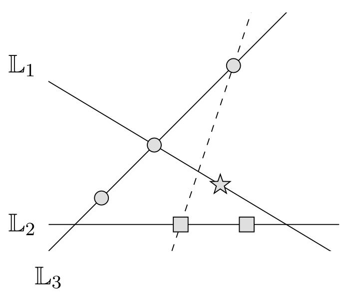

FIGURE 4. A $\Bbbk$-configuration of type $(1,2,3)$ with exactly one line with three points.

We begin with a lemma that allows us to break our argument into three separate cases. This lemma is similar to Lemma 2.9 in that it allows us to make some additional assumptions about the lines $\mathbb{L}_1, \dots, \mathbb{L}_s$ and points $\mathbb{X}_1, \dots, \mathbb{X}_s$ used to define the $\Bbbk$-configuration.

LEMMA 4.1. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^2$ is a $\Bbbk$-configuration of type $(1, 2, \dots, s)$ with $s \geq 2$. Let $\mathbb{X}_1, \dots, \mathbb{X}_s$ be the subsets of $\mathbb{X}$, and $\mathbb{L}_1, \dots, \mathbb{L}_s$ the lines used to define $\mathbb{X}$.*

Then one of the three disjoint cases must hold:

(i) *There are exactly $s+1$ lines that contain s points of $\mathbb{X}$, and the points of $\mathbb{X}$ are precisely the pairwise intersections of such lines.*

(ii) *There are exactly s lines that contain s points of $\mathbb{X}$, and we can assume that these lines are $\mathbb{L}_1, \dots, \mathbb{L}_s$. Moreover, for each $i = 1, \dots, s$, the set $\mathbb{X} \cap \mathbb{L}_i$ contains $s-1$ points located at the intersection of $\mathbb{L}_i$ and $\mathbb{L}_j$ (with $j \neq i$), and a single point P_i that does not belong to any line $\mathbb{L}_j$ for $j \neq i$.*

(iii) *There are $1 \leq r < s$ lines that contain s points, and furthermore, after a relabelling of the lines $\mathbb{L}_1, \dots, \mathbb{L}_s$ and subsets $\mathbb{X}_1, \dots, \mathbb{X}_s$, we can assume that none of these r lines pass through the point of $\mathbb{X}_1 = \{P\}$.*

Proof. By Corollary 2.8 there are at most $s+1$ lines that contain s points of $\mathbb{X}$. So, there are three cases: (i) exactly $s+1$ lines that contain s point of $\mathbb{X}$, (ii) exactly s lines that contain s points of $\mathbb{X}$, or (iii) $1 \leq r < s$ lines that contain s points of $\mathbb{X}$. We now show that in each case, we can label the $\mathbb{X}_i$'s and $\mathbb{L}_i$'s as described in the statement.

(i) Suppose that there are exactly $s+1$ lines that contain s points of $\mathbb{X}$. If $s = 2$, then the hypothesis that $\mathbb{X}$ is a $\Bbbk$-configuration implies that the three points of $\mathbb{X}$ are not colinear. Thus each pair of points of $\mathbb{X}$ uniquely determines a line, and the points of $\mathbb{X}$ are the intersections of such lines. Now suppose that $s > 2$, and let $\mathbb{H}_1, \dots, \mathbb{H}_s, \mathbb{H}_{s+1}$ be the $s+1$ lines each passing through s points of $\mathbb{X}$. Define the set of points $\mathbb{Y} := \mathbb{X} \setminus \mathbb{H}_{s+1}$, so that $\mathbb{X} = \mathbb{Y} \cup (\mathbb{X} \cap \mathbb{H}_{s+1})$. We have $|\mathbb{Y}| = \binom{s+1}{2} - s = \binom{s}{2}$. Each line $\mathbb{H}_1, \dots, \mathbb{H}_s$ passes through $s-1$ points of $\mathbb{Y}$, otherwise we would have $|\mathbb{Y}| > \binom{s}{2}$. By induction on s, the points of $\mathbb{Y}$ are the pairwise intersections of the lines $\mathbb{H}_1, \dots, \mathbb{H}_s$. By cardinality considerations, the s points of $\mathbb{X} \cap \mathbb{H}_{s+1}$ must be the intersections of $\mathbb{H}_{s+1}$ with the lines $\mathbb{H}_1, \dots, \mathbb{H}_s$. This shows that the points of $\mathbb{X}$ are precisely the pairwise intersections of the lines $\mathbb{H}_1, \dots, \mathbb{H}_s, \mathbb{H}_{s+1}$.

(ii) Suppose that there are exactly s lines that contain s points of $\mathbb{X}$. There are three subcases: (a) the s lines that contain s points are $\mathbb{L}_1, \dots, \mathbb{L}_s$; (b) $s-1$ of the lines that contain s points are among $\mathbb{L}_1, \dots, \mathbb{L}_s$ and there is one more line $\mathbb{L}$; and (c) $s-2$ of the lines that contain s points are among $\mathbb{L}_1, \dots, \mathbb{L}_s$, and there are two more lines that contain s points. Note that Lemma 2.7 implies that there is at most two lines not among the $\mathbb{L}_i$'s that will contain s points, so these are the only three cases. We will first show that if (c) is true, then we can relabel the lines and points so that we can assume case (b) is true. We will then show that in case (b), we can again relabel lines and points so we can assume case (a) is true.

Assume case (c) holds. By Lemma 2.7, the two lines that contain s points that are not among the $\mathbb{L}_i$'s are the lines $\mathbb{L}_{PQ_1}$ and $\mathbb{L}_{PQ_2}$ where $\mathbb{X}_1 = \{P\}$ and $\mathbb{X}_2 = \{Q_1, Q_2\}$. As argued in Corollary 2.8, the line $\mathbb{L}_1$ cannot contain s points. Since $\{P\} = \mathbb{X}_1 \subseteq \mathbb{L}_{PQ_1}$, we then have that the $\Bbbk$-configuration can also be defined by the same $\mathbb{X}_i$'s and the lines $\mathbb{L}_{PQ_1}, \mathbb{L}_2, \ldots, \mathbb{L}_s$. Note that we are in now case (b).

We now assume case (b), that is, $s-1$ of the lines that contain s points are among $\mathbb{L}_1, \ldots, \mathbb{L}_s$ and there is one additional line $\mathbb{L}$ that contains s points. Suppose that $\mathbb{L}_1$ does not contain s points. By Lemma 2.7, the additional line $\mathbb{L}$ contains $\mathbb{X}_1$, so as above, we replace $\mathbb{L}_1$ with $\mathbb{L}$, and the $\Bbbk$-configuration is defined by the same $\mathbb{X}_i$'s and the lines $\mathbb{L}, \mathbb{L}_2, \ldots, \mathbb{L}_s$, all of which contain s points. On the other hand, suppose $\mathbb{L}_1$ contains s points. Then there is exactly one line $\mathbb{L}_j \in \{\mathbb{L}_2, \ldots, \mathbb{L}_{s-1}\}$ that does not contain s points (note that $\mathbb{L}_s$ contains s points). Moreover, $\mathbb{L}_1, \ldots, \mathbb{L}_{j-1}$ must all intersect $\mathbb{L}_j$ at distinct points since each such $\mathbb{L}_i$ needs to contain s distinct points.

Set

$$\mathbb{T} = \mathbb{L} \cap (\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_j).$$

Since $\mathbb{L}$ contains s points of $\mathbb{X}$, we must have $\mathbb{L} \cap \mathbb{X}_i \neq \varnothing$ for all $i = 1, \ldots, s$, and in particular, $|\mathbb{T}| = j$. Then the $\Bbbk$-configuration $\mathbb{X}$ can also be defined using the subsets

$$\begin{aligned}
&\mathbb{X}'_i = (\mathbb{X}_i \setminus \mathbb{L}) \cup (\mathbb{L}_i \cap \mathbb{L}_j) \quad \text{and} \quad \mathbb{L}'_i = \mathbb{L}_i \quad \text{for } i = 1, \ldots, j-1,\\
&\mathbb{X}_j = \mathbb{T} \quad \text{and} \quad \mathbb{L}'_j = \mathbb{L}, \quad \text{and}\\
&\mathbb{X}'_i = \mathbb{X}_i \quad \text{and} \quad \mathbb{L}'_i = \mathbb{L}_i \quad \text{for } i = j+1, \ldots, s.
\end{aligned}$$

The verification of this fact is similar to the proof of Lemma 2.9. Note that the line $\mathbb{L}_j$ is no longer used to define the $\Bbbk$-configuration; moreover, the s lines that contain the s points are $\mathbb{L}'_1, \ldots, \mathbb{L}'_s$ after this relabelling, that is, we are now in case (a). We have now verified that we can assume that the lines that contain s points of $\mathbb{X}$ are exactly the lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$. We now verify the second part of (ii).

Now, for each $i = 1, \ldots, s$, the line $\mathbb{L}_i$ contains exactly s points of $\mathbb{X}$, so at least one of the s points in $\mathbb{X} \cap \mathbb{L}_i$ does not belong to $\mathbb{L}_j$ for $j \neq i$; call this point P_i. For $i = 1, \ldots, s-1$, set $\mathbb{Y}_i := \mathbb{X}_{i+1} \setminus \{P_{i+1}\}$. The set $\mathbb{Y} := \bigcup_{i=1}^{s-1} \mathbb{Y}_i$ is a $\Bbbk$-configuration of type $(1, \ldots, s-1)$ with supporting lines $\mathbb{L}_{i+1} \supseteq \mathbb{Y}_i$ (this follows from the fact that $\mathbb{X}$ is a $\Bbbk$-configuration with supporting lines $\mathbb{L}_i \supseteq \mathbb{X}_i$). Furthermore, there are exactly s lines that contain $s-1$ points of $\mathbb{Y}$, namely the lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$. Therefore, by part (i), all points of $\mathbb{Y}$ are precisely the pairwise intersections of the lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$. The statement in part (ii) follows.

(iii) Finally, suppose that there are $1 \leq r < s$ lines that contain s points of $\mathbb{X}$. Like case (ii), there are three subcases: (a) the r lines are among $\mathbb{L}_1, \ldots, \mathbb{L}_s$; (b) $r-1$ of the lines are among $\mathbb{L}_1, \ldots, \mathbb{L}_s$, and there is one additional line $\mathbb{L}$, or (c) $r-2$ of the lines are among $\mathbb{L}_1, \ldots, \mathbb{L}_s$, and there are two additional lines that contains s points. Like case (ii), we first show that we can relabel case (c) so case (b) is true. We then show that if case (b) is true, we can again relabel so case (a) is true.

If we assume case (c), we first apply Corollary 2.11 to relabel the lines so that $\mathbb{L}_1$ does not contain s points (since only the last $r-2$ lines will contain s points). Lemma 2.7 implies that the two additional lines are $\mathbb{L}_{PQ_1}, \mathbb{L}_{PQ_2}$. Since $\mathbb{X}_1 \subseteq \mathbb{L}_{PQ_1}$ we can still define the $\mathbb{k}$-configuration using the same $\mathbb{X}_i$'s, but with the lines $\mathbb{L}_{PQ_1}, \mathbb{L}_2, \ldots, \mathbb{L}_s$, that is, we are in case (b).

In case (b), we again first apply Corollary 2.11 to relabel our $\mathbb{k}$-configuration so that $\mathbb{L}_1$ does not contain s points. By Lemma 2.7, the additional line $\mathbb{L}$ is either $\mathbb{L}_{PQ_1}$ or $\mathbb{L}_{PQ_2}$. In either case, $\mathbb{X}_1 \subseteq \mathbb{L}$, so we again define the $\mathbb{k}$-configuration using the the same $\mathbb{X}_i$'s and the lines $\mathbb{L}, \mathbb{L}_2, \ldots, \mathbb{L}_s$. We have now relabelled the $\mathbb{k}$-configuration so case (a) holds.

Since we can assume that (a) holds, the $1 \leq r < s$ lines that contains s points are among $\mathbb{L}_1, \ldots, \mathbb{L}_s$. Again, by applying Corollary 2.11, we can assume that $\mathbb{L}_{s-r+1}, \ldots, \mathbb{L}_s$ are the r lines with s points, and in particular, none of these points contain $\mathbb{X}_1 = \{P\}$ by definition of a $\mathbb{k}$-configuration.

□

4.1. Case 1: Exactly $s+1$ lines. We will now consider the three cases of Lemma 4.1 separately. We first consider the case that there are exactly $s+1$ lines that contain s points of $\mathbb{X}$.

THEOREM 4.2. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a $\mathbb{k}$-configuration of type $d = (d_1, \ldots, d_s) = (1, 2, \ldots, s)$ with $s \geq 2$. Assume that there are exactly $s+1$ lines containing s points of $\mathbb{X}$. Then $s+1 = \Delta \mathbf{H}_{m\mathbb{X}}(md_s - 1)$ for all $m \geq 2$.*

Proof. Let $\mathbb{H}_1, \ldots, \mathbb{H}_s, \mathbb{H}_{s+1}$ be the lines containing s points of $\mathbb{X}$; by Lemma 4.1(i), the points of $\mathbb{X}$ are precisely the intersections of such lines.

To compute bounds on the Hilbert function of $m\mathbb{X}$, we apply Theorem 2.1 with the reduction vector $\mathbf{v}$ obtained from the sequence of lines

$$\mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1, \mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1, \ldots, \mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1,$$

where the subsequence $\mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1$ is repeated $\lceil \frac{m}{2} \rceil$ times.

We index the entries of the reduction vector by $v_{j(s+1)+i}$, where $j = 0, \ldots, \lceil \frac{m}{2} \rceil - 1$ is the number of times the subsequence of lines $\mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1$ has been completely removed, and $i = 1, \ldots, s+1$ indicates that we are going to remove the line $\mathbb{H}_{s-i+2}$. Note that each time the subsequence $\mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1$ is removed, the multiplicity of each point of $m\mathbb{X}$ decreases by two. If m is even, this process eventually reduces the multiplicity of each point to zero. If m is odd, then the process reduces the multiplicity of each point to one, so removing the sequence of lines $\mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1$ one more time reduces the multiplicity to zero. In particular, $m\mathbb{X}$ will be reduced to $\varnothing$ after removing the subsequence of lines $\mathbb{H}_{s+1}, \mathbb{H}_s, \ldots, \mathbb{H}_1$ $\lceil \frac{m}{2} \rceil$ times. At the step corresponding to $v_{j(s+1)+i}$, the line $\mathbb{H}_{s-i+2}$ contains:

- the points of intersection $\mathbb{H}_{s-i+2} \cap \mathbb{H}_k$ for $k > s-i+2$, with multiplicity $m-2j-1$;
- the points of intersection $\mathbb{H}_{s-i+2} \cap \mathbb{H}_k$ for $k < s-i+2$, with multiplicity $m-2j$.

This gives

$$\begin{aligned} v_{j(s+1)+i} &= (i-1)(m-2j-1) + (s-i+1)(m-2j) \\ &= (m-2j)s - i + 1. \end{aligned} \tag{4.1}$$

When $j = 0$, Equation (4.1) implies

$$v_i = ms - i + 1$$

for all $i = 1, \ldots, s+1$. For $j > 0$, we get

$$\begin{aligned} v_{j(s+1)+i} &= (m-2j)s - i + 1 = ms - 2js - i + 1 \\ &< ms - j(s+1) - i + 1 \end{aligned}$$

because $s > 1$. This shows that

$$v_{j(s+1)+i} \leq ms - \big(j(s+1) + i\big)$$

for all $j = 1, \ldots, \lceil \frac{m}{2} \rceil - 1$ and $i = 1, \ldots, s+1$.

Since $d_s = s$, we can summarize the results above by writing

$$\begin{aligned} &v_i = md_s - i + 1 \quad \text{for } i = 1, \ldots, s+1, \\ &v_i \leq md_s - i \quad \text{for } i = s+2, \ldots, \left\lceil \frac{m}{2} \right\rceil (s+1). \end{aligned}$$

Proceeding as in the proof of Theorem 3.1, we obtain

$$\mathbf{H}_{m\mathbb{X}}(md_s - 2) = \sum_{i=0}^{s} (md_s - 1 - i) + \sum_{i=s+2}^{\lceil \frac{m}{2} \rceil (s+1)-1} v_{i+1}.$$

Also as in the proof of Theorem 3.1, we have

$$\mathbf{H}_{m\mathbb{X}}(md_s - 1) = \deg(m\mathbb{X}) = \sum_{i=0}^{\lceil \frac{m}{2} \rceil (s+1)-1} v_{i+1} = \sum_{i=0}^{s} (md_s - i) + \sum_{i=s+2}^{\lceil \frac{m}{2} \rceil (s+1)-1} v_{i+1}.$$

We conclude that

$$\Delta\mathbf{H}_{m\mathbb{X}}(md_s - 1) = \mathbf{H}_{m\mathbb{X}}(md_s - 1) - \mathbf{H}_{m\mathbb{X}}(md_s - 2) = s + 1. \qquad \square$$

Remark 4.3. A $\Bbbk$-configuration of type $(1, 2, \ldots, s)$ which has exactly $s+1$ lines containing s points is also an example of a star configuration. When $m = 2$, Theorem 4.2 can be deduced from [6, Theorem 3.2].

4.2. Case 2: Exactly s lines. We next consider the case that there are exactly s lines containing s points. Reasoning as in the previous case, we may compute a reduction vector from these s lines, in order to calculate values of the Hilbert function. However, in this case, the bounds thus obtained may not be tight. The following example illustrates the issue, and a possible workaround.

EXAMPLE 4.4. Consider a $\mathbb{k}$-configuration $\mathbb{X}$ of type $(1,2,3,4)$ with exactly four lines that contain four points of $\mathbb{X}$. By Lemma 4.1(ii), $\mathbb{X}$ consists of the intersections of the lines $\mathbb{L}_1,\mathbb{L}_2,\mathbb{L}_3,\mathbb{L}_4$ defining the $\mathbb{k}$-configuration, and four non-colinear points P_1,P_2,P_3,P_4, with P_i belonging to $\mathbb{L}_i$. We have depicted such an $\mathbb{X}$ in Figure 5.

We proceed to compute bounds for $\mathbf{H}_{2\mathbb{X}}(2d_s-2)$ as we did in Example 2.2. First, we use the sequence of lines $\mathbb{L}_4,\mathbb{L}_3,\mathbb{L}_2,\mathbb{L}_1,\mathbb{L}_4,\mathbb{L}_3,\mathbb{L}_2,\mathbb{L}_1$. The table below compares the function $(2d_s-2)-i+1$ with the entries of the reduction vector $\mathbf{v}$; the minimum is in bold.

i	0	1	2	3	4	5	6	7
$6-i+1$	**7**	**6**	**5**	**4**	3	2	1	**0**
v_{i+1}	8	7	6	5	**1**	**1**	**1**	1

Summing the minimum values, we obtain the lower bound $\mathbf{H}_{2\mathbb{X}}(6) \geq f_{\mathbf{v}}(6) = 25$.

Now let $\mathbb{H}$ be the line through P_1 and P_2. In general, the line $\mathbb{H}$ could also contain P_3 or P_4, but not both. However in the $\mathbb{k}$-configuration depicted in Figure 5, $\mathbb{H}$ does not contain either P_3 or P_4. Consider the points of the set $\{P_1,P_2,P_3,P_4\} \setminus \mathbb{H}$, namely P_3 and P_4, and relabel them Q_1 and Q_2. Let $\mathbb{H}_1$ be a line through Q_1 not passing through Q_2, and let $\mathbb{H}_2$ be a line through Q_2. We compute a lower bound for $\mathbf{H}_{2\mathbb{X}}(2d_s-2)$ using the sequence

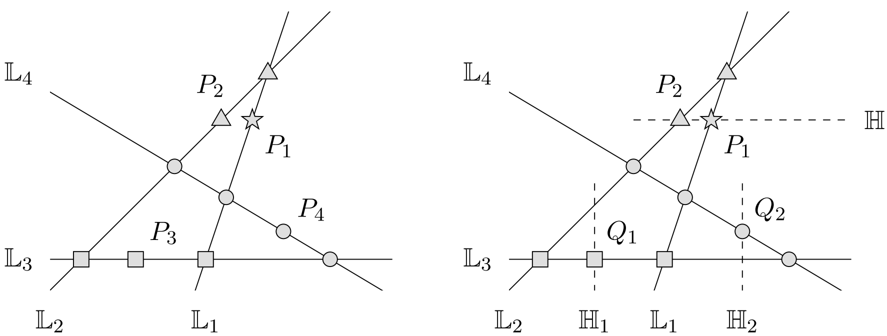

FIGURE 5. A $\mathbb{k}$-configuration of type $(1,2,3,4)$ with exactly 4 lines containing 4 points.

of lines $\mathbb{L}_4, \mathbb{L}_3, \mathbb{L}_2, \mathbb{L}_1, \mathbb{H}, \mathbb{H}_1, \mathbb{H}_2$. The table below summarizes the necessary information.

i	0	1	2	3	4	5	6
$6-i+1$	**7**	**6**	**5**	**4**	3	2	1
v_{i+1}	8	7	6	5	**2**	**1**	**1**

Summing the minimum values, we obtain the lower bound $\mathbf{H}_{2\mathbb{X}}(6) \geq f_{\mathbf{v}}(6) = 26$.

An easy computation with Equation (2.2) (using either reduction vector) leads to the upper bound $\mathbf{H}_{2\mathbb{X}}(6) \leq F_{\mathbf{v}}(6) \leq 26$. This shows that $\mathbf{H}_{2\mathbb{X}}(6) = 26$. In particular, the lower bound computed from the lines $\mathbb{L}_4, \mathbb{L}_3, \mathbb{L}_2, \mathbb{L}_1$ alone is not tight.

Using the above example as a guide, we prove our main result for the case under consideration.

THEOREM 4.5. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a $\Bbbk$-configuration of type $d = (d_1, \ldots, d_s) = (1, 2, \ldots, s)$ with $s \geq 2$. Assume that there are exactly s lines containing s points of $\mathbb{X}$. Then $s = \Delta\mathbf{H}_{m\mathbb{X}}(md_s - 1)$ for all $m \geq 2$.*

Proof. By Lemma 4.1(ii), we can assume that the lines containing s points of $\mathbb{X}$ are the lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$ that define the $\Bbbk$-configuration. Moreover, for each $i = 1, \ldots, s$, there is a point $P_i \in \mathbb{X} \cap \mathbb{L}_i$ that does not belong to $\mathbb{L}_j$ for any $j \neq i$. Then the points of $\mathbb{X}$ are the points of intersection of the lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$ together with the points $P_1, \ldots, P_s$.

To compute bounds on the Hilbert function of $m\mathbb{X}$, we apply Theorem 2.1 with the reduction vector $\mathbf{v}$ obtained from a sequence of lines

$$\mathbb{L}_s, \ldots, \mathbb{L}_1, \mathbb{L}_s, \ldots, \mathbb{L}_1, \ldots, \mathbb{L}_s, \ldots, \mathbb{L}_1, \mathbb{H}, \mathbb{H}_1, \ldots, \mathbb{H}_{s-u},$$

where the subsequence $\mathbb{L}_s, \ldots, \mathbb{L}_1$ is repeated $m-1$ times and the additional lines $\mathbb{H}, \mathbb{H}_1, \ldots, \mathbb{H}_{s-u}$ are constructed as follows.

Let $\mathbb{H}$ denote the line through P_1 and P_2. The line $\mathbb{H}$ contains u points of the set $\{P_1, \ldots, P_s\}$, where, by construction, $u \geq 2$. Furthermore, $\mathbb{H}$ is not one of the lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$, and therefore, it cannot contain s points of $\mathbb{X}$, that is, $u \leq s-1$. It follows that the set $\{P_1, \ldots, P_s\} \setminus \mathbb{H}$ is not empty, and must in fact contain $s-u$ points, which we denote $Q_1, \ldots, Q_{s-u}$. For each $i = 1, \ldots, s-u$, let $\mathbb{H}_i$ be a line passing through Q_i that does not contain any point Q_j for $j > i$.

Now we proceed to compute (or bound) the entries of the reduction vector $\mathbf{v}$. We claim that, for $i = 1, \ldots, s$, we have

$$v_i = ms - i + 1.$$

At the ith step, the line $\mathbb{L}_{s-i+1}$ contains:

- the points of intersection $\mathbb{L}_{s-i+1} \cap \mathbb{L}_k$ for $k > s-i+1$, with multiplicity $m-1$;
- the points of intersection $\mathbb{L}_{s-i+1} \cap \mathbb{L}_k$ for $k < s-i+1$, with multiplicity m; and
- the point P_{s-i+1} with multiplicity m.

This gives

$$v_i = (m-1)(i-1) + m(s-i) + m = ms - i + 1,$$

proving the claim.

Next, we claim that, for $l = s+1, \dots, (m-1)s + s - u + 1$, we have

$$v_l \leq ms - l. \tag{4.2}$$

We first prove this claim for entries v_{js+i}, where $j = 1, \dots, \lceil \frac{m}{2} \rceil - 1$ is the number of times the subsequence of lines $\mathbb{L}_s, \dots, \mathbb{L}_1$ has been completely removed, and $i = 1, \dots, s$ indicates that we are going to remove the line $\mathbb{L}_{s-i+1}$. At the step corresponding to v_{js+i}, the line $\mathbb{L}_{s-i+1}$ contains:

- the points of intersection $\mathbb{L}_{s-i+1} \cap \mathbb{L}_k$ for $k > s-i+1$, with multiplicity $m-2j-1$;
- the points of intersection $\mathbb{L}_{s-i+1} \cap \mathbb{L}_k$ for $k < s-i+1$, with multiplicity $m-2j$; and
- the point P_{s-i+1} with multiplicity $m-j$.

The $2j$ in the above multiplicities follows from the fact that points located at the intersections of the lines $\mathbb{L}_s, \dots, \mathbb{L}_1$ are removed twice with each full pass along the subsequence $\mathbb{L}_s, \dots, \mathbb{L}_1$. Thus, we obtain

$$\begin{aligned} v_{js+i} &= (m-2j-1)(i-1) + (m-2j)(s-i) + m - j \\ &= ms - i + j - 2js + 1 = ms - js - i + j(1-s) + 1 \\ &< ms - js - i + 1, \end{aligned}$$

from which the claim of Equation (4.2) follows for the chosen values of i and j.

Next, we prove the claim for v_{js+i}, where $j = \lceil \frac{m}{2} \rceil, \dots, m-2$, and $i = 1, \dots, s$. Since the lines $\mathbb{L}_s, \dots, \mathbb{L}_1$ have been removed $\lceil \frac{m}{2} \rceil$ times, the multiplicity of the points located at the intersections of the lines $\mathbb{L}_s, \dots, \mathbb{L}_1$ is now zero. Hence, at the step corresponding to v_{js+i}, the line $\mathbb{L}_{s-i+1}$ only contains the point P_{s-i+1} with multiplicity $m-j$. We get

$$v_{js+i} = m - j.$$

Since $j \leq m-2$, we have $m-j \geq 2$ and therefore

$$\frac{m-j}{m-j-1} = \frac{m-j-1+1}{m-j-1} = 1 + \frac{1}{m-j-1} \leq 2 \leq s.$$

This implies

$$v_{js+i} \leq (m-j-1)s = ms - js - s \leq ms - js - i,$$

thus proving the claim of Equation (4.2) for the given i and j.

At this stage, the multiplicity of the points $P_1, \dots, P_s$ has been reduced to one, because each line $\mathbb{L}_1, \dots, \mathbb{L}_s$ has been removed $m-1$ times. The next step is to find the value of $v_{(m-1)s+1}$, which corresponds to the line $\mathbb{H}$ defined at the beginning. By construction, $\mathbb{H}$ contains u points of the set $\{P_1, \dots, P_s\}$, with $u \le s-1$. Therefore

$$v_{(m-1)s+1} = u \le s-1 = ms - \big((m-1)s+1\big);$$

this shows that Equation (4.2) holds for this entry of $\mathbf{v}$.

Finally, we evaluate $v_{(m-1)s+h}$, for $h = 2, \dots, s-u+1$. For a given value of h, we assume that we have already removed $\mathbb{H}_1, \dots, \mathbb{H}_{h-2}$ and we are about to remove $\mathbb{H}_{h-1}$. The line $\mathbb{H}_{h-1}$ contains a single point of $\mathbb{X}$, namely Q_{h-1}. Moreover, Q_{h-1} is by definition one of the points in the set $\{P_1, \dots, P_s\} \setminus \mathbb{H}$, so its multiplicity is down to one. Thus, we have

$$v_{(m-1)s+h} = 1 \le s-h = ms - \big((m-1)s+h\big).$$

The inequality $1 \le s-h$ follows from $h \le s-u+1$ and $u \ge 2$. Thus, we have proved that Equation (4.2) holds for all the desired values.

To summarize, we showed that

$$\begin{aligned} &v_i = ms - i + 1 \quad \text{for } i = 1, \dots, s, \\ &v_i \le ms - i \quad \text{for } i = s+1, \dots, (m-1)s + s - u + 1. \end{aligned}$$

From here on, the proof proceeds as for Theorem 3.1, yielding

$$\begin{aligned} \Delta\mathbf{H}_{m\mathbb{X}}(ms-1) &= \mathbf{H}_{m\mathbb{X}}(ms-1) - \mathbf{H}_{m\mathbb{X}}(ms-2) \\ &= \sum_{i=0}^{s-1}(ms-i) - \sum_{i=0}^{s-1}(ms-1-i) = \sum_{i=0}^{s-1} 1 = s. \end{aligned}$$

□

4.3. Case 3: $1 \le r < s$ lines. We consider the final case when there are $1 \le r < s$ lines that contain s points of $\mathbb{X}$. Before going forward, we recall a result of Catalisano, Trung, and Valla [3, Lemma 3]; we have specialized this result to the case of points in $\mathbb{P}^2$.

LEMMA 4.6. *Let $P_1, \dots, P_k, P$ be distinct points in $\mathbb{P}^2$ and let I_P be the defining prime ideal of P. If $m_1, \dots, m_k$, and a are positive integers and $I = I_{P_1}^{m_1} \cap \cdots \cap I_{P_k}^{m_k}$, then*

(a) $\mathbf{H}_{R/(I+I_P^a)}(t) = \sum_{i=0}^{a-1} \dim_{\Bbbk}[(I+I_P^i)/(I+I_P^{i+1})]_t$ *for every $t > 0$, with $I_P^0 = R$.*

(b) *If $P = [1:0:0]$, then $[(I+I_P^i)/(I+I_P^{i+1})]_t = 0$ if and only if $i > t$ or $x_0^{t-i}M \in I + I_P^{i+1}$ for every monomial M of degree i in x_1, x_2.*

We now prove the remaining open case. Note that unlike Theorems 4.2 and 4.5, we need to assume that $m \ge s+1$ instead of $m \ge 2$.

THEOREM 4.7. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a $\Bbbk$-configuration of type $d = (d_1, \ldots, d_s) = (1, 2, \ldots, s)$ with $s \geq 2$. Assume that there are $1 \leq r < s$ lines containing s points of $\mathbb{X}$. Then $r = \Delta\mathbf{H}_{m\mathbb{X}}(md_s - 1)$ for all $m \geq s+1$.*

Proof. Let $\mathbb{L}_1, \ldots, \mathbb{L}_s$ be the s lines that define the $\Bbbk$-configuration. After using Lemma 4.1(iii) to relabel, we can assume that the r lines that contain s points are among $\mathbb{L}_2, \ldots, \mathbb{L}_s$, and thus the unique point P of $\mathbb{X}_1$ does not lie on any line containing s points of $\mathbb{X}$. So, we can write our $\Bbbk$-configuration as $\mathbb{X} = \mathbb{Y} \cup \{P\}$ where $\{P\} = \mathbb{X}_1$ is the point on the line $\mathbb{L}_1$ and $\mathbb{Y}$ is a $\Bbbk$-configuration of type $(2, 3, \ldots, s) = (d'_1, \ldots, d'_{s-1})$. Since there is no line containing s points of $\mathbb{X}$ that passes through the point P, the r lines that contain s points of $\mathbb{X}$ must also contain s points of $\mathbb{Y}$. So we can apply Theorem 3.1 to $\mathbb{Y}$.

Suppose that $\mathbb{X}_2 = \{Q_1, Q_2\}$. Since P, Q_1, Q_2 do not all lie on the same line, we can make a linear change of coordinates so that

$$P = [1:0:0], \qquad Q_1 = [0:1:0], \quad \text{and} \quad Q_2 = [0:0:1].$$

We let L_i denote the linear form that defines the line $\mathbb{L}_i$. Note that after we have made our change of coordinates, if $\mathbb{L}$, with defining form $L = ax_0 + bx_1 + cx_2$, is any line that does not pass through P, then $L \notin I_P = \langle x_1, x_2 \rangle$, i.e., $a \neq 0$.

With this setup, we make the following claim:

CLAIM. *For all $m \geq s+1$, $\mathbf{H}_{R/(I_{m\mathbb{Y}} + I_P^m)}(ms-2) = 0$.*

Proof. By Lemma 4.6, it suffices to show that for each $i = 0, \ldots, m-1$, the monomial $x_0^{ms-2-i}M \in [I_{m\mathbb{Y}} + I_P^{i+1}]_{ms-2}$ where M is any monomial of degree i in x_1, x_2. We will treat the cases $i \in \{0, \ldots, m-2\}$ and $i = m-1$ separately.

Fix an $i \in \{0, \ldots, m-2\}$ and let M be any monomial of degree i in x_1, x_2. Since none of the lines $\mathbb{L}_2, \ldots, \mathbb{L}_s$ pass through the point P, we have $L_k = a_{k,0}x_0 + a_{k,1}x_1 + a_{k,2}x_2$ with $a_{k,0} \neq 0$ for all $k = 2, \ldots, s$. Then

$$L_2^m L_3^m \cdots L_s^m = ax_0^{ms-m} + \sum_{k=1}^{ms-m} x_0^{ms-m-k} f_k(x_1, x_2)$$

with $a \neq 0$ and where $f_k(x_1, x_2)$ is a homogeneous polynomial of degree k only in x_1 and x_2. Since $L_2 \cdots L_s \in I_{\mathbb{Y}}$, it follows that

$$L_2^m \cdots L_s^m \in \left[(I_{\mathbb{Y}})^m\right]_{ms-m} \subseteq [I_{m\mathbb{Y}}]_{ms-m} \subseteq \left[I_{m\mathbb{Y}} + I_P^{i+1}\right]_{ms-m}$$

and thus

$$ax_0^{ms-m}M + \sum_{k=1}^{ms-m} x_0^{ms-m-k} f_k(x_1, x_2)M \in \left[I_{m\mathbb{Y}} + I_P^{i+1}\right]_{ms-m+i}.$$

But $I_P^{i+1} = \langle x_1, x_2 \rangle^{i+1}$, so $f_k(x_1, x_2)M \in I_P^{i+1}$ for each $k = 1, \ldots, ms-m$ since $f_k(x_1, x_2)M$ is a homogeneous polynomial only in x_1, x_2 of degree $i+k \geq i+1$.

But then this means that

$$a^{-1}ax_0^{ms-m}M = x_0^{ms-m}M \in \left[I_{m\mathbb{Y}} + I_P^{i+1}\right]_{ms-m+i}.$$

Since $i \leq m-2$, we thus have $x_0^{m-2-i}x_0^{ms-m}M = x_0^{ms-2-i}M \in [I_{m\mathbb{Y}} + I_P^{i+1}]_{ms-2}$.

Now suppose that $i = m-1$. Consider any monomial $M = x_1^a x_2^b$ with $a+b = m-1$ and $a,b \geq 1$. Since $I_{Q_1} = \langle x_0, x_2\rangle$ and $I_{Q_2} = \langle x_0, x_1\rangle$, this means that $x_1^a x_2^b \in I_{Q_1}^b \cap I_{Q_2}^a$. Because $\mathbb{L}_2$ is the line that passes through Q_1 and Q_2, we have $L_2^{m-1}M \in (I_{Q_1}^m \cap I_{Q_2}^m)$, and consequently,

$$\begin{aligned} L_2^{m-1}L_3^m \cdots L_s^m M &= ax_0^{ms-m-1}M + \sum_{k=1}^{ms-m-1} x_0^{ms-m-1-k} f_k(x_1,x_2)M \\ &\in [I_{m\mathbb{Y}}]_{ms-2} \subseteq \left[I_{m\mathbb{Y}} + I_P^m\right]_{ms-2}. \end{aligned}$$

Arguing as above, this implies that $x_0^{ms-m-1}M \in [I_{m\mathbb{Y}} + I_P^m]_{ms-2}$.

It remains to show that $x_0^{ms-m-1}x_1^{m-1}$ and $x_0^{ms-m-1}x_2^{m-1} \in [I_{m\mathbb{Y}} + I_P^m]_{ms-2}$. We only verify the second statement since the first statement is similar. Consider the line $\mathbb{L}$ through the point P and Q_2. Because $\mathbb{L}$ goes through P, it does not contain s points. In particular, there must be some $j \in \{3,\ldots,s\}$ such that $\mathbb{L} \cap \mathbb{X}_j = \varnothing$, that is, $\mathbb{L}$ does not intersect with any of the points of $\mathbb{X}$ on the line $\mathbb{L}_j$. Let $\mathbb{X}_j = \{S_1,\ldots,S_j\}$ be these j points, and let $\mathbb{H}_\ell$ be the line through Q_2 and S_ℓ for $\ell = 1,\ldots,j$. Furthermore, let H_ℓ denote the associated linear form. Note that none of the lines $\mathbb{H}_\ell$ can pass through the point P, so in particular, each H_ℓ has the form $H_\ell = a_\ell x_0 + b_\ell x_1 + c_\ell x_2$ with $a_\ell \neq 0$.

We now claim that

$$F := x_1^{m-1}H_1 \cdots H_j L_2^{m-j} L_3^m \cdots L_{j-1}^m L_j^{m-1} L_{j+1}^m \cdots L_s^m \in I_{m\mathbb{Y}}.$$

Because $j \leq s$ and $m \geq s+1$, $m-j \geq 1$. So, in particular, $x_1^{m-1}L_2^{m-j} \in I_{Q_1}^m$. Also, $H_1 \cdots H_j L_2^{m-j} \in I_{Q_2}^m$, so F vanishes at the points of $\{Q_1, Q_2\}$ to the correct multiplicity. Note that $H_1 \cdots H_j L_j^{m-1}$ vanishes at all the points on $\mathbb{L}_j$ to multiplicity at least m. Furthermore, for any other k, L_k^m vanishes at all the points on $\mathbb{L}_k$ to multiplicity at least m. So we have $F \in I_{m\mathbb{Y}}$. To finish the proof, we need to note that

$$\begin{aligned} &H_1 \cdots H_j L_2^{m-j} L_3^m \cdots L_{j-1}^m L_j^{m-1} L_{j+1}^m \cdots L_s^m \\ &\quad = ax_0^{ms-m-1} + \sum_{k=1}^{ms-m-1} x_0^{ms-m-1-k} f_k(x_1,x_2) \end{aligned}$$

with $a \neq 0$ and where each $f_k(x_1,x_2)$ is a homogeneous polynomial of degree k only in x_1, x_2. The rest of the proof now follows similar to the cases above. This ends the proof of the claim. $\square$

We now complete the proof. Let $m \geq s+1$ be any integer, and consider the short exact sequence

$$0 \to \left(I_{m\mathbb{Y}} \cap I_P^m\right) \to I_{m\mathbb{Y}} \oplus I_P^m \to I_{m\mathbb{Y}} + I_P^m \to 0.$$

Note that the ideal of $m\mathbb{X}$ is $I_{m\mathbb{X}} = I_{m\mathbb{Y}} \cap I_P^m$, so the short exact sequence implies

$$\mathbf{H}_{m\mathbb{X}}(t) = \mathbf{H}_{m\mathbb{Y}}(t) + \mathbf{H}_{mP}(t) - \mathbf{H}_{R/(I_{m\mathbb{Y}}+I_P^m)}(t)$$

for all $t \geq 0$. Note that $\mathbf{H}_{mP}(t) = \binom{m+1}{2}$ for all $t \geq m-1$. Using this fact, and the above claim we get

$$\begin{aligned}
\Delta\mathbf{H}_{m\mathbb{X}}(ms-1) &= \mathbf{H}_{m\mathbb{X}}(ms-1) - \mathbf{H}_{m\mathbb{X}}(ms-2) \\
&= \left(\mathbf{H}_{m\mathbb{Y}}(ms-1) + \mathbf{H}_{mP}(ms-1) - \mathbf{H}_{R/(I_{m\mathbb{Y}}+I_P^m)}(ms-1)\right) \\
&\quad - \left(\mathbf{H}_{m\mathbb{Y}}(ms-2) + \mathbf{H}_{mP}(ms-2) - \mathbf{H}_{R/(I_{m\mathbb{Y}}+I_P^m)}(ms-2)\right) \\
&= \Delta\mathbf{H}_{m\mathbb{Y}}(ms-1) + \left(\binom{m+1}{2} - \binom{m+1}{2}\right) - (0-0) \\
&= r.
\end{aligned}$$

The last equality comes from Theorem 3.1 since $\Delta\mathbf{H}_{m\mathbb{Y}}(ms-1) = r$ for all $m \geq 2$. $\square$

REMARK 4.8. Notice that in the proof of Theorem 4.7, the hypothesis that $m \geq s+1$ was only used in the proof of the claim to show that a particular monomial belonged to the ideal $I_{m\mathbb{Y}} + I_P^m$. However, there may be some room for improvement on the lower bound $s+1$. For example, for the $\Bbbk$-configuration of type $(1,2,3)$ given in Figure 4, computer tests have shown that $\Delta\mathbf{H}_{m\mathbb{X}}(m3-1) = 1$ for all $m \geq s = 3$, instead of $s+1 = 4$. Similarly, if we consider *standard linear configurations* of type $(1,2,\ldots,s)$ (as defined in [10, Definition 2.10]), then it can be shown that Theorem 4.7 holds for all $m \geq 2$. We omit this proof since it requires the special geometry of standard linear $\Bbbk$-configurations.

5. Concluding remarks

We conclude this paper with some observations. Following [3], we define the *regularity index* of a zero-dimensional scheme $\mathbb{Z} \subseteq \mathbb{P}^n$ to be

$$\operatorname{ri}(\mathbb{Z}) = \min\left(t \mid \mathbf{H}_{\mathbb{Z}}(t) = \deg(\mathbb{Z})\right).$$

Embedded in our proof of Theorem 1.2, we actually computed the regularity index of multiples of a $\Bbbk$-configuration. In particular, we proved that

COROLLARY 5.1. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a $\Bbbk$-configuration of type $d = (d_1,\ldots,d_s)$. Then for all integers $m \geq s+1$,*

$$\operatorname{ri}(m\mathbb{X}) = md_s - 1.$$

Proof. If $d=(1)$, then $\mathbb{X}=\{P\}$ is a single point. It is well-known that

$$\mathbf{H}_{m\mathbb{X}}(t)=\min\left(\binom{t+2}{2},\binom{m+1}{2}\right),$$

so the regularity index is $m-1$.

If $d\neq(1)$ and if $m\geq s+1$, Theorems 3.1, 4.2, 4.5, and 4.7, imply $\mathbf{H}_{m\mathbb{X}}(md_s-2)<\mathbf{H}_{m\mathbb{X}}(md_s-1)$. Moreover, as part of our proofs, we argued that $\mathbf{H}_{m\mathbb{X}}(md_s-1)=\deg(m\mathbb{X})$. □

The regularity index $\mathrm{ri}(\mathbb{Z})$ can also be defined as the maximal integer t such that $\Delta\mathbf{H}_{\mathbb{Z}}(t)\neq 0$. So, Theorem 1.2 can be restated as:

THEOREM 5.2. *Let $\mathbb{X}\subseteq\mathbb{P}^2$ be a $\Bbbk$-configuration of type $d=(d_1,\dots,d_s)\neq(1)$ and $m\geq s+1$. Then the number of lines containing exactly d_s points of $\mathbb{X}$ is the last non-zero value of $\Delta\mathbf{H}_{m\mathbb{X}}(t)$.*

As a final comment, we turn to a question posed by Geramita, Migliore, and Sabourin [10]:

QUESTION 5.3. What are all the possible Hilbert functions of fat point schemes in $\mathbb{P}^n$ whose support has a fixed Hilbert function $\mathbf{H}$?

As noted in [10], this question is quite difficult; in fact, [10] focused on the case of double points in $\mathbb{P}^2$. Using the work of this paper, we can give an interesting observation related Question 5.3.

THEOREM 5.4. *Fix integers $m\geq s+1\geq 3$. Then there are at least $s+1$ possible Hilbert functions of homogeneous fat points of multiplicity m in $\mathbb{P}^2$ whose support has the Hilbert function*

$$\mathbf{H}(t)=\min\left(\binom{t+2}{2},\binom{s+1}{2}\right).$$

Proof. Any $\Bbbk$-configuration $\mathbb{X}$ of type $(1,2,\dots,s)$ with $s\geq 2$ has Hilbert function $\mathbf{H}_{\mathbb{X}}(t)=\mathbf{H}(t)$ (see [13, Theorem 1.2]). By Theorem 5.2, $\mathbf{H}_{m\mathbb{X}}(ms-2)=\deg(m\mathbb{X})-r$ where r is the number of lines that contain s points of $\mathbb{X}$. As shown in Lemma 4.1, $1\leq r\leq s+1$. It suffices to show that each r is possible; this would imply that we have at least $s+1$ different Hilbert functions.

Fix $\mathbb{L}_1,\dots,\mathbb{L}_{s+1}$ distinct lines. If $r=s+1$, we take all pairwise intersections of these $s+1$ lines to get the desired set of points. So, suppose $1\leq r\leq s$. We construct a $\Bbbk$-configuration of type $(1,2,\dots,s)$ with exactly r lines containing s points as follows:

- let $\mathbb{X}_1$ be any point in $\mathbb{L}_1\setminus(\mathbb{L}_2\cup\cdots\cup\mathbb{L}_{s+1})$;
- let $\mathbb{X}_2$ be any two points $\mathbb{L}_2\setminus(\mathbb{L}_1\cup\mathbb{L}_3\cup\cdots\cup\mathbb{L}_{s+1})$;

$\vdots$

- let $\mathbb{X}_{s-r}$ be any $s-r$ points $\mathbb{L}_{s-r}\setminus(\mathbb{L}_1\cup\cdots\cup\widehat{\mathbb{L}}_{s-r}\cup\cdots\cup\mathbb{L}_s)$;

- let $\mathbb{X}_{s-r+1}$ be any $s-r+1$ points $\mathbb{L}_{s-r+1} \setminus (\mathbb{L}_1 \cup \cdots \cup \widehat{\mathbb{L}}_{s-r+1} \cup \cdots \cup \mathbb{L}_s)$;
- let $\mathbb{X}_{s-r+2}$ any $s-r+1$ points on $\mathbb{L}_{s-r+2} \setminus (\mathbb{L}_1 \cup \cdots \cup \widehat{\mathbb{L}}_{s-r+2} \cup \cdots \cup \mathbb{L}_s)$ and the point $\mathbb{L}_{s-r+2} \cap \mathbb{L}_{s-r+1}$;
- let $\mathbb{X}_{s-r+3}$ be any $s-r+1$ points on $\mathbb{L}_{s-r+3} \setminus (\mathbb{L}_1 \cup \cdots \cup \widehat{\mathbb{L}}_{s-r+3} \cup \cdots \cup \mathbb{L}_s)$ and the two points $\mathbb{L}_{s-r+3} \cap (\mathbb{L}_{s-r+1} \cup \mathbb{L}_{s-r+2})$;

$\vdots$

- let $\mathbb{X}_s$ be any $s-r+1$ points on $\mathbb{L}_s \setminus (\mathbb{L}_1 \cup \cdots \cup \widehat{\mathbb{L}}_s)$ and the $r-1$ points $\mathbb{L}_s \cap (\mathbb{L}_{s-r+1} \cup \cdots \cup \mathbb{L}_{s-1})$.

This configuration then gives the desired result. □

REMARK 5.5. As mentioned in the Introduction, $\mathbb{k}$-configurations of points can be defined in $\mathbb{P}^n$ (see, e.g., [8], [7], [11], [12]). It is natural to ask if a result similar to Theorem 1.2 also holds more generally. Based upon some calculations, it appears that this may be the case. For example, let $\mathbb{X}$ be the $\mathbb{k}$-configuration of points in $\mathbb{P}^3$ found in [11, Example 4.1] (see [11] for both the definition and a picture). For this example, one can see that there are three lines that contain four points. The Hilbert function of $2\mathbb{X}$ is given by

$$\mathbf{H}_{2\mathbb{X}} : 1\ 4\ 10\ 20\ 35\ 50\ 57\ 60 \rightarrow .$$

Note that $\mathrm{ri}(2\mathbb{X}) = 7$. Also, we have $\Delta\mathbf{H}_{2\mathbb{X}}(7) = 3$, that is, the same as the number of lines containing four points, which is similar to our statement in Theorem 5.2.

Although we suspect that a more general result holds, our proof relies on techniques developed in [5] that only give precise information when the points are in $\mathbb{P}^2$.

Acknowledgments. We would like to thank the referee for their helpful comments and suggestions. Work on this project began when the second and third authors visited Anthony (Tony) V. Geramita in Kingston, Ontario in the summer of 2015. Computer experiments using `CoCoA` [1] inspired our main result. Unfortunately, Tony became quite ill soon after our visit, and he passed away in June 2016. We would like to thank Tony for the input he was able to provide during the very initial stage of this project.

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