

## TUTTE RELATIONS, TQFT, AND PLANARITY OF CUBIC GRAPHS

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ABSTRACT. It has been known since the work of Tutte that the value of the chromatic polynomial of planar triangulations at  $(3 + \sqrt{5})/2$  has a number of remarkable properties. We investigate to what extent Tutte's relations characterize planar graphs. A version of the Tutte linear relation for the flow polynomial at  $(3 - \sqrt{5})/2$  is shown to give a planarity criterion for 3-connected cubic (trivalent) graphs. A conjecture is formulated that the golden identity for the flow polynomial characterizes planarity of cubic graphs as well. In addition, Tutte's upper bound on the chromatic polynomial of planar triangulations at  $(3 + \sqrt{5})/2$  is generalized to other Beraha numbers, and an exponential lower bound is given for the value at  $(3 - \sqrt{5})/2$ . The proofs of these results rely on the structure of the Temperley–Lieb algebra and more generally on methods of topological quantum field theory.

### 1. Introduction

In the late 1960s, W. T. Tutte observed [12], [11] that the value of the chromatic polynomial of planar triangulations at  $(3 + \sqrt{5})/2$  obeys a number of surprising relations. The Tutte golden identity states

$$(1.1) \quad \chi_T(\phi + 2) = (\phi + 2)\phi^{3V-10}(\chi_T(\phi + 1))^2,$$

where  $T$  is any planar triangulation,  $V$  is the number of its vertices, and  $\phi = (1 + \sqrt{5})/2$  denotes the golden ratio. Tutte also established a linear identity

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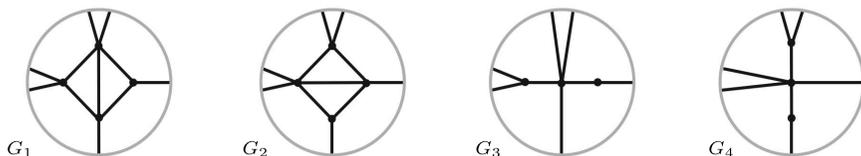


FIGURE 1. The graphs in the identity (1.2).

for the chromatic polynomial at  $\phi + 1$ :

$$(1.2) \quad \chi_{G_1}(\phi + 1) + \chi_{G_2}(\phi + 1) = \phi^{-3}(\chi_{G_3}(\phi + 1) + \chi_{G_4}(\phi + 1)),$$

where  $G_i$  are planar graphs which are identical outside a disk and are related within the disk as shown in Figure 1.

Using duality between the chromatic polynomial  $\chi_G$  of a planar graph  $G$  and the flow polynomial  $F_{G^*}$  of its dual, (1.1) and (1.2) may be restated as identities for the flow polynomial of planar cubic graphs.

This paper is motivated in part by the question of whether such identities for the chromatic and flow polynomials characterize planarity of graphs. We prove a converse to Tutte's theorem (1.2) for 3-connected cubic graphs: such a graph is planar if and only if a version of the linear relation (1.2) for the flow polynomial holds at each edge, see Theorem 3.2. We actually work with the Galois conjugate  $\frac{3-\sqrt{5}}{2}$  of  $\phi + 1$  roughly because the sign of the flow polynomial is determined by the number of edges and vertices (and components) on the interval  $(0, 1)$  which makes it possible to establish inequalities which feed into an induction. We also consider the identity (1.1) for the chromatic and flow polynomials. We conjecture that the golden identity for the flow polynomial characterizes planarity of cubic graphs, see Conjecture 5.1 for a precise statement. On the other hand, examples are given of non-planar graphs satisfying the chromatic polynomial version of the golden identity, and we explain the difference between the chromatic and flow settings in Section 5.

In [4], the identities (1.1), (1.2) were shown to fit in the framework of  $(2+1)$  dimensional topological quantum field theory (TQFT). In particular, these identities were proved to be a consequence of the structure of the Temperley–Lieb algebra at roots of unity, and the linear relation (1.2) was shown to have a counterpart at each value  $2 + 2\cos(\frac{2\pi j}{n})$ . We extend this framework and define a category for the flow polynomial which takes into account abstract (not necessarily planar) graphs. The structure of this category, together with combinatorial properties of non-planar cubic graphs (a strengthening of the Kuratowski criterion for planarity), give the characterization of planarity in Theorem 3.2.

We also use the TQFT framework to generalize Tutte’s bound on the value of the chromatic polynomial of a planar triangulations  $T$  with  $V$  vertices

$$(1.3) \quad |\chi_T(\phi + 1)| \leq \phi^{5-V}$$

to all Beraha numbers  $B_n = 2 + 2 \cos(\frac{2\pi}{n+1})$  and to real values  $x \geq 4$ , see theorem 4.1. The upper bound (1.3) is interesting in connection to the conjecture of Beraha that large planar triangulations have a real zero close to  $\phi + 1$ . This conjecture remains open. In contrast to (1.3), in Theorem 4.2 we give a lower bound at the Galois conjugate value  $(3 - \sqrt{5})/2$ , which is exponentially increasing in  $V$ .

## 2. The Tutte polynomial, the Temperley–Lieb algebra, and the Jones–Wenzl projectors

To provide background for the results of this paper and to fix the notation, this section recalls basic information about graph polynomials and the Temperley–Lieb algebra, and it summarizes the results of [4]. The Tutte polynomial  $T_G(x, y)$  of a graph  $G$  satisfies the contraction–deletion rule: for any edge  $e$  of  $G$  which is not a bridge or a loop,  $T_G = T_{G \setminus e} + T_{G/e}$ . The Tutte polynomial of a graph consisting of  $b$  bridges and  $l$  loops is defined to be  $x^b y^l$ . This paper will mainly concern two specializations, the chromatic polynomial  $\chi_G$  and the flow polynomial  $F_G$ :

$$\begin{aligned} \chi_G(x) &= (-1)^{V-c(G)} x^{c(G)} T_G(1-x, 0), \\ F_G(x) &= (-1)^{E+V+c(G)} T_G(0, 1-x), \end{aligned}$$

where  $V, E$  denote the number of vertices and edges of  $G$ , and  $c(G)$  is the number of connected components. For a planar graph  $G$  and its dual  $G^*$ , one has  $T_G(x, y) = T_{G^*}(y, x)$ . If  $G$  is connected, this implies

$$(2.1) \quad F_G(x) = x^{-1} \chi_{G^*}(x).$$

To describe the TQFT context for Tutte’s relations (1.1), (1.2), we summarize the relevant facts about the Temperley–Lieb algebra, the chromatic algebra and their structure at roots of unity. The reader is referred to [4] for more details.

**2.1. The Temperley–Lieb algebra.** The Temperley–Lieb algebra in degree  $n$ ,  $TL_n$ , is an algebra over  $\mathbb{C}[d]$  consisting of linear combinations of 1-dimensional submanifolds in a rectangle. Each submanifold meets both the top and the bottom of the rectangle in  $n$  points. The submanifolds are considered equivalent if they are isotopic rel boundary. Deleting a simple closed curve has the effect of multiplying the element by  $d$ . When  $d$  is specialized to a complex number, the algebra will be denoted  $TL_n^d$ . The multiplication is given by vertical stacking of rectangles. The usual generators of  $TL_4$  are illustrated in Figure 2.

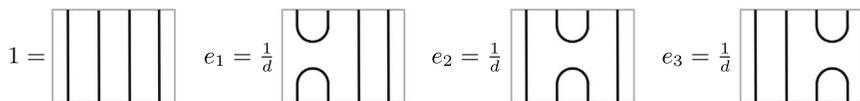


FIGURE 2. Generators of  $TL_4$ .

The trace  $tr_d: TL_n^d \rightarrow \mathbb{C}$  is defined on rectangular pictures (additive generators) by connecting the top and bottom endpoints by disjoint arcs in the complement of the rectangle in the plane, and then evaluating  $d^{\#circles}$ . The Hermitian product is defined by  $\langle a, b \rangle = tr(ab)$ , where the involution  $\bar{\phantom{a}}$  reflects pictures in a horizontal line and conjugates the coefficients.

The trace radical of  $TL_n^d$  is the ideal consisting of the elements  $a$  such that  $tr(ab) = 0$  for all  $b \in TL_n^d$ . An important fact, underlying the construction of  $SU(2)$  quantum invariants of knots and 3-manifolds, is that the trace radical in the Temperley–Lieb algebra is non-trivial precisely at the special values  $d = 2 \cos(\frac{\pi j}{n+1})$ . (In fact, these are the values of  $d$  for which there exists a non-trivial ideal in the Temperley–Lieb category, and the ideal equals the trace radical.) Moreover, for these values of  $d$ , (assuming  $j, n + 1$  are coprime) the trace radical is generated by a special element, the Jones–Wenzl projector  $P^{(n)}$  [17]. At primitive roots of unity ( $j = 1$ ) the Hermitian product descends to a positive definite inner product on the quotient of  $TL_n^d$  by the trace radical.

**2.2. The chromatic algebra.** Recall that a cubic graph is trivalent: every vertex has degree 3.  $\mathcal{C}_n$  is an algebra over  $\mathbb{C}[Q]$ , whose elements are formal linear combinations of isotopy classes of planar cubic graphs in a rectangle  $R$ , modulo the local relations shown in Figure 3. (The first relation is the version of the contraction–deletion rule for cubic graphs.)

The intersection of a graph with the boundary of the rectangle  $R$  consists of  $2n$  points:  $n$  points at the top and the bottom each, Figure 4. It is convenient to allow 2-valent vertices as well, and the value of a simple closed curve is set to be  $Q - 1$ . When  $Q$  is set to be a specific complex number, the algebra is denoted  $\mathcal{C}_n^Q$ .

The trace  $tr: \mathcal{C}_n^Q \rightarrow \mathbb{C}$  is defined by connecting the endpoints of  $G$  by disjoint arcs in the complement of the rectangle  $R$  in the plane and evaluating the flow polynomial of the resulting graph at  $Q$ . (Equivalently, the trace equals  $Q^{-1}$  times the chromatic polynomial of the dual graph. The focus in [4] was on the chromatic polynomial, explaining the name of the algebra.) For example, the traces of the elements in Figure 4 are  $0, (Q - 1)(Q - 2)^2$ . The trace is well-defined since the local relations are precisely the relations defining the flow polynomial of a planar cubic graph. The multiplication and the Hermitian product on  $\mathcal{C}_n^Q$  are defined analogously to the case of the Temperley–Lieb algebra.

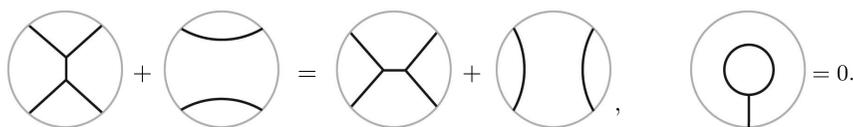


FIGURE 3. Defining relations in the the chromatic algebra.



FIGURE 4. Examples of graphs in the definition of  $\mathcal{C}_3$ .



FIGURE 5. The homomorphism  $\Phi: \mathcal{C}_n^Q \longrightarrow TL_{2n}^d, Q = d^2$ .

**2.3. The map  $\mathcal{C}_n \longrightarrow TL_{2n}$ .** Consider the homomorphism  $\Phi: \mathcal{C}_n^Q \longrightarrow TL_{2n}^d$ , where  $Q = d^2$ . This map replaces each edge of a graph with the second Jones–Wenzl projector, and each vertex is resolved as shown in Figure 5. Moreover, for a trivalent graph  $G$  there is an overall factor  $d^{V/2}$ , where  $V$  is the number of vertices of  $G$ .

One checks that  $\Phi$  induces a well-defined homomorphism of algebras  $\mathcal{C}_n^Q \longrightarrow TL_{2n}^d$ , where  $Q = d^2$ , and moreover it is trace-preserving: the diagram

$$(2.2) \quad \begin{array}{ccc} \mathcal{C}_n^{d^2} & \xrightarrow{\Phi} & TL_{2n}^d \\ \downarrow tr & & \downarrow tr_d \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

commutes. It follows that the pullback under  $\Phi$  of the trace radical in  $TL^d$  is in the trace radical of  $\mathcal{C}^{d^2}$ . Note that elements in the trace radical in the chromatic algebra are precisely local relations on graphs which preserve the flow polynomial, or equivalently the chromatic polynomial of the dual graph. When  $d = \phi$ , the trace radical of the Temperley–Lieb algebra is generated by the Jones–Wenzl projector  $P^{(4)}$ . The Tutte relation (1.2) then may be seen as a consequence of the structure of  $TL^\phi$  since it is mapped by  $\Phi$  to  $P^{(4)}$  [4, Section 2, p. 721].

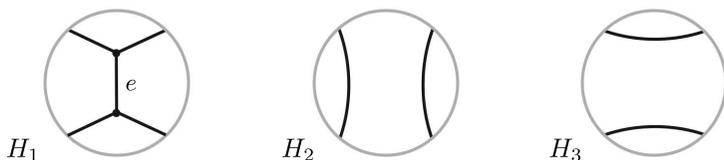


FIGURE 6. The graphs  $H_i$  in the identities (3.1), (3.2). The graphs are identical outside the disk.

### 3. A planarity criterion from TQFT

It follows from duality (2.1) that Tutte's linear relation (1.2) has an immediate analogue, a 4-term identity for the flow polynomial of dual graphs. Further, using the relations in Figure 3 (equivalent to the contraction–deletion rule), it can be restated as a 3-term identity

$$(3.1) \quad \phi F_{H_1}(\phi + 1) - F_{H_2}(\phi + 1) + \phi^{-1} F_{H_3}(\phi + 1) = 0,$$

where planar graphs  $H_i$  are shown in Figure 6.

We will use a version of this identity at the Galois conjugate value  $(3 - \sqrt{5})/2$ :

LEMMA 3.1. *Consider three planar graphs  $H_i$  which are locally related as shown in Figure 6. Then the values of their flow polynomials at  $\phi^{-2} = (3 - \sqrt{5})/2$  satisfy the linear identity*

$$(3.2) \quad F_{H_1}(\phi^{-2}) + \phi F_{H_2}(\phi^{-2}) + \phi^2 F_{H_3}(\phi^{-2}) = 0.$$

Recall from Section 2 that the identity (3.1) holds since its left-hand side is the pull-back of the 4th Jones–Wenzl projector  $P^{(4)}$  from the Temperley–Lieb algebra  $TL^d$  to the chromatic algebra  $\mathcal{C}^Q$ . Here  $d = \phi$ ,  $P^{(4)}$  generates the trace radical, and the corresponding value of  $Q$  is  $d^2 = \phi + 1$ . The identity (3.2) holds for the same reason, being a pull-back of  $P^{(4)}$  for  $d = (1 - \sqrt{5})/2 = -\phi^{-1}$  and  $Q = (3 - \sqrt{5})/2 = \phi^{-2}$ . We work with the Galois conjugate because the sign of the flow polynomial in the interval  $(0, 1)$  is determined by the number of vertices and edges, which is useful for proving inequalities.

Consider the question of whether (3.2) holds for a *ribbon* (not necessarily planar) graph. Recall that a ribbon graph is an abstract graph  $G$  together with an embedding into a surface  $S$  so that the complement  $S \setminus G$  consists of 2-cells. Given such an embedding, a small neighborhood  $N$  of  $G$  in  $S$  is a compact surface with boundary, which may be thought of as a choice of a 2-dimensional thickening of  $G$ . (Such a thickening of vertices and edges may be alternatively encoded using cyclic ordering of half-edges incident to every vertex.) For our applications, it does not matter whether the surface is orientable, and in fact only a local ribbon structure will be used, where a thickening of a given edge and of adjacent half-edges is specified. Ribbon



FIGURE 7. Two ribbon structures near an edge of a cubic graph.

graphs have been used both in the context of quantum topology (cf. [8]) and in combinatorics (cf. [3]).

A ribbon structure is needed to make sense of the relation (3.2). Specifically, an edge  $e$  of an abstract cubic graph  $H_1$ , shown on the left in Figure 6, uniquely determines  $H_3$ . However the graph denoted  $H_2$  in the figure depends on a choice of a ribbon structure near the edge  $e$ . There are two such choices, Figure 7.

A graph is  $k$ -connected if it has at least  $k + 1$  vertices, but does not contain a set of  $k - 1$  vertices whose removal (together with all of the edges adjacent to them) disconnects the graph.

We are in a position to state the main result of this section.

**THEOREM 3.2.** *A 3-connected cubic graph  $G$  is planar if and only if the identity (3.2) for the flow polynomial at  $(3 - \sqrt{5})/2$  holds for at least one of the two ribbon structures at every edge of  $G$ .*

The theorem follows from the following lemma. Note that in the first part of the statement the graph is not assumed to be cubic.

**LEMMA 3.3.** (1) *Let  $G$  be a graph with two adjacent trivalent vertices. Let  $e$  denote the edge connecting them, and fix one of the two ribbon structures near  $e$  (Figure 7). Denote by  $V, E$  the number of vertices, respectively edges, of  $G$ . Then*

$$(3.3) \quad (-1)^{V-E} [F_{H_1}(\phi^{-2}) + \phi F_{H_2}(\phi^{-2}) + \phi^2 F_{H_3}(\phi^{-2})] \geq 0,$$

where  $H_1 = G$  and  $H_2, H_3$  are locally related to  $H_1$  as in Figure 6.

(2) *Moreover, if  $G$  is a 3-connected cubic graph then  $G$  is non-planar if and only if there exists an edge  $e$  with a strict inequality in (3.3) for both ribbon structures at  $e$ .*

The proofs of Lemma 3.3 and Theorem 3.2 will be given in the context of flow category which is introduced next.

**3.1. The flow category.** Abstract (not necessarily planar) graphs may be used to define an algebra along the lines of the Temperley–Lieb and chromatic algebras in Section 2. However since the algebra structure will not be needed in the proof of Theorem 3.2, and also the graphs are no longer assumed to be planar, the notion of a category is more suitable.

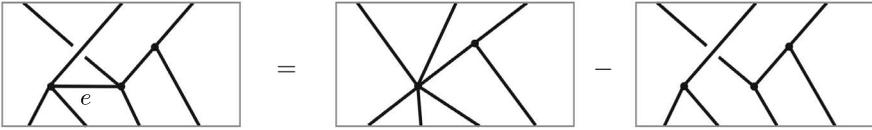


FIGURE 8. The contraction–deletion rule in  $\mathcal{F}_{4,3}$ .

The objects of the flow category are finite ordered sets  $\bar{n} = \{1, \dots, n\}$ . Consider  $\mathcal{G}_{m,n} = \{\text{finite graphs with } m + n \text{ marked univalent vertices}\}$ , where the marked vertices are divided into two ordered subsets of  $m$ , respectively  $n$  vertices. The edges incident to the marked vertices are called *boundary* edges and the rest are *internal* edges. Morphisms between  $\bar{m}, \bar{n}$  in the flow category are elements of  $\mathcal{F}_{m,n}^Q$ : formal  $\mathbb{C}$ -linear combinations of graphs  $\mathcal{G}_{m,n}$ , modulo the contraction–deletion relation (Figure 8) which applies to internal edges. In addition, the value of a loop is set to be  $Q - 1$ , and graphs which have a univalent vertex (other than the specified marked vertices) are set to be zero. It follows from the defining relations that a 2-valent vertex incident to an interior edge can be removed from a graph.

Graphs whose equivalence classes are elements of  $\mathcal{F}_{m,n}^Q$  may be represented geometrically as in Figure 8. (Over/under-crossings do not carry any information here since the figure is meant to represent only the abstract graph structure and not a specific planar projection.)

A graph without marked vertices (and so no boundary edges), considered in  $\mathcal{F}_{0,0}^Q \cong \mathbb{C}$ , evaluates to its flow polynomial at  $Q$ . The pairing  $\mathcal{F}_{k,m}^Q \times \mathcal{F}_{m,n}^Q \rightarrow \mathcal{F}_{k,n}^Q$  is obtained by gluing along  $m$  boundary edges. In particular, this pairing applied to two graphs  $A \in \mathcal{G}_{0,m}, B \in \mathcal{G}_{m,0}$  gives  $\langle A, B \rangle =$  the value of the flow polynomial  $F_{A \cup B}$  at  $Q$ .

REMARK 3.4. The flow category may be extended to a definition of a TQFT. This is a special case of a well-known construction [15] of TQFTs as fields (“pictures”) on a manifold modulo local relations. In particular, the example of the Tutte polynomial and of the local relation given by the contraction–deletion rule was discussed in [16]. Unlike the Temperley–Lieb and the chromatic algebra which give rise to the  $(2 + 1)$ -dimensional  $(\text{SU}(2)$  and  $\text{SO}(3)$  respectively) theories, this TQFT is not specific to dimension 2 since it is formulated in terms of abstract, not embedded graphs. We use elements of the structure of the flow category as a convenient setting for the proof of Theorem 3.2.

Given any graph representing an element of  $\mathcal{F}_{m,n}^Q$ , the contraction–deletion rule may be used to eliminate all internal edges. In particular, four graphs  $e_i$  in Figure 9 form a basis of  $\mathcal{F}_{0,4}^Q$ . Note that three of these graphs (viewed

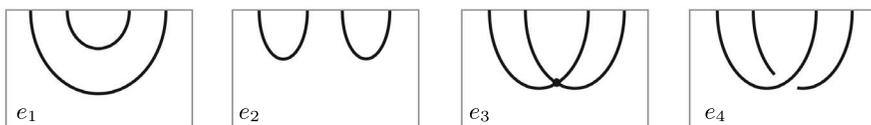


FIGURE 9. A basis of  $\mathcal{F}_{0,4}^Q$ .

relative to a fixed embedding of the marked vertices in the boundary of a rectangle) are planar and one, denoted  $e_4$ , is non-planar.

*Proof of Lemma 3.3.* Let  $G$  be a graph with two adjacent trivalent vertices (connected by an edge  $e$ ), as in the statement of the lemma, and fix  $Q = \phi^{-2} = (3 - \sqrt{5})/2$ . Cut each of the four edges adjacent to  $e$  in half, and consider the graph  $G$  as the union  $\overline{H}_1 \cup \overline{G}$ . Here  $\overline{H}_1$  is a “neighborhood” of the edge  $e$ , that is the part of  $G$  shown on the left in Figure 6, and  $\overline{G}$  is the rest of the graph  $G$ .  $\overline{H}_1$  is considered as an element of  $\mathcal{F}_{0,4}^Q$  and  $\overline{G}$  an element of  $\mathcal{F}_{4,0}^Q$ , and the union  $G = \overline{H}_1 \cup \overline{G}$  is taken along 4 pairs of marked points.

Consider the element  $P \in \mathcal{F}_{0,4}^Q$ ,

$$P := \overline{H}_1 + \phi \overline{H}_2 + \phi^2 \overline{H}_3.$$

where  $\overline{H}_i$  denotes the part of the graph  $H_i$  contained in the disk in Figure 6,  $i = 1, 2, 3$ . (In the planar context, this is an element of the trace radical of the chromatic algebra  $\mathcal{C}^Q$ , Section 2.3.) Lemma 3.3 may then be restated as saying that

$$(3.4) \quad (-1)^{V-E} \langle P, \overline{G} \rangle \geq 0.$$

The proof of (3.4) is by induction on the number of internal edges of  $\overline{G}$ . First, consider the base of the induction: no internal edges. Since the first three basis vectors in Figure 9 are planar and  $P$  is in the trace radical of the chromatic algebra,  $\langle P, e_i \rangle = 0$ ,  $i = 1, 2, 3$ . A direct calculation gives  $\langle P, e_4 \rangle = -\phi$ , so  $(-1)^{V-E} \langle P, e_4 \rangle > 0$ .

For the inductive step, consider the contraction–deletion rule

$$(3.5) \quad (-1)^{V-E} \langle P, G' \rangle = (-1)^{V-E} \langle P, G'/e' \rangle - (-1)^{V-E} \langle P, G' \setminus e' \rangle,$$

where  $e'$  is an edge (neither a bridge nor a loop) of  $G'$ , and  $V, E$  denote the number of vertices and edges of  $H_1 \cup G'$ . Both  $G'/e'$ ,  $G' \setminus e'$  have fewer edges than  $G'$ , so the inductive hypothesis applies. Moreover, the sign of  $(-1)^{V-E}$  of  $G'/e'$  agrees with that of  $G'$ , and it differs from the sign for  $G' \setminus e'$ . Therefore  $(-1)^{V-E} \langle P, G' \rangle \geq 0$ , being the sum of two non-negative terms.

To conclude the inductive step, we need to analyze the effect of removing a loop from  $G'$ . This changes the sign  $(-1)^{V-E}$ , but the loop value is  $-\phi^{-1}$ , so overall removing a loop does not affect the sign of  $(-1)^{V-E} \langle P, \overline{G} \rangle$ . This completes the proof of part (1) of the lemma.

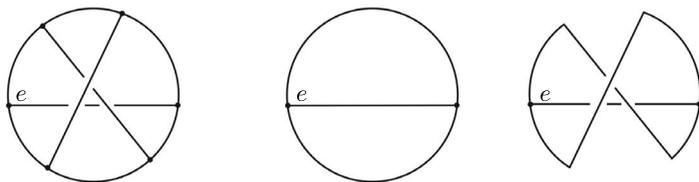


FIGURE 10. Left: a subdivision of  $K_{3,3}$  with three pairwise non-adjacent non-subdivided edges (drawn straight). The two figures on the right show the two relevant minors.

One of the implications in the second statement in lemma 3.3 is clear: if  $G$  is planar, then the equality in (3.3) follows from lemma 3.1. The proof of the other implication relies on the theorem [7, Theorem 5], [9, Theorem 3.1] that any 3-connected non-planar graph with at least six vertices contains a cycle with three pairwise crossing chords. In other words, the theorem asserts that any 3-connected non-planar graph, different from  $K_5$ , contains a subdivision of  $K_{3,3}$  with three pairwise non-adjacent non-subdivided edges, Figure 10.

We will use only a weaker version of the theorem, that one of the edges is unsubdivided, call it  $e$ . As in the proof above of part (1) of the lemma, decompose  $G = \overline{H}_1 \cup \overline{G}$ . The two ribbon structures near  $e$  correspond to the two cyclic orderings of the four univalent vertices of  $\overline{H}_1$  (and correspondingly of  $\overline{G}$ ). As illustrated in Figure 10, for either cyclic ordering  $\overline{G}$  has the non-planar basis element  $e_4$  (Figure 9) as a minor.

Fix either ribbon structure at  $e$ . Using the contraction–deletion rule on the internal edges of  $\overline{G}$ ,  $(-1)^{V-E} \langle P, \overline{G} \rangle$  equals the sum of  $2^k$  terms, where  $k$  is the number of internal edges. Each term is of the form  $(-1)^{V-E} \langle P, e_i \rangle$  where  $e_i$  is a basis element in Figure 9. Moreover, it is shown in the proof of (1) above that each of the  $2^k$  summands is non-negative. As indicated in the previous paragraph, at least one of the terms involves the non-trivial pairing with  $e_4$ . It follows that  $(-1)^{V-E} \langle P, \overline{G} \rangle > 0$ .  $\square$

#### 4. Bounds for the chromatic and flow polynomials and the Beraha numbers

It has been known since the work of Tutte [12], [11] and Beraha [2] that the Beraha numbers  $B_n = 2 + 2 \cos(\frac{2\pi}{n+1})$ , where  $n$  is a positive integer, play a special role in the theory of the chromatic polynomial of planar graphs.

**THEOREM 4.1.** *Given a planar triangulation  $T$ , let  $x$  be either a Beraha numbers  $B_n$  or a real number  $\geq 4$ . Then*

$$(4.1) \quad |\chi_T(x)| \leq x(x-1)(x-2)^{(V-2)}.$$

For  $x = B_4 = \phi + 1 = (3 + \sqrt{5})/2$  this upper bound coincides with the Tutte estimate

$$(4.2) \quad |\chi_T(\phi + 1)| \leq \phi^{5-V}.$$

Note that the upper bound (4.1) is exponentially decreasing only when  $x - 2 < 1$ . It is an interesting question whether the bound may be improved for 4-connected planar triangulations.

*Proof of Theorem 4.1.* When a given triangulation  $T$  has a multiedge or a triangle not bounding a face, one may decompose into two triangulations and induct to get the inequality. More precisely, the chromatic polynomial of a graph  $G$  which is the union of two subgraphs  $G_1, G_2$  such that  $G_1 \cap G_2$  is the complete graph  $K_n$ , satisfies

$$\chi_G(x) = \frac{\chi_{G_1}(x)\chi_{G_2}(x)}{\chi_{K_n}(x)}.$$

For planar graphs the relevant cases are  $n = 2, 3$ , and if both  $G_1, G_2$  satisfy (4.1), then so does  $G$ . Therefore, it suffices to prove the statement for 4-connected planar triangulations  $T$ . By a theorem of Whitney [18, Theorem 1]  $T$  has a Hamiltonian cycle. The cycle divides  $T$  into two outerplanar triangulations. The doubles of these triangulations have chromatic polynomial the same as that of the outerplanar graph, which is  $x(x - 1)(x - 2)^{(V-2)}$ . As explained in Section 2.3, the chromatic polynomial of a planar graph may be evaluated in the Temperley–Lieb algebra. The inner product in  $TL^d$  is positive semi-definite for  $d \in \{2 \cos(\frac{\pi}{n+1})\}$ , and it is positive definite for  $d > 2$  [6], [5, Corollary 7.1]. Then (4.1) follows from the Cauchy–Schwarz inequality.  $\square$

**THEOREM 4.2.** *Given a planar triangulation  $T$ , its chromatic polynomial satisfies*

$$(4.3) \quad |\chi_T(\phi^{-2})| \geq \phi^{2V-6}.$$

*Proof.* Consider the 3-connected dual cubic planar graph  $G = T^*$ , then  $|\chi_T(\phi^{-2})| = \phi^{-2}|F_G(\phi^{-2})|$ . It follows from an Euler characteristic calculation that the numbers  $n, V$  of vertices of  $G, T$  are related by  $n = 2V - 4$ . Then (4.3) is equivalent to the inequality

$$(4.4) \quad |F_G(\phi^{-2})| \geq \phi^{n-2}.$$

for the flow polynomial of 3-connected planar cubic graphs  $G$  with  $n$  vertices. Consider the operation of *removing an edge  $e$*  from  $G$ , where one disregards the 2-valent vertices (the boundary of the removed edge) in  $G \setminus e$ , so that the resulting graph is again cubic. In this terminology the graph  $H_3$  in Figure 6 is obtained by removing  $e$  from  $H_1$ . A theorem of Barnette–Grünbaum [1, Lemma 1] (and a closely related result of Tutte [10]) states that for any 3-connected graph with more than 6 edges there exists an edge whose removal yields a graph which is also 3-connected. Said differently, there is a sequence

of edge removals so that each graph in the sequence is 3-connected and the last graph is  $K_4$ .

Consider an inductive application of the identity (3.2) to such a sequence of edge removals. Let the graphs in Figure 6 show the inductive step: the edge  $e$  is removed from a planar 3-connected cubic graph  $H_1$ , giving rise to a planar 3-connected cubic graph  $H_3$ . In particular, it follows that all three graphs  $H_i$  are connected.

The flow polynomial of a connected bridgeless graph with  $n$  vertices and  $E$  edges is non-zero, of sign  $(-1)^{(E-n+1)}$  on the interval  $(-\infty, 1)$  [14, Theorem 3.11]. (This is the flow polynomial version of the analogous result for the chromatic polynomial [13, (2.3)].) It follows that the sign of  $F_{H_3}(\phi^{-2})$ , and also the sign of  $F_{H_2}(\phi^{-2})$  if  $H_2$  is bridgeless, are opposite of that of  $F_{H_1}(\phi^{-2})$ . Rewrite (3.2) as

$$(-1)^{E-n+1}F_{H_1}(\phi^{-2}) = (-1)^{E-n}[\phi F_{H_2}(\phi^{-2}) + \phi^2 F_{H_3}(\phi^{-2})],$$

where the first and the third terms are positive, and the second term is non-negative. Omitting the second term, one has

$$(-1)^{E-n+1}F_{H_1}(\phi^{-2}) \geq (-1)^{E-n}[\phi^2 F_{H_3}(\phi^{-2})].$$

To conclude the inductive step, note that  $H_3$  has two fewer vertices than  $H_1$ , and the inequality above gives a factor of  $\phi^2$ . The base of the induction is  $F_{K_4} = -\phi^2$ .  $\square$

It is interesting to compare the inequalities (4.2) and (4.3). Conceptually the difference between the upper bound (4.1) at Beraha numbers and the lower bound (4.3) at  $\phi^{-2} = (3 - \sqrt{5})/2$  may be understood using the structure of the Temperley–Lieb algebra  $TL^d$ . As discussed in Section 2, the value  $(3 + \sqrt{5})/2$  of the chromatic polynomial corresponds to the parameter  $d = 2 \cos(\pi/5)$  where  $TL^d$  is positive semi-definite and the Cauchy–Schwarz inequality applies. (This is true for all Beraha numbers, see the proof of Theorem 4.1.) On the other hand the value  $(3 - \sqrt{5})/2$  corresponds to  $d = 2 \cos(3\pi/5)$ . The 4th Jones–Wenzl projector is in the trace radical of  $TL^d$  for both  $d = 2 \cos(\pi/5)$  and  $d = 2 \cos(3\pi/5)$ , but the bilinear pairing on  $TL^d$  at the latter value of  $d$  has mixed signs.

## 5. The golden identity and inequality for non-planar graphs

The golden identity (1.1) for the chromatic polynomial of planar triangulations has an immediate counterpart stated dually for the flow polynomial of planar cubic graphs. Moreover, there are analogous identities for the chromatic and flow polynomials relating their values at the Galois conjugate  $(3 - \sqrt{5})/2 = \phi^{-2}$  and  $(5 - \sqrt{5})/2$ . In this section we consider the validity of these relations for non-planar graphs.

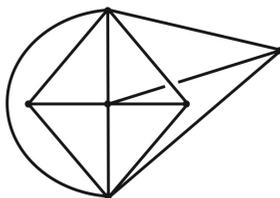


FIGURE 11. A non-planar graph  $G$  satisfying the golden identity (1.1).

**5.1. A conjecture for the flow polynomial.** The Galois conjugate of the golden identity, stated for the flow polynomial of planar cubic graphs, reads

$$F_G((5 - \sqrt{5})/2) = (-\phi^{-1})^E F_G((3 - \sqrt{5})/2)^2,$$

where  $E$  is the number of edges of  $G$ .

Based on numerical evidence, we conjecture that in fact this identity characterizes planar cubic graphs:

CONJECTURE 5.1. *For any cubic bridgeless graph  $G$ ,*

$$(5.1) \quad (-\phi)^E F_G((5 - \sqrt{5})/2) \geq F_G((3 - \sqrt{5})/2)^2,$$

*Moreover,  $G$  is planar if and only if (5.1) is an equality.*

Similarly, numerical evidence points to the inequality  $F_G(\phi + 2) \leq \phi^E F_G(\phi + 1)^2$  for any cubic graph  $G$ , with an equality if and only if  $G$  is planar.

**5.2. The chromatic polynomial.** In contrast to Conjecture 5.1, there are non-planar graphs satisfying the chromatic version (1.1) of the golden identity. This section describes a specific example (see Figure 11), and it suggests an explanation of the apparent difference with the behavior of the flow polynomial.

The main point is that the *chromatic* Tutte linear relation (1.2) at  $\phi + 1$ , unlike its flow counterpart (3.1), still holds when coupled with certain non-planar graphs. (Note that in (1.2) there is an arbitrary number of edges intersecting the boundary of the disk, so in fact this is an infinite collection of local relations for the chromatic polynomial of planar graphs.)

Consider the graph in Figure 11, obtained from  $K_{3,3}$  by adding three edges. Endow it with the ribbon structure induced from the shown planar projection. A copy of this graph, denoted  $G_2$ , is also included in Figure 12. Consider the intersection of  $G_2$  with the shaded region indicated in the figure as a particular example of the second graph in Figure 1. The graphs  $G_1, G_3, G_4$ , shown below, are locally related to  $G_2$  as in Figure 1. Note that  $G_1, G_3, G_4$  are all planar, so they satisfy (1.1).

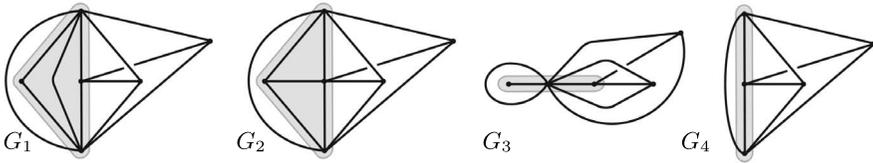


FIGURE 12. A non-planar graph  $G_2$  satisfying the golden identity (1.1). Also shown are the graphs locally related to  $G_2$  as in Figure 1.

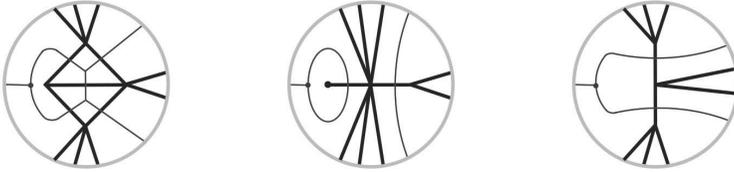


FIGURE 13. Parts of the graphs  $G_2, G_3, G_4$  (and their duals) in the shaded disks in Figure 12.

A key ingredient in the proof of the golden identity (1.1) in [11] (and also in [4]) is the linear identity (1.2) which holds as a consequence of the vanishing of the projector  $P^{(4)}$  in the Temperley–Lieb algebra  $TL^\phi$  (see Section 2.3). This identity is intrinsic to planar graphs, and it certainly does not hold for general ribbon graphs. However, we will now show that it still holds for *some* non-planar graphs.

In the relation (1.2) and Figure 1 there is an arbitrary number of edges leading to the boundary of the disk. In the special case in Figure 12 the internal vertex on the left has *no* edges leading to the boundary. The map to the Temperley–Lieb algebra, described in Section 2.3, is defined in terms of dual trivalent graphs. The relevant dual graphs are shown in Figure 13. One checks that the image equals zero in the Temperley–Lieb algebra at  $d = \phi$ . (Indeed, every term involves the projector  $P^{(4)}$  capped with a turnback, or the partial trace of  $P^{(4)}$ , see Figure 14. Such elements equal zero in  $TL^\phi$  [4, Section 5.4].) Consequently, the identity (3.1) and its chromatic dual:

$$(5.2) \quad \phi \chi_{G_2}(\phi + 1) = \chi_{G_3}(\phi + 1) + (1 - \phi) \chi_{G_4}(\phi + 1)$$

hold for the graphs in Figure 12. (More generally, the identity still holds for an arbitrary number of edges leading to the boundary of the disk at the other three vertices in Figure 13, and these local pictures may be paired with an arbitrary non-planar graph.)

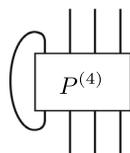


FIGURE 14. The partial trace of the projector  $P^{(4)}$  equals zero at  $d = \phi$  (and also at  $d = -\phi^{-1}$ ).

The chromatic polynomial of these graphs satisfies the contraction–deletion rule pictured in Figure 3:

$$(5.3) \quad \chi_{G_1}(\phi + 2) + \chi_{G_3}(\phi + 2) = \chi_{G_2}(\phi + 2) + \chi_{G_4}(\phi + 2).$$

As shown in the proof of Lemma 3.1 in [4], (5.2) yields

$$(5.4) \quad \phi^3(\chi_{G_1}(\phi + 1))^2 + (\chi_{G_3}(\phi + 1))^2 = \phi^3(\chi_{G_2}(\phi + 1))^2 + (\chi_{G_4}(\phi + 1))^2.$$

Equations (5.3), (5.4), combined with the fact that three of the graphs satisfy the golden identity, imply that the remaining graph,  $G_2$ , also obeys (1.1).

The argument above suggests a strategy for generating families of non-planar graphs satisfying the chromatic golden identity. It would be interesting to find a characterization of all such graphs, and in particular to see whether the argument using the 4th Jones–Wenzl projector considered here generates all of them. Another open question is whether there exist 4-connected non-planar graphs with the number of edges and vertices satisfying  $E = 3V - 6$  (analogous to the case of a planar triangulation) and satisfying the golden identity (1.1).

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