NORMALITY PRESERVING OPERATIONS FOR CANTOR SERIES EXPANSIONS AND ASSOCIATED FRACTALS, I

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ABSTRACT. It is well known that rational multiplication preserves normality in base b. We study related normality preserving operations for the Q-Cantor series expansions. In particular, we show that while integer multiplication preserves Q-distribution normality, it fails to preserve Q-normality in a particularly strong manner. We also show that Q-distribution normality is not preserved by non-integer rational multiplication on a set of zero measure and full Hausdorff dimension.

1. Introduction

Let $\mathscr{N}(b)$ be the set of numbers normal in base b and let f be a function from $\mathbb{R}$ to $\mathbb{R}$. We say that f *preserves b-normality* if $f(\mathscr{N}(b)) \subseteq \mathscr{N}(b)$. We can make a similar definition for preserving normality with respect to the regular continued fraction expansion, β-expansions, the Lüroth series expansion, etc.

Several authors have studied b-normality preserving functions. They naturally arise in H. Furstenberg's work on disjointness in ergodic theory [13]. V. N. Agafonov [1], T. Kamae [15], T. Kamae and B. Weiss [16], and W. Merkle and J. Reimann [22] studied b-normality preserving selection rules.

For a real number r, define real functions π_r and σ_r by $\pi_r(x) = rx$ and $\sigma_r(x) = r + x$. In 1949 D. D. Wall proved in his Ph.D. thesis [29] that for nonzero rational r the function π_r is b-normality preserving for all b and that the function σ_r is b-normality preserving for all b whenever r is rational. These results were also independently proven by K. T. Chang in 1976 [8]. D. D. Wall's method relies on the well-known characterization that a real number x is normal in base b if and only if the sequence $(b^n x)$ is uniformly distributed mod 1 [17].

D. Doty, J. H. Lutz, and S. Nandakumar took a substantially different approach from D. D. Wall and strengthened his result. They proved in [9] that for every real number x and every non-zero rational number r the b-ary expansions of $x, \pi_r(x)$, and $\sigma_r(x)$ all have the same finite-state dimension and the same finite-state strong dimension. It follows that π_r and σ_r preserve b-normality. It should be noted that their proof uses different methods from those used by D. D. Wall and is unlikely to be proven using similar machinery.

C. Aistleitner generalized D. D. Wall's result on σ_r in [4]. Suppose that q is a rational number and that the digits of the b-ary expansion of z are non-zero on a set of indices of density zero. He proved that the function σ_{qz} is b-normality preserving. We will show as a consequence of Theorem 3.1 that C. Aistleitner's result does not generalize to at least one notion of normality for some of the Cantor series expansions. G. Rauzy obtained a complete characterization of

$$\Sigma_b := \big\{ r : \sigma_r\big(\mathscr{N}(b)\big) = \mathscr{N}(b) \big\}$$

in [24]. M. Bernay used this characterization to prove that Σ_b has zero Hausdorff dimension [5].

There are still many open questions relating to the functions π_r and σ_r. For example, M. Mendés France asked in [21] if the function π_r preserves simple normality with respect to the regular continued fraction for every non-zero rational r. The authors are unaware of any theorems that state that either π_r or σ_r preserve any other form of normality than b-normality.

We will focus on the normality preserving properties of π_r for the Q-Cantor series expansion as well as two other related functions. We will show that while π_r is Q-distribution normality preserving for every non-zero integer r, the set of x where $\pi_r(x)$ is not Q-distribution normal has full Hausdorff dimension whenever $r \in \mathbb{Q} \setminus \mathbb{Z}$ and Q is infinite in limit. Our main theorem will show that the function π_r is so far from preserving Q-normality that there exist basic sequences Q and real numbers x that are Q-normal and Q-distribution normal where $\pi_r(x)$ is not Q-normal for every integer $r \geq 2$. In the sequel to this paper [3], the authors and J. Vandehey prove that for a class of basic sequences Q, the set of real numbers x where $\pi_r(x)$ is Q-normal for all non-zero rationals r but where x is not Q-distribution normal has full Hausdorff dimension.

2. Cantor series expansions

The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was first done by P. Erdős and A. Rényi in [10] and [11] and by A. Rényi in [25], [26], and [27] and by P. Turán in [28].

The Q-Cantor series expansions, first studied by G. Cantor in [7], are a natural generalization of the b-ary expansions.[1] Let $\mathbb{N}_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in \mathbb{N}_2^{\mathbb{N}}$, then we say that Q is a *basic sequence*. Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$, the Q-*Cantor series expansion* of a real number x is the (unique)[2] expansion of the form

$$(2.1) \qquad\qquad x = E_0 + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n},$$

where $E_0 = \lfloor x \rfloor$ and E_n is in $\{0, 1, \ldots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (2.1) with the notation $x = E_0.E_1 E_2 E_3 \ldots$ w.r.t. Q. We define $N_n^Q(B, x)$ as the number of occurrences of the block B in the Q-Cantor series expansion of x up to position n.

A *block* is an ordered tuple of non-negative integers, a *block of length k* is an ordered k-tuple of integers, and a *block of length k in base b* is an ordered k-tuple of integers in $\{0, 1, \ldots, b - 1\}$.

Let

$$Q_n^{(k)} := \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}} \quad \text{and} \quad T_{Q,n}(x) := \left(\prod_{j=1}^{n} q_j \right) x \pmod 1.$$

A. Rényi [26] defined a real number x to be *normal* with respect to Q if for all blocks B of length 1,

$$(2.2) \qquad\qquad \lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(1)}} = 1.$$

If $q_n = b$ for all n and we restrict B to consist of only digits less than b, then (2.2) is equivalent to *simple normality in base b*, but not equivalent to *normality in base b*. A basic sequence Q is *k-divergent* if $\lim_{n \to \infty} Q_n^{(k)} = \infty$ and *fully divergent* if Q is k-divergent for all k. A basic sequence Q is *infinite in limit* if $q_n \to \infty$.

DEFINITION 2.1. A real number x is *Q-normal of order k* if for all blocks B of length k,

$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = 1.$$

We let $\mathscr{N}_k(Q)$ be the set of numbers that are Q-normal of order k. The real number x is *Q-normal* if $x \in \mathscr{N}(Q) := \bigcap_{k=1}^{\infty} \mathscr{N}_k(Q)$. A real number x is *Q-distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed mod 1. Let $\mathscr{DN}(Q)$ be the set of Q-distribution normal numbers.

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[1] G. Cantor's motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number $e = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [14].
[2] Uniqueness can be proven in the same way as for the b-ary expansions.

It follows from a well known result of H. Weyl [30], [31] that $\mathscr{D}\mathscr{N}(Q)$ is a set of full Lebesgue measure for every basic sequence Q. We will need the following result of the second author [19] later in this paper.

THEOREM 2.2. *Suppose that Q is infinite in limit. Then $\mathscr{N}_k(Q)$ (resp. $\mathscr{N}(Q)$) is of full measure if and only if Q is k-divergent (resp. fully divergent).*

Note that in base b, where $q_n = b$ for all n, the corresponding notions of Q-normality and Q-distribution normality are equivalent. This equivalence is fundamental in the study of normality in base b.

Another definition of normality, Q-ratio normality, has also been studied (we call the set of Q-ratio normal number $\mathscr{R}\mathscr{N}(Q)$ see [20]). We do not introduce this notion here as this set contains the set of Q-normal numbers and all results in this paper that hold for Q-normal numbers also hold for Q-ratio normal numbers. The complete containment relation between the sets of these normal numbers and pair-wise intersections thereof is proven in [20]. The Hausdorff dimensions of difference sets such as $\mathscr{R}\mathscr{N}(Q) \cap \mathscr{D}\mathscr{N}(Q) \setminus \mathscr{N}(Q)$ are computed in [2].

A surprising property of Q-normality of order k is that we may not conclude that $\mathscr{N}_k(Q) \subseteq \mathscr{N}_j(Q)$ for all $j < k$ like we may for the b-ary expansions. In fact, it was shown in [18] that for every k there exists a basic sequence Q such that $\mathscr{N}_k(Q) \setminus \bigcup_{j=1}^{k-1} \mathscr{N}_j(Q)$ is non-empty. Thus, rather than showing that some functions do not preserve Q-normality of order k, we will show that they do not preserve Q-normality of any order.

3. Results

We note the following theorem which may be stated in terms of Q-normality preserving functions. Instead, we present it in its current form for simplicity.

THEOREM 3.1. *Suppose that Q is infinite in limit and that $x = E_0.E_1E_2\cdots$ w.r.t. Q is Q-normal of order k. Then there exists a real number $y = F_0.F_1F_2\cdots$ w.r.t. Q where $E_n \neq F_n$ on a set of density zero and $y \notin \bigcup_{j=1}^{\infty} \mathscr{N}_j(Q)$.*

Proof. Define (F_n) by

$$F_n = \begin{cases} 1 & \text{if } E_n = 0, \\ E_n & \text{if } E_n \neq 0. \end{cases}$$

Note that $E_n \neq F_n$ along a set of density zero since

$$\lim_{n\to\infty} \frac{\#\{i \leq n : E_i \neq F_i\}}{n} = \lim_{n\to\infty} \frac{N_n^Q(0,x)}{n} = \lim_{n\to\infty} \frac{N_n^Q(0,x)}{P_n^{(1)}} \frac{P_n^{(1)}}{n} = 0.$$

The number $y = F_0.F_1F_1\cdots$ w.r.t. Q is not Q-normal of any order since the digit 0 never occurs. $\qquad\square$

This shows that C. Aistleitner's [4] result does not generalize to Q-normality for all Q-Cantor series expansions by letting $q = 1$ and by letting $z = 0.G_1 G_2 \cdots$ w.r.t. Q where G_n is equal to 1 along a set of indices of zero density and 0 otherwise and setting $y = x + z$.

THEOREM 3.2. *Suppose Q is a basic sequence and that $x = E_0.E_1 E_2 \cdots$ w.r.t. Q is Q-distribution normal. If $y = F_0.F_1 F_2 \cdots$ w.r.t. Q and*

$$(3.1) \qquad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{|E_n - F_n| + 1}{q_n} = 0,$$

then $y \in \mathscr{DN}(Q)$.

COROLLARY 3.3. *Suppose Q satisfies*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{q_n} = 0$$

and that $x = E_0.E_1 E_2 \cdots$ w.r.t. Q is Q-distribution normal. If $y = F_0.F_1 F_2 \cdots$ w.r.t. Q and

$$(3.2) \qquad \lim_{n \to \infty} \frac{|E_n - F_n|}{q_n} = 0,$$

then $y \in \mathscr{DN}(Q)$.

Together, Theorem 3.1 and Theorem 3.2 suggest that Q-distribution normality is a far more robust notion than Q-normality. We wish to give the following example demonstrating that distribution normality is not preserved in all bases by rational multiplication.

EXAMPLE 3.4. Define the sequences $E = (E_n)$ and $Q = (q_n)$ by

$$E = (0, 0, 2, 0, 2, 4, 0, 2, 4, 6, \ldots);$$
$$Q = (2, 4, 4, 6, 6, 6, 8, 8, 8, 8, \ldots).$$

Set $x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}$. Then $\left(\frac{E_n}{q_n}\right) = (0, 0, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, \ldots)$ is u.d. mod 1 so $x \in \mathscr{DN}(Q)$. However, $\frac{1}{2} x = 0.\frac{E_1}{2} \frac{E_2}{2} \cdots$ w.r.t. Q and $\frac{E_n/2}{q_n} \leq \frac{1}{2}$. Thus, $\pi_{1/2}(x) \notin \mathscr{DN}(Q)$.

However, a much stronger result holds.

THEOREM 3.5. *For all basic sequences Q, Q-distribution normality is preserved by non-zero integer multiplication. If Q is infinite in limit and $r \in \mathbb{Q} \backslash \mathbb{Z}$, then*

$$(3.3) \qquad \lambda\big(\big\{x \in \mathscr{DN}(Q) : \pi_r(x) \notin \mathscr{DN}(Q)\big\}\big) = 0;$$
$$(3.4) \qquad \dim_{\mathrm{H}}\big(\big\{x \in \mathscr{DN}(Q) : \pi_r(x) \notin \mathscr{DN}(Q)\big\}\big) = 1,$$

where λ is the Lebesgue measure on $\mathbb{R}$ and $\dim_H$ is the Hausdorff dimension.

We wish to define an equivalence relation $\sim$ on the set of basic sequences as follows. If $P = (p_n)$ and $Q = (q_n)$ are basic sequences then we write $P \sim Q$ if $p_n \neq q_n$ on a set of density zero.

Suppose that Q is infinite in limit and fully divergent. A nonempty subset $S_Q \subseteq \mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ was shown to exist in Theorem 3.12 in [20]. The members of S_Q have the following property. If $x \in S_Q$, then for any integer $n \geq 2$, the real number $\pi_n(x)$ is not Q-distribution normal and not Q-normal of any order. Since Q-distribution normality is preserved by integer multiplication, it is natural to ask if there exists a basic sequence Q and $x \in \mathcal{N}(Q) \cap \mathcal{DN}(Q)$ such that for any integer $n \geq 2$ the number $\pi_n(y)$ is not Q-normal of any order. The following theorem answers this question.

THEOREM 3.6. *Let $k \in \mathbb{N}$. If $P = (p_n)$ is eventually non-decreasing, infinite in limit, and k-divergent (resp. fully divergent), then there exists a basic sequence $Q = (q_n)$ and a real number x where the following hold.*

(1) *The basic sequence Q is infinite in limit, k-divergent (resp. fully divergent), and $P \sim Q$.*
(2) *The real number x is Q-normal of all orders 1 through k (resp. Q-normal) and Q-distribution normal.*
(3) *For every integer $n \geq 2$, the real number $\pi_n(x)$ is not Q-normal of any order.*

For a sequence of real numbers $X = (x_n)$ with $x_n \in [0, 1)$ and an interval $I \subseteq [0, 1]$, define $A_n(I, X) = \#\{i \leq n : x_i \in I\}$. We will first prove Theorem 3.2. To do this, we will need the following standard definition and lemma that we quote from [17].

DEFINITION 3.7. *Let $X = (x_1, \ldots, x_N)$ be a finite sequence of real numbers. The number*
$$D_N = D_N(X) = \sup_{0 \leq \alpha \leq \beta \leq 1} \left| \frac{A_N([\alpha, \beta), X)}{N} - (\beta - \alpha) \right|$$
is called the discrepancy *of the sequence X.*

It is well known that a sequence X is uniformly distributed mod 1 if and only if $D_N(X) \to 0$.

LEMMA 3.8. *Let $x_1, x_2, \ldots, x_N$ and $y_1, y_2, \ldots, y_N$ be two finite sequences in $[0, 1)$. Suppose $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ are non-negative numbers such that $|x_n - y_n| \leq \varepsilon_n$ for $1 \leq n \leq N$. Then, for any $\varepsilon \geq 0$, we have*
$$\left| D_N(x_1, \ldots, x_N) - D_N(y_1, \ldots, y_N) \right| \leq 2\varepsilon + \frac{\overline{N}(\varepsilon)}{N},$$
where $\overline{N}(\varepsilon)$ denotes the number of n, $1 \leq n \leq N$, such that $\varepsilon_n > \varepsilon$.

Proof of Theorem 3.2. Define the sets

$$S(i) = \left\{ j : \frac{|E_j - F_j| + 1}{q_j} \geq \frac{1}{i} \right\}$$

and the sequence

$$N_i = \min\left\{ n : \frac{\#S(i) \cap \{1, \ldots, j\}}{j} < \frac{1}{i}, \forall j \geq n \right\}.$$

Note that the density of the sets $S(i)$ must be 0 by (3.1) and therefore N_i is well-defined. Set $x_i = T_{Q,i}(x)$ and $y_i = T_{Q,i}(y)$. Note that $|x_i - y_i| \leq \frac{|E_i - F_i| + 1}{q_i}$. Then for any $n > N_i$ we have by Lemma 3.8 that

$$\left| D_n(x_1, \ldots, x_n) - D_n(y_1, \ldots, y_n) \right| \leq 2\frac{1}{i} + \frac{\#S(i) \cap \{1, \ldots, N\}}{N} < \frac{3}{i}.$$

Thus, $\lim_{n\to\infty} |D_n(x_1, \ldots, x_n) - D_n(y_1, \ldots, y_n)| = 0$. This implies that y is Q-distribution normal if x is Q-distribution normal. $\qquad\square$

For the remainder of this paper, let $E_{R,i}(\xi)$ be the ith digit of the R-Cantor series expansion of ξ.

Proof of Theorem 3.5. The first part is trivial as any integer multiple of a uniformly distributed sequence is uniformly distributed. For any null set E the preimage $\pi_r^{-1}(E)$ is also a null set. Therefore, $\mathscr{DN}(Q) \cap \pi_r^{-1}(\mathscr{DN}(Q))$ has full measure so (3.3) holds.

Let $r = a/b \in \mathbb{Q} \setminus \mathbb{Z}$ for relatively prime integers a and b. We now wish to show (3.4). Let (x_n) be a sequence of real numbers that is uniformly distributed mod 1. Define

$$C(n) = \max\left\{ m \in b\mathbb{Z} : \frac{m}{q_n} \leq x_n \right\}.$$

Set

$$f(n) = \frac{\min\{\log q_n, \log q_1 q_2 \cdots q_{n-1}\}}{\log q_n} \quad \text{and} \quad \omega(n) = \left\lfloor \frac{q_n^{1-f(n)}}{b} \right\rfloor.$$

Note that $\lim_{n\to\infty} \frac{\omega(n)}{q_n} = 0$ since

$$\lim_{n\to\infty} f(n) \log(q_n) \to \infty.$$

Define the sets

$$V(n) = \left[C(n) - \frac{b}{2}\omega(n), C(n) + \frac{b}{2}\omega(n) \right] \cap b\mathbb{Z}.$$

Consider the set

$$\Phi_{Q,b} = \left\{ x = 0.E_1 E_2 \cdots \text{ w.r.t. } Q : E_n \in V(n) \right\}.$$

Note that for any $x \in \Phi_{Q,b}$, we have that

$$\lim_{n\to\infty} \left| \frac{E_n}{q_n} - x_n \right| \leq \lim_{n\to\infty} \left| \frac{E_n}{q_n} - \frac{C(n)}{q_n} \right| + \left| \frac{C(n)}{q_n} - x_n \right| \leq \lim_{n\to\infty} \left| \frac{b\omega(n)}{2q_n} \right| + 0 = 0.$$

Therefore, $(\frac{E_n}{q_n})$ is uniformly distributed mod 1. Since $|T_{Q,n}(x) - \frac{E_n}{q_n}| \leq \frac{1}{q_n} \to 0$ we also get that the sequence $(T_{Q,n}(x))$ is uniformly distributed. Hence, $\Phi_{Q,b} \subseteq \mathscr{DN}(Q)$. Furthermore, every digit of $x \in \Phi_{Q,b}$ is divisible by b. Thus,

$$\frac{E_{Q,n}(\frac{1}{b}x)}{q_n} \in \left[0, \frac{1}{b}\right)$$

which implies that

$$T_{Q,n}\left(\frac{1}{b}x\right) \in \left[0, \frac{1}{b} + \frac{1}{q_n}\right).$$

Furthermore, $\frac{E_{Q,n}(x/b)}{q_n}$ is uniformly distributed in $[0, 1/b)$, so $T_{Q,n}(x/b)$ is uniformly distributed in $[0, 1/b)$ as well since $q_n \to 0$. Note that

$$T_{Q,n}\left(\frac{a}{b}x\right) = aT_{Q,n}\left(\frac{1}{b}x\right) \bmod 1.$$

Let $c \equiv a \bmod b$. Then

$$\lim_{n\to\infty} \frac{A_n([0, \frac{c}{b}), (T_{Q,n}(\pi_r(x))))}{n} = \frac{(\frac{1}{b} - \frac{\lfloor a/b\rfloor}{a})\lfloor a/b\rfloor}{\frac{1}{b}} = \left\lfloor \frac{a}{b} \right\rfloor \frac{c}{a} \neq \frac{c}{b},$$

so $\pi_r(x) \notin \mathscr{DN}(Q)$.

Following the notation of [12], $\Phi_{Q,b}$ is a homogeneous Moran set with $c_k = \frac{1}{q_k}$ and $n_k = \omega(k)$. By Theorem 2.1 in [12], we have that

$$\begin{aligned}
\dim_{\mathrm{H}}(\Phi_{Q,b}) &\geq \liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}} \\
&= \liminf_{k\to\infty} \frac{\log \omega(1)\omega(2)\cdots\omega(k)}{-\log(\frac{1}{q_1}\frac{1}{q_2}\cdots\frac{1}{q_{k+1}}\omega(k+1))} \\
&= \liminf_{k\to\infty} \frac{\log(q_1^{1-f(i)} q_2^{1-f(2)} \cdots q_k^{1-f(k)}) - k\log b}{\log q_1 q_2 \cdots q_k - f(k+1)\log q_{k+1}} \\
&= \liminf_{k\to\infty} \frac{\log q_1 \cdots q_k}{\log q_1 \cdots q_k - f(k+1)\log q_{k+1}} \\
&= \liminf_{k\to\infty} \frac{1}{1 - \frac{f(k+1)\log q_{k+1}}{\log q_1 \cdots q_k}} = 1. \qquad \square
\end{aligned}$$

We may now turn our attention to Theorem 3.6. Let $(P, Q) \in \mathbb{N}_2^{\mathbb{N}} \times \mathbb{N}_2^{\mathbb{N}}$ and suppose that $x = E_0.E_1E_2\cdots$ w.r.t. P. We define $\psi_{P,Q} : \mathbb{R} \to [0,1]$ by

$$\psi_{P,Q}(x) := \sum_{n=1}^{\infty} \frac{\min(E_n, q_n - 1)}{q_1 \cdots q_n}.$$

The following theorem of [20] will be critical in proving Theorem 3.6.

THEOREM 3.9. *Suppose that* $Q_1 = (q_{1,n}), Q_2 = (q_{2,n}), \ldots, Q_j = (q_{j,n})$ *are basic sequences and infinite in limit. If* $x = E_0.E_1E_2\cdots$ *w.r.t.* Q_1 *satisfies* $E_n < \min_{2 \le r \le j}(q_{r,n} - 1)$ *for infinitely many* n, *then for every block* B

$$N_n^{Q_j}\big(B, (\psi_{Q_{j-1},Q_j} \circ \psi_{Q_{j-2},Q_{j-1}} \circ \cdots \circ \psi_{Q_1,Q_2})(x)\big) = N_n^{Q_1}(B, x) + O(1).$$

Here we state a theorem of C. T. Rajagopal [23] that will be needed later to estimate a weighted density.

THEOREM 3.10. *Let* (w_n) *and* (s_n) *be sequences of positive real numbers such that* $\sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} s_n = \infty$. *If* $\frac{w_n}{s_n}$ *is non-increasing, then for every sequence* (a_n) *of real numbers, we have that*

$$\liminf_{n \to \infty} \frac{\sum_{k=1}^{n} s_k a_k}{\sum_{k=1}^{n} s_k} \le \liminf_{n \to \infty} \frac{\sum_{k=1}^{n} w_k a_k}{\sum_{k=1}^{n} w_k}$$

$$\le \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} w_k a_k}{\sum_{k=1}^{n} w_k} \le \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} s_k a_k}{\sum_{k=1}^{n} s_k}.$$

We also note the following basic lemma.

LEMMA 3.11. *Let* L *be a real number and* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be two sequences of positive real numbers such that*

$$\sum_{n=1}^{\infty} b_n = \infty \quad and \quad \lim_{n \to \infty} \frac{a_n}{b_n} = L.$$

Then

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = L.$$

From now on if $B = (k)$, then we write $N_n^Q(B, x)$ as $N_n^Q(k, x)$.

Proof of Theorem 3.6. Let P be a basic sequence that is infinite in limit and k-divergent. The proof of the statement for when P is fully divergent will follow similarly. Let y be a P-distribution normal number of all orders $1, \ldots, k$ such that ny is P-normal of all orders $1, \ldots, k$ for all natural numbers n. Note that such a y exists by Theorem 2.2.

Define the sequence $\{\ell_n\}$ as follows:

$$\ell_1 = 1;$$

$$\ell_i = \min\left\{t : \sum_{m=1}^{i} N_n^P\big(1, \pi_m(y)\big) < \frac{n}{i}, \forall n \ge t\right\}.$$

Define $M(n) = \max\{c : \ell_c < n\}$. We have that $M(n)$ tends to infinity since for a fixed i

$$\frac{\sum_{m=1}^{i} N_n^P(1, \pi_m(y))}{iP_n^{(1)}} \to 1 \quad \text{and} \quad \frac{iP_n^{(1)}}{n} \to 0.$$

Furthermore, for any $i \leq M(N)$ and $n \geq N$ we have that

$$\frac{\sum_{m=1}^{M(N)} N_n^P(1, \pi_m(y))}{n} < \frac{1}{M(N)}.$$

Construct $Q = (q_n)$ as follows: If there is a $i \in \{1, \ldots, M(n)\}$ such that $E_{P,n}(\pi_i(y)) = 1$, then set $q_n = M(n)p_n$ and $q_{n+1} = M(n+1)p_{n+1}$. Otherwise, set $q_n = p_n$. If $E_{P,n}(\pi_i(y)) = E_{P,n-1}(\pi_j(y)) = 1$ for some $i, j \leq M(n)$, put $q_{n-1} = M(n-1)p_{n-1}$, $q_n = M(n)p_n$ and $q_{n+1} = M(n+1)p_{n+1}$ and so on.

Put $x = \psi_{P,Q}(y)$. Note that $E_{Q,n}(\pi_m(x)) = 1$ only if

$$\frac{mE_{P,n}(y)}{q_n} + \frac{mE_{P,n+1}(y)}{q_n q_{n+1}} + \cdots \in \left[\frac{1}{q_n}, \frac{2}{q_n}\right) \quad \text{or}$$

$$\frac{mE_{P,n}(y)}{M(n)p_n} + \frac{mE_{P,n+1}(y)}{M(n)M(n+1)p_n p_{n+1}} + \cdots \in \left[\frac{1}{M(n)p_n}, \frac{2}{M(n)p_n}\right).$$

But for this to happen, we must have that $E_{P,n}(y) = 0$, and $E_{P,i}(y) \geq p_i - 1$ for all $i > n$. However, this cannot happen since $y = E_{P,0}(y).E_{P,1}(y)E_{P,2}(y)\ldots$ w.r.t. P is the P-Cantor series expansion of y. Therefore, at position n we have $E_{Q,n}(\pi_m(x)) \neq 1$ when $m \in \{2, 3, \ldots, M(n)\}$. Then for all $m \in \mathbb{N}_2$, the limit $\lim_{n\to\infty} N_n^Q(1, \pi_m(x))$ is finite. Therefore, $\pi_m(x)$ is not Q-normal of any order for all $m \in \mathbb{N}_2$.

Let $A \subseteq \mathbb{N}$ be the set of indices where $q_n \neq p_n$. Since

$$\frac{\sum_{m=1}^{M(N)} N_n^P(1, \pi_m(y))}{n} < \frac{1}{M(N)},$$

for sufficiently large n this set has density zero. Note that $\left(\frac{E_{Q,i}(x)}{q_i}\right)$ differs from $\left(\frac{E_{P,i}(x)}{p_i}\right)$ if and only if $n \in A$. Since $\left(\frac{E_{P,i}(y)}{p_i}\right)$ is uniformly distributed mod 1, the sequence $\left(\frac{E_{Q,i}(x)}{q_i}\right)$ is uniformly distributed mod 1 by Lemma 3.8. Since Q is infinite in limit, the number x is Q-distribution normal.

Fix some $j \in \{1, 2, \ldots, k\}$. Note that

$$\left|Q_n^{(j)} - P_n^{(j)}\right| \leq \sum_{i=1}^{n} \frac{\chi_{\mathbb{N}\setminus A}(i)}{p_i \cdots p_{i+j-1}}.$$

Using the notation of Theorem 3.10, set $a_n = \chi_A(n)$, $w_n = \frac{1}{p_n \cdots p_{n+j-1}}$, and $s_n = 1$. Since P is non-decreasing and k-divergent, we can apply Theorem 3.10. Thus, we have

$$\limsup_{n \to \infty} \frac{\sum_{i=1}^n \frac{\chi_A(i)}{p_i \cdots p_{i+j-1}}}{P_n^{(j)}} \leq \limsup_{n \to \infty} \frac{\# A \cap \{1, \ldots, n\}}{n} = 0,$$

so

$$\lim_{n \to \infty} \frac{Q_n^{(j)}}{P_n^{(j)}} = 1.$$

Since $N_n^P(B, y) = N_n^Q(B, x)$ for all blocks B, we have that x is Q-normal of orders $1, 2, \ldots, k$. $\qquad \square$

4. Further problems

Theorem 3.6 suggests some natural further investigations.

PROBLEM 4.1. Can the condition that Q is monotone be weakened or replaced by other conditions in Theorem 3.6?

We consider the following two conditions on a basic sequence Q and a real number x.

$$(4.1) \qquad x \in \mathscr{D}\mathscr{N}(Q) \cap \bigcap_{j=1}^k \mathscr{N}_j(Q) \quad \text{and} \quad nx \notin \bigcup_{j=1}^\infty \mathscr{N}_j(Q) \quad \forall n \in \mathbb{N}_2;$$

$$(4.2) \qquad x \in \mathscr{D}\mathscr{N}(Q) \cap \mathscr{N}(Q) \quad \text{and} \quad nx \notin \bigcup_{j=1}^\infty \mathscr{N}_j(Q) \quad \forall n \in \mathbb{N}_2.$$

PROBLEM 4.2. Is it true that for all Q that are k-divergent and infinite in limit that there exists a real number x satisfying (4.1)? If Q is fully divergent and infinite in limit must there exist an x that satisfies (4.2)? If not, what must we assume about Q?

We note that the use of Theorem 2.2 in the proof of Theorem 3.6 means we have not given any explicit examples of the basic sequence or real number x mentioned in Theorem 3.6. There exist some basic sequences Q where the set $\mathscr{D}\mathscr{N}(Q)$ does not contain any computable real numbers. See [6]. Thus, it is reasonable to ask the following question.

PROBLEM 4.3. Give an example of a computable basic sequence and a computable real number x that satisfies the conditions (4.1) or (4.2). Can this be done for every computable basic sequence Q?

The authors strongly believe that sets of numbers that satisfy conditions (4.1) or (4.2) must have full Hausdorff dimension. Thus, we ask the following.

PROBLEM 4.4. For k-divergent Q that are infinite in limit, compute
$$\dim_{\mathrm{H}}\big(\{x : \text{condition (4.1) holds}\}\big).$$
If Q is fully divergent and infinite in limit, compute
$$\dim_{\mathrm{H}}\big(\{x : \text{condition (4.2) holds}\}\big).$$

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