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A HALF-SPACE THEOREM FOR GRAPHS OF CONSTANT MEAN CURVATURE $0 < H < \frac{1}{2}$ IN $\mathbb{H}^2 \times \mathbb{R}$

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ABSTRACT. We study a half-space problem related to graphs in $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane, having constant mean curvature H defined over unbounded domains in $\mathbb{H}^2$.

1. Introduction

The half-space theorem by Hoffman and Meeks [10] states that if a properly immersed minimal surface S in $\mathbb{R}^3$ lies on one side of some plane P, then S is a plane parallel to P. As a consequence, they proved the strong half-space theorem which says that two properly immersed minimal surfaces in $\mathbb{R}^3$ that do not intersect must be parallel planes.

These theorems have been generalized to some other ambient simply connected homogeneous manifolds with dimension 3. For example, we have half-space theorems with respect to horospheres in $\mathbb{H}^3$ [19], vertical minimal planes in Nil_3 and Sol_3 [3], [4] and entire minimal graph in Nil_3 [4]. It is known that there is no half-space theorem for horizontal slices in $\mathbb{H}^2 \times \mathbb{R}$, since rotational minimal surfaces (catenoids) are contained in a slab [15], [16], but one has half-space theorems for constant mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [8].

In [12], the first author proved a general half-space theorem for constant mean curvature surfaces. Under certain hypothesis, he proved that in a Riemannian 3-manifold of bounded geometry, a constant mean curvature H surface on one side of a parabolic constant mean curvature H surface Σ is an equidistant surface to Σ.

In Euclidian spaces of dimension higher than 4, there is no strong half-space theorem, since there exist rotational proper minimal hypersurfaces contained in a slab.

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In [14], Menezes proves a half-space theorem for some complete vertical minimal graphs, more precisely, she looks at some particular graphs $\Sigma \subset M \times \mathbb{R}$ over an unbounded domain $D \subset M$, where M is a Hadamard surface with bounded curvature, these graphs are called ideal Scherk graphs and their existence was proved by Collin and Rosenberg in [2] for $\mathbb{H}^2$ and by Galvez and Rosenberg in [6] in the general case.

THEOREM 1 (Menezes [14]). *Let M denote a Hadamard surface with bounded curvature and let $\Sigma = \mathrm{Graph}(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If S is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ, then S is a vertical translate of Σ.*

In this paper, we are interested in the case where the graph has constant mean curvature. More precisely, we consider graphs over unbounded domains of $\mathbb{H}^2$ with constant mean curvature $0 < H < \frac{1}{2}$ (the domains are some "ideal polygons" with edges of constant curvatrure). In that case, we prove a result similar to the one of Menezes. We notice that the value $H = \frac{1}{2}$ is critical in this setting (see [13], [17] for the $H = \frac{1}{2}$ case).

The graphs that we will work with are graphs of functions u defined in an unbounded domain $D \subset \mathbb{H}^2$ whose boundary ∂D is composed of complete arcs $\{A_i\}$ and $\{B_j\}$ whose curvatures with respect to the domain are $\kappa(A_i) = 2H$ and $\kappa(B_j) = -2H$. These graphs will have constant mean curvature and u will assume the value $+\infty$ on each A_i and $-\infty$ on each B_j. These domains D will be called *Scherk type domains* and the functions u *Scherk type solutions.* The existence of these graphs is assured by A. Folha and S. Melo in [5] (for bounded domains see [9]). There, the authors give necessary and sufficient conditions on the geometry of the domain D to prove the existence of such a solution. In this context, we prove the following result.

THEOREM 2. *Let $D \subset \mathbb{H}^2$ be a Scherk type domain and u be a Scherk type solution over D (for some value $0 < H < \frac{1}{2}$). Denote by $\Sigma = \mathrm{Graph}(u)$. If S is a properly immersed CMC H surface contained in $D \times \mathbb{R}$ and above Σ, then S is a vertical translate of Σ.*

The original idea of Hoffman and Meeks is to use the 1-parameter family of catenoids as a priori barriers to control minimal surfaces on one side of a plane (here a priori means that the choice of catenoids is independent of the particular minimal surface you want to control). In more general situations, it is not easy to construct such a continuous family of barriers so some authors use a discrete family (see, for example, [4], [20]). Menezes works also with such a discrete family. In our case, it does not seem possible to construct such a family in an easy way. Our approach is based on the existence of only one barrier whose construction depends on the particular surface S.

This paper is organized as follows. In Section 2, we will give a brief presentation of the Scherk type graphs and the result of Folha and Melo. Section 3 contains the proof of Theorem 2, so one of the main step is the existence of the barriers which uses the Perron method. We also prove a uniqueness result for the constant mean curvature equation.

2. Constant mean curvature Scherk type graphs

In this section, we present the theorem by A. Folha and S. Melo in [5] that assures the existence of constant mean curvature graphs which take the boundary value $+\infty$ on certain arcs A_i and $-\infty$ on arcs B_j. All along this section H will be a real constant in $(0, \frac{1}{2})$.

First, let us fix some notations. Let $\mathbb{H}^2$ be the hyperbolic plane, and $\mathbb{H}^2 \times \mathbb{R}$ be endowed with the product metric. Let D be a simply connected domain in $\mathbb{H}^2$ and $u : D \longrightarrow \mathbb{R}$ a function. Denote by

$$\Sigma = \mathrm{Graph}(u) = \big\{ \big(x, u(x)\big), x \in D \big\}.$$

The upward unit normal to Σ is given by

$$N = \frac{1}{W}(\partial_t - \nabla u), \tag{1}$$

where

$$W = \sqrt{1 + |\nabla u|^2}. \tag{2}$$

The graph Σ has mean curvature H if u satisfies the equation

$$\mathcal{L}u := \operatorname{div} \frac{\nabla u}{W} - 2H = 0, \tag{3}$$

where the divergence and the gradient are taken with respect to the metric on $\mathbb{H}^2$. Let us now give some definitions.

Definition 1. The boundary of an unbounded domain D in $\mathbb{H}^2$ is a $2H$-*polygon* if its boundary is made of a finite number of complete arcs with constant curvature $2H$ and the cluster points of D in $\partial_\infty \mathbb{H}^2$ are the endpoints of these arcs. The arcs are called the edges of D and the cluster points are the vertices of D.

We notice that a complete curve in $\mathbb{H}^2$ with constant curvature $2H$ is proper.

If Ω is a domain whose boundary is a $2H$-polygon, we will denote by A_i (resp. B_i) the arcs of the boundary whose curvature is $2H$ (resp. $-2H$) with respect to the inward pointing unit normal.

Definition 2. We say that an unbounded domain D in $\mathbb{H}^2$ is a *Scherk type domain* if its boundary is a $2H$-polygon and if each vertex is the end point of one arc A_i and one arc B_j.

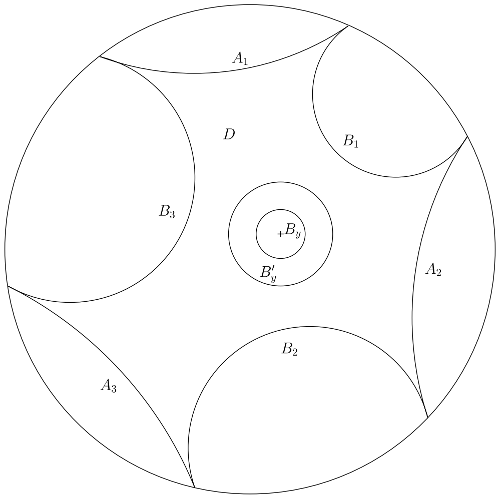

FIGURE 1. The Scherk type domain D and the balls B_y and B'_y.

Such a domain D is drawn in Figure 1.

DEFINITION 3. Let Ω be a Scherk type domain. We say that P is an *admissible inscribed polygon* if $P \subset \Omega$ is an unbounded domain whose boundary is a $2H$-polygon and its vertices are among the ones of Ω.

Let D be a Scherk type domain, in [5], Folha and Melo study the following Dirichlet problem

$$\begin{cases} \mathcal{L}(u) = 0, & \text{in } D, \\ u = +\infty, & \text{on } A_i, \\ u = -\infty, & \text{on } B_i. \end{cases} \tag{4}$$

In order to state the result of Folha and Melo, let us introduce some notations. Let P be an admissible inscribed polygon in D and let $\{d_i\}_{i\in I}$ denote the vertices of P. Consider the set

$$\Theta = \big\{(\mathcal{H}_i)_{i\in I} | \mathcal{H}_i \text{ is a horodisk at } d_i \text{ and } \mathcal{H}_i \cap \mathcal{H}_j = \emptyset \text{ if } i \neq j\big\}.$$

We notice that, by choosing sufficiently small horodisks, Θ is not empty.

Let $(\mathcal{H}_i)_{i\in I}$ be in Θ such that the following is true: each arc A_i and B_j meets exactly two of these horodisks. Denote by $\tilde{A}_i$ the compact arc of A_i which is the part of A_i outside these two horodisks. Let $|A_i|$ denote the length of $\tilde{A}_i$. We introduce the same notations for the B_j. For each arc $\eta_j \in \partial P$, we also define $\tilde{\eta}_j$ and $|\eta_j|$ in the same way.

We define

$$\alpha(\partial P) = \sum_{A_i \in \partial P} |A_i|, \qquad \beta(\partial P) = \sum_{B_i \in \partial P} |B_i| \quad \text{and} \quad \ell(\partial P) = \sum_j |\eta_j|,$$

where $\partial P = \bigcup_j \eta_j$. We remark that a Scherk type domain has finite area. So we can introduce $\mathcal{A}(D)$ the area of D and $\mathcal{A}(P)$ the area of P.

With these definitions we can state the main theorem of [5].

THEOREM 3. *Let D be a Scherk type domain. Then there exists a solution u for the Dirichlet problem* (4) *in D if and only if for some choice of the horodisks* (*in* Θ) *at the vertices,*

$$\alpha(\partial D) = \beta(\partial D) + 2H\mathcal{A}(D)$$

and for any admissible inscribed polygons $P \neq D$,

$$2\alpha(\partial P) < \ell(\partial P) + 2H\mathcal{A}(\Omega) \quad \textit{and} \quad 2\beta(\partial P) < \ell(\partial P) - 2H\mathcal{A}(\Omega).$$

It could seem that the conditions depend on the choice of the horodisks in Θ, actually they are independent of that choice if the horodisks are small enough. The details and the proof of this theorem can be found in [5].

3. The main result

In this section, we will prove the following result.

THEOREM 2. *Let $D \subset \mathbb{H}^2$ be a Scherk type domain and u be a Scherk type solution over D* (*for some value* $0 < H < \frac{1}{2}$). *Denote by* $\Sigma = \mathrm{Graph}(u)$. *If S is a properly immersed CMC H surface contained in $D \times \mathbb{R}$ and above Σ, then S is a vertical translate of Σ.*

The proof of the theorem consists in constructing barriers to control the surface S. Before starting the proof, let us give some notations and preliminary results that we will use.

So we fix a value of $H \in (0, \frac{1}{2})$, a Scherk type domain D and a Scherk type solution u. Let $y \in D$ and B_y and B'_y be open balls centered in y such that $B_y \subsetneq B'_y \subsetneq D$ (see Figure 1). The following result consists in constructing a first barrier to control S.

LEMMA 1. *There exists a constant $\varepsilon > 0$ such that for all $t \in [0, \varepsilon)$ there exists $v \in C^2(\overline{B'_y \setminus B_y})$ such that v solves* (3) *and $v = u$ on $\partial B'_y$ and $v = u + t$ on ∂B_y.*

Proof. Consider the operator $F : C^{2,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y)) \longrightarrow C^{0,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y))$ given by

$$F(v, \phi) = (\mathcal{L}v, v - \phi).$$

Observe that

$$F(u, u) = 0.$$

Moreover, consider the operator

$$T := D_1F(u,u) : C^{2,\alpha}\big(\overline{B'_y \setminus B_y}\big) \longrightarrow C^{0,\alpha}\big(\overline{B'_y \setminus B_y}\big) \times C^{2,\alpha}\big(\partial\big(B'_y \setminus B_y\big)\big),$$
$$h \longmapsto \lim_{t \longrightarrow 0} \frac{F(u+th,u) - F(u,u)}{t}.$$

We have that

$$T(h) = \left(\mathrm{Div}\left(\frac{\nabla h - \nabla u/W \langle \nabla u/W, \nabla h\rangle}{W}\right), h\right).$$

Observe that T is a linear operator, of the form $T = (T_1, T_2)$ where T_1 is an elliptic operator of the form

$$T_1(v) = a^{ij}(x)D_{ij}v + b^i(x)D_iv; \quad a^{ij} = a^{ji}.$$

Moreover, since $|\nabla u| \le C$, we have that $\frac{|\nabla u|}{W} \le C' < 1$, this implies that T_1 is uniformly elliptic. We also have that the coefficients of T_1 belong to $C^{0,\alpha}(\overline{B'_y \setminus B_y})$. It follows by Theorem 6.14 in [7] that if $g \in C^{0,\alpha}(\overline{B'_y \setminus B_y})$ and $\phi \in C^{2,\alpha}(\partial(B'_y \setminus B_y))$, then there exists a unique $w \in C^{2,\alpha}(\overline{B'_y \setminus B_y})$ such that $T_1(w) = g$ in $B'_y \setminus B_y$ and $w = \phi$ on $\partial(B'_y \setminus B_y)$.

We conclude that T is invertible. It follows by the implicit function theorem that for all ϕ close to u there exists a solution of $\mathcal{L}v = 0$ in $B'_y \setminus \overline{B_y}$ with $v = \phi$ in $\partial(B'_y \setminus B_y)$. In other words, it exists $\varepsilon > 0$ such that for all $t \in [0, \varepsilon)$ there exists v such that v solves (3) and $v = u$ over $\partial B'_y$ and $v = u + t$ over ∂B_y. $\square$

Let S be as in Theorem 2. Give $p \in D$, define $g(p)$ by

$$g(p) = \inf\big\{t \in \mathbb{R}; (p,t) \in S\big\} \in \mathbb{R} \cup \{+\infty\}.$$

Observe that g is a lower semicontinuous functions and $g \ge u$. From now on, we will assume that $g > u$ (the case where $g(p) = u(p)$ for a point $p \in D$ will be considered in the proof of the theorem). Then for $\varepsilon' > 0$ sufficiently small, we have that

$$g > u + \varepsilon' \quad \text{on } \partial B_y. \tag{5}$$

Now, let ε be as in Lemma 1, fix $\varepsilon' < \varepsilon$ where ε' satisfies (5) and v given by Lemma 1 associated to ε'. We will construct a second barrier to control the surface S. More precisely, we will prove the existence of a function $\beta \le g$ that satisfies

$$\mathcal{L}\beta = 0 \quad \text{in } D \setminus \overline{B_y}, \tag{6}$$
$$\beta = u + \varepsilon' \quad \text{in } \partial B_y. \tag{7}$$

PROPOSITION 1. *There is a solution $\beta \in C^2(D \setminus B_y)$ for the Dirichlet problem (6)–(7) such that* $\max(u,v) \le \beta \le \min(u+\varepsilon', g)$ *(v is defined just above).*

Proof. To prove this proposition we will use the Perron method. Let us recall the framework of this method (see, for example, Theorem 6.11 in [7]). A function $w \in C^0(D \setminus B_y)$ is called a *subsolution* for $\mathcal{L}$ if, for any compact subdomain $U \subset D \setminus B_y$ and any solution h of (3) with $w \leq h$ on the boundary ∂U, we have $w \leq h$ on U.

First, observe that u is a subsolution for (3). Moreover, if w' and w are subsolutions, the continuous function $\max(w', w)$ is also a subsolution.

Let $\Delta \subset D \setminus B_y$ be a geodesic disk of small radius such that $\kappa(\partial\Delta) \geq 2H$. Theorem 3.2 in [9] implies that the Dirichlet problem for Equation (3) can be solved in Δ. So, for any such disk Δ and subsolution w, we can define a continuous function $M_\Delta(w)$ as

$$M_\Delta(w)(x) = \begin{cases} w(x), & \text{if } x \in D \setminus \Delta, \\ \nu(x), & \text{if } x \in \Delta, \end{cases}$$

where ν is the solution of $\mathcal{L}\nu = 0$ in Δ, with $\nu = w$ in $\partial\Delta$.

Also, define $u_+ = \min(u + \varepsilon', g)$ and $u_- = \max(u, v)$. Denote by Γ the set of all subsolutions w such that $w \leq u_+$ on $D \setminus B_y$,

Claim 1. *If $w \in \Gamma$ and $\Delta \in D \setminus B_y$ is a geodesic disk, then $M_\Delta(w) \in \Gamma$.*

Proof. First, we have to prove that $M_\Delta(w)$ is a subsolution. So, take a arbitrary compact subdomain $U \subset D \setminus B_y$, and let h be a solution of (3) in U with $M_\Delta(w) \leq h$ on ∂U. Since $w = M_\Delta(w)$ in $U \setminus \Delta$, we have that $M_\Delta(w) \leq h$ in $U \setminus \Delta$.

Moreover, $M_\Delta(w)$ is a solution of (3) in Δ. Then, by the maximum principle, we have that $M_\Delta(w) \leq h$ in $U \cap \Delta$. So, $M_\Delta(w) \leq h$ in U. Since U is arbitrary, it follows that $M_\Delta(w)$ is a subsolution in $D \setminus B_y$.

Now we have to prove that $M_\Delta(w) \leq u_+$. Observe that in $(D \setminus B_y) \setminus \Delta$, $M_\Delta(w) = w$, since $w \in \Gamma$, then $w \leq u_+$, and so, $M_\Delta(w) \leq u_+$ in $D \setminus \Delta$.

On the other hand, $M_\Delta(w) = \nu$ in Δ, where ν is a solution of $\mathcal{L}(\nu) = 0$ in Δ and $\nu = w$ on $\partial\Delta$. Thus $\nu \leq u_+ = \min(u + \varepsilon', g)$ on $\partial\Delta$. It follows by the maximum principle that $\nu \leq u + \varepsilon'$ in Δ. So, we have to prove that $\nu \leq g$ in Δ.

Suppose that there exists $q \in \Delta$ such that $(\nu - g)(q) > 0$. Then, there exists $p \in \Delta$ such that $(\nu - g)(p) = \max(\nu - g) = C > 0$.

Now, observe that the graph of g in Δ is a piece of the surface S, let us denote it by S_g. Since $g \geq \nu - C$, then the graph $\Sigma_{\nu - C}$ of $\nu - C$ is a CMC $2H$ surface which is bellow the surface S_g. Moreover, $(p, g(p))$ is a point of contact of $\Sigma_{\nu-C}$ and S_g, and by the maximum principle, $S_g = \Sigma_{\nu-C}$. It follows that $g = \nu - C$ in Δ, since $\nu \leq g$ on $\partial\Delta$ then $C \leq 0$, and this contradicts $C > 0$. Then $\nu \leq g$ in Δ. ☐

For $q \in D \setminus B_y$, we define our solution by the following formula

$$\beta(q) = \sup_{w \in \Gamma} w(q).$$

Observe that u_- is a subsolution, since u and v are subsolutions. Also, $u \leq u_+ = \min(u+\varepsilon', g)$. Moreover, in the proof of Claim 1 we see that $v \leq u_+$. Then $u_- = \max(u,v) \leq u_+$. We conclude that $u_- \in \Gamma$, then Γ is non empty, and u_+ is an upper bound for any w in Γ, thus β is well defined. Besides $\beta = u_+ = u_- = u+\varepsilon'$ on ∂B_y.

The method of Perron claims that β is a solution of Equation (3).

Claim 2. *The function β is a solution of (3) in $D \setminus \overline{B_y}$.*

Proof. Let $p \in D \setminus B_y$ and $\Delta \subset D \setminus B_y$ be a geodesic disk of small radius centered at p as above. By definition of β there exists a sequence of subsolutions (w_n) such that $w_n(p) \longrightarrow \beta(p)$. Then, consider the sequence of subsolutions $M_\Delta(w_n)$, we have that $M_\Delta(w_n)(p) \longrightarrow \beta(p)$. Also, we have that $M_\Delta(w_n)$ is a bounded sequence of solutions of (3) in Δ, so, by considering a subsequence if necessary, we can assume that it converges to a solution $\overline{w}$ on Δ with $\beta \geq \overline{w}$ and $\overline{w}(p) = \beta(p)$. Let us prove that $\beta = \overline{w}$ on Δ, then β will be a solution of (3).

We have that $\beta \geq \overline{w}$. Suppose that there is a point $q \in \Delta$ where $\beta(q) > \overline{w}(q)$. So, there is a subsolution s such that $s(q) > \overline{w}(q)$. Now consider the sequence of subsolutions $M_\Delta(\max(s, w_n))$. We have that $M_\Delta(\max(s, w_n))$ is a sequence of solutions of (3) in Δ. Thus, considering a subsequence, it converges to a solution $\overline{s} \geq \overline{w}$ of (3) in Δ.

So, we have $\overline{w}$ and $\overline{s}$ solutions of (3) in Δ, with $\overline{w}(p) = \beta(p) = \overline{s}(p)$, thus by the maximum principle we have that $\overline{w} = \overline{s}$ in Δ.

But, since $M_\Delta(\max(s, w_n)) \geq s$, we have that $\overline{s} \geq s$. This implies that $\overline{s}(q) \geq s(q) > \overline{w}(q)$, which contradicts $\overline{w} = \overline{s}$ in Δ. □

Until now, we know the function β is in $C^2(D \setminus \overline{B_y})$ and is a solution of (3) in $D \setminus \overline{B_y}$ such that $u \leq \beta \leq \min(u+\varepsilon', g)$. But we don't have any information about the regularity of β on the boundary ∂B_y. So, the next step is prove that β is continuous up to ∂B_y.

Claim 3. *The function β is continuous up to the boundary ∂B_y. It takes the value $u+\varepsilon'$ on ∂B_y.*

Proof. We have that $\beta(q) = \sup_{w\in\Gamma} w(q)$, then, $\beta(q) \leq u_+(q) \leq u+\varepsilon'$. On the other hand, $u_-(q) = \max(u,v) \in \Gamma$, then $u_-(q) \leq \beta(q)$. Moreover, in ∂B_y we have that $u_-(q) = u+\varepsilon'$. Thus, let $p \in \partial B_y$, and $\{x_n\} \in D \setminus B_y$ a sequence such that $x_n \longrightarrow p$. We have

$$u_-(x_n) \leq \beta(x_n) \leq u_+(x_n),$$

since

$$\lim_{x_n \longrightarrow p} u_-(x_n) = \lim_{x_n \longrightarrow p} u_+(x_n) = u(p) + \varepsilon',$$

we have that

$$\lim_{x_n \longrightarrow p} \beta(x_n) = u(p) + \varepsilon'.$$

Then β is continuous at $p \in D \setminus B_y$ and $\beta = u + \varepsilon'$ in ∂B_y. $\square$

We have proved the existence of a function β defined on $D \setminus B_y$ such that $\max(u, v) \le \beta \le \min(u + \varepsilon', g)$ and $\beta \in C^2(D \setminus \overline{B_y}) \cap C^0(D \setminus B_y)$. The fact that β is C^2 up to the boundary will come from the following claim.

CLAIM 4. *$\nabla\beta$ is bounded in a neighborhood of ∂B_y.*

Proof. Theorem 1.1 in [21] says that there is a continuous function f of two variables such that, for any positive solution w of (3) in a geodesic disk of radius ρ centered at q, if $|w| \le M$ in the disk then $|\nabla w(q)| \le f(M, \frac{M}{\rho})$.

Take $q \in D \setminus \overline{B_y}$ with $d(q, B_y)$ small and consider Δ_q the disk centered at q and radius $d(q, B_y)$. On Δ_q, $v \le \beta \le u + \varepsilon'$ and the three functions coincide on ∂B_y. v and u have bounded gradient near ∂B_y, so $0 \le \sup_{\Delta_q} \beta - \inf_{\Delta_q} \beta \le C d(q, B_y)$ for some constant C that does not depend on q. Applying Theorem 1.1 in [21] to $\beta - \inf_{\Delta_q} \beta$ on Δ_q, we get $|\nabla\beta(q)| \le f(Cd(q, B_y), C)$, *that is*, $\nabla\beta$ is bounded near ∂B_y. $\square$

The above claim allows us to apply Theorem 4.6.3 in [11] to obtain that $\beta \in C^{2,\alpha}(D \setminus B_y)$. This concludes the proof of Proposition 1. $\square$

In the next result, we will prove a uniqueness result for CMC graphs in $\mathbb{H}^2 \times \mathbb{R}$ defined over unbounded domain in $\mathbb{H}^2$ whose existence was proved by A. Folha and S. Melo in [5].

PROPOSITION 2. *Let $\beta \in C^2(D \setminus B_y)$ be a solution of the Dirichlet problem (6)–(7) such that $u \le \beta \le u + \varepsilon'$. Then, $\beta = u + \varepsilon'$ on $D \setminus B_y$.*

Proof. Let us first analyze the boundary ∂D. As in Section 2, let $A_1, B_1, \ldots, A_k, B_k$ be the edges of the ∂D with $u(A_i) = +\infty = \beta(A_i)$ and $u(B_i) = -\infty = \beta(B_i)$.

For each i, let $\mathcal{H}_i(n)$ be a horodisk asymptotic to the vertex d_i of D such that $\mathcal{H}_i(n+1) \subset \mathcal{H}_i(n)$ and $\bigcap_n \mathcal{H}_i(n) = \emptyset$. For each side A_i, let us denote by A_i^n the compact subarc of A_i which is the part of A_i outside the two horodisks, and by $|A_i^n|$ the length of A_i^n. Analogously, we define B_i^n for each side B_i. Denote by C_i^n the compact arc of $\partial\mathcal{H}_i(n)$ contained in the domain D and let P^n be the subdomain of D bounded by the closed curve formed by the arcs A_i^n, B_i^n and C_i^n and let us denote $\Gamma^n = \partial(P^n \setminus \overline{B_y})$.

We have by the theorem of Stokes that

$$\begin{aligned} 0 &= \int_{P^n \setminus B_y} \mathrm{Div}\left(\frac{\nabla u}{W_u} - \frac{\nabla\beta}{W_\beta}\right) \\ &= \int_{\Gamma^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla\beta}{W_\beta}\right) \cdot \eta, \end{aligned}$$

where $W_u^2 = 1 + |\nabla u|^2$, $W_\beta^2 = 1 + |\nabla\beta|^2$ and η is the outward unit normal.

Thus

$$0 = \sum_i \int_{A_i^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta + \sum_i \int_{B_i^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta$$
$$+ \sum_i \int_{C_i^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta + \int_{\partial B_y} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta.$$

Using $\frac{|\nabla u|}{W_u} < 1$, $\frac{|\nabla \beta|}{W_\beta} < 1$ on C_i^n and Theorem 5.1 in [5], these integrals can be estimated. We have

$$0 < \sum_i \left(|A_i^n| - |A_i^n| + |B_i^n| - |B_i^n| + 2|C_i^n| \right) + \int_{\partial B_y} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta.$$

Then

$$\int_{\partial B_y} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta > -2 \sum_i |C_i^n|.$$

Since $|C_i^n| \longrightarrow 0$ when $n \longrightarrow \infty$, we get $\int_{\partial B_y} (\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta}) \cdot \eta \geq 0$. Now, we have that $\beta \leq u + \varepsilon'$, this implies that the normal derivative $\partial_\eta (u + \varepsilon' - \beta) \leq 0$ and, by Lemmas 1 and 2 in [1], $(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta}) \cdot \eta \leq 0$. As a consequence, the integral of this quantity must vanish and $\partial_\eta (u + \varepsilon' - \beta) = 0$ on ∂B_y. $u + \varepsilon - \beta$ is non-negative and solves a linear elliptic equation so the boundary maximum principle (Theorem 7 in [18]) implies $u + \varepsilon' - \beta = 0$. □

Now we are able to prove our main theorem.

Proof of Theorem 2. We know that S is a properly immersed CMC surface contained in $D \times \mathbb{R}$ above Σ. Then, let $y \in D$, $B_y \subset D$ and ε' as above. We have three cases to analyze

(1) Suppose that there exists $p \in D$ such that $g(p) = u(p)$ (is this the case we had let aside before Proposition 1). In this case, by the maximum principle, we conclude that $u = g$ and $S = \Sigma$.

(2) Suppose that $g > u$ and $\inf(g - u) = 0$. In this case, by Proposition 1 there exists β solution of (6)–(7) defined over $D \setminus B_y$ such that $u \leq \beta \leq g$. Moreover, Proposition 2 assures that $\beta = u + \varepsilon'$. This yields a contradiction, since we assume that $\inf(g - u) = 0$.

(3) Finally, suppose that $g > u$ and $\inf(g - u) = \alpha > 0$. Then, pushing up Σ by a vertical translations, that is, by considering $u + \alpha$ instead of u, we have now that $g \geq u + \alpha$ and $\inf(g - \alpha - u) = 0$, this case reduces to cases (1) and (2) and we conclude that $g = u + \alpha$, where α is a constant. □

References

[1] P. Collin and R. Krust, *Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non bornés*, Bull. Soc. Math. France **119** (1991), 443–462. MR 1136846

[2] P. Collin and H. Rosenberg, *Construction of harmonic diffeomorphisms and minimal graphs*, Ann. of Math. (2) **172** (2010), no. 3, 1879–1906. MR 2726102

[3] B. Daniel and L. Hauswirth, *Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 2, 445–470. MR 2481955

[4] B. Daniel, W. H. Meeks III and H. Rosenberg, *Halfspace theorems for minimal surfaces in Nil_3 and Sol_3*, J. Differential Geom. **88** (2011), no. 1, 41–59. MR 2819755

[5] A. Folha and S. Melo, *The Dirichlet problem for constant mean curvature graphs in $\mathbb{H}\times\mathbb{R}$ over unbounded domains*, Pacific J. Math. **251** (2011), no. 1, 37–65. MR 2794614

[6] J. Galvez and H. Rosenberg, *Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces*, Amer. J. Math. **132** (2010), no. 5, 1249–1273. MR 2732346

[7] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer, New York, 2001. MR 1814364

[8] L. Hauswirth, H. Rosenberg and J. Spruck, *On complete mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Comm. Anal. Geom. **16** (2008), no. 5, 989–1005. MR 2471365

[9] L. Hauswirth, H. Rosenberg and J. Spruck, *Infinite boundary value problems for constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*, Amer. J. Math. **131** (2009), no. 1, 195–226. MR 2488489

[10] D. Hoffman and W. H. Meeks III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990), no. 1, 373–377. MR 1062966

[11] O. Ladyzhenskaya and N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968. MR 0244627

[12] L. Mazet, *A general halfspace theorem for constant mean curvature surfaces*, Amer. J. Math. **135** (2013), no. 3, 801–834. MR 3068403

[13] L. Mazet, *The half space property for cmc $1/2$ graphs in $\mathbb{E}(-1,\tau)$*, Calc. Var. Partial Differential Equations **52** (2015), no. 3–4, 661–680. MR 3311909

[14] A. Menezes, *A half-space theorem for ideal Scherk graphs in $M \times \mathbb{R}$*, Michigan Math. J. **63** (2014), 675–685. MR 3286665

[15] B. Nelli and H. Rosenberg, *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 2, 263–292. MR 1940353

[16] B. Nelli and H. Rosenberg, *Errata. "Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$"* [Bull. Braz. Math. Soc. (N.S.) **33** (2002), no. 2, 263–292], Bull. Braz. Math. Soc. (N.S.) **38** (2007), no. 4, 661–664. MR 2371953

[17] B. Nelli and R. Sa Earp, *A halfspace theorem for mean curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, J. Math. Anal. Appl. **365** (2010), no. 1, 167–170. MR 2585087

[18] M. Protter and H. Weinberger, *Maximum principles in differential equations*, Springer, New York, 1984. MR 0762825

[19] L. Rodriguez and H. Rosenberg, *Half-space theorems for mean curvature one surfaces in hyperbolic space*, Proc. Amer. Math. Soc. **126** (1998), no. 9, 2755–2762. MR 1458259

[20] H. Rosenberg, F. Schulze and J. Spruck, *The half-space property and entire positive minimal graphs in $M \times \mathbb{R}$*, J. Differential Geom. **95** (2013), no. 2, 321–336. MR 3128986

[21] J. Spruck, *Interior gradient estimates and existence theorems for constant mean curvature graphs in $\mathbb{H}^n \times \mathbb{R}$*, Pure Appl. Math. Q. **3** (2007), no. 3, 785–800. MR 2351645

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