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P-MAPPING SPACES FOR P-OPERATOR SPACES ON L_p SPACES

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ABSTRACT. In this paper, we introduce p-mapping spaces for p-operator spaces on L_p spaces, which can be regarded as p-generalization of mapping spaces for operator spaces. We then apply p-mapping spaces to study the p-local reflexivity for p-operator spaces on L_p spaces.

1. Introduction

Throughout this writing, we always assume $1 < p < \infty$ unless stated otherwise. Given p, its conjugate exponent is denoted by p' so that $1/p + 1/p' = 1$. Some fundamental results (p-completely bounded maps, p-Haagerup and p-projective tensor products) for p-operator spaces have been studied by Pisier [13], Le Merdy [9], and Daws [2]. In [1], the p-injective tensor product was introduced for p-operator spaces, and various properties related to this tensor product were studied, including the p-approximation property for p-operator spaces on L_p spaces. The p-operator space tensor products are crucial in this paper.

In Section 2, we recall some basic notations and properties of p-operator spaces developed by Le Merdy [9] and Daws [2]. Obviously, certain operator space properties may fail for general p-operator spaces. For instance, there is non-existence of the corresponding Arveson–Wittstock–Hahn–Banach theorem for p-completely bounded maps (see Lee [12]). The p-Haagerup tensor product for p-operator spaces is not injective anymore (see Le Merdy [9]).

The theory of mapping spaces for operator spaces arose from [5], [6], [7], [4] and [8]. The most successful application of mapping spaces in operator spaces is to show that the dual of every C^*-algebra is locally reflexive

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in [4]. We first in Section 3 introduce a p-complete isometry $T_n(V) \cong T_n \overset{\wedge_p}{\otimes} V$. The analysis of p-completely 1-summing mappings rests upon a careful study of $T_n(V)$. Here we respectively explore the p-completely nuclear mappings in Section 3, the p-completely integral mappings in Section 4 and the p-completely 1-summing and ∞-summing mappings in Section 5, for p-operator spaces on L_p spaces. In Section 6, we then apply these p-mapping spaces to the study of the p-local reflexivity for p-operator spaces on L_p spaces. We prove in Theorem 6.2 the equivalence with the isometric conditions. However, due to the lack of the corresponding Arveson–Wittstock–Hahn–Banach theorem for p-completely bounded maps, it is not clear whether this is true for the p-completely isometric conditions. Finally, we end the section by an observation on p-completely 1-summing and ∞-summing mappings in the condition of the p-local reflexivity.

2. P-operator spaces

Let $1 < p < \infty$. A p-operator space is a Banach space V together with a matrix norm, that is, a norm $\|\cdot\|_n$ on each matrix space $M_n(V)$, which satisfies the following two conditions $\mathcal{D}_\infty$: $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$ for $x \in M_n(V)$ and $y \in M_m(V)$, $\mathcal{M}_p$: $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$ for $x \in M_n(V)$ and $\alpha, \beta \in M_n = B(l_p^n)$.

When V is a p-operator subspace of some $B(L_p(\mu))$, then we say that V is a p-operator space on L_p space. Unlike operator spaces, there exists a p-operator space V such that the inclusion $\kappa_V : V \to V^{**}$ is not p-completely isometric (see Daws [2]). By Proposition 4.9 in [2], however, κ_V is a p-complete isometry if and only if V is a p-operator space on L_p space.

In [2], Daws defined and studied the p-projective tensor product. The p-projective tensor product preserves most of properties of operator space projective tensor product. For instance, the tensor product of p-complete contractions (respectively, p-complete quotients) is again a p-complete contraction (respectively, a p-complete quotient). The p-projective tensor product is associative, that is, $(V \overset{\wedge_p}{\otimes} W) \overset{\wedge_p}{\otimes} Z = V \overset{\wedge_p}{\otimes} (W \overset{\wedge_p}{\otimes} Z)$, and commutative, that is, $V \overset{\wedge_p}{\otimes} W = W \overset{\wedge_p}{\otimes} V$. We also have the p-completely isometric identifications

$$\mathrm{CB}_p(X \overset{\wedge_p}{\otimes} Y, Z) = \mathrm{CB}_p(X \times Y, Z) = \mathrm{CB}_p\big(X, \mathrm{CB}_p(Y, Z)\big).$$

In particular,

$$(X \overset{\wedge_p}{\otimes} Y)^* = \mathrm{CB}_p\big(X, Y^*\big).$$

In [1], the authors introduced the p-injective tensor product. The tensor product of p-complete contractions under the p-injective tensor product is again a p-complete contraction. In particular, if V and W are p-operator spaces, the bilinear mapping

$$V \times W \to V \otimes_{\vee_p} W : (v, w) \mapsto v \otimes w$$

is p-completely contractive, and thus determines a p-complete contraction

$$\Phi : V \overset{\wedge_p}{\otimes} W \to V \overset{\vee_p}{\otimes} W.$$

Let V, W be p-operator spaces on L_p spaces. It was known from [1] that for each $u \in M_n(V \otimes W)$, the p-injective tensor norm $\|u\|_{\vee_p}$ can be expressed by

$$\|u\|_{\vee_p} = \sup\big\{\big\|(\varphi \otimes \psi)_n(u)\big\| : \varphi \in M_m\big(V^*\big)_1, \psi \in M_k\big(W^*\big)_1, m, k \in \mathbb{N}\big\}.$$

If $V \subseteq B(L_p(\mu))$, then we have a p-completely isometric isomorphism

$$M_n(V) = M_n \overset{\vee_p}{\otimes} V.$$

Let V, W be p-operator spaces on L_p spaces, then the canonical inclusion

$$V^* \overset{\vee_p}{\otimes} W \hookrightarrow \mathrm{CB}_p(V, W)$$

is a p-completely isometric injection. We do not know whether the p-injective tensor product is injective. But if all p-operator spaces under consideration are on L_p spaces, then the p-injective tensor product is injective (see [11]).

THEOREM 2.1. *Suppose that V, W, and X are p-operator spaces. Then the natural mappings*

$$V \otimes_{\wedge_p} (W \otimes_{\vee_p} X) \to (V \otimes_{\wedge_p} W) \otimes_{\vee_p} X$$

are p-completely contractive.

Proof. we let $Z = W \otimes_{\vee_p} X$. Given $u \in M_n(V \otimes Z)$ and $\varepsilon > 0$, we may assume that

$$u = \alpha(v \otimes z)\beta = \Bigg[\sum_{i,j,k,l} \alpha_{g,(i,k)}(v_{ij} \otimes z_{kl})\beta_{(j,l),h}\Bigg],$$

where $v \in M_r(Z)$, $z \in M_q(Z)$, $\alpha \in M_{n,r\times q}$, and $\beta \in M_{r\times q,n}$ satisfy

$$\|\alpha\|\|v\|\|z\|_{\vee_p}\|\beta\| < \|u\|_{\wedge_p} + \varepsilon.$$

We let $z = [z_{kl}]$, where

$$z_{kl} = \sum_t w_{kl}^{(t)} \otimes x_{kl}^{(t)},$$

with $w_{kl}^{(t)} \in W$ and $x_{kl}^{(t)} \in X$. Then we have

$$u = \Bigg[\sum_{i,j,k,l,t} \alpha_{g,(i,k)}\big(\big(v_{ij} \otimes w_{kl}^{(t)}\big) \otimes x_{kl}^{(t)}\big)\beta_{(j,l),h}\Bigg].$$

If $\|v\|$ =0, it is easy see that

$$\|u\|_{V\otimes_{\wedge_p}(W\otimes_{\vee_p}X)} = \|u\|_{(V\otimes_{\wedge_p}W)\otimes_{\vee_p}X}.$$

So here, we can assume $\|v\| > 0$.

From the definition of p-operator space injective tensor product norm in [11],

$$\begin{aligned}
&\|u\|_{(V\otimes_{\wedge_p}W)\otimes_{\vee_p}X}\\
&\quad=\sup\Big\{\Big\|\Big[\sum_{i,j,k,l,t}\alpha_{g,(i,k)}e_{st}\big(v_{ij}\otimes w_{kl}^{(t)}\big)x_{kl}^{(t)}\beta_{(j,l),h}\Big]\Big\|_{M_{mn}(X)}:\\
&\qquad m\in\mathbb{N}, e=[e_{st}]\in M_m\big((V\otimes_{\wedge_p}W)^*\big)_1\Big\},
\end{aligned}$$

where $M_m((V\otimes_{\wedge_p}W)^*)_1$ denotes the closed unit ball of

$$M_m\big((V\otimes_{\wedge_p}W)^*\big)=\mathrm{CB}_p\big((V\otimes_{\wedge_p}W),M_m\big).$$

If we fix such element e, e determines a p-complete contraction

$$E\in\mathrm{CB}_p\big(V,\mathrm{CB}_p(W,M_m)\big),$$

where

$$E(v_0)(w_0)=e(v_0\otimes w_0)$$

for any $v_0\in V$ and $w_0\in W$. Thus, if $f_{ij}=E(v_{ij})/\|v\|$, then

$$f=[f_{ij}]\in M_r\big(\mathrm{CB}_p(W,M_m)\big)=\mathrm{CB}_p(W,M_{r\times m})$$

satisfies

$$\|f\|_{pcb}\le 1.$$

So we have

$$\begin{aligned}
&\Big\|\Big[\sum_{i,j,k,l,t}\alpha_{g,(i,k)}e_{st}\big(v_{ij}\otimes w_{kl}^{(t)}\big)x_{kl}^{(t)}\beta_{(j,l),h}\Big]\Big\|_{M_{mn}(X)}\\
&\quad=\Big\|\Big[\sum_{i,j,k,l,t}\alpha_{g,(i,k)}E(v_{ij})\big(w_{kl}^{(t)}\big)x_{kl}^{(t)}\beta_{(j,l),h}\Big]\Big\|\\
&\quad=\Big\|\Big[\sum_{i,j,k,l}\alpha_{g,(i,k)}\Big(\sum_t f_{ij}\big(w_{kl}^{(t)}\big)x_{kl}^{(t)}\Big)\beta_{(j,l),h}\Big]\Big\|\|v\|\\
&\quad\le\|\alpha\|\|z\|_{\vee_p}\|\beta\|\|v\|\\
&\quad<\|u\|_{\wedge_p}+\varepsilon.
\end{aligned}$$

It follows that

$$\|u\|_{(V\otimes_{\wedge_p}W)\otimes_{\vee_p}X}\le\|u\|_{V\otimes_{\wedge_p}(W\otimes_{\vee_p}X)}.$$

Thus we obtain the desired inequality. □

THEOREM 2.2. *Let V, W, and X be p-operator spaces on L_p spaces. Then we have the p-completely isometric isomorphisms*

$$V\overset{\vee_p}{\otimes}W\cong W\overset{\vee_p}{\otimes}V$$

and

$$(V \overset{\vee_p}{\otimes} W) \overset{\vee_p}{\otimes} X \cong V \overset{\vee_p}{\otimes} (W \overset{\vee_p}{\otimes} X).$$

Proof. Given any index set I, J, and K, we have the natural isometries

$$\big(l_p(I) \otimes_p l_p(J)\big) \otimes_p l_p(K) \cong l_p(I) \otimes_p \big(l_p(J) \otimes_p l_p(K)\big)$$

and

$$l_p(I) \otimes_p l_p(J) \cong l_p(J) \otimes_p l_p(I).$$

Thus, the results follow from Proposition 3.3 in [1]. □

THEOREM 2.3. *Let V, W be p-operator spaces on L_p spaces with V or W finite-dimensional. Then we have the p-complete isometry*

$$V^* \overset{\vee_p}{\otimes} W \cong \mathrm{CB}_p(V, W).$$

Proof. We have the p-completely isometric inclusion

$$V^* \overset{\vee_p}{\otimes} W \hookrightarrow \mathrm{CB}_p(V, W).$$

Hence to prove the identification, it suffices to show

$$\varphi : V^* \overset{\vee_p}{\otimes} W \hookrightarrow \mathrm{CB}_p(V, W)$$

is surjective. Since V or W is finite-dimensional, we have the identification $V^* \otimes W \cong \mathrm{FCB}_p(V, W)$. Thus, we obtain that φ is surjective. □

3. P-completely nuclear mappings

DEFINITION 3.1. Let V, W, U, X be p-operator spaces on L_p spaces. A p-operator space mapping ideal $\mathcal{O}$ is an assignment to each pair of p-operator spaces V, W of a linear space $\mathcal{O}$ of p-completely bounded mappings $\varphi : V \to W$, together with a p-operator space matrix norm $\|\cdot\|_{\mathcal{O}}$, such that for each $\varphi \in M_n(\mathcal{O})$,

(a) $\|\varphi\|_{pcb} \le \|\varphi\|_{\mathcal{O}}$ and
(b) for any linear mappings $r : U \to V$ and $s : W \to X$,

$$\|s_n \circ \varphi \circ r\|_{\mathcal{O}} \le \|s\|_{pcb} \|\varphi\|_{\mathcal{O}} \|r\|_{pcb}.$$

We say the p-operator space mapping ideal $\mathcal{O}$ is local if for each linear mapping $\varphi : V \to W$,

$$\|\varphi\|_{\mathcal{O}} = \sup\big\{\|\varphi|_L\|_{\mathcal{O}} : \text{ for any } \textit{finite-dimensional} \text{ subspace } L \subseteq V\big\}.$$

DEFINITION 3.2. Let V, W be p-operator spaces on L_p spaces. Guided by operator spaces, we define the p-completely nuclear mappings $\mathcal{N}_p(V, W)$ to be the image of the mapping

$$\Phi : V^* \overset{\wedge_p}{\otimes} W \to V^* \overset{\vee_p}{\otimes} W \subseteq \mathrm{CB}_p(V, W)$$

with the quotient p-operator space structure determined by the identification

$$\mathcal{N}_p(V,W) \cong \frac{V^* \overset{\wedge_p}{\otimes} W}{\ker \Phi}.$$

Let ν_n^p be the matrix norm on $M_n(\mathcal{N}_p(V,W))$.

For exploring the identifications in the p-completely nuclear mappings, we define the following spaces with a norm similar to $\|\cdot\|_1$ in operator space theory, which has been introduced by Lee [10].

DEFINITION 3.3. For a p-operator space V, let $T_n(V)$ denote a Banach space

$$\big(M_n(V), \|\cdot\|_{1,n}\big),$$

where $\|\cdot\|_{1,n}$ is defined by

$$\|v\|_{1,n} = \inf\big\{\|\alpha\|_{p'}\|w\|\|\beta\|_p : r \in \mathbb{N}, v = \alpha w \beta, \\ \alpha \in M_{n,r}, \beta \in M_{r,n}, w \in M_r(V)\big\},$$

where $\|\alpha\|_{p'} = (\sum_{i=1}^n \sum_{j=1}^r |\alpha_{ij}|^{p'})^{1/p'}$ and $\|\beta\|_p = (\sum_{k=1}^r \sum_{l=1}^n |\beta_{kl}|^p)^{1/p}$.

For a p-operator space V, $T_n(V)^* \cong M_n(V^*) \cong \mathrm{CB}_p(V, M_n)$ are isometric isomorphisms ([11], Lemma 3.4). Also, these identifications are p-completely isometric isomorphisms. Let nuclear operators $\mathcal{N}(l_p^n)$ to be the image of the mapping

$$\Phi : \big(l_p^n\big)^* \overset{\wedge_p}{\otimes} l_p^n \to \big(l_p^n\big)^* \overset{\vee_p}{\otimes} l_p^n \subseteq B\big(l_p^n\big)$$

with the quotient norm coming from $\mathcal{N}(l_p^n) \cong \frac{(l_p^n)^* \overset{\wedge_p}{\otimes} l_p^n}{\ker \Phi}$. If we use

$$T_n \cong \mathcal{N}\big(l_p^n\big), \qquad M_n(\mathbb{C}) \cong B\big(l_p^n\big),$$

then by Proposition 2.2 in [1], we have

$$T_n \cong M_n^*, \qquad M_n \cong T_n^*,$$

and

$$T_\infty \cong K_\infty^*, \qquad M_\infty \cong T_\infty^*.$$

Let V be a p-operator space on L_p space. By Theorem 3.6 in [1], we have the isometric isomorphism

$$(M_n \overset{\vee_p}{\otimes} V)^* \cong T_n \overset{\wedge_p}{\otimes} V^*.$$

LEMMA 3.4. *Let V, W be p-operator spaces on L_p spaces. Given linear mappings $\varphi_n : M_n(V) \to M_n(W)$ and $T_n(\varphi) : T_n(V) \to T_n(W)$ for each $n \in \mathbb{N}$. If $T_n(\varphi)$ is an isometric injection for each $n \in \mathbb{N}$, then so is φ_n.*

Proof. We may prove that if φ_n is a quotient mapping for each $n \in \mathbb{N}$, then so is $T_n(\varphi)$. Let us suppose that φ_n is a quotient mapping for each $n \in \mathbb{N}$. For any $w \in T_n(W)$ with $\|w\|_{1,n} < 1$, we may assume that $w = \alpha \widetilde{w} \beta$, where $\widetilde{w} \in M_r(W)$, $\alpha \in M_{n,r}$ and $\beta \in M_{r,n}$ satisfy $\|\widetilde{w}\|, \|\alpha\|_{p'}, \|\beta\|_p < 1$. By hypothesis, we may choose an element $\widetilde{v} \in M_r(V)$ with $\|\widetilde{v}\| < 1$, for which $\varphi_r(\widetilde{v}) = \widetilde{w}$. If we let $v = \alpha \widetilde{v} \beta$, then it follows that $\|v\|_{1,n} < 1$ and $T_n(\varphi)(v) = w$. So $T_n(\varphi)$ is a quotient mapping for each $n \in \mathbb{N}$.

We have the isometric isomorphisms

$$M_n(V)^* \cong (M_n \overset{\vee_p}{\otimes} V)^* \cong T_n \overset{\wedge_p}{\otimes} V^* \cong T_n\big(V^*\big).$$

We can note that $T_n(\varphi)^* = (\varphi^*)_n$ and $(\varphi_n)^* = T_n(\varphi^*)$. Thus from (A.2.1) in [8], $T_n(\varphi)$ is an isometric injection for each $n \in \mathbb{N} \Rightarrow (\varphi^*)_n$ is a quotient mapping for each $n \in \mathbb{N} \Rightarrow T_n(\varphi^*)$ is a quotient mapping for each $n \in \mathbb{N} \Rightarrow \varphi_n$ is an isometric injection for each $n \in \mathbb{N}$. □

For any p-operator space V, we have the p-complete isometries

$$(T_n \overset{\wedge_p}{\otimes} V)^* \cong \mathrm{CB}_p(V, M_n) \cong M_n\big(V^*\big) \cong \big(T_n(V)\big)^*.$$

Then, we obtain a natural isometry $T_n(V) \cong T_n \overset{\wedge_p}{\otimes} V$.

COROLLARY 3.5. *Let V be a p-operator space. The natural isometry $T_n(V) \cong T_n \overset{\wedge_p}{\otimes} V$ is a p-completely isometric isomorphism.*

Proof. We have the p-complete isometries

$$(T_n \overset{\wedge_p}{\otimes} V)^* \cong \mathrm{CB}_p(V, M_n) \cong M_n\big(V^*\big) \cong \big(T_n(V)\big)^*.$$

Then for each $r \in \mathbb{N}$, we have the isometries

$$\big(T_r(T_n \overset{\wedge_p}{\otimes} V)\big)^* \cong M_r\big((T_n \overset{\wedge_p}{\otimes} V)^*\big) \cong M_r\big(\big(T_n(V)\big)^*\big) \cong \big(T_r\big(T_n(V)\big)\big)^*,$$

and thus $T_r(T_n \overset{\wedge_p}{\otimes} V) \cong T_r(T_n(V))$. From Lemma 3.4, for each $r \in \mathbb{N}$ we have the isometry $M_r(T_n \overset{\wedge_p}{\otimes} V) \cong M_r(T_n(V))$. Then, $T_n(V) \cong T_n \overset{\wedge_p}{\otimes} V$ is a p-completely isometric isomorphism. □

THEOREM 3.6. *Let V be a p-operator space on L_p space. We have the following p-completely isometric identifications*

$$M_n(V)^{**} \cong M_n\big(V^{**}\big)$$

and

$$M_n(V)^* \cong T_n\big(V^*\big).$$

Proof. We have the isometric isomorphisms

$$M_n(V)^* \cong (M_n \overset{\vee_p}{\otimes} V)^* \cong T_n \overset{\wedge_p}{\otimes} V^* \cong T_n\big(V^*\big).$$

It is easy to see that $M_n(V)^{**} \cong M_n(V^{**})$ is a p-completely isometric isomorphism. Then we just need to show that

$$M_r\big(T_n\big(V^*\big)\big) \to M_r\big(M_n(V)^*\big)$$

is isometric for each $r \in \mathbb{N}$. To see this, it suffices to show that the correspond mapping

$$T_r \overset{\wedge_p}{\otimes} T_n \overset{\wedge_p}{\otimes} V^* \to T_r \overset{\wedge_p}{\otimes} M_n(V)^*$$

is isometric for each $r \in \mathbb{N}$. This is apparent from the commutative diagram

$$\begin{array}{ccc} T_r \overset{\wedge_p}{\otimes} T_n \overset{\wedge_p}{\otimes} V^* & \longrightarrow & T_r \overset{\wedge_p}{\otimes} M_n(V)^* \\ \downarrow & & \downarrow \\ T_{r\times n} \overset{\wedge_p}{\otimes} V^* & \longrightarrow & (M_{r\times n}(V))^* \end{array}$$

since we can obtain that the bottom and vertical mappings are isometric, we have $M_n(V)^* \cong T_n(V^*)$ is a p-completely isometric isomorphism. $\square$

We can obtain the p-completely isometric identifications

$$T_n\big(\mathcal{N}_p(V,W)\big) \cong \mathcal{N}_p\big(V, T_n(W)\big) \cong \mathcal{N}_p\big(M_n(V), W\big)$$

which is evident from the diagram

$$\begin{array}{ccccc} T_n \overset{\wedge_p}{\otimes} (V^* \overset{\wedge_p}{\otimes} W) & = & V^* \overset{\wedge_p}{\otimes} T_n(W) & = & M_n(V)^* \overset{\wedge_p}{\otimes} W \\ \downarrow & & \downarrow & & \downarrow \\ T_n \overset{\wedge_p}{\otimes} \mathcal{N}_p(V,W) & & \mathcal{N}_p(V, T_n(W)) & & \mathcal{N}_p(M_n(V), W) \end{array}$$

in which the column mappings are p-complete quotient mappings, and their null spaces are the same.

THEOREM 3.7. *$\mathcal{N}_p$ is a p-operator space mapping ideal.*

Proof. Let us suppose that we are given $\varphi \in M_n(\mathcal{N}_p(V,W))$ and linear mappings $r : U \to V$ and $s : W \to X$. Since $\Phi : V^* \overset{\wedge_p}{\otimes} W \to V^* \overset{\vee_p}{\otimes} W$ is p-completely contractive, we have $\|\varphi\|_{pcb} \leq \nu^p(\varphi)$. If we choose

$$u \in M_n\big(V^* \overset{\wedge_p}{\otimes} W\big)$$

with $\varphi = \Phi_n(u)$, it follows that

$$s_n \circ \varphi \circ r = \Phi_n\big(u'\big),$$

where

$$u' = \big(r^* \otimes s\big)_n(u) \in M_n\big(U^* \overset{\wedge_p}{\otimes} X\big),$$

and thus

$$\nu_n^p(s_n \circ \varphi \circ r) \le \left\|u'\right\|_{\wedge_p} \le \|s\|_{pcb}\|u\|_{\wedge_p}\|r\|_{pcb}.$$

Taking the infimum over all u with $\varphi = \Phi_n(u)$, we have that

$$\nu_n^p(s_n \circ \varphi \circ r) \le \|s\|_{pcb}\nu_n^p(\varphi)\|r\|_{pcb}.$$

So we conclude that $\mathcal{N}_p$ is a p-operator space mapping ideal. $\square$

LEMMA 3.8. *Let V, W be p-operator spaces on L_p spaces. If $\varphi^* : W^* \to V^*$ is a p-complete quotient mapping, then $\varphi : V \to W$ is a p-complete isometry.*

Proof. By Lemma 4.6 in [2], the mapping $\varphi^{**} : V^{**} \to W^{**}$ is p-completely isometric. We have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & W \\ \Big\downarrow & & \Big\downarrow \\ V^{**} & \longrightarrow & W^{**} \end{array},$$

where the columns are p-completely isometric inclusions, and the bottom row is p-completely isometric. It follows that $\varphi : V \to W$ is p-completely isometric. $\square$

LEMMA 3.9. *Let V, W be p-operator spaces on L_p spaces. Then the usual inclusion mapping $\iota : V \hookrightarrow V^{**}$ induces the p-completely isometric injection $V \overset{\wedge_p}{\otimes} W \hookrightarrow V^{**} \overset{\wedge_p}{\otimes} W$.*

Proof. By Lemma 4.5 in [2], the mapping $\iota^* : V^{***} \to V^*$ is p-completely contractive.

For any $n \in \mathbb{N}$, the mapping $\varphi \to (\iota^*)_n \circ \varphi$ provides us with a quotient mapping in the top row of the diagram

$$\begin{array}{ccc} \mathrm{CB}_p(W, M_n(V^{***})) & \longrightarrow & \mathrm{CB}_p(W, M_n(V^*)) \\ \Big\| & & \Big\| \\ M_n((V^{**} \overset{\wedge_p}{\otimes} W)^*) & \longrightarrow & M_n((V \overset{\wedge_p}{\otimes} W)^*) \end{array},$$

since we are given a p-complete contraction $\psi \in \mathrm{CB}_p(W, M_n(V^*))$, then $(\iota_{V^*})_n \circ \psi$ is the p-completely contractive preimage. Thus, the bottom row is also a quotient mapping. It follows that $(V^{**} \overset{\wedge_p}{\otimes} W)^* \to (V \overset{\wedge_p}{\otimes} W)^*$ is a p-complete quotient mapping.

Owing to Lemma 3.8, we have that $V \overset{\wedge_p}{\otimes} W \hookrightarrow V^{**} \overset{\wedge_p}{\otimes} W$ is p-completely isometric. $\square$

PROPOSITION 3.10. *Let V, W be p-operator spaces on L_p spaces and $\varphi : V \to W$ is a p-completely bounded mapping, then $\varphi^* : W^* \to V^*$ satisfies*

$$\nu^p(\varphi^*) \le \nu^p(\varphi).$$

If V or W is finite-dimensional, then $\nu^p(\varphi^*) = \nu^p(\varphi)$.

Proof. The result follows from Lemma 3.9 and a commutative diagram

$$\begin{array}{ccc} V^* \overset{\wedge p}{\otimes} W & \longrightarrow & V^* \overset{\wedge p}{\otimes} W^{**} \\ \big\downarrow & & \big\downarrow \\ \mathcal{N}_p(V,W) & \longrightarrow & \mathcal{N}_p(W^*,V^*) \end{array}.$$

□

PROPOSITION 3.11. *Suppose that L is a finite-dimensional p-operator space on L_p space. Then for any p-operator space W on L_p space, the natural injection*

$$\mathcal{N}_p(L,W) \to \mathcal{N}_p\big(L,W^{**}\big)$$

is p-completely isometric.

Proof. The result follows from Lemma 3.9 and a commutative diagram

$$\begin{array}{ccc} L^* \overset{\wedge p}{\otimes} W & \longrightarrow & L^* \overset{\wedge p}{\otimes} W^{**} \\ \big\downarrow & & \big\downarrow \\ \mathcal{N}_p(L,W) & \longrightarrow & \mathcal{N}_p(L,W^{**}) \end{array}.$$

□

4. P-completely integral mappings

DEFINITION 4.1. Let V,W be p-operator spaces on L_p spaces. We define the mapping $\varphi : V \to W$ with a p-operator space matrix norm $\iota^p(\cdot)$ to be p-completely integral, which

$$\iota^p(\varphi) = \sup\big\{\nu^p(\varphi_{|L}) : \text{ for any } \textit{finite-dimensional} \text{ subspace } L \subseteq V\big\} < \infty.$$

And let $\mathcal{I}_p(V,W)$ denote the p-completely integral mapping spaces.

Given $\varphi \in M_n(\mathcal{I}_p(V,W))$, we define

$$\iota^p_n(\varphi) = \sup\big\{\nu^p_n(\varphi_{|L}) : \text{ for any } \textit{finite-dimensional} \text{ subspace } L \subseteq V\big\} < \infty.$$

Given a linear mapping $\varphi : V \to W$ and a finite-dimensional subspace L of V we have $\varphi|_L = \varphi \circ r$, where $r : L \to V$ is the inclusion mapping, and thus

$$\|\varphi|_L\|_{pcb} \le \nu^p(\varphi|_L) \le \nu^p(\varphi)\|r\|_{pcb} = \nu^p(\varphi).$$

From this, we infer that

$$\|\varphi\|_{pcb} \le \iota^p(\varphi) \le \nu^p(\varphi).$$

If V is finite-dimensional, then from the definition $\nu^p(\varphi) \le \iota^p(\varphi)$. So we have an isometric identification $\mathcal{I}_p(V,W) \cong \mathcal{N}_p(V,W)$.

THEOREM 4.2. *$\mathcal{I}_p$ is a local p-operator space mapping ideal.*

Proof. To see this, let us suppose that we are given $\varphi \in M_n(\mathcal{I}_p(V,W))$ and linear mappings $r : U \to V$ and $s : W \to X$. If K is a finite-dimensional subspace of U and we let $L = r(K)$, then

$$\nu_n^p(s_n \circ \varphi \circ r|_K) \leq \|s\|_{pcb} \nu_n^p(\varphi|_L) \|r\|_{pcb} \leq \|s\|_{pcb} \iota_n^p(\varphi) \|r\|_{pcb},$$

and thus

$$\iota_n^p(s_n \circ \varphi \circ r) \leq \|s\|_{pcb} \iota_n^p(\varphi) \|r\|_{pcb}.$$

Since $\|\varphi\|_{pcb} \leq \iota^p(\varphi)$, we have that $\mathcal{I}_p$ is a p-operator space mapping ideal. Then from Definition 4.1, we have that this p-mapping ideal is local. □

THEOREM 4.3. *Let V, W be p-operator spaces on L_p spaces. The natural mapping $\mathcal{I}_p(V,W) \to \mathcal{I}_p(V,W^{**})$ is p-completely isometric.*

Proof. Since $\mathcal{I}_p$ is a p-mapping ideal, this mapping is a p-complete contraction. On the other hand, letting $\lambda : W \to W^{**}$ be the canonical injection, let us suppose that $\iota_n^p(\lambda_n \circ \varphi) \leq 1$. Given a finite-dimensional subspace $L \subseteq V$, it follows from Proposition 3.11 that

$$\nu_n^p(\varphi|_L) = \nu_n^p(\lambda_n \circ \varphi|_L) \leq 1,$$

and thus $\iota_n^p(\varphi) \leq 1$. □

Let V, W be p-operator spaces on L_p spaces. We have a natural diagram of p-complete contractions

$$\begin{array}{ccccc} \mathcal{N}_p(V,W^*) & \subseteq \mathcal{I}_p(V,W^*) \subseteq & & & \mathrm{CB}_p(V,W^*) \\ \widehat{\Phi}\uparrow & & & & \downarrow S \\ V^* \overset{\wedge_p}{\otimes} W^* & \overset{\theta}{\to} & (V \overset{\vee_p}{\otimes} W)^* & \overset{\Phi^*}{\to} & (V \overset{\wedge_p}{\otimes} W)^* \end{array},$$

where $S : \mathrm{CB}_p(V,W^*) \cong (V \overset{\wedge_p}{\otimes} W)^*$ is a p-complete isometry determined by

$$S(\varphi) : V \otimes W \to \mathbb{C} : v \otimes w \mapsto \varphi(v)(w),$$

$\Phi : V \overset{\wedge_p}{\otimes} W \to V \overset{\vee_p}{\otimes} W$ and $\widehat{\Phi} : V^* \overset{\wedge_p}{\otimes} W^* \to \mathcal{N}_p(V,W^*)$ are the canonical mappings. The map θ is determined by the fact that the bilinear mapping

$$V^* \times W^* \to (V \overset{\wedge_p}{\otimes} W)^*$$

is p-completely contractive in the sense that

$$\|f \otimes g\| \leq \|f\| \|g\|$$

for $f \in M_r(V^*)$ and $g \in M_s(W^*)$. The diagram commutes since it is immediate that $S(\widehat{\Phi}(F)) = \Phi^*(\theta(F))$ for $F = f \otimes g$ $(f \in V^*,\ g \in W^*)$, and extending linearly and using continuity, we find that this relation holds for all $F \in V^* \overset{\wedge_p}{\otimes} W^*$.

LEMMA 4.4. *Let V, W be p-operator spaces on L_p spaces. There is a p-completely contractive mapping*

$$S_{\mathrm{int}} : \mathcal{I}_p\big(V, W^*\big) \to (V \overset{\vee_p}{\otimes} W)^*$$

for which the following diagram commutes

$$\begin{array}{ccccc} \mathcal{N}_p(V, W^*) & \subseteq & \mathcal{I}_p(V, W^*) & \subseteq & \mathrm{CB}_p(V, W^*) \\ \widehat{\Phi}\uparrow & & \downarrow S_{\mathrm{int}} & & \downarrow S \\ V^* \overset{\wedge_p}{\otimes} W^* & \xrightarrow{\theta} & (V \overset{\vee_p}{\otimes} W)^* & \xrightarrow{\Phi^*} & (V \overset{\wedge_p}{\otimes} W)^* \end{array}.$$

Proof. Our task is to show that for any $\phi \in M_n(\mathcal{I}_p(V, W^*))$,

$$S_p(\phi) : V \otimes_{\vee_p} W \to M_n$$

satisfies $\|S_n(\phi)\|_{pcb} \leq \iota_n^p(\phi)$. From this, it will follow that the restriction S_{int} of S is p-completely contractive.

Given $\phi \in M_n(\mathrm{CB}_p(V, W^*))$ with $\iota_n^p(\phi) \leq 1$, there is by definition a net $\psi_\alpha(v) \in M_n(W^*)$ converges to $\phi(v)$ in norm for all $v \in V$. It follows that the net of scalar matrices $(S_n(\psi_\alpha))_m(u)$ converges to $(S_n(\phi))_m(u)$ for any $u \in M_m(V \otimes W)$. Letting $\psi_\alpha = \widehat{\Phi}_n(F_\alpha)$ with $\|F_\alpha\|_{\wedge_p} \leq 1$, we have

$$\big\|\big(S_n(\psi_\alpha)\big)_m(u)\big\| = \big\|\big(\theta_n(F_\alpha)\big)_m(u)\big\| \leq \|F_\alpha\|_{\wedge_p}\|u\|_{\vee_p} < \|u\|_{\vee_p}.$$

Taking the limit, we see that $\|S_n(\phi)(u)\|_{pcb} \leq \|u\|_{\vee_p}$, and thus

$$\big\|S_n(\phi)\big\|_{pcb} \leq 1. \qquad \square$$

LEMMA 4.5. *Let V, W be p-operator spaces on L_p spaces. Then the composition*

$$S_0 : \mathcal{I}_p(V, W) \hookrightarrow \mathcal{I}_p\big(V, W^{**}\big) \to \big(V \overset{\vee_p}{\otimes} W^*\big)^*$$

is isometric.

Proof. By Theorem 4.3 and Lemma 4.4, we have the composition is contractive. Let us suppose that $\varphi \in \mathcal{I}_p(V, W)$ satisfies $\|S_0(\varphi)\| \leq 1$.

Since

$$W^* \overset{\vee_p}{\otimes} V \cong V \overset{\vee_p}{\otimes} W^* \hookrightarrow \mathrm{CB}_p\big(W, V^{**}\big) \cong \big(V^* \overset{\wedge p}{\otimes} W\big)^*$$

are p-completely isometric, we may identify $W^* \overset{\vee_p}{\otimes} V$ with a p-operator subspace of $(V^* \overset{\wedge p}{\otimes} W)^*$. It follows from the Hahn–Banach theorem that $S_0(\varphi)$ has a contractive extension

$$F_\varphi \in \big(V^* \overset{\wedge p}{\otimes} W\big)^{**}.$$

From the bipolar theorem, we may choose a net of elements

$$u_\lambda \in V^* \overset{\wedge p}{\otimes} W$$

such that

$$\|u_\lambda\|_{V^* \overset{\wedge p}{\otimes} W} < 1$$

and u_λ converges to F_φ in the point-norm topology on $(V^* \overset{\wedge p}{\otimes} W)^{**}$. It follows that

$$\varphi_\lambda = \Phi(u_\lambda) \in \mathcal{N}_p(V, W)$$

is a net with $\nu^p(\varphi_\lambda) < 1$, and for each $v \in V$ and $g \in W^*$,

$$\varphi_\lambda(v)(g) = u_\lambda(v \otimes g) = S_0(\varphi)(v \otimes g) = \varphi(v)(g).$$

Therefore, φ_λ converges to φ in the point-weak topology, and thus $\iota^p(\varphi) \leq 1$. We conclude that the composition is isometric. □

Theorem 4.6. *If L is a finite-dimensional p-operator space on L_p space, then for any p-operator spaces on L_p space V we have the isometry*

$$S_{\text{int}} : \mathcal{I}_p\big(V, L^*\big) \cong (V \overset{\vee p}{\otimes} L)^*.$$

Proof. It is immediate from Lemma 4.5. □

Given p-operator spaces V and W, Lee defined $V^{**} :\overset{\vee p}{\otimes}: W^{**}$, $V \overset{\vee p}{\otimes}: W^{**}$ and $V^{**} :\overset{\vee p}{\otimes} W$, which were called the p-augmented, p-right augmented and p-left augmented injective tensor products, respectively (see [11]).

Theorem 4.7. *For any p-operator spaces V and W on L_p spaces, the mapping*

$$S_{\text{int}} : \mathcal{I}_p\big(V, W^*\big) \to (V \overset{\vee p}{\otimes} W)^*$$

is an isometric surjection if and only if we have the natural isometric isomorphism

$$V \overset{\vee p}{\otimes}: W^{**} \cong V \overset{\vee p}{\otimes} W^{**}.$$

Proof. Let us suppose that we have $V \overset{\vee p}{\otimes}: W^{**} \cong V \overset{\vee p}{\otimes} W^{**}$. For any

$$\varphi \in \mathcal{I}_p\big(V, W^*\big),$$

$F_\varphi = S_{\text{int}}(\varphi) = S(\varphi)$ is determined by $\langle F_\varphi, v \otimes w\rangle = \varphi(v)(w)$ (see Lemma 4.4). From Lemma 4.5, we have the natural isometry

$$S_0 : \mathcal{I}_p\big(V, W^*\big) \hookrightarrow \big(V \overset{\vee p}{\otimes} W^{**}\big)^*.$$

It follows that

$$\begin{aligned}\iota^p(\varphi) &= \sup\big\{\big|\langle F_\varphi, u\rangle\big| : u \in V \otimes W^{**}, \|u\|_{V \overset{\vee p}{\otimes} W^{**}} \leq 1\big\} \\ &= \sup\big\{\big|\langle F_\varphi, u\rangle\big| : u \in V \otimes W^{**}, \|u\|_{V \overset{\vee p}{\otimes} : W^{**}} \leq 1\big\}.\end{aligned}$$

Since the closed unit ball of $V \otimes_{\vee_p} W$ is weak* dense in the closed unit ball of $(V \overset{\vee_p}{\otimes} W)^{**}$,

$$\iota^p(\varphi) = \sup\left\{\left|\langle F_\varphi, u\rangle\right| : u \in V \otimes W, \|u\|_{V \overset{\vee_p}{\otimes} W} \leq 1\right\} = \|F_\varphi\|.$$

To prove that S_{int} is a surjection, let us suppose that $f \in (V \overset{\vee_p}{\otimes} W)^*$. Then since the mapping $S : \mathrm{CB}_p(V, W^*) \cong (V \overset{\wedge_p}{\otimes} W)^*$ is a p-completely isometric surjection and $\Phi : V \overset{\wedge_p}{\otimes} W \to V \overset{\vee_p}{\otimes} W$ is contractive, there is a p-complete contraction $\varphi : V \to W^*$ such that $S(\varphi) = \Phi^*(f)$. Restricting to the algebraic tensor product $V \otimes W$, we have $F_\varphi = f$, and thus from the above calculations we obtain $\iota^p(\varphi) = \|f\| < \infty$. We conclude that $\varphi \in \mathcal{I}_p(V, W^*)$ and $S_{\text{int}}(\varphi) = f$.

Conversely, let us suppose that

$$S_{\text{int}} : \mathcal{I}_p\big(V, W^*\big) \to (V \overset{\vee_p}{\otimes} W)^*$$

is an isometric surjection. Then we have the commutative diagram

$$\begin{tikzcd}
\mathcal{I}_p(V, W^*) \arrow[r, "S_{\text{int}}"] \arrow[d, "\widetilde{J}"'] & (V \overset{\vee_p}{\otimes} W)^* \arrow[r, "\Phi^*"] & (V \overset{\wedge_p}{\otimes} W)^* \cong \mathrm{CB}_p(V, W^*) \arrow[d] \\
\mathcal{I}_p(V, W^{***}) \arrow[r, "\widetilde{S}_{\text{int}}"] & (V \overset{\vee_p}{\otimes} W^{**})^* \arrow[r] & (V \overset{\wedge_p}{\otimes} W^{**})^* \cong \mathrm{CB}_p(V, W^{***})
\end{tikzcd},$$

where $\widetilde{J}$ is the isometry described in Theorem 4.3, and the right column is the obvious isometric inclusion. Thus, if we let

$$\eta = \widetilde{S}_{\text{int}} \circ \widetilde{J} \circ S_{\text{int}}^{-1},$$

then we obtain a diagram of contractions

$$\begin{tikzcd}
(V \overset{\vee_p}{\otimes} W)^* \arrow[r, "\Phi^*"] \arrow[d, "\eta"'] & (V \overset{\wedge_p}{\otimes} W)^* \cong \mathrm{CB}_p(V, W^*) \arrow[d] \\
(V \overset{\vee_p}{\otimes} W^{**})^* \arrow[r] & (V \overset{\wedge_p}{\otimes} W^{**})^* \cong \mathrm{CB}_p(V, W^{***})
\end{tikzcd}.$$

If we take the adjoints of the mappings in this diagram, then we obtain the commutative diagram

$$\begin{tikzcd}
V \otimes W^{**} \arrow[r] \arrow[dr] & (V \overset{\wedge_p}{\otimes} W^{**})^{**} \arrow[r] \arrow[d] & (V \overset{\vee_p}{\otimes} W^{**})^{**} \arrow[d] \\
& (V \overset{\wedge_p}{\otimes} W)^{**} \arrow[r] & (V \overset{\vee_p}{\otimes} W)^{**}
\end{tikzcd}.$$

The bottom composition has range $V \otimes :_{\vee_p} W^{**}$. On the other hand, $V \otimes W^{**}$ in $(V \otimes :_{\vee_p} W^{**})^{**}$, and thus the algebraic identification $V \otimes_{\vee_p} W^{**} = V \otimes :_{\vee_p} W^{**}$ is an isometric isomorphism. $\square$

The conditions C_p, C'_p and C''_p of p-operator spaces on L_p spaces have been studied by Lee (see [11]). Let V be a p-operator space on L_p space. We say V satisfies condition C_p if we have the isometry $V^{**} :\overset{\vee_p}{\otimes}: W^{**} \cong V^{**} \overset{\vee_p}{\otimes} W^{**}$ for all p-operator spaces W on L_p spaces. It is equivalent to suppose that the isometry is a p-complete isometry, since Theorem 3.6 and the isometry imply that

$$M_n\big(V^{**} :\overset{\vee_p}{\otimes}: W^{**}\big) \cong V^{**} :\overset{\vee_p}{\otimes}: M_n(W)^{**} \cong M_n\big(V^{**} \overset{\vee_p}{\otimes} W^{**}\big).$$

Similarly, we say V satisfies condition $C_p^{'}$ if we have the isometry $V \overset{\vee_p}{\otimes}: W^{**} \cong V \overset{\vee_p}{\otimes} W^{**}$ for all p-operator spaces W on L_p spaces. We say V satisfies condition $C_p^{''}$ if we have the isometry $V^{**} :\overset{\vee_p}{\otimes} W \cong V^{**} \overset{\vee_p}{\otimes} W$ for all p-operator spaces W on L_p spaces. Once again, these conditions are stable in the sense that if they hold, then these identifications are p-completely isometric isomorphisms.

COROLLARY 4.8. *Let V be a p-operator space on L_p space.*

(1) *V satisfies condition C'_p if and only $\mathcal{I}_p(V, W^*) \cong (V \overset{\vee_p}{\otimes} W)^*$ is an isometry for all p-operator spaces W on L_p spaces*;

(2) *V satisfies condition C''_p if and only if $\mathcal{I}_p(W, V^*) \cong (V \overset{\vee_p}{\otimes} W)^*$ is an isometry for all p-operator spaces W on L_p spaces.*

Proof. This is an immediate consequence of Theorem 4.7 and the definitions of the conditions C'_p and C''_p. □

5. P-completely 1-summing and ∞-summing mappings

Completely 1-summing mappings have been studied by Effros and Ruan [6] and completely ∞-summing mappings have been considered by Dong [3]. In this section, we will define and study p-completely 1-summing and ∞-summing mappings.

DEFINITION 5.1. If $\varphi : V \to W$ is a linear mapping of p-operator spaces on L_p spaces, then we define $\pi_1^p(\varphi)$ in $[0, \infty]$ by

$$\begin{aligned}\pi_1^p(\varphi) &= \| \mathrm{id}_{T_\infty} \otimes \varphi : T_\infty \overset{\vee_p}{\otimes} V \to T_\infty \overset{\wedge p}{\otimes} W \| \\ &= \sup\big\{ \| \mathrm{id}_{T_r} \otimes \varphi : T_r \overset{\vee_p}{\otimes} V \to T_r \overset{\wedge p}{\otimes} W \| : r \in \mathbb{N} \big\}.\end{aligned}$$

If $\pi_1^p(\varphi) < \infty$, we say that φ is p-completely 1-summing and we refer to $\pi_1^p(\varphi)$ as the p-completely 1-summing norm of φ. We let $\Pi_1^p(V, W)$ denote the space of all p-completely 1-summing mappings from V into W.

THEOREM 5.2. *For any p-operator spaces on L_p spaces V and W, a linear mapping $\varphi : V \to W$ satisfies $\pi_1^p(\varphi) < 1$ if and only if for each $n \in \mathbb{N}$ and p-complete contraction $\theta : M_n \to V$, $\nu^p(\varphi \circ \theta) \le 1$.*

Proof. This is apparent from the commutative diagram

$$\begin{array}{ccc} T_n \overset{\vee_p}{\otimes} V & \xrightarrow{\mathrm{id}\otimes\varphi} & T_n \overset{\wedge p}{\otimes} W \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{CB}_p(M_n, V) & \longrightarrow & \mathcal{N}_p(M_n, W) \end{array}.$$

□

COROLLARY 5.3. *Let V and W be p-operator spaces on L_p spaces. The bifunctor $\Pi_1^p : (V, W) \mapsto (\Pi_1^p(V, W), \Pi_1^p)$ is a local p-operator space mapping ideal, and for any linear mapping $\varphi : V \to W$, $\pi_1^p(\varphi) \le \iota^p(\varphi)$.*

Proof. If $r = 1$, we have $\| \mathrm{id}_{T_r} : T_r \overset{\vee_p}{\otimes} V \to T_r \overset{\wedge p}{\otimes} W\|$. Then we have $\|\varphi\| \le \pi_1^p(\varphi)$. Suppose linear mappings $r : U \to V$ and $s : W \to X$. Then it is apparent from the diagram

$$T_\infty \overset{\vee_p}{\otimes} U \overset{\mathrm{id}\otimes r}{\to} T_\infty \overset{\vee_p}{\otimes} V \overset{\mathrm{id}\otimes\varphi}{\to} T_\infty \overset{\wedge p}{\otimes} W \overset{\mathrm{id}\otimes s}{\to} T_\infty \overset{\wedge p}{\otimes} X$$

that

$$\pi_1^p(s \circ \varphi \circ r) \le \|s\| \pi_1^p(\varphi) \|r\|.$$

Therefore Π_1^p is a p-mapping ideal. Since Π_1^p has the p-ideal property, it is clear that for every finite-dimensional p-operator subspace $L \subseteq V$,

$$\pi_1^p(\varphi|_L) \le \pi_1^p(\varphi).$$

On the other hand, suppose that for any finite-dimensional p-operator subspace $L \subseteq V$, $\pi_1^p(\varphi|_L) \le 1$. For any $n \in \mathbb{N}$ and p-complete contraction $\psi : M_n \to V$, we set $L = \psi(M_n)$. Since $\pi_1^p(\varphi|_L) \le 1$, it follows from Theorem 5.2 that

$$\nu^p(\varphi \circ \psi) = \nu^p(\varphi|_L \circ \psi) \le 1.$$

Theorem 5.2 shows that $\pi_1^p(\varphi) \le 1$ and therefore Π_1^p is local.

If $\nu^p(\varphi) \le 1$, then for any $n \in \mathbb{N}$ and each p-complete contraction $\psi : M_n \to V$

$$\nu^p(\varphi \circ \psi) \le \nu^p(\varphi) \cdot \|\psi\|_{pcb} \le 1$$

and from Theorem 5.2,

$$\pi_1^p(\varphi) \le \nu^p(\varphi).$$

Since Π_1^p and $\mathcal{I}_p$ are local,

$$\begin{aligned} \pi_1^p(\varphi) &= \sup\{\pi_1^p(\varphi|_L) : \text{ for any } \textit{finite-dimensional} \text{ subspace } L \subseteq V\} \\ &\le \sup\{\nu^p(\varphi|_L) : \text{ for any } \textit{finite-dimensional} \text{ subspace } L \subseteq V\} \\ &= \iota^p(\varphi). \end{aligned}$$

□

DEFINITION 5.4. If $\varphi : V \to W$ is a linear mapping of p-operator spaces on L_p spaces, then we define $\pi_\infty^p(\varphi)$ in $[0, \infty]$ by

$$\begin{aligned} \pi_\infty^p(\varphi) &= \| \mathrm{id}_{M_\infty} \otimes \varphi : M_\infty \overset{\vee_p}{\otimes} V \to M_\infty \overset{\wedge p}{\otimes} W\| \\ &= \sup\{\| \mathrm{id}_{M_r} \otimes \varphi : M_r \overset{\vee_p}{\otimes} V \to M_r \overset{\wedge p}{\otimes} W\| : r \in \mathbb{N}\}. \end{aligned}$$

This definition is 'stable' in the sense that we may replace the bounded norms with p-completely bounded norms. To see this, let us suppose that $\pi_\infty^p(\varphi) \leq 1$. Let us fix r. We have

$$\begin{aligned}\|\operatorname{id}_{M_r} \otimes \varphi\|_{pcb} = \sup\big\{\big\|&\operatorname{id}_{M_n} \otimes \operatorname{id}_{M_r} \otimes \varphi : \\ &M_n \overset{\vee_p}{\otimes} (M_r \overset{\vee_p}{\otimes} V) \to M_n \overset{\vee_p}{\otimes} (M_r \overset{\wedge p}{\otimes} W)\big\| : n \in \mathbb{N}\big\}.\end{aligned}$$

From Theorem 2.1 and the definition of π_∞^p, the two mappings in the diagram

$$\begin{aligned}M_n \overset{\vee_p}{\otimes} (M_r \overset{\vee_p}{\otimes} V) = M_{nr} \overset{\vee_p}{\otimes} V \to M_{nr} \overset{\wedge p}{\otimes} W \\ = (M_n \overset{\vee_p}{\otimes} M_r) \overset{\wedge p}{\otimes} W \to M_n \overset{\vee_p}{\otimes} (M_r \overset{\wedge p}{\otimes} W)\end{aligned}$$

are contractions, and thus $\|\operatorname{id}_{M_r} \otimes \varphi\|_{pcb} \leq 1$. If we let $r = 1$, then $\|\varphi\|_{pcb} \leq 1$, and thus $\|\varphi\|_{pcb} \leq \pi_\infty^p(\varphi)$. If $\pi_\infty^p(\varphi) < \infty$, we say that φ is p-completely ∞-summing and we refer to $\pi_\infty^p(\varphi)$ as the p-completely ∞-summing norm of φ. We let $\Pi_\infty^p(V, W)$ denote the space of all p-completely ∞-summing mappings from V into W.

THEOREM 5.5. *For any p-operator spaces V and W on L_p spaces, a linear mapping $\varphi : V \to W$ satisfies $\pi_\infty^p(\varphi) < 1$ if and only if for each $n \in \mathbb{N}$ and p-complete contraction $\theta : T_n \to V$, $\nu^p(\varphi \circ \theta) \leq 1$.*

Proof. This is apparent from the commutative diagram

$$\begin{array}{ccc} M_n \overset{\vee_p}{\otimes} V & \xrightarrow{\operatorname{id} \otimes \varphi} & M_n \overset{\wedge p}{\otimes} W \\ \big\downarrow & & \big\downarrow \\ \mathrm{CB}_p(T_n, V) & \longrightarrow & \mathcal{N}_p(T_n, W) \end{array}.$$

□

COROLLARY 5.6. *Let V and W be p-operator spaces on L_p spaces. The bifunctor $\Pi_\infty^p : (V, W) \mapsto (\Pi_\infty^p(V, W), \Pi_\infty^p)$ is a local p-operator space mapping ideal, and for any linear mapping $\varphi : V \to W$, $\pi_\infty^p(\varphi) \leq \iota^p(\varphi)$.*

Proof. We may use the argument for the p-completely 1-summing norm. □

THEOREM 5.7. *Given p-operator spaces on L_p spaces V, W and a linear mapping $\varphi : V \to W$, we have $\pi_1^p(\varphi) \leq \pi_\infty^p(\varphi^*)$. Moreover, we have $\pi_1^p(\varphi) = \pi_\infty^p(\varphi^*)$ for any p-operator space W and linear mapping $\varphi : V \to W$ if and only if $\mathcal{I}_p(V, M_n) = \mathcal{N}_p(V, M_n)$ for any $n \in \mathbb{N}$.*

Proof. Since $M_n \overset{\wedge p}{\otimes} V^* \to (T_n \overset{\vee_p}{\otimes} V)^*$ is norm-decreasing, we conclude that

$$\begin{aligned}\pi_1^p(\varphi) &= \sup\big\{\|\operatorname{id}_{T_n} \otimes \varphi : T_n \overset{\vee_p}{\otimes} V \to T_n \overset{\wedge p}{\otimes} W\| : n \in \mathbb{N}\big\} \\ &= \sup\big\{\big\|(\operatorname{id}_{T_n} \otimes \varphi)^* : (T_n \overset{\wedge p}{\otimes} W)^* \to (T_n \overset{\vee_p}{\otimes} V)^*\big\| : n \in \mathbb{N}\big\}\end{aligned}$$

$$\leq \sup\{\|\mathrm{id}_{M_n} \otimes \varphi^* : M_n \overset{\vee_p}{\otimes} W^* \to M_n \overset{\wedge p}{\otimes} V^*\| : n \in \mathbb{N}\}$$
$$= \pi_\infty^p(\varphi^*).$$

If $\mathcal{I}_p(V, M_n) = \mathcal{N}_p(V, M_n)$, then $M_n \overset{\wedge p}{\otimes} V^* \to (T_n \overset{\vee_p}{\otimes} V)^*$ is isometric, and the above calculation implies that $\pi_1^p(\varphi) = \pi_\infty^p(\varphi^*)$.

Conversely, we first prove $\Pi_\infty^p(T_n, V^*) = \mathcal{N}_p(T_n, V^*)$. In fact, it follows from Corollary 5.6 that $\pi_\infty^p(\psi) \leq \iota^p(\psi) \leq \nu^p(\psi)$ for any $\psi : T_n \to V^*$. Suppose that $\pi_\infty^p(\psi) \leq 1$ for any $\psi : T_n \to V^*$. Theorem 5.5 shows that for $\mathrm{id}_{T_n} : T_n \to T_n$,

$$\nu^p(\psi) = \nu^p(\psi \circ \mathrm{id}_{T_n}) \leq 1.$$

Therefore, $\nu^p(\psi) = \pi_\infty^p(\psi)$ and $\Pi_\infty^p(T_n, V^*) = \mathcal{N}_p(T_n, V^*)$.

Thus we have the isometries

$$\Pi_1^p(V, M_n) = \Pi_\infty^p\big(T_n, V^*\big) = \mathcal{N}_p\big(T_n, V^*\big) = \mathcal{N}_p(V, M_n),$$

where the first equation follows from the hypothesis and the third from Proposition 3.10. Then, it follows from Corollary 5.6 we easily have

$$\Pi_1^p(V, M_n) = \mathcal{I}_p(V, M_n) = \mathcal{N}_p(V, M_n). \qquad \square$$

6. P-local reflexivity

Definition 6.1. We say that a p-operator space W on L_p space is p-locally reflexive if for any finite-dimensional p-operator space L on L_p space, every p-complete contraction $\varphi : L \to W^{**}$ is the point-weak* limit of a net of linear mappings $\varphi_\alpha : L \to W$ with $\|\varphi_\alpha\|_{pcb} \leq 1$.

Theorem 6.2. *Suppose that W is a p-operator space on L_p space. Then the following are equivalent*:

(1) *W is p-locally reflexive*;

(2) *For any finite-dimensional p-operator space L on L_p space, we have the isometry*

$$L^* \overset{\wedge p}{\otimes} W^* \cong (L \overset{\vee_p}{\otimes} W)^*;$$

(2)′ *For any finite-dimensional p-operator space L on L_p space, we have the isometry*

$$\mathcal{I}_p\big(W, L^*\big) \cong \mathcal{N}_p\big(W, L^*\big);$$

(3) *For any p-operator space V on L_p space, we have the isometry*

$$\mathcal{I}_p\big(V, W^*\big) \cong (V \overset{\vee_p}{\otimes} W)^*;$$

(4) *W satisfies condition C_p''.*

Proof. We have already proved (3) $\Leftrightarrow$ (4) (see Corollary 4.8).

(2) $\Leftrightarrow$ (2)$'$ It is immediate from Theorem 4.6.

(1) $\Leftrightarrow$ (2) Since for any finite-dimensional p-operator space L on L_p space,

$$\big(L^* \overset{\wedge_p}{\otimes} W^*\big)^* \cong \mathrm{CB}_p\big(L^*, W^{**}\big) \cong L \overset{\vee_p}{\otimes} W^{**},$$

(2) holds if and only if we have the natural isometric isomorphism

$$L \overset{\vee_p}{\otimes} W^{**} \cong (L \overset{\vee_p}{\otimes} W)^{**}.$$

The corresponding is explicitly given by the norm-increasing linear isomorphism

$$\tau : L \overset{\vee_p}{\otimes} W^{**} \to (L \overset{\vee_p}{\otimes} W)^{**}.$$

Thus, the relation is isometric if and only if

$$\varphi \in \big(L \overset{\vee_p}{\otimes} W^{**}\big)_{\|\cdot\|\le 1} \cong \mathrm{CB}_p\big(L^*, W^{**}\big)_{\|\cdot\|_{pcb}\le 1}$$

implies that

$$\varphi \in (L \overset{\vee_p}{\otimes} W)^{**}_{\|\cdot\|\le 1}.$$

From the bipolar theorem, the latter is the case if and only if φ is a weak* limit of elements in

$$(L \overset{\vee_p}{\otimes} W)_{\|\cdot\|\le 1} \cong \mathrm{CB}_p\big(L^*, W\big)_{\|\cdot\|_{pcb}\le 1}.$$

Since it is evident that

$$\tau : \mathrm{CB}_p\big(L^*, W^{**}\big) \to (L \overset{\vee_p}{\otimes} W)^{**}$$

is a homeomorphism in the point-weak* and weak* topologies, we are done.

(3) $\Rightarrow$ (2) For any finite-dimensional p-operator space L on L_p space, we have the isometries

$$L^* \overset{\wedge_p}{\otimes} W^* \cong \mathcal{N}_p\big(L, W^*\big) \cong \mathcal{I}_p\big(L^*, W^*\big) \cong (L \overset{\vee_p}{\otimes} W)^*.$$

(2) $\Rightarrow$ (3) From Lemma 4.4, we have seen that

$$S_{\mathrm{int}} : \mathcal{I}_p\big(V, W^*\big) \to (V \overset{\vee_p}{\otimes} W)^*$$

is a contractive injection. Let us suppose that the mapping in (2) is isometric. If we have a contractive functional $F \in (V \overset{\vee_p}{\otimes} W)^*$, then $F = S(\varphi)$ for some $\varphi : V \to W^*$(see Lemma 4.4). For any finite-dimensional subspace $L \subseteq V$ and p-complete contraction $\psi : L \to V$, we have

$$F \circ (\psi \otimes \mathrm{id}_W) \in (V \overset{\vee_p}{\otimes} W)^* \quad \text{and} \quad \varphi \circ \psi : L \to W^*.$$

Since for any $x \in L, y \in W$

$$\big(F \circ (\psi \otimes \mathrm{id}_W)\big)(x \otimes y) = F\big(\psi(x) \otimes y\big) = \varphi\big(\psi(x)\big)(y),$$

we have $F \circ (\psi \otimes \mathrm{id}_W) = S(\varphi \circ \psi)$. Thus from (2) and $L^* \overset{\wedge_p}{\otimes} W^* \cong \mathcal{N}_p(L, W^*)$,

$$\nu^p(\varphi \circ \psi) = \big\|F \circ (\psi \otimes \mathrm{id}_W)\big\| \le \|F\|.$$

From the definition of $\iota^p(\varphi)$, we have $\iota^p(\varphi) \le \|F\|$. Therefore, $\iota^p(\varphi) = \|F\|$ for $\varphi \in \mathcal{I}_p(V, W^*)$ and thus $\mathcal{I}_p(V, W^*) \cong (V \overset{\vee_p}{\otimes} W)^*$. □

Corollary 6.3. *Suppose that W is a p-operator space on L_p space. If W is p-locally reflexive, then any subspace $X \subseteq W$ is p-locally reflexive.*

Proof. For any finite-dimensional p-operator space L on L_p space, from Theorem 6.2, we have the isometry

$$L^* \overset{\wedge_p}{\otimes} W^* \cong (L \overset{\vee_p}{\otimes} W)^*.$$

Since

$$\big(L^* \overset{\wedge_p}{\otimes} W^*\big)^* \cong \mathrm{CB}_p\big(L^*, W^{**}\big) \cong L \overset{\vee_p}{\otimes} W^{**},$$

$L^* \overset{\wedge_p}{\otimes} W^* \cong (L \overset{\vee_p}{\otimes} W)^*$ holds if and only if we have the natural isometric isomorphism

$$L \overset{\vee_p}{\otimes} W^{**} \cong (L \overset{\vee_p}{\otimes} W)^{**}.$$

Then X is p-locally reflexive from Theorem 6.2 and the commutative diagram

$$\begin{tikzcd}
L \overset{\vee_p}{\otimes} X^{**} \arrow[r] \arrow[d] & (L \overset{\vee_p}{\otimes} X)^{**} \arrow[d] \\
L \overset{\vee_p}{\otimes} W^{**} \arrow[r] & (L \overset{\vee_p}{\otimes} W)^{**}
\end{tikzcd}$$

in which the columns are isometric. □

Corollary 6.4. *Suppose that W is a p-operator space on L_p space. If W is p-locally reflexive, then $\Pi_1^p(W, V) \cong \Pi_\infty^p(V^*, W^*)$ for any p-operator space V on L_p space and linear mapping $\varphi : W \to V$.*

Proof. It follows from Theorem 5.7 and Theorem 6.2 immediately. □

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