

INVOLUTIONS AND TRIVOLUTIONS IN ALGEBRAS RELATED TO SECOND DUALS OF GROUP ALGEBRAS

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ABSTRACT. We define a trivolution on a complex algebra A as a non-zero conjugate-linear, anti-homomorphism τ on A , which is a generalized inverse of itself, that is, $\tau^3 = \tau$. We obtain characterizations of trivolutions and show with examples that they appear naturally on many Banach algebras, particularly those arising from group algebras. We give several results on the existence or non-existence of involutions on the dual of a topologically introverted space. We investigate conditions under which the dual of a topologically introverted space admits trivolutions.

1. Introduction and preliminaries

By a well-known result of Civin and Yood [8, Theorem 6.2], if A is a Banach algebra with an involution $\rho : A \rightarrow A$, then the second (conjugate-linear) adjoint $\rho^{**} : A^{**} \rightarrow A^{**}$ is an involution on A^{**} (with respect to either of the Arens products) if and only if A is Arens regular; when this is the case, ρ^{**} is called the canonical extension of ρ . Grosser [23, Theorem 1] has shown that if A is a Banach algebra with a bounded right [left] approximate identity, then a necessary condition for the existence of an involution on A^{**} with respect to the first [second] Arens product is that $A^* \cdot A = A^*$ [$A \cdot A^* = A^*$]. The above results applied to the group algebra $L^1(G)$ imply that a necessary condition for $L^1(G)^{**}$ to have an involution (with respect to either of the Arens products) is that G is discrete (Grosser [23, Theorem 2]); moreover, the natural involution of $L^1(G)$ has the canonical extension to $L^1(G)^{**}$ if and only if $L^1(G)$ is Arens regular, and, by a well-known result of Young [39], $L^1(G)$ is Arens regular if and only if G is finite.

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In general, since A is not norm dense in A^{**} , an involution on A may have extensions to A^{**} which are different from the canonical extension. A necessary and sufficient condition for the existence of such extensions does not seem to be known. However, for the special case of the group algebra $L^1(G)$, Farhadi and Ghahramani [15, Theorem 3.2(a)] have shown that if a locally compact group G has an infinite amenable subgroup, then $L^1(G)^{**}$ does not have any involution extending the natural involution of $L^1(G)$. (See also the related paper by Neufang [33], answering a question raised in [15].) It is unknown whether for a discrete group which does not contain an infinite amenable group, there is an involution on $L^1(G)^{**}$ extending the natural involution of $L^1(G)$.

In Singh [37], the third author introduced the concept of α -amenability for a locally compact group G . Given a cardinal α , a group G is called α -amenable if there exists a subset $\mathcal{F} \subset L^1(G)^{**}$ containing a mean M (not necessarily left invariant) such that $|\mathcal{F}| \leq \alpha$ and the linear span of \mathcal{F} is a left ideal of $L^1(G)^{**}$. The group G is called subamenable if G is α -amenable for some cardinal $1 \leq \alpha < 2^{2^{\kappa(G)}}$, where $\kappa(G)$ denotes the compact covering number of G , that is, the least cardinality of a compact covering of G . (We remark that at the time of the proof-reading of the paper Singh [37], the author did not know that the term subamenable had already been used in different contexts, viz., subamenable semigroups by Lau and Takahashi in [31], [32], and initially subamenable groups by Gromov in [22]. She thanks A. T.-M. Lau for interesting exchange of mathematical points in this regard.) It follows that 1-amenable of G is equivalent to the amenability of G , and α -amenability implies β -amenability for every $\beta \geq \alpha$. If G is a non-compact locally compact group, then every non-trivial right ideal in $L^1(G)^{**}$ or in $\text{LUC}(G)^*$ has (vector space) dimension at least $2^{2^{\kappa(G)}}$ (Filali and Pym [18, Theorem 5] and Filali and Salmi [19, Theorem 6], Filali [16], Filali, Neufang, and Sangani Monfared [17]). It follows from this lower bound that if G is a subamenable, non-compact, locally compact group, then $L^1(G)^{**}$ has no involution (Singh [37, Theorem 2.2]). Singh [37, Theorem 2.9(i)] also showed that every discrete group G is a subgroup of a subamenable discrete group G_σ with $|G_\sigma| \leq 2^{|G|}$. It is not known whether there exists any non-subamenable group, and in particular, it remains an open question whether the free group on 2 generators is subamenable.

All the above results show that the existence of involutions on second dual Banach algebras imposes strong conditions on A . So it seems natural to consider involution-like operators on Banach algebras and their second duals. In this paper, we relax the condition of bijectivity on an involution ρ and of ρ being its own inverse to that of ρ being a generalized inverse of itself, namely, $\rho^3 = \rho$, and call them trivolutions (Definition 2.1) following the terminology of Degen [12]. It follows from the definition that every involution is a trivolution,

but as we shall show later there are many naturally arising trivolutions which are not involutions.

In Section 2, we start with a general study of trivolutions on algebras and we give several characterizations of trivolutions in Theorem 2.3. We use the decompositions given in Theorem 2.3(iii) to characterize trivolutional homomorphisms (Theorem 2.5). In Theorem 2.10, we show that, unlike involutions, a trivolution can have various extensions to the unitized algebra A^\sharp , and we give a complete characterization of all such extensions. We show that concepts such as Hermitian, normal, and unitary elements, usually associated with involutions, can be naturally defined in the context of trivolutional algebras. In Section 3, we study involutions on the dual of a topologically introverted space. In Theorem 3.1, we extend the result of Civin and Yood (discussed above) to the dual of a topologically introverted space, and obtain the result of Farhadi and Ghahramani [15, Theorem 3.2(a)] as a corollary. In Theorem 3.3, we investigate the relationship between the existence of topologically invariant elements and the existence of involutions on the dual of a topologically introverted space. As a corollary, we show that under fairly general conditions, neither of the Banach algebras $PM_p(G)^*$ and $UC_p(G)^*$, $1 < p < \infty$, have involutions (for the definitions of these spaces see below as well as the discussion prior to Corollary 3.4). In Section 4, we give some sufficient conditions under which a second dual Banach algebra A^{**} admits trivolutions (Theorem 4.1). In Theorem 4.3, we show that for G non-discrete, $L^1(G)^{**}$ does not admit any trivolutions with range $L_0^\infty(G)^*$. However, the space $L_0^\infty(G)^*$ itself always admits trivolutions (Theorem 4.7).

We close this section with a few preliminary definitions and notation. Given a Banach algebra A , the dual space A^* can be viewed as a Banach A -bimodule with the canonical operations:

$$\langle \lambda \cdot a, b \rangle = \langle \lambda, ab \rangle, \quad \langle a \cdot \lambda, b \rangle = \langle \lambda, ba \rangle,$$

where $\lambda \in A^*$ and $a, b \in A$. Let X be a norm closed A -submodule of A^* . Then given $\Psi \in X^*$, $\lambda \in X$, we may define $\Psi \cdot \lambda \in A^*$ by $\langle \Psi \cdot \lambda, a \rangle = \langle \Psi, \lambda \cdot a \rangle$. If $\Psi \cdot \lambda \in X$ for all choices of $\Psi \in X^*$ and $\lambda \in X$, then X is called a left topologically introverted subspace of A^* . The dual of a left topologically introverted subspace X can be turned into a Banach algebra if, for all $\Phi, \Psi \in X^*$, we define $\Phi \square \Psi \in X^*$ by $\langle \Phi \square \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle$. In particular, by taking $X = A^*$, we obtain the first (or the left) Arens product on A^{**} , defined by Arens [1], [2]. The space X^* can be identified with the quotient algebra A^{**}/X° , where $X^\circ = \{\Phi \in A^{**} : \Phi|_X = 0\}$. If X is faithful (that is, $a = 0$ whenever $\lambda(a) = 0$ for all $\lambda \in X$), then the natural map of A into X^* is an embedding, and we will regard A as a subalgebra of (X^*, \square) . The space X^* has a canonical A -bimodule structure defined by $\langle a \cdot \Phi, \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle$, $\langle \Phi \cdot a, \lambda \rangle = \langle \Phi, a \cdot \lambda \rangle$ ($\Phi \in X^*, \lambda \in X, a \in A$). One can then verify that $a \cdot \Phi =$

$a \square \Phi, \Phi \cdot a = \Phi \square a$ for each $a \in A$ and $\Phi \in X^*$. For each $\Phi \in X^*$, the map $\Psi \mapsto \Psi \square \Phi, X^* \rightarrow X^*$, is continuous in the w^* -topology of X^* .

Right topologically introverted subspaces of A^* are defined similarly; for these spaces the second (or the right) Arens product on X^* will be denoted by $\Phi \diamond \Psi$. A space which is both left and right topologically introverted is called topologically introverted. When A is commutative, there will be no distinction between left and right topologically introverted spaces.

Let A be a Banach algebra. The space of [weakly] almost periodic functionals on A , denoted by [WAP(A^*)] $AP(A^*)$, is defined as the set of all $\lambda \in A^*$ such that the linear map $A \rightarrow A^*, a \mapsto a \cdot \lambda$, is [weakly] compact. The spaces $A^*, WAP(A^*)$, and $AP(A^*)$ are examples of topologically introverted spaces. The space of left uniformly continuous functionals on A defined by $LUC(A) = \overline{\text{lin}}(A^* \cdot A)$ (the closure is in norm topology), is an example of a left topologically introverted space. If G is a locally compact group, then $LUC(L^1(G))$ coincides with $LUC(G)$, the space of left uniformly continuous functions on G (cf. Lau [28]). For more information and additional examples one may consult [9], [10], [13], [20], [27], [38].

If $f : X \rightarrow X$ is a map on a set X , for simplicity and when no confusion arises, we write f^n to denote the n -times composition of f with itself, that is, $f^n := f \circ \dots \circ f$ (n -times). Also when no confusion arises, we write $\Phi \Psi$ to denote the first Arens product $\Phi \square \Psi$.

2. Trivolutions

DEFINITION 2.1. A *trivolution* on a complex algebra A is a non-zero, conjugate linear, anti-homomorphism $\tau : A \rightarrow A$, such that $\tau^3 = \tau$. When A is a normed algebra, we shall assume that $\|\tau\| = 1$. The pair (A, τ) will be called a *trivolutive algebra*.

REMARKS 2.2. (i) It follows from the definition that every involution on a non-zero complex algebra is a trivolution. Conversely, a trivolution which is either injective or surjective, is an involution. Thus, for a trivolution injectivity is equivalent to surjectivity; in particular, a trivolution is surjunctive in the sense that it is either surjective or non-injective (see, for instance, [22] and the references therein for history and more on this term). If (A, τ) is a trivolutive normed algebra, then $\|\tau(x)\| \leq \|x\|$ for every $x \in A$; in particular, when τ is an involution, we have $\|\tau(x)\| = \|x\|$ for every x .

(ii) If (A, τ) is a trivolutive algebra, then $\tau(A)$ is a subalgebra of A . Moreover, if (A, τ) is a trivolutive normed algebra, then $\tau(A)$ is closed subalgebra of A : if (x_n) is a sequence such that $\tau(x_n) \rightarrow x$ for some $x \in A$, then $\tau(x_n) = \tau^3(x_n) \rightarrow \tau^2(x)$, and hence $x = \tau^2(x) \in \tau(A)$.

The next result gives several characterizations of trivolutions. We remark that by the equivalence of (i) and (iii) in the following theorem, it follows

that every trivolutive [Banach] algebra is a [strongly] splitting extension of an involutive [Banach] algebra.

THEOREM 2.3. *Let A be a complex algebra, $\tau : A \rightarrow A$ a non-zero map, and $B := \tau(A)$ a subalgebra of A . Let \mathcal{S}_B denote the set of all involutions on B . Then the following statements are equivalent.*

- (i) τ is a trivolution.
- (ii) τ is a conjugate-linear, anti-homomorphism, and $\tau|_B \in \mathcal{S}_B$.
- (iii) There exist a projection $p : A \rightarrow B$ with p an algebra homomorphism, a two sided ideal I of A , and an involution ρ on B such that

$$(1) \quad A = I \oplus B, \quad \tau = \rho \circ p.$$

- (iv) \mathcal{S}_B is non-empty, and for each $\rho_2 \in \mathcal{S}_B$, there is a surjective homomorphism $\rho_1 : A \rightarrow B$ that satisfies $\tau = \rho_2 \circ \rho_1$ and $\rho_1 \circ \rho_2 \in \mathcal{S}_B$.

The statements (i)–(iv) remain equivalent if A is a normed algebra and the maps in (i)–(iv) are assumed to be contractive.

Proof. (i) \implies (ii): Since τ is a conjugate-linear anti-homomorphism, the identity $\tau^3 = \tau$ implies that $(\tau|_B)^2$ is the identity map on B , and therefore $\tau|_B \in \mathcal{S}_B$.

(ii) \implies (iii): Let $\rho := \tau|_B$ and $p := \tau \circ \tau : A \rightarrow B$. We leave it for the reader to verify the easy facts that $\tau = \rho \circ p$, and p and ρ satisfy the requirements in (iii). If we let $I = \ker \tau = \ker p$, then I is a two-sided ideal of A , and since p is a projection we have $A = I \oplus B$. Moreover, since $I = \ker \tau$ and $\rho = \tau|_B$, it follows that for each $x = y + z \in I \oplus B$, we have $\tau(x) = \tau(y) + \tau(z) = \rho(z)$.

(iii) \implies (iv): The first two statements of (iv) are immediate consequences of (iii). Let $\rho_2 \in \mathcal{S}_B$ be given and define $\rho_1 := \rho_2 \circ \tau$. Then ρ_1 is a surjective homomorphism from A to B and furthermore, $\rho_2 \circ \rho_1 = \rho_2 \circ \rho_2 \circ \tau = I_B \circ \tau = \tau$ (where I_B denotes the identity map on B). Since $\rho_1 \circ \rho_2$ is a conjugate-linear anti-homomorphism on B , it remains to show that $(\rho_1 \circ \rho_2)^2 = I_B$. This follows from the fact that $\tau^2 = p$, since then

$$(\rho_1 \circ \rho_2)^2 = \rho_2 \circ \tau \circ \rho_2 \circ \rho_2 \circ \tau \circ \rho_2 = \rho_2 \circ \tau^2 \circ \rho_2 = \rho_2 \circ p \circ \rho_2 = I_B.$$

(iv) \implies (i): Since $\tau = \rho_2 \circ \rho_1$, with ρ_1, ρ_2 as in (iv), it follows that τ is a surjective, conjugate-linear anti-homomorphism. Now using the assumption that $\rho_1 \circ \rho_2$ is an involution on B , we have

$$\tau^3 = \rho_2 \circ (\rho_1 \circ \rho_2) \circ (\rho_1 \circ \rho_2) \circ \rho_1 = \rho_2 \circ \rho_1 = \tau;$$

hence τ is a trivolution. □

COROLLARY 2.4. *Suppose that $I = \ker \tau \neq \{0\}$. Let $B = \tau(A)$ and $A = I \oplus B$ as in (1). Then the following hold.*

- (i) $\tau = \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix}$, where $\rho = \tau|_B$.

(ii) If I has an involution J , then there are an algebra C , a homomorphism $\lambda : A \rightarrow C$, an involution σ on C , and a homomorphism $\mu : C \rightarrow A$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tau} & A \\ \lambda \downarrow & & \uparrow \mu \\ C & \xrightarrow{\sigma} & C \end{array}$$

commutes, namely, $\tau = \mu \circ \sigma \circ \lambda$.

Proof. Statement (i) follows immediately from the fact that for each $x = y + z \in I \oplus B$, we have $\tau(x) = \tau(y) + \tau(z) = \rho(z)$.

To prove (ii), let $\text{pr}_1 = I_A - \tau^2$ be the projection of A onto I . We define the algebra \mathcal{A} as the vector space $I \oplus B$ equipped with coordinatewise multiplication, therefore, if $x = y + z \in \mathcal{A}$ and $x' = y' + z' \in \mathcal{A}$ (with $y, y' \in I$ and $z, z' \in B$), then $xx' = yy' + zz' \in \mathcal{A}$. Now we let $C = \mathcal{A} \times \mathcal{A}$, equipped with coordinatewise operations, and we define

$$\lambda(x) = (\tau^2(x), 0), \quad \mu(x, x') = \tau^2(x)$$

and

$$\begin{aligned} \sigma(x, x') &= \begin{pmatrix} \tau & J \circ \text{pr}_1 \\ J \circ \text{pr}_1 & \tau \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \\ &= (J \circ \text{pr}_1(x') + \tau(x), J \circ \text{pr}_1(x) + \tau(x')). \end{aligned}$$

It is easy to check that both λ and μ are homomorphisms, σ is a conjugate-linear anti-homomorphism, and $\tau = \mu \circ \sigma \circ \lambda$. Moreover, since

$$\begin{aligned} \sigma^2 &= \begin{pmatrix} \tau^2 + J \circ \text{pr}_1 \circ J \circ \text{pr}_1 & \tau \circ J \circ \text{pr}_1 + J \circ \text{pr}_1 \circ \tau \\ \tau \circ J \circ \text{pr}_1 + J \circ \text{pr}_1 \circ \tau & \tau^2 + J \circ \text{pr}_1 \circ J \circ \text{pr}_1 \end{pmatrix} \\ &= \begin{pmatrix} \tau^2 + \text{pr}_1 & 0 \\ 0 & \tau^2 + \text{pr}_1 \end{pmatrix} = \begin{pmatrix} I_A & 0 \\ 0 & I_A \end{pmatrix}, \end{aligned}$$

it follows that σ is an involution on C , completing the proof of (b). □

If (A, τ) is a trivolutive algebra, then we shall call the identities in (1) of Theorem 2.3, namely, $A = I \oplus B$, $\tau = \rho \circ p$, in which $p = \tau^2$, $B = p(A) = \tau(A)$, $I = \ker p = \ker \tau$, and $\rho = \tau|_B$, the *canonical decompositions* of (A, τ) . As an interesting application of these ideas, we shall give a characterization of trivolutive homomorphisms $\pi : (A_1, \tau_1) \rightarrow (A_2, \tau_2)$. These are algebra homomorphisms such that $\pi(\tau_1(x)) = \tau_2(\pi(x))$, for all $x \in A_1$.

THEOREM 2.5. *Let (A_i, τ_i) , $i = 1, 2$, be two trivolutive [Banach] algebras and $\pi : A_1 \rightarrow A_2$ be a [continuous] trivolutive homomorphism. Let $A_i = I_i \oplus B_i$ and $\tau_i = \rho_i \circ p_i$ be the canonical decompositions of (A_i, τ_i) . Then*

$$\pi = \begin{pmatrix} \pi_{11} & 0 \\ 0 & \pi_{22} \end{pmatrix},$$

where $\pi_{11} : I_1 \rightarrow I_2$ and $\pi_{22} : B_1 \rightarrow B_2$ are both [continuous] homomorphisms, and π_{22} is involutive.

Proof. We confine our attention to the case when I_1 and I_2 are both non-zero. Following (1) of Theorem 2.3, we can write $\tau_1 = \begin{pmatrix} 0 & 0 \\ 0 & \rho_1 \end{pmatrix}$, and $\tau_2 = \begin{pmatrix} 0 & 0 \\ 0 & \rho_2 \end{pmatrix}$. Let $\pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$ be the block matrix representation of π . The identity $\pi \circ \tau_1 = \tau_2 \circ \pi$ implies that

$$\begin{pmatrix} 0 & \pi_{12} \circ \rho_1 \\ 0 & \pi_{22} \circ \rho_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \rho_2 \circ \pi_{21} & \rho_2 \circ \pi_{22} \end{pmatrix}.$$

Since both ρ_1 and ρ_2 are bijective, the identities $\pi_{12} \circ \rho_1 = 0$ and $\rho_2 \circ \pi_{21} = 0$, imply that $\pi_{12} = 0$ and $\pi_{21} = 0$. Now it is easy to check that both π_{11} and π_{22} must be algebra homomorphisms. Moreover, the identity $\pi_{22} \circ \rho_1 = \rho_2 \circ \pi_{22}$, implies that $\pi_{22} : (B_1, \rho_1) \rightarrow (B_2, \rho_2)$ is involutive, as we wanted to show. \square

There are various automatic continuity results for homomorphisms and for homomorphisms intertwining with involutions in the literature (cf. Dales [9] and Palmer [34], [35]). We can use these results to prove various automatic continuity results involving trivolutions. We illustrate just one such result. By a famous result of B. E. Johnson if A and B are Banach algebras with B semisimple, then any surjective homomorphism $\varphi : A \rightarrow B$ is continuous (Palmer [34, Theorem 6.1.3]). Moreover, it is well known that if A is an involutive Banach algebra and B is a C^* -algebra, then every involutive algebra homomorphism $\pi : A \rightarrow B$ is continuous (Dales [9, Corollary 3.2.4]). As an immediate consequence of these results and Theorem 2.5, we can state the following automatic continuity result for trivolution homomorphisms.

COROLLARY 2.6. *Let (A_i, τ_i) , $i = 1, 2$, be two trivolution Banach algebras and $\pi : A_1 \rightarrow A_2$ be a trivolution homomorphism. If $\ker \tau_2$ is semisimple and is contained in $\pi(A_1)$, then π is continuous provided that for every $b \in \tau_2(A_2)$ we have $\|\tau_2(b)b\| = \|b\|^2$.*

Next, we give a few examples of trivolutions.

EXAMPLES 2.7. (a) Let (X, μ) be a measure space and $K \subset X$ be a measurable subset of X with $\mu(K) > 0$. Let $L_K^\infty(X, \mu)$ be the subalgebra of $L^\infty(X, \mu)$ consisting of all those functions which vanish locally almost everywhere on $X \setminus K$. Let χ_K be the characteristic function of K and $p : L^\infty(X, \mu) \rightarrow L_K^\infty(X, \mu)$ be the homomorphism $f \mapsto \chi_K f$. If ρ is the usual complex conjugation on $L_K^\infty(X, \mu)$, then the map τ given by $\tau(f) := \rho \circ p(f) = \chi_K \bar{f}$ is a trivolution on $L^\infty(X, \mu)$. We next give an abstract alternative.

(b) Let M be a von Neumann algebra and $P \in M$ be a central projection. Then $\tau(T) = PT^*$ is a trivolution on M .

(c) The quotient of a trivlutive [normed] algebra (A, τ) by a two-sided [closed] ideal I such that $\tau(I) \subset I$ and $\tau(A) \not\subset I$, is a trivlutive [normed] algebra. Finite products of trivlutive [normed] algebras, the completion of a trivlutive normed algebra, and the opposite of a trivlutive [normed] algebra, are all trivlutive [normed] algebras in canonical ways.

THEOREM 2.8. *Let τ be an anti-homomorphism on an algebra A and let $B = \tau(A)$.*

- (i) *If $e \in B$ is a right identity of A , then $\tau(e) = e$ and e is the identity of B .*
- (ii) *The set B can contain at most one right identity of A .*
- (iii) *Let A be a subalgebra of an algebra C of the form eC with e being a right identity of C . If τ is a trivluton on A , and if ℓ_e denotes the left multiplication map by e on C , then $\tau_1 := \tau \circ \ell_e$ is a trivluton on C and $\tau_1(C) = \tau(A) = B$.*

Proof. (i) If $a \in A$, then $a = ae$ and hence $\tau(a) = \tau(e)\tau(a)$, which shows that $\tau(e)$ is a left identity for B . Since $e \in B$, $e = \tau(e)e = \tau(e)$, proving that $e = \tau(e)$ and e is the identity for B .

(ii) This is an immediate consequence of (i).

(iii) Since e is a right identity of C and $B \subset A = eC$, it follows that $\ell_e|_B$ is the identity map on B : in fact, given $b \in B$, we can write $b = ec$ for some $c \in C$, and hence $\ell_e(b) = e(ec) = ec = b$. It follows that $\ell_e \circ \tau = \tau$, and hence

$$\tau_1^3 = (\tau \circ \ell_e)^3 = \tau \circ (\ell_e \circ \tau) \circ (\ell_e \circ \tau) \circ \ell_e = \tau^3 \circ \ell_e = \tau \circ \ell_e = \tau_1.$$

In addition, since e is a right identity of C , we have $e(c_1c_2) = (ec_1)(ec_2)$, for all $c_1, c_2 \in C$. Hence, ℓ_e is a homomorphism on C which implies that $\tau_1 = \tau \circ \ell_e$ is a conjugate-linear, anti-homomorphism, completing the proof that τ_1 is a trivluton on C . The fact that $\tau_1(C) = B$ is now immediate. \square

REMARKS 2.9. (i) Similar results hold if a right identity is replaced by a left identity in Theorem 2.8; we leave the formulation of the results and their proofs for the readers.

(ii) Let τ be a trivluton on A and let $B = \tau(A)$. If A has the identity e , then $\tau(e)$ is the identity of B , which we may denote by e_B . Clearly $e = e_B$ if and only if $e \in B$. This however may not always be the case: let $A = \mathbf{C}^2$, $B = \mathbf{C} \times \{0\}$, and $\tau(z_1, z_2) = (\bar{z}_1, 0)$; then $e = (1, 1)$ but $e_B = (1, 0)$.

Next, we consider the problem of extending a trivluton to the unitized algebra $A^\sharp = \mathbf{C} \times A$. Let (A, τ) be a trivlutive algebra and $\tau^\sharp : A^\sharp \rightarrow A^\sharp$ be a trivluton extending τ , namely, $\tau^\sharp(0, x) = (0, \tau(x))$, for all $x \in A$. If $\tau^\sharp(1, 0) = (\lambda_0, x_0)$, then, from conjugate linearity of τ^\sharp we obtain:

$$(2) \quad \tau^\sharp(\lambda, x) = \tau^\sharp(\lambda(1, 0) + (0, x)) = (\bar{\lambda}\lambda_0, \bar{\lambda}x_0 + \tau(x)).$$

If $(\lambda_0, x_0) = (1, 0)$, then $\tau^\sharp(\lambda, x) = (\bar{\lambda}, \tau(x))$. We call this map the canonical extension of τ to A^\sharp . We note that by Theorem 2.8(i), the condition $\tau^\sharp(1, 0) =$

$(1, 0)$ is equivalent to $(1, 0)$ being in the range of τ^\sharp , and the latter condition can be shown to be equivalent to $\lambda_0 \neq 0$ and $x_0 \in \tau(A)$.

While every involution has only the canonical extension to an involution on the unitized algebra, the situation is different for trivolutions, as the following theorem shows.

THEOREM 2.10. *Let τ be a trivolution on a complex algebra A . The map τ^\sharp in (2) is a trivolution extending τ if and only if either of the following conditions hold:*

- (i) $\tau^\sharp(\lambda, x) = (\bar{\lambda}, \bar{\lambda}x_0 + \tau(x))$, where $x_0 \in A$ is such that
- (3) $x_0^2 = -x_0, \quad x_0\tau(A) = \tau(A)x_0 = \{0\}, \quad \tau(x_0) = 0.$
- (ii) $\tau^\sharp(\lambda, x) = (0, \bar{\lambda}x_0 + \tau(x))$, where $x_0 \in \tau(A)$ is the identity of $\tau(A)$.

Proof. The proof that both (i) and (ii) define trivolutions on A^\sharp extending τ , is routine and is left for the reader. We prove the necessity part of the theorem. Using the idempotence of $(1, 0)$, we get

$$(\lambda_0, x_0) = \tau^\sharp(1, 0) = \tau^\sharp(1, 0)^2 = (\lambda_0^2, 2\lambda_0x_0 + x_0^2),$$

which implies that either $\lambda_0 = 1$ and $x_0^2 = -x_0$; or $\lambda_0 = 0$ and $x_0^2 = x_0$. We consider these two cases.

Case I: $\lambda_0 = 1$ and $x_0^2 = -x_0$. In this case $\tau^\sharp(\lambda, x) = (\bar{\lambda}, \bar{\lambda}x_0 + \tau(x))$. Applying τ^\sharp to the identity $(1, 0)(0, x) = (0, x)$, we obtain

$$(0, \tau(x) + \tau(x)x_0) = (0, \tau(x)),$$

which implies that $\tau(x)x_0 = 0$ for all $x \in A$. Similarly, starting from the identity $(0, x)(1, 0) = (0, x)$ we can show that $x_0\tau(x) = 0$ for all $x \in A$. Moreover it follows from $(\tau^\sharp)^3(1, 0) = \tau^\sharp(1, 0)$, that

$$(1, x_0 + \tau(x_0) + \tau^2(x_0)) = (1, x_0),$$

which is equivalent to $\tau(x_0) + \tau^2(x_0) = 0$. Therefore,

$$0 = x_0\tau(x_0) = \tau^2(x_0)\tau(x_0) = -\tau(x_0)^2 = -\tau(x_0^2) = \tau(x_0).$$

Thus x_0 satisfies all the conditions in (3).

Case II: $\lambda_0 = 0$. In this case $\tau^\sharp(\lambda, x) = (0, \bar{\lambda}x_0 + \tau(x))$. Applying τ^\sharp to the identities $(1, 0)(0, x) = (0, x)$ and $(0, x)(1, 0) = (0, x)$, we obtain respectively, $\tau(x)x_0 = \tau(x)$, $x_0\tau(x) = \tau(x)$, for all $x \in A$. Moreover, from $(\tau^\sharp)^3(1, 0) = \tau^\sharp(1, 0)$, it follows that $x_0 = \tau^2(x_0) \in \tau(A)$. Thus x_0 is the identity of $\tau(A)$. □

COROLLARY 2.11. *Let (A, τ) be a trivolution normed algebra and A^\sharp be the unitized algebra with the norm $\|(\lambda, x)\| = |\lambda| + \|x\|$. A map $\tau^\sharp : A^\sharp \rightarrow A^\sharp$, is a trivolution extending τ if and only if either of the following conditions hold:*

- (i) $\tau^\sharp(\lambda, x) = (\bar{\lambda}, \tau(x))$;

(ii) $\tau^\sharp(\lambda, x) = (0, \overline{\lambda}x_0 + \tau(x))$, where $x_0 \in \tau(A)$ is the identity of $\tau(A)$ with $\|x_0\| = 1$.

Proof. If τ^\sharp is of the form given in Theorem 2.10(i), then $\tau^\sharp(\lambda, 0) = (\overline{\lambda}, \overline{\lambda}x_0)$ for all $\lambda \in \mathbf{C}$. Since we must have $\|\tau^\sharp\| = 1$, we obtain $|\lambda| + |\lambda|\|x_0\| \leq |\lambda|$, implying that $x_0 = 0$.

If however τ^\sharp is of the form given in Theorem 2.10(ii), then $\tau^\sharp(1, 0) = (0, x_0)$. Hence the condition $\|\tau^\sharp\| = 1$ implies that $\|x_0\| \leq 1$. The inequality $\|x_0\| \geq 1$ is immediate since x_0 is a non-zero idempotent. \square

REMARK 2.12. Let τ be an involution on a complex algebra A . Then $\tau(A) = A$, and hence any extension of τ of the form given in Theorem 2.10(i), satisfies $\tau^\sharp(\lambda, x) = (\overline{\lambda}, \tau(x))$, since $\tau(x_0) = 0$ implies that $x_0 = 0$. It should be noted that if A has no identity, then τ has no extension of the form given in Theorem 2.10(ii).

We can define the concepts of normality, hermiticity, and positivity for elements of trivolutional algebras.

DEFINITION 2.13. Let (A, τ) be a trivolutional algebra and let $x \in A$. Then x is called

- (i) *Hermitian* if $\tau(x) = x$;
- (ii) *normal* if $x\tau(x) = \tau(x)x$ and $x\tau^2(x) = \tau^2(x)x$;
- (iii) *projection* if x is Hermitian and $x^2 = x$;
- (iv) *unitary* if A is unital with identity e and $x\tau(x) = \tau(x)x = e$;
- (v) *positive* if x is Hermitian and $x = \tau(y)y$ for some $y \in A$.

We denote the set of all Hermitian (respectively, unitary, positive) elements of A , by A_h (respectively, A_u, A^+). It follows that A_h is a real vector subspace of A , and A_u is a group under multiplication (the unitary group of A). It follows from the definition that if x is Hermitian, then $x \in A^+$ if and only if $x = z\tau(z)$ for some $z \in A$. It should be noted that for trivolutional algebras in general, A^+ need not form a positive cone. The definition of normality is designed to have the τ -invariant algebra generated by x (and therefore, containing both $\tau(x)$ and $\tau^2(x)$) commutative (since $\tau(x)\tau^2(x) = \tau(\tau(x)x) = \tau(x\tau(x)) = \tau^2(x)\tau(x)$). If x is unitary in A , then by letting $B = \tau(A)$ and $e_B = \tau(e)$, we see that $\tau^2(x)$ is the inverse of $\tau(x)$ in B , and $x \in B$ implies that $e \in B$ (and $e = e_B$). Thus, $e \in B$ if and only if B contains at least one unitary element.

Let (A, τ) be a trivolutional algebra and τ^* be the conjugate-linear adjoint of τ defined by $\langle \tau^*(f), a \rangle = \overline{\langle f, \tau(a) \rangle}$ ($f \in A^*, a \in A$). If $f : A \rightarrow \mathbf{C}$ is a linear functional on A , then $f^\tau := \tau^*(f)$, is also a linear functional on A . One can easily check that the map $f \rightarrow f^\tau$, is conjugate-linear and in general $f^{\tau\tau} = f$. If (A, τ) is normed, then $\|f^\tau\| \leq \|f\|$. We call f Hermitian if

$f^\tau = f$. Clearly if χ is a (Gelfand) character on A , then χ^τ is also a character on A .

We close this section by stating the following two results whose straightforward proofs are omitted for brevity.

THEOREM 2.14. *Let (A, τ) be a unital trivolutive algebra and $B = \tau(A)$. Let $x \in A$.*

- (i) *If x is invertible in A , then $\tau(x)$ is invertible in B and $\tau(x)^{-1} = \tau(x^{-1})$.*
- (ii) *If $\tau(x)$ is invertible in A , then $\tau(x)$ is invertible in B .*
- (iii) *$\text{Sp}_B(\tau(x)) \subset \overline{\text{Sp}_A(x)}$ (where the bar denotes the complex conjugate).*

THEOREM 2.15. *Let (A, τ) be a trivolutive algebra.*

- (i) *$x \in A$ can be written uniquely in the form $x = x_1 + ix_2$, with x_1, x_2 Hermitian, if and only if $x \in \tau(A)$.*
- (ii) *$f \in A^*$ can be written uniquely in the form $f = f_1 + if_2$, with f_1, f_2 Hermitian, if and only if $f \in \tau^*(A^*)$.*
- (iii) *A linear functional f is Hermitian if and only if f is real valued on A_h and it vanishes on $\ker \tau$.*
- (iv) *The map $f \rightarrow f|_{A_h}$ is an isomorphism between the real vector space of all Hermitian linear functionals and the dual vector space of the real space A_h .*

3. Involution on the dual of a topologically introverted space

The following theorem is an extension of a result of Civin and Yood [8, Theorem 6.2] to the dual of a topologically introverted space.

THEOREM 3.1. *Let A be a Banach algebra and X , a faithful, topologically (left and right) introverted subspace of A^* .*

- (i) *If there is a w^* -continuous, injective, anti-homomorphism (with respect to either of the Arens products) $\Theta : X^* \rightarrow X^*$ such that $\Theta(A) \subset A$, then the two Arens products coincide on X^* .*
- (ii) *Let $\theta : A \rightarrow A$ be an involution on A and let $\theta^* : A^* \rightarrow A^*$ be its conjugate-linear adjoint. If $\theta^*(X) \subset X$ and if the two Arens products coincide on X^* , then $\Theta = (\theta^*|_X)^* : X^* \rightarrow X^*$, is an involution on X^* , extending θ .*

Proof. (i) Let $\mu, \nu \in X^*$, and let $(a_\alpha), (b_\beta)$ be two nets in A such that $a_\alpha \rightarrow \mu, b_\beta \rightarrow \nu$, in the w^* -topology. Let us assume Θ is an anti-homomorphism with respect to the first Arens product. Then $\Theta(a_\alpha) \rightarrow \Theta(\mu)$, and $\Theta(b_\beta) \rightarrow \Theta(\nu)$. Hence,

$$\begin{aligned} \Theta(\mu \square \nu) &= \Theta(\nu) \square \Theta(\mu) \\ &= w^*\text{-}\lim_{\beta} \Theta(b_\beta) \square \Theta(\mu) \end{aligned}$$

$$\begin{aligned}
&= w^* \text{-} \lim_{\beta} \left(w^* \text{-} \lim_{\alpha} \Theta(b_{\beta}) \square \Theta(a_{\alpha}) \right) \\
&= w^* \text{-} \lim_{\beta} \left(w^* \text{-} \lim_{\alpha} \Theta(b_{\beta}) \Theta(a_{\alpha}) \right) \\
&= w^* \text{-} \lim_{\beta} \left(w^* \text{-} \lim_{\alpha} \Theta(a_{\alpha} b_{\beta}) \right) \\
&= w^* \text{-} \lim_{\beta} \Theta(\mu \diamond b_{\beta}) \\
&= \Theta(\mu \diamond \nu),
\end{aligned}$$

where \diamond denotes the second Arens product. Since Θ is injective, $\mu \square \nu = \mu \diamond \nu$, which is what we wanted to show.

The claim in (ii) follows by a similar argument as in (i). \square

It is well known that the two Arens products coincide on $WAP(A^*)^*$ and $AP(A^*)^*$ (Dales and Lau [10, Proposition 3.11]). It is also straightforward to check that both of these spaces are invariant under the conjugate-linear adjoint of any involution of A . Therefore if either of these spaces is faithful (which is the case, for example, if the spectrum of A separates the points of A ; see Dales and Lau [10, p. 32]), then its dual has an involution extending that of A . Hence as a corollary of the above theorem, we obtain the following result due to Farhadi and Ghaharamani [15, Theorem 3.5].

COROLLARY 3.2. *Suppose that A is an involutive Banach algebra and X is either of the topologically introverted spaces $AP(A^*)^*$, $WAP(A^*)^*$. If X is faithful, then X^* has an involution extending the involution of A .*

Let $X \subset A^*$ be a faithful, topologically left introverted subspace of A^* . Let $\sigma(A)$ denote the spectrum of A and let $\varphi \in \sigma(A) \cap X$. We call an element $m \in X^*$ a φ -topological invariant mean (φ -TIM) if $\langle m, \varphi \rangle = 1$ and $a \cdot m = m \cdot a = \varphi(a)m$ for all $a \in A$.

The following theorem is an extension of a result of Farhadi and Ghaharamani [15, Theorem 3.2(a)] (see the Introduction) to the dual of topologically left introverted spaces.

THEOREM 3.3. *Let A be a Banach algebra and $X \subset A^*$ be a faithful, topologically left introverted subspace of A^* . Let $\varphi \in \sigma(A) \cap X$. If X^* contains at least two φ -TIMs, then X^* cannot have an involution $*$ such that $\varphi(a^*) = \overline{\varphi(a)}$ for every $a \in A$.*

Proof. Let us suppose that X^* has an involution as in the statement of the theorem. Let $m \in X^*$ be an arbitrary φ -TIM. Then we have

$$(4) \quad a \cdot m^* = m^* \cdot a = \varphi(a)m^* \quad (a \in A).$$

To prove the above identities, we note that for every $a \in A$:

$$a \cdot m^* = (m \cdot a^*)^* = (\varphi(a^*)m)^* = (\overline{\varphi(a)}m)^* = \varphi(a)m^*.$$

The other identity in (4) is proved similarly. Using the w^* -continuity of the product $n \square m$ on the variable n , it follows from (4) that

$$(5) \quad n \square m^* = \langle n, \varphi \rangle m^* \quad (n \in X^*).$$

Thus using (5) and the fact that $\langle m, \varphi \rangle = 1$, we get

$$(6) \quad m = (m^*)^* = (m \square m^*)^* = m \square m^* = \langle m, \varphi \rangle m^* = m^*.$$

Now if m_1, m_2 are two distinct φ -TIMs, then using (5) and (6), we have

$$m_1 = m_2 \square m_1 = (m_2 \square m_1)^* = m_1^* \square m_2^* = m_1 \square m_2 = m_2;$$

which is a contradiction. □

To state a corollary of the above theorem, we first recall a few definitions. Let G be a locally compact group, $1 < p < \infty$, and $\mathcal{L}(L^p(G))$ be the space of continuous linear operators on $L^p(G)$. Let $\lambda_p : M(G) \rightarrow \mathcal{L}(L^p(G))$, $\lambda_p(\mu)(g) = \mu * g$, where $\mu * g(x) = \int_G g(y^{-1}x) d\mu(y)$, be the left regular representation of $M(G)$ on $L^p(G)$. The space $\text{PM}_p(G)$ is the w^* -closure of $\lambda_p(M(G))$ in $\mathcal{L}(L^p(G))$. This space is the dual of the Herz–Figà-Talamanca algebra $A_p(G)$, consisting of all functions $u \in C_0(G)$, such that $u = \sum_{i=1}^\infty g_i * \check{f}_i$, where $f_i \in L^p(G), g_i \in L^q(G)$, $1/p + 1/q = 1$, and $\sum_{i=1}^\infty \|f_i\|_p \|g_i\|_q < \infty$ (Herz [25]). When $p = 2$, $A_2(G)$ and $\text{PM}_2(G)$ coincide respectively, with the Fourier algebra $A(G)$ and the group von Neumann algebra $\text{VN}(G)$ studied by Eymard in [14]. In the following, for simplicity of notation, we denote $\text{UC}(A_p(G))$ by $\text{UC}_p(G)$; when $p = 2$, this space is also denoted by $\text{UC}(\widehat{G})$ in the literature.

COROLLARY 3.4. *Let $1 < p < \infty$, G be a non-discrete locally compact group, and $X = \text{PM}_p(G)$ or $X = \text{UC}_p(G)$. Let $e \in G$ be the identity of G . Then X^* does not have any involution $*$ such that $u^*(e) = \overline{u(e)}$, for every $u \in A_p(G)$.*

Proof. Let $\varphi_e \in \sigma(A_p(G)) \cap X$ be the evaluation functional at e . Let $\text{TIM}(X^*)$ denote the set of all φ_e -TIMs on X^* . Granirer [21, Theorem, p. 3400] has shown that if G is non-discrete, then $|\text{TIM}(X^*)| \geq 2^{\mathfrak{c}}$, where \mathfrak{c} is the cardinality of real numbers. Therefore our result follows from Theorem 3.3. □

REMARK 3.5. For $p = 2$, the cardinality of $\text{TIM}(\text{PM}_2(G))$ was determined for second countable groups by Chou [7], and in full generality by Hu [26]. For additional information on the cardinality of topological invariant means, see Chou [5], [6], Lau and Paterson [29].

Let G be a locally compact group and $\text{LUC}(G)$ the space of all left uniformly continuous functions on G . It is known, and easy to verify, that the natural restriction map $\pi : L^\infty(G)^* \rightarrow \text{LUC}(G)^*$ is a continuous algebra homomorphism (with respect to the first Arens product). Using this fact we can prove the following analogue of Singh’s result [37, Theorem 2.2] for the

non-existence of involutions on $\text{LUC}(G)^*$. Our result extends Farhadi and Ghahramani [15, Theorem 3.2(b)], from amenable to subamenable groups (see the Introduction).

THEOREM 3.6. *If G is a non-compact subamenable group, then $\text{LUC}(G)^*$ has no involutions.*

Proof. Since G is subamenable, there exists a subset \mathcal{F} of $L^\infty(G)^*$ containing a mean m such that $|\mathcal{F}| < 2^{2^{\kappa(G)}}$, and the linear span of \mathcal{F} is a left ideal J of $L^\infty(G)^*$. Since $\pi : L^\infty(G)^* \rightarrow \text{LUC}(G)^*$ (defined above) is a continuous homomorphism, it follows that $\pi(J)$ is a non-trivial left ideal in $\text{LUC}(G)^*$; $\pi(J)$ is non-trivial since $\pi(m) \neq 0$. Since the dimension of $\pi(J)$ is less than or equal to the dimension of J , it follows that $\text{LUC}(G)^*$ has a non-trivial left ideal of dimension less than $2^{2^{\kappa(G)}}$. By Filali and Pym [18, Theorem 5], if G is a non-compact locally compact group, every non-trivial right ideal of $\text{LUC}(G)^*$ has dimension at least $2^{2^{\kappa(G)}}$. It follows that for non-compact subamenable groups G , $\text{LUC}(G)^*$ cannot have any involutions. \square

4. Trivolutions on the duals of introverted spaces

In Corollary 3.4 and Theorem 3.6, we saw several examples of topologically left introverted spaces X for which there can be no involution on X^* . Our objective in this section is to consider some cases for which A^{**} or X^* admits trivolutions.

THEOREM 4.1. *Let A be a non-zero Banach algebra with an involution θ . Under each of the following conditions, A^{**} admits a trivolution.*

- (i) *There exists a non-zero topologically introverted, faithful subspace $X \subset A^*$, such that the two Arens products coincide on X^* , $\theta^*(X) \subset X$, and $A^{**} = X^\circ \oplus X^*$.*
- (ii) *A is a dual Banach algebra.*
- (iii) *A has a bounded two-sided approximate identity and is a right ideal in (A^{**}, \square) .*
- (iv) *$A^{**} = I \oplus A$, as a topological direct sum of closed subspaces I and A , with I as an ideal in A^{**} , the projection p on A^{**} onto A being a homomorphism.*

Proof. (i) This follows from Theorem 3.1(ii) and Theorem 2.3(iii).

(ii) Let A_* be a predual of A , and consider the canonical Banach space decomposition $A^{**} = (A_*)^\circ \oplus A$ (cf. Dales [9, p. 241]). It is easy to verify that A_* is a topologically introverted subspace of A^* , and clearly the two Arens products coincide on $(A_*)^* = A$. Hence (ii) follows from Theorem 2.3(iii).

(iii) Since A has a bounded two-sided approximate identity we have the decomposition $A^{**} \cong (A^* \cdot A)^\circ \oplus (A^* \cdot A)^*$, and since A is a right ideal in its second dual, we have $\text{WAP}(A^*) = A^* \cdot A$ (cf. [3, Corollary 1.2, Theorem 1.5]). Therefore, $A^{**} \cong \text{WAP}(A^*)^\circ \oplus \text{WAP}(A^*)^*$. It is easy to check that under

these conditions, $\text{WAP}(A^*)$ is faithful, and therefore our result follows by Corollary 3.2 and Theorem 2.3(iii).

(iv) An appeal to Theorem 2.3 gives the result. □

Next, we study trivolutions on the Banach algebra $L^\infty(G)^*$, equipped with its first Arens product \square . Here G is a locally compact group. For simplicity of notation, in the following we shall denote $E\square F$ by EF , whenever $E, F \in L^\infty(G)^*$. If $K \subset G$ is measurable and $f \in L^\infty(G)$, let

$$\|f\|_K = \text{ess sup } \{|f(x)| : x \in K\},$$

and let $L_0^\infty(G)$ be the closed ideal of $L^\infty(G)$ consisting of all $f \in L^\infty(G)$ such that for given $\varepsilon > 0$, there exists a compact set $K \subset G$ such that $\|f\|_{G \setminus K} < \varepsilon$.

In [30, Theorems 2.7 and 2.8], Lau and Pym showed that $L_0^\infty(G)$ is a faithful, topologically introverted subspace of $L^\infty(G)$ and $L^\infty(G)^*$ is the Banach space direct sum

$$(7) \quad L^\infty(G)^* = L_0^\infty(G)^\circ \oplus L_0^\infty(G)^*.$$

In this decomposition $L_0^\infty(G)^*$ is identified with the closed subalgebra of $L^\infty(G)^*$ defined as the norm closure of elements in $L^\infty(G)^*$ with compact carriers ($F \in L^\infty(G)^*$ has compact carrier if for some compact set K , $F(f) = F(\chi_K f)$ for every $f \in L^\infty(G)$). In addition, Lau and Pym showed that if $\pi : L^\infty(G)^* \rightarrow \text{LUC}(G)^*$ is the natural restriction map, then $\pi(L_0^\infty(G)^*) = M(G)$. Lau and Pym [30] make a case for the study of $L_0^\infty(G)^*$ for general G (in place of $L^1(G)^{**}$). In [36], the third named author has expressed $L_0^\infty(G)^*$ as the second dual of $L^1(G)$ with a locally convex topology similar to the strict topology (see also [24]). Let $\mathcal{E}(G)$ denote the set of all right identities of $L^\infty(G)^*$, and $\mathcal{E}_1(G)$ the set of those with norm one. In $L^\infty(G)^*$, when G is not discrete, there is an abundance of such right identities, a fact noted and well-utilized in [23], [30], [36], [15], for instance.

For the convenience of our readers, we shall now state the following result of Lau and Pym [30, Theorems 2.3 and 2.11] which will be needed in the last three theorems of this paper. In the following results all products are with respect to the first Arens product.

THEOREM 4.2 (Lau and Pym [30]). *Let G be a locally compact group and let the map $\pi : L^\infty(G)^* \rightarrow \text{LUC}(G)^*$ be the natural restriction map. Then the following hold.*

- (i) $\mathcal{E}_1(G) \subset L_0^\infty(G)^*$.
- (ii) For each $E \in \mathcal{E}(G)$, $\pi|_{EL^\infty(G)^*}$ is a continuous isomorphism from $EL^\infty(G)^*$ to $\text{LUC}(G)^*$, and if $\|E\| = 1$, the isomorphism is an isometry.
- (iii) For each $E \in \mathcal{E}_1(G)$, $L_0^\infty(G)^* = EL_0^\infty(G)^* + (\ker \pi \cap L_0^\infty(G)^*)$, and the algebra $EL_0^\infty(G)^*$ is isometrically isomorphic with $M(G)$ via π .

THEOREM 4.3. *Let G be a non-discrete locally compact group and X and Y be subalgebras of $L^\infty(G)^*$ with $L_0^\infty(G)^* \subset Y \subset X$. Then there are no trivolutions of X onto Y . In particular, $L^\infty(G)^*$ has no trivolutions with range $L_0^\infty(G)^*$.*

Proof. If G is compact, then $X = Y = L^\infty(G)^*$, and hence any trivolution of X onto Y is an involution on $L^\infty(G)^*$. Such an involution does not exist if G is non-discrete by Grosser [23, Theorem 2].

Let G be non-compact. To obtain a contradiction, let ρ be a trivolution from X onto Y . By Theorem 4.2(i), $\mathcal{E}_1(G) \subset Y$. By Theorem 2.8(i), each E in $\mathcal{E}_1(G)$ is the identity for Y , and therefore, also the identity for $L_0^\infty(G)^*$. But the identity for $L_0^\infty(G)^*$ is clearly in the topological centre of $L_0^\infty(G)^*$ and so it belongs to $L^1(G)$ (cf. Budak, Işık and Pym [4, Proposition 5.4]), which is not possible since G is not discrete. \square

THEOREM 4.4. *The algebra $L_0^\infty(G)^*$ has an involution if and only if G is discrete. Further, if G is discrete, $L^\infty(G)^*$ has a trivolution with range $L^1(G)$, extending the natural involution on $L^1(G)$.*

Proof. If G is discrete, then $L_0^\infty(G)^* = C_0(G)^* = L^1(G)$ has a natural involution, and hence by (7) and by Theorem 2.3(iii), $L^\infty(G)^*$ has a trivolution with range $L^1(G)$, extending the involution of $L^1(G)$.

If G is not discrete, then the result follows from Theorem 4.3 upon taking $X = Y = L_0^\infty(G)^*$. \square

REMARK 4.5. In the above theorem we used the natural involution of $\ell^1(G)$ to obtain a trivolution on $\ell^\infty(G)^*$ with range $\ell^1(G)$. Alternatively, we can view $\ell^\infty(G)^*$ as the Banach algebra $M(\beta G)$, where βG is the Stone-Ćech compactification of an infinite discrete group G , recalling that $M(\beta G) = M(\beta G \setminus G) \oplus \ell^1(G)$ where $M(\beta G \setminus G)$ is an ideal in $M(\beta G)$ (Dales, Lau and Strauss [11, Theorem 7.11 and (7.4)]) and then use Theorem 4.1(iv).

THEOREM 4.6. *If G is compact, then for each $E \in \mathcal{E}(G)$, there are trivolutions of $L^\infty(G)^*$ onto $EL^\infty(G)^*$.*

Proof. Let $E \in \mathcal{E}(G)$. The compactness of G implies that $L_0^\infty(G) = L^\infty(G)$ and $\text{LUC}(G)^* = M(G)$. Let ρ be any involution on $\text{LUC}(G)^*$ and let $\pi' = \pi|_{EL^\infty(G)^*}$. It follows from Theorem 4.2(iii) that $\rho' := (\pi')^{-1} \circ \rho \circ \pi'$ is an involution on $EL^\infty(G)^*$. Let $\ell_E : L^\infty(G)^* \rightarrow EL^\infty(G)^*$ be the left multiplication by E . Then by Theorem 2.8(iii), $\tau := \rho' \circ \ell_E$ is a trivolution of $L^\infty(G)^*$ onto $EL^\infty(G)^*$, as required. \square

THEOREM 4.7. *Let G be a locally compact group. For each $E \in \mathcal{E}_1(G)$, there exists a trivolution of $L_0^\infty(G)^*$ onto $EL_0^\infty(G)^*$.*

Proof. By Theorem 4.2, $\mathcal{E}_1(G) \subset L_0^\infty(G)^*$, and for each $E \in \mathcal{E}_1(G)$, $EL_0^\infty(G)^* \cong M(G)$. If ρ is an involution on $M(G)$, then it is easily checked

that $\rho' := (\pi|_{EL_0^\infty(G)^*})^{-1} \circ \rho \circ (\pi|_{EL_0^\infty(G)^*})$ is an involution on $EL_0^\infty(G)^*$, and hence by Theorem 2.8(iii), $\tau := \rho' \circ \ell_E$ is a trivolution of $L_0^\infty(G)^*$ onto $EL_0^\infty(G)^*$. \square

REMARK 4.8. If ρ in the proofs of Theorems 4.6 or 4.7 restricts to an involution ρ_0 on $L^1(G)$, then in view of the fact that π is the identity on $L^1(G)$, the trivolution τ constructed in the respective proofs will be an extension of ρ_0 .

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