Illinois Journal of Mathematics
Volume 57, Number 2, Summer 2013, Pages 577–594
S 0019-2082

CUSPED SURFACES AND BOUNDARY BEHAVIOR OF MAPPINGS OF FINITE DISTORTION

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ABSTRACT. We study bounded quasiregular mappings and mappings of finite distortion $f : \mathbb{H}^n \to \mathbb{R}^n$, $n \geq 3$. We show that almost every k-dimensional cone-like cusp with vertex in $\partial\mathbb{H}^n$ maps to a set of finite k-dimensional, $k \in \{2, \dots, n-1\}$, measure under these mappings.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. A mapping $f : \Omega \to \mathbb{R}^n$ is called a mapping of finite distortion if the following conditions are satisfied:

(1) $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$,
(2) $J_f(x) = \det(Df) \in L^1_{\text{loc}}(\Omega)$,
(3) and there exists a measurable $K_f : \Omega \to [1, \infty)$ so that for almost every $x \in \Omega$ we have

$$\left|Df(x)\right|^n \leq K_f(x) J_f(x).$$

If in addition $K_f \in L^\infty(\Omega)$, $\exp(\lambda K_f) \in L^1_{\text{loc}}(\Omega)$ for some $\lambda > 0$ or $K_f \in L^p_{\text{loc}}(\Omega)$ for some $p > n-1$ then we say that f is a quasiregular mapping, f has exponentially integrable distortion or that f has p-integrable distortion, respectively. All of the aforementioned assumptions give nice properties for the mappings in consideration. Under these assumptions, f is continuous, open, discrete, differentiable almost everywhere and satisfies Lusin's condition (N). For these properties to hold in the p-integrable distortion case we also have to assume $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$. For the basic theory on quasiregular mappings and mappings of finite distortion, see [5], [6], [7], [13], [14] and [16].

One interesting open question in the field of quasiregular mappings is the generalization of Fatou's theorem: does a bounded quasiregular mapping $f : B^n(0,1) \to \mathbb{R}^n$ have radial limits at almost every point in $S^{n-1}(0,1)$? For

Received November 20, 2012; received in final form October 2, 2013.
2010 *Mathematics Subject Classification.* 30C65.

planar quasiregular mappings this is not true. Nevertheless, the radial limits exist in a set with positive Hausdorff dimension, but this dimension can be made arbitrarily small, see [10]. In higher dimensions, it is not even known whether the radial limits exist for any point in $S^{n-1}(0,1)$. If f is a quasiregular mapping and there exist constants $C>0$ and $a\in(0,n-1)$ so that for all $0<r<1$

$$\int_{B^n(0,r)} J_f(x)\,dx \le C(1-r)^{-a}, \tag{4}$$

then f has radial limits almost everywhere in $S^{n-1}(0,1)$, see [8]. One also has a Hausdorff dimension estimate for the set where radial limits do not exist, see [9] and [1]. It is well known that every bounded quasiregular mapping satisfies (4) with $a=n-1$, see Section 4. In [1], the author has shown the existence of radial limits almost everywhere if we assume

$$\int_{B^n(0,r)} J_f(x)\,dx \le C(1-r)^{1-n}\log^{\beta}\left(\frac{1}{1-r}\right)$$

for $\beta<-1-n$. For more results on boundary behavior of quasiregular mappings we refer the reader to see [14] and [16].

The boundary behavior of mappings with exponentially or p-integrable distortion is not yet so well understood. In the planar case, we loose the existence of radial limits, since there exists a bounded mapping with exponentially integrable distortion having no radial limits: Assume $\mathcal{A}:[1,\infty)\to[1,\infty)$ is strictly increasing, and let $g:\mathbb{D}(0,1)\to\mathbb{D}(0,1)$ be a homeomorphism of finite distortion given in polar coordinates by $g(r,\theta)=(r,\theta+\xi(r))$, where $\xi:[0,1]\to[0,\infty)$ is such that

$$\xi(r)=\int_0^r\left(\mathcal{A}^{-1}\left(\frac{1}{r(1-r)^{\frac{1}{5}}}\right)\right)^{\frac{1}{2}}dr.$$

Notice that $\xi'(r)\to\infty$ as $r\to1$ and thus the image of each radial segment under g is tangential to $\partial\mathbb{D}(0,1)$. Furthermore, let $h:\mathbb{D}(0,1)\to\mathbb{C}$ be the bounded analytic function given by [3, Theorem 2.22] which does not have a limit along $\{g(r,\theta):r\in[0,1)\}$ for any $\theta\in[0,2\pi)$. Finally set $f=h\circ g$, then f does not have limits along any radial segment and

$$\int_{\mathbb{D}(0,1)}\mathcal{A}\big(K_f(x)\big)\,dx\le\pi\int_0^1\mathcal{A}\big(\xi'(r)^2\big)r\,dr=\pi\int_0^1\frac{1}{r^{\frac{1}{5}}}\,dr<\infty.$$

This shows that the quasiregularity assumption is sharp for the existence of at least one radial limit. In all dimensions $n\ge2$, we know that mappings with exponentially integrable distortion satisfying (4) have radial limits almost everywhere, see [1]. Moreover, we know that for this conclusion it suffices to assume that

$$\int_{B^n(0,r)} J_f(x)\,dx \le C(1-r)^{1-n}\log^{\beta}\left(\frac{1}{1-r}\right) \quad \text{for all } r\in(0,1)$$

for $\beta < -2-n$. On the other hand, in Section 4, we show that every bounded mapping with exponentially integrable distortion satisfies

$$\int_{B^n(0,r)} J_f(x)\,dx \le C(1-r)^{1-n}\log^{n-1}\left(\frac{1}{1-r}\right).$$

Similarly, one has the existence of radial limits almost everywhere for mappings in $W^{1,n}_{\mathrm{loc}}$ with p-integrable distortion satisfying (4) with $a\in(0,n-1-n/p)$.

A mapping f has a radial limit along some radial segment, if the image of that radial segment is rectifiable. We study the behavior of mappings on k-dimensional sets that are symmetric with respect to radial segments. We extend the results in [11], where Rajala proved that there is a family of $(n-1)$-dimensional cusps, symmetric with respect to radial segments, with vertices in $S^{n-1}(0,1)$ such that they are mapped to sets of finite $(n-1)$-measure under quasiregular mapping. We extend this theorem to lower dimensional cusps and prove similar results for mappings with exponentially and p-integrable distortion. In [15], Rudin has constructed an example of a bounded analytic function defined in the unit disc, so that the image of almost every radial segment is non-rectifiable. Thus the corresponding results for bounded analytic mappings f, that is, $n=2$ and $K_f\equiv 1$, are not valid. Throughout the paper we keep $k\in\{2,\dots,n-1\}$ fixed. If $x\in\mathbb{R}^n$ we write $x=(\bar{x},x_{k+1},\dots,x_n)$, where $\bar{x}=(x_1,\dots,x_k)$. To state our main theorem, define a mapping $\phi:\mathbb{R}^k\to\mathbb{R}^n$,

$$\phi(\bar{x})=\big(\bar{x},0,\dots,0,g(|\bar{x}|)\big),$$

where $g:(0,1)\to(0,1/2)$ is a diffeomorphism satisfying $|(g^{-1})'(t)|\le 1$ and

$$\lim_{t\to 0^+} g(t)=0.$$

Our standard surface is defined as

$$\Omega^k=\phi\big(B^k(0,1)\setminus\{0\}\big).$$

For $x\in\partial\mathbb{H}^n\cap Q(0,1)$ define

$$\Omega^k_x=\Omega^k+x.$$

Our main theorem is the following theorem.

THEOREM 1.1. *Let $n\ge 3$, and let $f:\mathbb{H}^n\to\mathbb{R}^n$ be a bounded mapping of finite distortion. Assume that g is as above and also satisfies*

$$g^{-1}(t)\le Ct^{\zeta}\log^{-\sigma}\left(\frac{1}{t}\right).$$

If f meets one of the following conditions:

(i) *f is quasiregular,* $\zeta = 1$ *and*

$$\sigma > \frac{n+k}{n(k-1)},$$

(ii) *there is* $\lambda > 0$ *such that* $\int_{\mathbb{H}^n \cap B(a,r)} \exp(\lambda K_f) < \infty$ *for some* $r > 0$ *and every* $a \in \partial \mathbb{H}^n$, $\zeta = 1$ *and*

$$\sigma > \frac{n(k+1)+k}{n(k-1)},$$

(iii) $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{H}^n, \mathbb{R}^n)$, *there is* $p > n-1$ *such that* $\int_{\mathbb{H}^n \cap B(a,r)} K_f^p < \infty$ *for some* $r > 0$ *and every* $a \in \partial \mathbb{H}^n$, $\zeta = 1 + \frac{k}{p(k-1)}$ *and*

$$\sigma > \frac{p(n+k)+k}{pn(k-1)},$$

then $\mathcal{H}^k(f(\Omega^k_x)) < \infty$ *for almost every* $x \in \partial \mathbb{H}^n$.

Notice that if the distortion is p-integrable, then the results are weaker than in the exponentially integrable distortion case and at the limit $p \to \infty$ we recover the quasiregular case. For the proof, we use the modulus of k-dimensional surfaces and a generalization of the K_O-inequality for path families. In the quasiregular case, if $k = n-1$ we have $\sigma > \frac{2n-1}{n(n-2)}$. This gives a better result than the proof in [11]. The sharpness of these results would be interesting to know, but there is a lack of examples even in the quasiregular case.

2. Notation and preliminaries

We denote the upper half space of the n-dimensional Euclidean space $\mathbb{R}^n$ by $\mathbb{H}^n$. Euclidean ball and a sphere of dimension k, with center x and radius r, are denoted by $B^k(x,r)$ and $S^k(x,r)$, respectively. We also define

$$\mathbb{S}^{k-1}(x,r) := \big\{ y \in \mathbb{R}^n : y = (\bar{y}, 0, \dots, 0), |\bar{y}| = r \big\} + x.$$

By $Q(x,r)$ we mean a closed cube with center x and side length $2r$. The symbol $|\cdot|$ denotes Euclidean norm or operator norm depending on the input. It will be clear from the context which norm we mean. By $A \lesssim B$ we mean $A \leq CB$, where C only depends on the data. Moreover, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. We denote k-dimensional Hausdorff measure by $\mathcal{H}^k$. Notice that Ω^k_x is symmetric with respect to the line $\{x + te_n : t > 0\}$ and

$$\{x_n = t\} \cap \Omega^k_x = \mathbb{S}^{k-1}\big(x + te_n, g^{-1}(t)\big).$$

For each $i \in \mathbb{N}$, we set $t_i = 2^{-i}$ and $H(i) = \{t_{i+1} \leq x_n \leq t_i\}$. Moreover, define

$$\Omega^k_{x,i} = \Omega^k_x \cap H(i).$$

Assume that $E \subset \partial\mathbb{H}^n$ is Borel-measurable, and define a collection of k-dimensional surfaces

$$\Gamma_E^i = \{\Omega_{x,i}^k : x \in E\}.$$

Let $s \geq 1$, and let $\omega : \mathbb{R}^n \to [0, \infty]$ be a measurable function. A basic tool we use in proving Theorem 1.1 is the concept of modulus for k-dimensional sets. Let Γ be a collection of sets such that $\mathcal{H}^k(\gamma) > 0$, for all $\gamma \in \Gamma$. We say that a Borel-measurable function $\rho : \mathbb{R}^n \to [0, \infty]$ is admissible for Γ, denoted by $\rho \in \mathrm{Adm}(\Gamma)$, if

$$\int_\gamma \rho \, d\mathcal{H}^k \geq 1$$

for all $\gamma \in \Gamma$. The weighted s-modulus of Γ with weight ω, is defined as

$$\mathrm{Mod}_s^\omega(\Gamma) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_{\mathbb{H}^n} \rho^{\frac{s}{k}}(x)\omega(x) \, dx.$$

If $\omega \equiv 1$ then we just write Mod_s. One should notice that Mod_n is invariant under conformal mappings ($K_f \equiv 1$). A collection Γ is said to be s-exceptional if $\mathrm{Mod}_s(\Gamma) = 0$. This is equivalent to the fact that there exists $\frac{s}{k}$-integrable $\rho : \mathbb{R}^n \to [0, \infty]$ such that

$$\int_\gamma \rho \, d\mathcal{H}^k = \infty$$

for all $\gamma \in \Gamma$, see [4, Theorem 2].

The following lemma is very useful in what follows. This is a special case of [12, Lemma 4.6]. Denote

$$m_{k-1}(u, r) = \mathcal{H}^{k-1}\big(\mathbb{S}^{k-1}(u, r) \cap E\big),$$

when $u \in \partial\mathbb{H}^n$ and $r > 0$.

Lemma 2.1. *Assume $\rho : \mathbb{R}^n \to [0, \infty]$ Borel-measurable, and that $E \subset \partial\mathbb{H}^n$ is a Borel set. Then*

$$\int_E \int_{\mathbb{S}^{k-1}(x,r)} \rho(y) \, d\mathcal{H}^{k-1}(y) \, d\mathcal{H}^{n-1}(x) = \int_{\partial\mathbb{H}^n} \rho(u) m_{k-1}(u, t) \, d\mathcal{H}^{n-1}(u).$$

Proof. First, we observe that

$$m_{k-1}(u, r) = \mathcal{H}^{k-1}\big(\mathbb{S}^{k-1}(0, r) \cap (E - u)\big) = \int_{\partial\mathbb{H}^n} \chi_E(z + u) \, d\mathcal{H}^{k-1}_{|_{\mathbb{S}^{k-1}(0,r)}}(z).$$

This leads to the following chain of equalities:

$$\begin{aligned}
&\int_{\partial\mathbb{H}^n} \rho(u) m_{k-1}(u, r) \, d\mathcal{H}^{n-1}(u) \\
&\quad = \int_{\partial\mathbb{H}^n} \int_{\partial\mathbb{H}^n} \rho(u) \chi_E(z + u) \, d\mathcal{H}^{k-1}_{|_{\mathbb{S}^{k-1}(0,r)}}(z) \, d\mathcal{H}^{n-1}(u) \\
&\quad = \int_{\partial\mathbb{H}^n} \int_{\partial\mathbb{H}^n} \rho(x - z) \chi_E(x) \, d\mathcal{H}^{n-1}(x) \, d\mathcal{H}^{k-1}_{|_{\mathbb{S}^{k-1}(0,r)}}(z)
\end{aligned}$$

$$= \int_E \int_{\mathbb{S}^{k-1}(0,r)} \rho(x-z)\, d\mathcal{H}^{k-1}(z)\, d\mathcal{H}^{n-1}(x)$$
$$= \int_E \int_{\mathbb{S}^{k-1}(x,r)} \rho(y)\, d\mathcal{H}^{k-1}(y)\, d\mathcal{H}^{n-1}(x). \qquad \square$$

3. Modulus bounds for Γ_E^i and K_O-inequality

For the proof of Theorem 1.1, we need lower bounds for $\mathrm{Mod}_s^{\omega}(\Gamma_E^i)$. These are derived using change of variables, Lemma 2.1 and some basic inequalities. The next elementary inequality is needed in the proof of Lemma 3.2. Fix $i \in \mathbb{N}$ for this and the next section.

LEMMA 3.1. *Assume $a, b \geq 0$ and $\alpha > 0$. Then*

$$ab \leq a \log^{\frac{1}{\alpha}}(a+1) + e^{(1+\frac{1}{\alpha})b^{\alpha}}.$$

Proof. First, notice that if $f : [0,\infty[\to [0,\infty[$ is strictly increasing and $f(0) = 0$, then

$$ab \leq \int_0^a f(t)\, dt + \int_0^b f^{-1}(t)\, dt.$$

Set $f(t) = \log^{\frac{1}{\alpha}}(t+1)$. Then $f^{-1}(t) = e^{t^{\alpha}} - 1$ and using above inequality and the fact that $b \leq e^{\frac{1}{\alpha}b^{\alpha}}$, we have

$$ab \leq \int_0^a \log^{\frac{1}{\alpha}}(t+1)\, dt + \int_0^b e^{t^{\alpha}} - 1\, dt$$
$$\leq a \log^{\frac{1}{\alpha}}(a+1) + be^{b^{\alpha}} \leq a \log^{\frac{1}{\alpha}}(a+1) + e^{(1+\frac{1}{\alpha})b^{\alpha}}. \qquad \square$$

The next lemma contains lower bounds for $\mathrm{Mod}_s^{\omega}(\Gamma_E^i)$ with no weight and with p- and exponentially integrable weights.

LEMMA 3.2. *Assume $E \subset \partial\mathbb{H}^n \cap Q(0,1)$ and $s > n-1$. Then*

$$\mathrm{Mod}_s\big(\Gamma_E^i\big) \gtrsim g^{-1}(t_i)^{\frac{s}{k}(1-k)} 2^{\frac{i(s-k)}{k}} \mathcal{H}^{n-1}(E).$$

Assume that $K : \mathbb{R}^n \to [1,\infty]$ is measurable function. If there is $\lambda > 0$ such that $\exp(\lambda K) \in L^1(\mathbb{H}^n \cap Q(0,5))$, then we have the following:

$$\mathrm{Mod}_s^{K^{-1}}\big(\Gamma_E^i\big) \gtrsim g^{-1}(t_i)^{\frac{s}{k}(1-k)} 2^{\frac{i(s-k)}{k}} \left(\log\left(\frac{2^i}{\mathcal{H}^{n-1}(E)} + 1\right) + 1\right)^{-1} \mathcal{H}^{n-1}(E).$$

If instead of exponential integrability we assume that $K \in L^p(\mathbb{H}^n \cap Q(0,5))$ for $p > n-1$, then

$$\mathrm{Mod}_s^{K^{-1}}\big(\Gamma_E^i\big) \gtrsim g^{-1}(t_i)^{\frac{s}{k}(1-k)} 2^{i\frac{p(s-k)-k}{pk}} \mathcal{H}^{n-1}(E)^{\frac{p+1}{p}}.$$

Proof. Define

$$A(u,t) = m_{k-1}\big(u, g^{-1}(t)\big)$$

and fix $x \in E$. If $\rho \in \mathrm{Adm}(\Gamma_E^i)$ then

$$1 \le \int_{\Omega_{x,i}^k} \rho(y)\, d\mathcal{H}^k(y). \tag{5}$$

Set $h : \mathbb{R}^n \to \mathbb{R}$, $h(x) = x_n$. Using the Co-area formula on rectifiable sets [2, Theorem 2.93], we have

$$\int_{\Omega_{x,i}^k} \rho(y) C_k D^{\Omega_{x,i}^k} h(y)\, d\mathcal{H}^k = \int_{t_{i+1}}^{t_i} \int_{\Omega_{x,i}^k \cap h^{-1}(t)} \rho(y)\, d\mathcal{H}^{k-1}(y)\, dt,$$

where $D^{\Omega_{x,i}^k} h(y) = \nabla h(y)|_{D\phi(\phi^{-1}(y))(\mathbb{R}^k)}$ and $C_k D^{\Omega_{x,i}^k} h(y)$ is the Co-area factor defined as $C_k L = \sqrt{\det LL^{\mathsf{T}}}$ for a linear map $L : \mathbb{R}^k \to \mathbb{R}^m$, $k \ge m$. Since $|(g^{-1})'(t)| \le 1$ we know that $C_k D^{\Omega_{x,i}^k} h(y)$ is bounded below by constant depending only on the dimension. Using this with (5) and integrating both sides over E, we get

$$\mathcal{H}^{n-1}(E) \lesssim \int_{t_{i+1}}^{t_i} \int_E \int_{\mathbb{S}^{k-1}(x, g^{-1}(t))} \rho(y + te_n)\, d\mathcal{H}^{k-1}(y)\, d\mathcal{H}^{n-1}(x)\, dt.$$

Now we can use Lemma 2.1 to get

$$\mathcal{H}^{n-1}(E) \lesssim \int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n} \rho(u + te_n) A(u,t)\, d\mathcal{H}^{n-1}(u)\, dt. \tag{6}$$

From here we use different tools depending on the weight associated to the modulus we are looking at. In the non-weighted case, we use Hölder's inequality and Lemma 2.1 to get the following chain of inequalities:

$$\begin{aligned}
\mathcal{H}^{n-1}(E) &\lesssim \int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n} \rho(u + te_n) A(u,t)\, d\mathcal{H}^{n-1}(u)\, dt \\
&\le \left(\int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n} \rho^{\frac{s}{k}}(u + te_n)\, d\mathcal{H}^{n-1}(u)\, dt \right)^{\frac{k}{s}} \\
&\quad \times \left(\int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n} A(u,t)^{\frac{s}{s-k}}\, d\mathcal{H}^{n-1}(u)\, dt \right)^{\frac{s-k}{s}} \\
&\lesssim \left(\int_{H(i)} \rho^{\frac{s}{k}}(x)\, dx \right)^{\frac{k}{s}} \\
&\quad \times \left(\int_{t_{i+1}}^{t_i} g^{-1}(t)^{\frac{k(k-1)}{s-k}} \int_{\partial\mathbb{H}^n} A(u,t)\, d\mathcal{H}^{n-1}(u)\, dt \right)^{\frac{s-k}{s}} \\
&\lesssim \left(\int_{H(i)} \rho^{\frac{s}{k}}(x)\, dx \right)^{\frac{k}{s}} \left(\int_{t_{i+1}}^{t_i} g^{-1}(t)^{\frac{s(k-1)}{s-k}}\, dt \right)^{\frac{s-k}{s}} \big(\mathcal{H}^{n-1}(E)\big)^{\frac{s-k}{s}}.
\end{aligned}$$

From this we have the claim since g is increasing and $\rho \in \mathrm{Adm}(\Gamma_E^i)$ was arbitrary. Next assume that we have a weight $\omega = \frac{1}{K}$ on our modulus, where

$$\int_{\mathbb{H}^n \cap Q(0,5)} \exp(\lambda K)\,dx < \infty.$$

Now we continue from (6) by Hölder's inequality as in the non-weighted case:

$$\begin{aligned}
\mathcal{H}^{n-1}(E) &\lesssim \int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n} \rho(u+te_n) A(u,t)\,d\mathcal{H}^{n-1}(u)\,dt \\
&\le \left(\int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n} \rho^{\frac{s}{k}}(u+te_n) K^{-1}(u+te_n)\,d\mathcal{H}^{n-1}(u)\,dt \right)^{\frac{k}{s}} \\
&\quad \times \left(\int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n \cap Q(0,2)} K^{\frac{k}{s-k}}(u+te_n) A(u,t)^{\frac{s}{s-k}}\,d\mathcal{H}^{n-1}(u)\,dt \right)^{\frac{s-k}{s}}.
\end{aligned}$$

Let $\phi_i > 0$ constant which will be given later. Using Lemma 3.1, we can estimate the last term in the above inequality so that

$$\begin{aligned}
&\int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n \cap Q(0,2)} K^{\frac{k}{s-k}}(u+te_n) A(u,t)^{\frac{s}{s-k}}\,d\mathcal{H}^{n-1}(u)\,dt \\
&\quad \le \tilde{C} \int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n \cap Q(0,2)} A(u,t)^{\frac{s}{s-k}} \log^{\frac{k}{s-k}}\big(\tilde{C} A(u,t)^{\frac{s}{s-k}}/\phi_i + 1\big)\,d\mathcal{H}^{n-1}(u)\,dt \\
&\qquad + \phi_i \int_{\mathbb{H}^n \cap Q(0,5)} \exp\big(\lambda K(x)\big)\,dx.
\end{aligned}$$

Here $\tilde{C} = s/\lambda(s-k)$. Now we choose ϕ_i so that

$$\phi_i \int_{\mathbb{H}^n \cap Q(0,5)} \exp\big(\lambda K(x)\big)\,dx = \hat{C} g^{-1}(t_i)^{\frac{s(k-1)}{s-k}} 2^{-i} \mathcal{H}^{n-1}(E),$$

where $\hat{C} = C(n,p,k,\lambda)$. For simplicity, set

$$L = \int_{\mathbb{H}^n \cap Q(0,5)} \exp\big(\lambda K(x)\big)\,dx.$$

Now we can estimate

$$\begin{aligned}
&\int_{t_{i+1}}^{t_i} \int_{\partial\mathbb{H}^n \cap Q(0,2)} K^{\frac{k}{s-k}}(u+te_n) A(u,t)^{\frac{s}{s-k}}\,d\mathcal{H}^{n-1}(u)\,dt \\
&\quad \lesssim \log^{\frac{k}{s-k}}\left(\frac{L2^i}{\mathcal{H}^{n-1}(E)} + 1 \right) \int_{t_{i+1}}^{t_i} g^{-1}(t)^{\frac{s(k-1)}{s-k}-1} \int_{\partial\mathbb{H}^n} A(u,t)\,d\mathcal{H}^{n-1}(u)\,dt \\
&\qquad + g^{-1}(t_i)^{\frac{s(k-1)}{s-k}} 2^{-i} \mathcal{H}^{n-1}(E) \\
&\quad \lesssim \mathcal{H}^{n-1}(E) \log^{\frac{k}{s-k}}\left(\frac{L2^i}{\mathcal{H}^{n-1}(E)} + 1 \right) \int_{t_{i+1}}^{t_i} g^{-1}(t)^{\frac{s(k-1)}{s-k}}\,dt
\end{aligned}$$

$$
\begin{aligned}
&+g^{-1}(t_i)^{\frac{s(k-1)}{s-k}}2^{-i}\mathcal{H}^{n-1}(E)\\
&\lesssim g^{-1}(t_i)^{\frac{s(k-1)}{s-k}}2^{-i}\left(\log^{\frac{k}{s-k}}\left(\frac{2^i}{\mathcal{H}^{n-1}(E)}+1\right)+1\right)\mathcal{H}^{n-1}(E).
\end{aligned}
$$

Here we used Lemma 2.1 and the fact that g^{-1} is increasing. Since ρ was arbitrary, combining the above estimate with the earlier one gives

$$
\mathcal{H}^{n-1}(E)^{\frac{k}{s}}\lesssim\left(\mathrm{Mod}_p^{K^{-1}}\left(\Gamma_E^i\right)\right)^{\frac{k}{s}}g^{-1}(t_i)^{k-1}2^{\frac{-i(s-k)}{s}}\left(\log^{\frac{k}{s}}\left(\frac{2^i}{\mathcal{H}^{n-1}(E)}+1\right)+1\right).
$$

Rearranging and raising to power $\frac{s}{k}$ gives the claim

$$
\mathrm{Mod}_p^{K^{-1}}\left(\Gamma_E^i\right)\gtrsim g^{-1}(t_i)^{\frac{s}{k}(1-k)}2^{\frac{i(s-k)}{k}}\left(\log\left(\frac{2^i}{\mathcal{H}^{n-1}(E)}+1\right)+1\right)^{-1}\mathcal{H}^{n-1}(E).
$$

If $K\in L^p(\mathbb{H}^n)$ for some $p>n-1$, then we can continue from (6) applying Hölder's inequality with $\beta=\frac{ps}{k}$:

$$
\mathcal{H}^{n-1}(E)^{1-\frac{1}{\eta}}\lesssim\left(\int_{H(i)}\rho^{\frac{s}{k}}(x)K^{-1}(x)\,dx\right)^{\frac{k}{s}}\|K\|_{L^p}^{\frac{1}{\beta}}\left(\int_{t_{i+1}}^{t_i}g^{-1}(t)^{\eta(k-1)}\,dt\right)^{\frac{1}{\eta}},
$$

where $\eta=\frac{ps}{p(s-k)-k}$ and the claim follows as in the non-weighted case. Thus, the lemma is proved. $\square$

In the remaining parts of this section, we will prove an analog of the K_O-inequality for path families for $\mathrm{Mod}_s^{\omega}(\Gamma_E^i)$. By $D^kf(x)$, we mean the $\binom{n}{k}\times\binom{n}{k}$ matrix whose entries are the $k\times k$-minors of $Df(x)$. Also, we denote the Hilbert–Schmidt norm of a matrix A by $|A|_{\mathrm{HS}}$. In the next lemma, we prove a change of variables formula for $f\in W^{1,s}_{\mathrm{loc}}(\mathbb{H}^n\cap Q(0,5),\mathbb{R}^n)$, $s>n-1$, on k-dimensional sets $\Omega_{x,i}^k$.

LEMMA 3.3. *Let $E\subset\partial\mathbb{H}^n\cap Q(0,1)$ be a Borel set. Assume that $\rho:\mathbb{H}^n\cap Q(0,5)\to[0,\infty]$ is continuous and bounded. If $f\in W^{1,s}_{\mathrm{loc}}(\mathbb{H}^n\cap Q(0,5),\mathbb{R}^n)$ for some $s>n-1$ then*

$$
\sqrt{n}\int_{\Omega_{x,i}^k}\rho\big(f(z)\big)\big|D^kf(z)\big|\,d\mathcal{H}^k(z)\geq\int_{f(\Omega_{x,i}^k)}\rho(y)\,d\mathcal{H}^k(y)\tag{7}
$$

for almost every $x\in E$.

Proof. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of smooth convolution approximations of f. First, we show that

$$
\lim_{j\to\infty}\int_{\Omega_{x,i}^k}\rho\big(f_j(z)\big)\big|D^kf_j(z)\big|\,d\mathcal{H}^k(z)=\int_{\Omega_{x,i}^k}\rho\big(f(z)\big)\big|D^kf(z)\big|\,d\mathcal{H}^k(z),\tag{8}
$$

for almost every $x \in E$. Denote $Q = H(i) \cap Q(0,2)$. We make the following observation:

$$
\begin{aligned}
&\int_Q \big|\rho\big(f_j(x)\big)\big|D^k f_j(x)\big| - \rho\big(f(x)\big)\big|D^k f(x)\big|\big|^{\frac{s}{k}}\,dx \\
&\quad\lesssim \int_Q \big|\rho\big(f_j(x)\big) - \rho\big(f(x)\big)\big|^{\frac{s}{k}}\big|D^k f_j(x)\big|^{\frac{s}{k}}\,dx \\
&\qquad + \int_Q \big|\rho\big(f(x)\big)\big|^{\frac{s}{k}}\big|\big|D^k f(x)\big| - \big|D^k f_j(x)\big|\big|^{\frac{s}{k}}\,dx \\
&\quad\le \int_Q \big|\rho\big(f_j(x)\big) - \rho\big(f(x)\big)\big|^{\frac{s}{k}}\big|Df_j(x)\big|^{s}\,dx \\
&\qquad + \|\rho\|_\infty^{\frac{s}{k}} \int_Q \big|\big|D^k f(x)\big| - \big|D^k f_j(x)\big|\big|^{\frac{s}{k}}\,dx \\
&\quad\lesssim \int_Q \big|\rho\big(f_j(x)\big) - \rho\big(f(x)\big)\big|^{\frac{s}{k}}\big|Df_j(x)\big|^{s}\,dx \\
&\qquad + \|\rho\|_\infty^{\frac{s}{k}} \int_Q \big|D^k f(x) - D^k f_j(x)\big|_{\mathrm{HS}}^{\frac{s}{k}}\,dx.
\end{aligned}
$$

Notice that

$$
\int_Q \big|\rho\big(f_j(x)\big) - \rho\big(f(x)\big)\big|^{\frac{s}{k}}\big|Df_j(x)\big|^{s}\,dx \to 0
$$

as $j \to \infty$, since ρ is continuous and $f_j \to f$ uniformly on Q. If the latter term in the above inequality vanishes as $j \to \infty$, then we have the claim by Fuglede's lemma [4, Theorem 3] for all $x \in E \setminus F$, where $\mathrm{Mod}_s(\Gamma_F^i) = 0$. Then by Lemma 3.2 we know that $\mathcal{H}^{n-1}(F) = 0$. Thus it suffices to show

$$
\int_Q \big|D^k f(x) - D^k f_j(x)\big|_{\mathrm{HS}}^{\frac{n}{k}}\,dx \to 0
$$

as $j \to \infty$. For this, we use the language of differential forms. Notice that the following holds:

$$
\big|D^k f(x) - D^k f_j(x)\big|_{\mathrm{HS}}^2 \le \sum_{1 \le i_1 \le \cdots \le i_k \le n} \big|df_j^{i_1} \wedge \cdots \wedge df_j^{i_k} - df^{i_1} \wedge \cdots \wedge df^{i_k}\big|^2.
$$

Writing the right-hand side of this equality as a telescoping sum and using Hadamard's inequality, we have

$$
\begin{aligned}
&\sum_{1 \le i_1 \le \cdots \le i_k \le n} \big|df_j^{i_1} \wedge \cdots \wedge df_j^{i_k} - df^{i_1} \wedge \cdots \wedge df^{i_k}\big|^2 \\
&\quad= \sum_{1 \le i_1 \le \cdots \le i_k \le n} \left|\sum_{l=1}^{k} df^{i_1} \wedge \cdots \wedge df^{i_{l-1}} \wedge \big(df_j^{i_l} - df^{i_l}\big) \wedge \cdots \wedge df_j^{i_k}\right|^2
\end{aligned}
$$

$$\lesssim \sum_{1\le i_1\le\cdots\le i_k\le n}\left(\sum_{l=1}^{k}|df^{i_1}|\cdots|df^{i_{l-1}}||df_j^{i_l}-df^{i_l}|\cdots|df_j^{i_k}|\right)^2$$
$$\lesssim \left(\sum_{l=1}^{k}|Df(x)|^{l-1}|Df_j(x)|^{k-l}|Df_j(x)-Df(x)|\right)^2.$$

Now we may deduce by Hölder's inequality that

$$\begin{aligned}
&\int_Q \big||D^kf(x)|-|D^kf_j(x)|\big|^{\frac{s}{k}}\,dx\\
&\quad\lesssim \sum_{l=1}^{k}\Big[\int_Q |Df(x)|^{\frac{s(l-1)}{k}}|Df_j(x)-Df(x)|^{\frac{s(k-(l-1))}{k}}\,dx\\
&\qquad+\int_Q |Df(x)|^{\frac{s(k-1)}{k}}|Df_j(x)-Df(x)|^{\frac{s}{k}}\,dx\Big]\\
&\quad\lesssim \sum_{l=1}^{k}\Big[\|Df(x)\|_s^{\frac{s(l-1)}{k}}\|Df_j(x)-Df(x)\|_s^{\frac{s(k-(l-1))}{k}}\\
&\qquad+\|Df(x)\|_s^{\frac{s(k-1)}{k}}\|Df_j(x)-Df(x)\|_s^{\frac{s}{k}}\Big]\to 0
\end{aligned}$$

as $j\to\infty$. Thus we have shown (8). Define the Area factor as $J_kL=\sqrt{\det L^{\mathsf{T}}L}$ for a linear map $L:\mathbb{R}^k\to\mathbb{R}^n$, $k\le n$. Using (8), the Area and Cauchy–Binet formulas [2, Theorem 2.71 and Proposition 2.69] we have for almost every $x\in E$

$$\begin{aligned}
&\sqrt{n}\int_{\Omega^k_{x,i}}\rho\big(f(z)\big)|D^kf(z)|\,d\mathcal{H}^k(z)\\
&\ge \int_{\Omega^k_{x,i}}\rho\big(f(z)\big)|D^kf(z)|_{\mathrm{HS}}\,d\mathcal{H}^k(z)\\
&=\lim_{j\to\infty}\int_{\Omega^k_{x,i}}\rho\big(f_j(z)\big)|D^kf_j(z)|_{\mathrm{HS}}\,d\mathcal{H}^k(z)\\
&\ge\lim_{j\to\infty}\int_{\Omega^k_{x,i}}\rho\big(f_j(z)\big)J_kD^{\Omega^k_{x,i}}f_j(z)\,d\mathcal{H}^k(z)\\
&=\lim_{j\to\infty}\int_{B^k(0,1)\setminus\{0\}}\rho\big(f_j\circ\phi(w)\big)J_kD^{\Omega^k_{x,i}}f_j\big(\phi(w)\big)J_kD\phi(w)\,d\mathcal{H}^k(w)\\
&=\lim_{j\to\infty}\int_{f_j(\Omega^k_{x,i})}\rho(y)N\big(y,f_j\circ\phi,B^k(0,1)\setminus\{0\}\big)\,d\mathcal{H}^k(y)\\
&\ge\int_{f(\Omega^k_{x,i})}\rho(y)\,d\mathcal{H}^k(y).
\end{aligned}$$

Here $N(y,f,U)$ is the number of preimages that y has with mapping f in the set U. $\square$

Now we are in the position to prove an analog of the well-known K_O-inequality for path families [14].

LEMMA 3.4. *Suppose that $E\subset\partial\mathbb{H}^n\cap Q(0,1)$ is a Borel set and F is the set where the change of variables formula (7) does not hold. Assume further that $\rho\in \mathrm{Adm}(f(\Gamma^i_{E\setminus F}))\cap C^0(\mathbb{H}^n\cap Q(0,5))\cap L^\infty(\mathbb{H}^n\cap Q(0,5))$. Then we have for quasiregular mapping f*

$$\mathrm{Mod}_n\big(\Gamma^i_{E\setminus F}\big)\le n^{\frac{n}{2k}}\|K\|_\infty\int_{\mathbb{H}^n}\rho(y)^{\frac{n}{k}}N\big(y,f,H(i)\cap Q(0,2)\big)\,dy.$$

If instead f has exponentially integrable distortion or $f\in W^{1,n}_{\mathrm{loc}}(\mathbb{H}^n\cap Q(0,5))$ and has p-integrable distortion, $p>n-1$, then

$$\mathrm{Mod}_n^{\frac{1}{K}}\big(\Gamma^i_{E\setminus F}\big)\le n^{\frac{n}{2k}}\int_{\mathbb{H}^n}\rho(y)^{\frac{n}{k}}N\big(y,f,H(i)\cap Q(0,2)\big)\,dy.$$

Proof. Define $\hat\rho:\mathbb{H}^n\to\mathbb{R}^n$ s.t.

$$\hat\rho(z)=\sqrt{n}\rho\big(f(z)\big)\big|D^kf(z)\big|\chi_{H(i)\cap Q(0,2)}(z).$$

Then by Lemma 3.3 we have that $\hat\rho\in\mathrm{Adm}(\Gamma^i_{E\setminus F})$. Thus by distortion inequality (3) and change of variables we have

$$\begin{aligned}\mathrm{Mod}_n\big(\Gamma^i_{E\setminus F}\big)&\le n^{\frac{n}{2k}}\int_{H(i)\cap Q(0,2)}\rho\big(f(z)\big)^{\frac{n}{k}}\big|D^kf(z)\big|^{\frac{n}{k}}\,dz\\&\le n^{\frac{n}{2k}}\|K\|_\infty\int_{\mathbb{H}^n}\rho(y)^{\frac{n}{k}}N\big(y,f,H(i)\cap Q(0,2)\big)\,dy.\end{aligned}$$

This proves the claim in the quasiregular case and the p-integrable and exponential integrable distortion cases are proved similarly. $\square$

4. Estimates for the integral of the Jacobian determinant

For the proof of our main theorem, we need to establish bounds for the growth of the L^1-norm of the Jacobian determinant of a mapping of finite distortion.

THEOREM 4.1. *Assume that $f:\mathbb{H}^n\cap Q(0,5)\to\mathbb{R}^n$ is a bounded mapping of finite distortion. Then*

$$\int_{H(i)\cap Q(0,2)}J_f(x)\,dx\lesssim\theta_f(i),$$

where $\theta_f(i)=2^{i(n-1)}$ if f is quasiregular, $\theta_f(i)=2^{i(n-1)}i^{n-1}$ if f has exponentially integrable distortion and $\theta_f(i)=2^{\frac{p+1}{p}i(n-1)}$ if $f\in W^{1,n}_{\mathrm{loc}}(\mathbb{H}^n\cap Q(0,5))$ and has p-integrable distortion.

Proof. With all of the above assumptions on f, J_f fulfills the following integration by parts formula:

$$\int_{\mathbb{H}^n} \varphi(x) J_f(x)\, dx = -\int_{\mathbb{H}^n} f_1(x) J_{(\varphi, f_2, \ldots, f_n)}(x)\, dx$$

for all $\varphi \in C_0^\infty(\mathbb{H}^n \cap Q(0,5))$ cf. [5, Theorem 7.2.1]. Let $\varphi = \psi^n$, where $\psi \geq 0$, $\psi \in C_0^\infty(\mathbb{H}^n \cap Q(0,5))$. Thus using integration by parts, distortion inequality and Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{H}^n} \psi^n J_f\, dx &\leq \int_{\mathbb{H}^n} |f_1| \big| d\big(\psi^n\big) \wedge df_2 \wedge \cdots \wedge df_n \big|\, dx \\
&\leq n \int_{\mathbb{H}^n} |f| |\psi|^{n-1} |\nabla \psi| |Df|^{n-1}\, dx \\
&\leq n \int_{\mathbb{H}^n} |f| |\psi|^{n-1} |\nabla \psi| K^{\frac{n-1}{n}} J_f^{\frac{n-1}{n}}\, dx \\
&\leq n \|f\|_\infty \bigg(\int_{\mathbb{H}^n} |\nabla \psi|^n K^{n-1}\, dx \bigg)^{\frac{1}{n}} \bigg(\int_{\mathbb{H}^n} |\psi|^n J_f\, dx \bigg)^{\frac{n-1}{n}}.
\end{aligned}$$

This implies that

$$\int_{\mathbb{H}^n} |\psi|^n J_f\, dx \leq n^n \|f\|_\infty^n \int_{\mathbb{H}^n} |\nabla \psi|^n K^{n-1}\, dx. \tag{9}$$

Next choose ψ such that

$$\begin{aligned}
\psi(x) &= 1 \quad \text{if } x \in H(i) \cap Q(0,2), \\
\psi(x) &= 0 \quad \text{if } x \notin \hat{Q}, \\
\big|\nabla \psi(x)\big| &\leq C(n) 2^i \quad \text{if } x \in \hat{Q} \setminus H(i) \cap Q(0,2),
\end{aligned}$$

where $\hat{Q}$ is the expansion of the set $H(i) \cap Q(0,2)$ by a factor of 2^{-i-2}. We observe that $|\hat{Q} \setminus H(i) \cap Q(0,2)| \leq C(n) 2^{-i}$. First, assume that f has exponentially integrable distortion. Using Jensen's inequality for the convex function $\psi(x) = \exp(\lambda x^{\frac{1}{n-1}})$ yields

$$\begin{aligned}
\int_{\hat{Q}} K(x)^{n-1}\, dx &= |\hat{Q}| \psi^{-1} \bigg(\psi \bigg(\fint_{\hat{Q}} K(x)^{n-1}\, dx \bigg) \bigg) \\
&\leq |\hat{Q}| \psi^{-1} \bigg(\fint_{\hat{Q}} \psi\big(K(x)^{n-1}\big)\, dx \bigg) \\
&\lesssim 2^{-i} \log^{n-1} \bigg(2^i \int_{\mathbb{H}^n \cap Q(0,5)} \exp\big(\lambda K(x)\big)\, dx \bigg) \\
&\lesssim 2^{-i} \log^{n-1}\big(2^i\big) \lesssim 2^{-i} i^{n-1}.
\end{aligned}$$

Combining this with (9) gives

$$\int_{H(i)\cap Q(0,2)} J_f(x)\,dx \lesssim 2^{in} \int_{\hat{Q}\setminus H(i)\cap Q(0,2)} K(x)^{n-1}\,dx$$
$$\lesssim 2^{in} \int_{\hat{Q}} K(x)^{n-1}\,dx \lesssim 2^{i(n-1)} i^{n-1}.$$

Next, assume f is quasiregular. Then from (9) it follows that

$$\int_{H(i)\cap Q(0,2)} J_f(x)\,dx \lesssim 2^{in} \int_{\hat{Q}\setminus H(i)\cap Q(0,2)} K(x)^{n-1}\,dx$$
$$\lesssim 2^{i(n-1)}.$$

If f has p-integrable distortion then apply Hölder's inequality to the right-hand side of (9) and thus

$$\int_{H(i)\cap Q(0,2)} J_f(x)\,dx \lesssim 2^{in} \left(\int_{\hat{Q}\setminus H(i)\cap Q(0,2)} K(x)^p\,dx \right)^{\frac{n-1}{p}}$$
$$\times \left| \hat{Q} \setminus H(i) \cap Q(0,2) \right|^{\frac{p-(n-1)}{p}}$$
$$\lesssim 2^{i\frac{np-p+(n-1)}{p}}.$$

□

5. Proof of the main theorem

Now we have the tools to prove our main theorem.

Proof of Theorem 1.1. Let $f : \mathbb{H}^n \to \mathbb{R}^n$ be a bounded mapping of finite distortion. It is sufficient to show the claim in $\partial\mathbb{H}^n \cap Q(0,1)$. Notice that Lemma 3.3 fails in a set of measure zero, thus we may restrict to those $x \in \partial\mathbb{H}^n \cap Q(0,1)$ for which Lemma 3.3 holds. We may also assume that

$$\int_{\Omega^k_{x,i}} \left| D^k f(z) \right| d\mathcal{H}^k(z) < \infty.$$

This can be deduced as follows: for each assumption (i)–(iii) in Theorem 1.1, there exists $s_f > n-1$ such that $f \in W^{1,s_f}_{\mathrm{loc}}(\mathbb{H}^n \cap Q(0,5), \mathbb{R}^n)$. Then we observe that

$$\int_{H(i)\cap Q(0,2)} \left| D^k f(z) \right|^{\frac{s_f}{k}} dz \le \int_{H(i)\cap Q(0,2)} \left| Df(z) \right|^{s_f} dz < \infty.$$

Thus by the definition of $\frac{s_f}{k}$-exceptional sets we know that the modulus of those $\Omega^k_{x,i} \subset \mathbb{H}^n \cap Q(0,2)$ for which

$$\int_{\Omega^k_{x,i}} \left| D^k f(z) \right| d\mathcal{H}^k(z) = \infty$$

is zero. Then by Lemma 3.2 the corresponding collections of vertex points has $(n-1)$-dimensional measure zero.

Now assume that f is a quasiregular mapping. Let σ be as in the statement of the theorem. Define

$$E_i = \{x \in \partial\mathbb{H}^n \cap Q(0,1) : \mathcal{H}^k\big(f\big(\Omega^k_{x,i}\big)\big) > i^{-1-\alpha}\},$$

where

$$\alpha = \frac{\sigma n(k-1)}{n+k} - 1 > 0.$$

By the definition of E_i, we know that $i^{1+\alpha} \in \mathrm{Adm}(f(\Gamma^i_{E_i}))$. Moreover, by Lemmas 3.2 and 3.4, we have

$$\mathcal{H}^{n-1}(E_i) \lesssim g^{-1}(t_i)^{\frac{n}{k}(k-1)} 2^{-i\frac{n-k}{k}} i^{(i+\alpha)\frac{n}{k}} \int_{\mathbb{H}^n} N\big(y, f, H(i) \cap Q(0,2)\big)\, dy.$$

Using change of variables, Theorem 4.1, and the assumption on g^{-1}, we have

$$\begin{aligned}\mathcal{H}^{n-1}(E_i) &\lesssim t_i^{\frac{n}{k}(k-1)+\frac{n-k}{k}} \log\left(\frac{1}{t_i}\right)^{-\frac{\sigma n}{k}(k-1)} i^{(1+\alpha)\frac{n}{k}} 2^{i(n-1)} \\ &= i^{-\frac{\sigma n}{k}(k-1)+(1+\alpha)\frac{n}{k}} = i^{-1-\alpha},\end{aligned}$$

where the last equality follows from the definition of α. Now set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_i.$$

Then by above calculations we have that

$$\mathcal{H}^{n-1}(E) \le \lim_{k\to\infty} \sum_{i=k}^{\infty} \mathcal{H}^{n-1}(E_i) \lesssim \lim_{k\to\infty} \sum_{i=k}^{\infty} i^{-1-\alpha} = 0.$$

Thus, for almost every $x \in \partial\mathbb{H}^n \cap Q(0,1)$, there exists $N_x \in \mathbb{N}$ such that

$$x \in \big(\partial\mathbb{H}^n \cap Q(0,1)\big) \setminus \bigcup_{i=N_x}^{\infty} E_i.$$

Fix such x. Then

$$\sum_{i=N_x}^{\infty} \mathcal{H}^k\big(f\big(\Omega^k_{x,i}\big)\big) \le \sum_{i=N_x}^{\infty} i^{-1-\alpha} < \infty.$$

We still need to show that

$$\mathcal{H}^k\big(f\big(\Omega^k_{x,i}\big)\big) < \infty$$

for all $i = 1, \ldots, N_x$. This follows from the fact that for all these i's,

$$\int_{\Omega^k_{x,i}} \big|D^k f(z)\big|\, d\mathcal{H}^k(z) < \infty,$$

and thus by Lemma 3.3

$$\mathcal{H}^k\big(f\big(\Omega^k_{x,i}\big)\big) \le \int_{\Omega^k_{x,i}} \big|D^k f(z)\big|\, d\mathcal{H}^k(z) < \infty.$$

Thus, the quasiregular case is proved. Next, assume that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{H}^n)$ and

$$\int_{\mathbb{H}^n \cap B(a,r)} K_f^p \, dx < \infty$$

for some $r > 0$ and every $a \in \partial\mathbb{H}^n$. By considering

$$h(x) = f\left(\frac{r}{5\sqrt{n}} x + a\right)$$

we may assume that $a = 0$ and that $\int_{\mathbb{H}^n \cap Q(0,5)} K_f^p \, dx < \infty$. Thus all the results in Sections 3 and 4 are applicable to this situation. Let σ be as in the statement of the theorem and define

$$\tilde{\alpha} = \frac{\sigma p n(k-1)}{p(n+k)+k} - 1 > 0.$$

Now by Lemmas 3.2 and 3.4, and Theorem 4.1, we have

$$\begin{aligned}\mathcal{H}^{n-1}(E_i)^{\frac{p+1}{p}} &\lesssim g^{-1}(t_i)^{\frac{n}{k}(k-1)} \left(2^{-i}\right)^{\frac{p(n-k)-k}{pk}} \int_{\mathbb{H}^n} N\big(y, f, H(i) \cap Q(0,2)\big)\, dy \\ &\le i^{\frac{n}{k}(1+\tilde{\alpha}-\sigma(k-1))} \left(2^{-i}\right)^{\zeta \frac{n}{k}(k-1) + \frac{p(n-k)-k}{pk} + \frac{p+1}{p}(1-n)}.\end{aligned}$$

By our choice of ζ, we have

$$\zeta \frac{n}{k}(k-1) + \frac{p(n-k)-k}{pk} + \frac{p+1}{p}(1-n) = 0.$$

Thus by the choice of $\tilde{\alpha}$ we have that

$$\mathcal{H}^{n-1}(E_i) \lesssim i^{\frac{p}{p+1}(1+\tilde{\alpha}-\sigma(k-1))} = i^{-1-\tilde{\alpha}}.$$

The claim follows from this just as in the quasiregular case. Finally, assume that

$$\int_{\mathbb{H}^n \cap B(a,r)} \exp(\lambda K_f)\, dx < \infty$$

for some $r > 0$ and every $a \in \partial\mathbb{H}^n$. Again, arguing as above we may assume that the results in Sections 3 and 4 are applicable. Then we define

$$\hat{\alpha} = \frac{\sigma n(k-1) - nk}{n+k} - 1 > 0.$$

By the distortion and Hölder's inequalities, we have that $f \in W^{1,s}_{\mathrm{loc}}(\mathbb{H}^n \cap Q(0,5))$, for some $n-1 < s < n$. Thus, Lemma 3.3 holds for f, and using the corresponding parts of Lemmas 3.2 and 3.4, we have

$$\begin{aligned}\mathcal{H}^{n-1}(E_i) \lesssim g^{-1}(t_i)^{\frac{n}{k}(k-1)} 2^{\frac{-i(n-k)}{k}} &\left(\log\left(2^i/\mathcal{H}^{n-1}(E_i)+1\right)+1\right) \\ &\times \int_{\mathbb{H}^n} N\big(y, f, H(i) \cap Q(0,2)\big)\, dy.\end{aligned}$$

We may also assume that

$$\mathcal{H}^{n-1}(E_i) \ge i^{-1-\hat{\alpha}},$$

since otherwise we could use the same arguments as in the quasiregular case. Thus, the above with Theorem 4.1 yields

$$\begin{aligned}\mathcal{H}^{n-1}(E_i) &\lesssim g^{-1}(t_i)^{\frac{n}{k}(k-1)}2^{\frac{-i(n-k)}{k}}\big(\log\big(2^i i^{1+\hat{\alpha}}+1\big)+1\big)\\ &\quad\times i^{(1+\hat{\alpha})\frac{n}{k}}i^{n-1}2^{i(n-1)}\\ &\lesssim i^{(1+\hat{\alpha})\frac{n}{k}+n-1+1-\frac{\sigma n}{k}(k-1)} \approx i^{-1-\hat{\alpha}}.\end{aligned}$$

From this, we can continue just as in the quasiregular case. □

Acknowledgments. The author wishes to thank Prof. Kai Rajala for many discussions and comments on the manuscript. Part of the work was done while the author was visiting the University of Michigan at Ann Arbor. The author wishes to thank the department for the hospitality. The author was financially supported by the foundation of Vilho, Yrjö and Kalle Väisälä.

References

[1] T. Äkkinen, *Radial limits of mappings of bounded and finite distortion*, J. Geom. Anal. **24** (2014), 1298–1322.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, Oxford University Press, New York, 2000. MR 1857292

[3] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 56, Cambridge University Press, Cambridge, 1966. MR 0231999

[4] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219. MR 0097720

[5] T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2001. MR 1859913

[6] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: Discreteness and openness*, Arch. Ration. Mech. Anal. **160** (2001), no. 2, 135–151. MR 1864838

[7] J. Manfredi and E. Villamor, *Mappings with integrable dilatation in higher dimensions*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), no. 2, 235–240. MR 1313107

[8] O. Martio and S. Rickman, *Boundary behavior of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I **507** (1972), 17 pp. MR 0379846

[9] O. Martio and U. Srebro, *Locally injective automorphic mappings in $\mathbb{R}^n$*, Math. Scand. **85** (1999), no. 1, 49–70. MR 1707745

[10] K. Noshiro, *Cluster sets*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., vol. 28, Springer, Berlin, 1960. MR 0133464

[11] K. Rajala, *Surface families and boundary behavior of quasiregular mappings*, Illinois J. Math. **49** (2005), no. 4, 1145–1153. MR 2210356

[12] K. Rajala and S. Wenger, *An upper gradient approach to weakly differentiable cochains*, J. Math. Pures Appl. (9) **100** (2013), 868–906. MR 3125271

[13] Y. G. Reshetnyak, *Space mappings with bounded distortion*, Translations of Mathematical Monographs, vol. 73, American Mathematical Society, Providence, RI, 1989. MR 0994644

[14] S. Rickman, *Quasiregular mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 26, Springer, Berlin, 1993. MR 1238941

[15] W. Rudin, *The radial variation of analytic functions*, Duke Math. J. **22** (1955), 235–242. MR 0079093

[16] M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Mathematics, vol. 1319, Springer, Berlin, 1988. MR 0950174

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