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THE HOMOTOPY TYPES OF GAUGE GROUPS OF NONORIENTABLE SURFACES AND APPLICATIONS TO MODULI SPACES

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Abstract. We determine the homotopy types of gauge groups of principle G-bundles over closed, connected nonorientable surfaces for $G = U(n)$ and G a simply-connected, compact Lie group. Applications are made to moduli spaces of stable vector bundles.

1. Introduction

In their seminal paper [AB], Atiyah and Bott studied the moduli space of stable vector bundles over a compact Riemann surface. This led to a great deal of work to better understand these moduli spaces by studying their cohomology [AB], [HN], [JK] and their homotopy groups [DU]. Recently, a great deal of attention has been paid to establishing analogues of Atiyah and Bott's results for nonorientable surfaces, such as in [HL1], [HL2], [R].

In [T2], the author used decomposition methods in homotopy theory to determine the homotopy types of the gauge groups of principle $U(n)$-bundles over a compact Riemann surface for certain n. This allowed for information to be deduced about the homotopy groups of the moduli space of stable vector bundles over a compact Riemann surface, through an appropriate dimensional range. In this paper we apply the same approach to study the case of nonorientable surfaces.

Let M be a closed, connected surface and let G be a compact, connected Lie group. Let P be a principle G-bundle over M. The *gauge group* of P is the group $\mathcal{G}(P)$ of G-equivariant automorphisms of P which fix M. By the classification of surfaces, if M is nonorientable then it is homeomorphic to a connected sum of m copies of $\mathbb{R}P^2$ for some $m > 0$, which we denote by M_m. It is straightforward to see (a calculation is included in Section 3) that if G is simply-connected then any principle G-bundle over M_m must be trivial,

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and if G is $U(n)$ for $n \geq 1$ then there are two distinct equivalence classes of principle G-bundles over M_m, one represented by the trivial bundle and the other by a nontrivial bundle. In either option for G, let $\mathcal{G}_0(M_m)$ be the gauge group of the trivial principle G-bundle over M_m, and let $\mathcal{G}_1(M_m)$ be the gauge group of the nontrivial principle $U(n)$-bundle.

We begin by decomposing $\mathcal{G}_0(M_m)$ and $\mathcal{G}_1(M_m)$. Let $G\{2\}$ be the homotopy fibre of the 2nd-power map on G. Let $S^3\langle 3\rangle$ be the three-connected cover of S^3.

THEOREM 1.1. *Let M_m be the connected sum of m copies of $\mathbb{R}P^2$ and let P be a principle G-bundle over M_m, where G is either $U(n)$ or a simply-connected, compact Lie group. The following hold*:

(a) *if P is the trivial bundle, then there is a homotopy decomposition*

$$\mathcal{G}_0(M_m) \simeq G \times \Omega G\{2\} \times \left(\prod_{i=2}^{m} \Omega G\right);$$

(b) *if $G = U(n)$ for $n \geq 1$ and P is the nontrivial bundle, then there is a homotopy decomposition*

$$\mathcal{G}_1(M_m) \simeq \mathcal{G}_1\big(\mathbb{R}P^2\big) \times \left(\prod_{i=2}^{m} \Omega U(n)\right)$$

and a nontrivial fibration $\Omega U(n)\{2\} \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$;

(c) *in dimensions $\leq 2n-3$ the nontrivial fibration in part* (b) *splits as*

$$\mathcal{G}_1\big(\mathbb{R}P^2\big) \simeq U(n) \times \Omega U(n)\{2\},$$

and if $n \geq 3$ there is a splitting of homotopy groups

$$\pi_m\big(\mathcal{G}_1\big(\mathbb{R}P^2\big)\big) \cong \pi_m\big(U(n)\big) \oplus \pi_m\big(\Omega U(n)\{2\}\big)$$

for $m \leq 2n-1$ if n is odd and for $m \leq 2n-2$ if n is even;

(d) *if the nontrivial fibration in part* (b) *is localized at an odd prime or rationally, then there is a homotopy equivalence $\mathcal{G}_1(\mathbb{R}P^2) \simeq U(n)$*;

(e) *if the nontrivial fibraiton in part* (b) *is localized at* 2 *and $n = 2$, then there is a homotopy decomposition*

$$\Omega\mathcal{G}_1\big(\mathbb{R}P^2\big) \simeq \mathbb{Z}/2\mathbb{Z} \times S^1 \times \Omega^2 S^3\langle 3\rangle \times \Omega^3 S^5.$$

The decompositions in Theorem 1.1 express the gauge groups in terms of simpler spaces which are easier to analyze. For example, the homotopy groups of the Lie groups G have been determined through a range, and by [N] the homotopy groups of $\Omega G\{2\}$ are all $\mathbb{Z}/2\mathbb{Z}$-summands. Consequently, the homotopy groups of $\mathcal{G}_0(M_m)$ are known to the same extent as those of G, the homotopy group calculations for $\mathcal{G}_1(M_m)$ are reduced to studying the

2-primary effect in the twisted fibration $\Omega U(n)\{2\} \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$—which is trivial in dimensions $\leq 2n-2$ for $n \geq 3$, and in the case of $U(2)$ the homotopy groups of $\mathcal{G}_1(\mathbb{R}P^2)$ are known to the same extents as those of S^3 and S^5.

We now turn to moduli spaces. Let P be a principle $U(n)$-bundle over M_m, where $m, n > 1$. Let $\mathcal{N}_0(P)$ be the space of flat connections on P. Ramras [R] showed that $\mathcal{N}_0(P)$ is $((m-1)(n-1)-1)$-connected. Let $\mathcal{M}(P) = \mathcal{N}_0(P)/\mathcal{G}(P)$ be the moduli space of gauge equivalence classes of flat connections on P. The connectivity property of $\mathcal{N}_0(P)$ implies that $\mathcal{M}(P)$ is homotopy equivalent to the classifying space $B\mathcal{G}(P)$ of $\mathcal{G}(P)$ in dimensions $\leq (m-1)(n-1)-1$. Thus, there is an isomorphism of homotopy groups $\pi_t(\mathcal{M}(P)) \cong \pi_{t-1}(\mathcal{G}(P))$ for $t \leq (m-1)(n-1)-1$. Consequently, if P is trivial then the homotopy groups of $\mathcal{M}(P)$ can be determined explicitly in dimensions $\leq (m-1)(n-1)-1$. The same holds if P is nontrivial and $n = 2$. If P is nontrivial, $n \geq 3$ and n is even, then the homotopy groups of $\mathcal{M}(P)$ can be determined explicitly in dimensions $\leq n-2$ if $m = 2$, $\leq 2n-3$ if $m = 3$, and $\leq 2n-1$ if $m > 3$. If P is nontrivial, $n \geq 3$ and n is odd, then the homotopy groups of $\mathcal{M}(P)$ can be determined explicitly in the same range if $m \in \{2, 3\}$ and in dimensions $\leq 2n$ if $m > 3$. This improves significantly on [R], where K-theory was used to calculate π_1 of $\mathcal{M}(P)$.

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2. A decomposition of ΣM_m and consequences

Since M_m is a connected sum of m copies of $\mathbb{R}P^2$, there is a homotopy cofibration sequence

$$S^1 \xrightarrow{f} \bigvee_{i=1}^{m} S^1 \xrightarrow{i} M_m \xrightarrow{q} S^2,$$

where f is the attaching map of the top cell of M_m, i is the inclusion of the 1-skeleton, and q is the pinch map to the top cell. Further, if $p_i : \bigvee_{i=1}^{m} S^1 \longrightarrow S^1$ is the pinch map to the ith-wedge summand, then the composite $p_i \circ f$ is of degree 2. In Lemma 2.1, we give a decomposition of the suspension of M_m.

For $m \geq 1$, let $\underline{2} : S^m \longrightarrow S^m$ be the map of degree 2. Notice that the degree 2 map on S^m is homotopic to the suspension of the degree 2 map on S^{m-1}. Thus, as $\mathbb{R}P^2$ is the cofiber of the degree 2 map on S^1, for $m \geq 1$ there is a homotopy cofibration $S^m \xrightarrow{\underline{2}} S^m \longrightarrow \Sigma^{m-1}\mathbb{R}P^2$.

Focus on the pinch map $\bigvee_{i=1}^{m} S^1 \xrightarrow{p_1} S^1$ onto the first wedge summand. Note that it is the cofibre of the map $\bigvee_{i=2}^{m} S^1 \longrightarrow \bigvee_{i=1}^{m} S^1$ which includes the 2nd through mth wedge summands. Since $p_1 \circ f$ has degree 2, we obtain a homotopy pushout diagram

$$\begin{tikzcd}
 & \bigvee_{i=2}^{m} S^1 \arrow[r, equal] \arrow[d] & \bigvee_{i=2}^{m} S^1 \arrow[d] \\
S^1 \arrow[r, "f"] \arrow[d, equal] & \bigvee_{i=1}^{m} S^1 \arrow[r] \arrow[d, "p_1"] & M_m \arrow[d, "\theta"] \\
S^1 \arrow[r, "\underline{2}"] & S^1 \arrow[r] & \mathbb{R}P^2
\end{tikzcd} \tag{1}$$

which defines the map θ.

LEMMA 2.1. *The map* $\Sigma M_m \xrightarrow{\Sigma\theta} \Sigma\mathbb{R}P^2$ *has a right homotopy inverse. Consequently, the homotopy cofibration* $\bigvee_{i=2}^{m} S^2 \longrightarrow \Sigma M_m \xrightarrow{\Sigma\theta} \Sigma\mathbb{R}P^2$ *splits, inducing a homotopy equivalence*

$$\Sigma M_m \simeq \Sigma\mathbb{R}P^2 \vee \left(\bigvee_{i=2}^{m} S^2\right).$$

Proof. By the Hilton–Milnor theorem, $\pi_2(\bigvee_{i=1}^{m} S^2) \cong \bigoplus_{i=1}^{m} \mathbb{Z}$, where the ith-generator is determined by the pinch map $\bigvee_{i=1}^{m} S^2 \xrightarrow{\Sigma p_i} S^2$ onto the ith-wedge summand. Thus, any map $S^2 \xrightarrow{g} \bigvee_{i=1}^{m} S^2$ is homotopic to the composite

$$S^2 \xrightarrow{\ \sigma\ } \bigvee_{i=1}^{m} S^2 \xrightarrow{\bigvee_{i=1}^{m} \Sigma p_i \circ g} \bigvee_{i=1}^{m} S^2,$$

where σ is the suspension of the m-fold iteration of the equatorial pinch map $S^1 \longrightarrow S^1 \vee S^1$. In our case, consider $S^2 \xrightarrow{\Sigma f} \bigvee_{i=1}^{m} S^2$. Since $p_i \circ f$ has degree 2 for each $1 \leq i \leq m$, we have $\Sigma p_i \circ \Sigma f \simeq \underline{2}$. Thus, Σf is homotopic to the composite

$$S^2 \xrightarrow{\ \sigma\ } \bigvee_{i=1}^{m} S^2 \xrightarrow{\bigvee_{i=1}^{m} \underline{2}} \bigvee_{i=1}^{m} S^2.$$

Since σ is a suspension it commutes with degree maps, so Σf is homotopic to the composite

$$S^2 \xrightarrow{\underline{2}} S^2 \xrightarrow{\sigma} \bigvee_{i=1}^{m} S^2.$$

From this homotopy, we obtain a homotopy pushout diagram

$$\begin{tikzcd} S^2 \arrow[r, "2"] \arrow[d, equal] & S^2 \arrow[r] \arrow[d, "\sigma"] & \Sigma\mathbb{R}P^2 \arrow[d, "h"] \\ S^2 \arrow[r, "\Sigma f"] & \bigvee_{i=1}^{m} S^2 \arrow[r] & \Sigma M_m \end{tikzcd}$$

which defines the map h. Juxtaposing this diagram with (1), we obtain a homotopy cofibration diagram

(2)
$$\begin{tikzcd} S^2 \arrow[r, "2"] \arrow[d, equal] & S^2 \arrow[r] \arrow[d, "\sigma"] & \Sigma\mathbb{R}P^2 \arrow[d, "h"] \\ S^2 \arrow[r, "\Sigma f"] \arrow[d, equal] & \bigvee_{i=1}^{m} S^2 \arrow[r] \arrow[d, "\Sigma p_1"] & \Sigma M_m \arrow[d, "\Sigma\theta"] \\ S^2 \arrow[r, "2"] & S^2 \arrow[r] & \Sigma\mathbb{R}P^2. \end{tikzcd}$$

Observe that in the middle column, $\Sigma p_1 \circ \sigma$ is homotopic to the identity map. Since the top row of (2) maps to the bottom row by a morphism of cofibration sequences, we obtain an induced morphism of long exact sequences in homology. As the left and middle columns induce the identity map in homology, the five-lemma implies that $(\Sigma\theta \circ h)_*$ is an isomorphism. Thus, $\Sigma\theta \circ h$ is a homotopy equivalence. The lemma now follows. $\square$

Consider the cofibration sequence $\bigvee_{i=2}^{m} S^2 \longrightarrow \Sigma M_m \xrightarrow{\Sigma\theta} \Sigma\mathbb{R}P^2$ induced by suspending the right column in (1). For any pointed, path-connected space X, applying $\mathrm{Map}^*(\cdot, X)$ to this homotopy cofibration we obtain a fibration

(3)
$$\mathrm{Map}^*\big(\Sigma\mathbb{R}P^2, X\big) \xrightarrow{(\Sigma\theta)^*} \mathrm{Map}^*(\Sigma M_m, X) \longrightarrow \mathrm{Map}^*\left(\bigvee_{i=2}^{m} S^2, X\right).$$

By Lemma 2.1, $\Sigma\theta$ has a right homotopy inverse $\phi : \Sigma\mathbb{R}P^2 \longrightarrow \Sigma M_m$. Thus, the induced map

$$\phi^* : \mathrm{Map}^*(\Sigma M_m, X) \longrightarrow \mathrm{Map}^*\big(\Sigma\mathbb{R}P^2, X\big)$$

is a left homotopy inverse for $(\Sigma\theta)^*$.

We now rewrite the fibration (3) and the left inverse for $(\Sigma\theta)^*$ in an equivalent way. The pointed exponential law implies that there is a homotopy equivalence $\mathrm{Map}^*(\Sigma M, X) \simeq \Omega\mathrm{Map}^*(M, X)$, and under this equivalence a map $\Sigma M \xrightarrow{\Sigma f} \Sigma N$ has the property that $(\Sigma f)^* \simeq \Omega f^*$. As well, there is a homotopy equivalence $\mathrm{Map}^*(M \vee N, X) \simeq \mathrm{Map}^*(M, X) \times \mathrm{Map}^*(N, X)$. So (3) can be rewritten as a fibration

$$\Omega\mathrm{Map}^*\big(\mathbb{R}P^2, X\big) \xrightarrow{\Omega\theta^*} \Omega\mathrm{Map}^*(M_m, X) \longrightarrow \prod_{i=2}^{m} \mathrm{Map}^*\big(S^2, X\big) \tag{4}$$

and $\Omega\theta^*$ has a left homotopy inverse

$$\phi^* : \Omega\mathrm{Map}^*(M_m, X) \longrightarrow \Omega\mathrm{Map}^*\big(\mathbb{R}P^2, X\big).$$

The existence of such a left inverse in (4) implies the following.

LEMMA 2.2. *Let X be a pointed, path-connected space. Then there is a homotopy equivalence*

$$\Omega\mathrm{Map}^*(M_m, X) \simeq \Omega\mathrm{Map}^*\big(\mathbb{R}P^2, X\big) \times \prod_{i=2}^{m} \mathrm{Map}^*\big(S^2, X\big).$$

Next, we turn from a decomposition of $\Omega\mathrm{Map}^*(M_m, X)$ to a decomposition of $\Omega\mathrm{Map}(M_m, X)$. Observe that there is a fibration $\mathrm{Map}^*(M_m, X) \longrightarrow \mathrm{Map}(M_m, X) \xrightarrow{ev} X$ where ev evaluates a map at the basepoint of M_m. By naturality, the map $M_m \xrightarrow{\theta} \mathbb{R}P^2$ induces a pullback diagram

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & \\
\mathrm{Map}^*\big(\mathbb{R}P^2, X\big) \arrow[r] \arrow[d, "\theta^*"] & \mathrm{Map}\big(\mathbb{R}P^2, X\big) \arrow[r, "ev"] \arrow[d, "\Theta^*"] & X \arrow[d, equal] \\
\mathrm{Map}^*(M_m, X) \arrow[r] & \mathrm{Map}(M_m, X) \arrow[r, "ev"] & X
\end{tikzcd}$$

which defines the space Y. Continuing the fibration sequences vertically, we obtain a homotopy pullback

$$\begin{tikzcd}
\Omega\mathrm{Map}^*\big(\mathbb{R}P^2, X\big) \arrow[r] \arrow[d, "\Omega\theta^*"] & \Omega\mathrm{Map}\big(\mathbb{R}P^2, X\big) \arrow[r, "\Omega ev"] \arrow[d, "\Omega\Theta^*"] & \Omega X \arrow[d, equal] \\
\Omega\mathrm{Map}^*(M_m, X) \arrow[r] \arrow[d] & \Omega\mathrm{Map}\big(M_m, X\big) \arrow[r, "\Omega ev"] \arrow[d] & \Omega X \\
Y \arrow[r, equal] & Y. &
\end{tikzcd} \tag{5}$$

First, since $\Omega\theta^*$ has a left homotopy inverse, there is a homotopy equivalence $\Omega\mathrm{Map}^*(M_m, X) \simeq \Omega\mathrm{Map}^*(\mathbb{R}P^2, X) \times Y$. The bottom square in (5) therefore implies that Y retracts off $\Omega\mathrm{Map}(M_m, X)$. Thus, the fibration in the middle column of (5) splits, so we can use the loop multiplication on $\Omega\mathrm{Map}(M_m, X)$ to obtain a homotopy equivalence $\Omega\mathrm{Map}(M_m, X) \simeq \Omega\mathrm{Map}(\mathbb{R}P^2, X) \times Y$. Next, by (4) and (5), there are fibrations $\Omega\mathrm{Map}^*(\mathbb{R}P^2,X)\xrightarrow{\Omega\theta^*}\Omega\mathrm{Map}^*(M_m,X)\longrightarrow\prod_{i=2}^m\mathrm{Map}^*(S^2,X)$ and $\Omega\mathrm{Map}^*(\mathbb{R}P^2,X)\xrightarrow{\Omega\theta^*}\Omega\mathrm{Map}^*(M_m,X)\longrightarrow Y$. The left homotopy inverse for $\Omega\theta^*$ therefore implies that $Y\simeq\prod_{i=2}^m\mathrm{Map}^*(S^2,X)$. Combining this with the decomposition for $\Omega\mathrm{Map}(M_m,X)$ above, we obtain the following.

LEMMA 2.3. *Let X be a pointed, path-connected space. Then there is a homotopy equivalence*

$$\Omega\mathrm{Map}(M_m,X)\simeq\Omega\mathrm{Map}\big(\mathbb{R}P^2,X\big)\times\prod_{i=2}^m\mathrm{Map}^*\big(S^2,X\big).$$

Now we specialize to $X=BG$, where G is a connected, compact Lie group. We wish to identify the space $\mathrm{Map}^*(\mathbb{R}P^2,BG)$ appearing in Lemma 2.2.

LEMMA 2.4. *If G is a connected, compact Lie group then* $\mathrm{Map}^*(\mathbb{R}P^2,BG)\simeq G\{2\}$.

Proof. Consider the cofibration $S^1\xrightarrow{2}S^1\longrightarrow\mathbb{R}P^2$. Applying $\mathrm{Map}^*(\cdot,BG)$ we obtain a homotopy fibration $\mathrm{Map}^*(\mathbb{R}P^2,BG)\longrightarrow\mathrm{Map}^*(S^1,BG)\xrightarrow{2^*}\mathrm{Map}^*(S^1,BG)$. We have $\mathrm{Map}^*(S^1,BG)=\Omega BG\simeq G$. As 2^* induces the multiplication by 2 on the loop structure, the previous fibration is equivalent to the homotopy fibration $G\{2\}\longrightarrow G\xrightarrow{2}G$. In particular, $\mathrm{Map}^*(\mathbb{R}P^2,BG)\simeq G\{2\}$. □

By definition, $\mathrm{Map}^*(S^2,BG)=\Omega^2BG\simeq\Omega G$. So Lemmas 2.2 and 2.4 combine to imply the following.

LEMMA 2.5. *Let G be a connected, compact Lie group. There is a homotopy equivalence*

$$\Omega\mathrm{Map}^*(M_m,BG)\simeq\Omega G\{2\}\times\left(\prod_{i=2}^m\Omega G\right).$$

In what follows, we will be considering $G=U(n)$ or G a simply-connected, compact Lie group. If G is simply-connected and compact then it is actually 2-connected, so $\Omega G\{2\}$ is connected. If $G=U(n)$, there are potentially different components in $\Omega U(n)\{2\}$. However, we will show in Corollary 2.8 that $\Omega U(n)\{2\}$ is also connected.

LEMMA 2.6. *There is a homotopy fibration $\mathbb{Z}/2\mathbb{Z}\longrightarrow S^1\xrightarrow{2}S^1$.*

Proof. One way to see this is to observe that S^1 is the Eilenberg–MacLane space $K(\mathbb{Z},1)$ and classify the fibration of interest. Consider the 2nd-power map on $\mathbb{C}P^\infty=K(\mathbb{Z},2)$. Define F by the homotopy fibration $F\longrightarrow\mathbb{C}P^\infty\xrightarrow{2}\mathbb{C}P^\infty$. Since $\pi_m(\mathbb{C}P^\infty)$ is $\mathbb{Z}$ if $m=2$ and 0 if $m\neq 2$, the long exact sequence of homotopy groups induced by the fibration defining F implies that $\pi_m(F)$ is

$\mathbb{Z}/2\mathbb{Z}$ if $m=1$ and 0 if $m\neq 1$. That is, $F\simeq K(\mathbb{Z}/2\mathbb{Z},1)$. Therefore, looping the fibration defining F, we obtain a homotopy fibration $\mathbb{Z}/2\mathbb{Z}\longrightarrow S^1\xrightarrow{2} S^1$. $\square$

LEMMA 2.7. *For each $n\geq 1$, there is a homotopy fibration $SU(n)\{2\}\longrightarrow U(n)\{2\}\longrightarrow \mathbb{Z}/2\mathbb{Z}$.*

Proof. Consider the fibration of groups $SU(n)\longrightarrow U(n)\xrightarrow{d} S^1$, where d is the determinant homomorphism. As this is a sequence of groups and group homomorphisms, the 2nd-power map induces a fibration diagram

$$\begin{tikzcd}
SU(n)\{2\} \arrow[r] \arrow[d] & SU(n) \arrow[r, "2"] \arrow[d] & SU(n) \arrow[d] \\
U(n)\{2\} \arrow[r] \arrow[d] & U(n) \arrow[r, "2"] \arrow[d, "d"] & U(n) \arrow[d, "d"] \\
\mathbb{Z}/2\mathbb{Z} \arrow[r] & S^1 \arrow[r, "2"] & S^1,
\end{tikzcd}$$

where Lemma 2.6 has been used to identify the homotopy fibre along the bottom row. The left column of this fibration diagram is the fibration asserted by the lemma. (Note also that if $n=1$ then $SU(1)\simeq *$, so $SU(1)\{2\}\simeq *$, and therefore $U(1)\{2\}\simeq\mathbb{Z}/2\mathbb{Z}$.) $\square$

COROLLARY 2.8. *For each $n\geq 1$, there is a homotopy equivalence $\Omega U(n)\{2\}\simeq \Omega SU(n)\{2\}$.*

3. Classification of principal G-bundles over M_m

We classify the principle G-bundles over M_m, so we know for which bundles we will be taking gauge groups. These results are surely well known, but we include them for the sake of completeness. Let BG be the classifying space of G.

LEMMA 3.1. *Let M_m be the connected sum of m copies of $\mathbb{R}P^2$. The following hold:*

(a) *if G is simply-connected, then $[M_m,BG]\cong 0$;*
(b) *if $G=U(n)$ for $n\geq 1$, then $[M_m,BG]\cong\mathbb{Z}/2\mathbb{Z}$.*

Proof. If G is simply-connected then BG is 2-connected. Therefore, as M_m is 2-dimensional, we have $[M_m,BG]\cong 0$.

Next, consider the cofibration sequence $\bigvee_{i=2}^m S^1\longrightarrow M_m\xrightarrow{\theta}\mathbb{R}P^2\xrightarrow{\delta}\bigvee_{i=2}^m S^2$ from (1). If X is a pointed, path-connected space, this cofibration sequence induces an exact sequence $[\bigvee_{i=2}^m S^2,X]\xrightarrow{\delta^*}[\mathbb{R}P^2,X]\xrightarrow{\theta^*}[M_m,X]\longrightarrow[\bigvee_{i=2}^m S^1,X]$. Since $\Sigma\theta$ has a right homotopy inverse, the map δ is null homotopic. Thus $\delta^*=0$. If we also assume that X is simply-connected, then

$[\bigvee_{i=2}^m S^1, X] \cong 0$, so θ^* induces an isomorphism $[\mathbb{R}P^2, X] \cong [M_m, X]$. So to prove part (b) we are reduced to showing that $[\mathbb{R}P^2, BU(n)] \cong \mathbb{Z}/2\mathbb{Z}$.

Observe that the 3-skeleton of $BU(n)$ is homotopy equivalent to S^2. Since $\mathbb{R}P^2$ is 2-dimensional, the inclusion $S^2 \longrightarrow BU(n)$ of the bottom cell induces an isomorphism $[\mathbb{R}P^2, S^2] \cong [\mathbb{R}P^2, BU(n)]$. So we are reduced to calculating $[\mathbb{R}P^2, S^2]$. The cofibration sequence $S^1 \longrightarrow \mathbb{R}P^2 \longrightarrow S^2 \xrightarrow{2} S^2$ induces an exact sequence $[S^2, S^2] \xrightarrow{(2)^*} [S^2, S^2] \longrightarrow [\mathbb{R}P^2, S^2] \longrightarrow [S^1, S^2]$, that is, an exact sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow [\mathbb{R}P^2, S^2] \longrightarrow 0$. Thus $[\mathbb{R}P^2, S^2]$ is isomorphic to the cokernel of $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, which is $\mathbb{Z}/2\mathbb{Z}$. $\square$

Since $[M_m, BG]$ classifies the equivalence classes of principle G-bundles over M_m, Lemma 3.1 immediately implies the following.

COROLLARY 3.2. *Let M_m be the connected sum of m copies of $\mathbb{R}P^2$. The following hold*:

(a) *if G is simply-connected, then there is one equivalence class of principle G-bundles over M_m, that of the trivial bundle*;

(b) *if $G = U(n)$ for $n \geq 1$, then there are two distinct equivalence classes of principle G-bundles over M_m, represented by the trivial bundle and a nontrivial bundle.*

4. Gauge groups

Recall from the Introduction that the *gauge group* $\mathcal{G}(P)$ of a principle G-bundle $P \longrightarrow M$ is the group of G-equivariant automorphisms of P that fix M. The *based gauge group* $\mathcal{G}^*(P)$ of $P \longrightarrow M$ is the subgroup of $\mathcal{G}(P)$ which fixes the fibre over the basepoint of M. Let $B\mathcal{G}(P)$ and $B\mathcal{G}^*(P)$ be the classifying spaces of $\mathcal{G}(P)$ and $\mathcal{G}^*(P)$, respectively. By [AB], there is a homotopy commutative diagram

$$\begin{tikzcd} B\mathcal{G}^*(P) \arrow[r] \arrow[d, "\simeq"] & B\mathcal{G}(P) \arrow[d, "\simeq"] \\ \mathrm{Map}^*_P(M, BG) \arrow[r] & \mathrm{Map}_P(M, BG), \end{tikzcd}$$

where BG is the classifying space of G, and $\mathrm{Map}_P(M, BG)$ and $\mathrm{Map}^*_P(M, BG)$ are the components of $\mathrm{Map}(M, BG)$ and $\mathrm{Map}^*(M, BG)$ respectively which contain the map inducing P. Consider the fibration $\mathrm{Map}^*_P(M, BG) \longrightarrow \mathrm{Map}_P(M, BG) \xrightarrow{ev} BG$ where ev is the map which evaluates at the basepoint of M. The previous diagram implies that we obtain a homotopy fibration sequence

$$G \xrightarrow{\partial_P} B\mathcal{G}^*(P) \longrightarrow B\mathcal{G}(P) \longrightarrow BG, \tag{6}$$

where ∂_P is the fibration connecting map.

In our case, we have $M = M_m$. Let $P \longrightarrow M_m$ be a principal G-bundle. If P is the trivial bundle, let $\mathcal{G}_0(M_m)$ be its gauge group. If $G = U(n)$, then Corollary 3.2 states that there is a unique class of nontrivial principal G-bundles. Choosing a representative nontrivial bundle, let $\mathcal{G}_1(M_m)$ be its gauge group. By (6), for $k \in \{1,2\}$ there are homotopy fibration sequences

$$G \xrightarrow{\overline{\partial}_k} B\mathcal{G}_k^*(M_m) \longrightarrow B\mathcal{G}_k(M_m) \longrightarrow BG.$$

By Lemma 3.1, there is a one-to-one correspondence between the components of $\mathrm{Map}^*(M_m, BG)$ and $\mathrm{Map}^*(\mathbb{R}P^2, BG)$. This correspondence passes to a one-to-one correspondence between the components of $\mathrm{Map}(M_m, BG)$ and $\mathrm{Map}(\mathbb{R}P^2, BG)$ since BG is simply-connected. Thus, the decompositions in Lemmas 2.2, 2.3, and 2.5 imply the following.

Proposition 4.1. *For $k \in \{0,1\}$, there are homotopy equivalences*

$$\mathcal{G}_k^*(M_m) \simeq \mathcal{G}_k^*\big(\mathbb{R}P^2\big) \times \left(\prod_{i=2}^{m} \Omega G\right),$$

$$\mathcal{G}_k(M_m) \simeq \mathcal{G}_k\big(\mathbb{R}P^2\big) \times \left(\prod_{i=2}^{m} \Omega G\right)$$

and for $k = 0$ there is a homotopy equivalence

$$\mathcal{G}_0^*(M_m) \simeq \Omega G\{2\} \times \left(\prod_{i=2}^{m} \Omega G\right).$$

Further, when $G = U(n)$ then Lemma 3.1 implies that $\mathrm{Map}^*(\mathbb{R}P^2, BU(n))$ has two components. By Lemma 2.4, $\mathrm{Map}^*(\mathbb{R}P^2, BU(n)) \simeq U(n)\{2\}$, and by Lemma 2.7, each component of $U(n)\{2\}$ is homotopy equivalent to $SU(n)\{2\}$. Thus, for $k \in \{0,1\}$ we have $B\mathcal{G}_k^*(\mathbb{R}P^2) \simeq SU(n)\{2\}$. Corollary 2.8 states that after looping we have $\mathcal{G}_k^*(\mathbb{R}P^2) \simeq \Omega SU(n)\{2\} \simeq \Omega U(n)\{2\}$. Thus from the fibrations $B\mathcal{G}_k^*(\mathbb{R}P^2) \longrightarrow B\mathcal{G}_k(\mathbb{R}P^2) \longrightarrow BU(n)$, we obtain the following.

Lemma 4.2. *For $k \in \{0,1\}$, there is a homotopy fibration*

$$\Omega U(n)\{2\} \longrightarrow \mathcal{G}_k\big(\mathbb{R}P^2\big) \longrightarrow U(n).$$

5. A decomposition of the gauge group of the trivial bundle

In this section, we refine the decomposition of $\mathcal{G}_0(M_m)$ in Proposition 4.1. We begin with some general observations regarding gauge groups of trivial bundles.

Lemma 5.1. *Let G be a connected topological group and M a connected space. Let $P_0 \longrightarrow M$ be the trivial principle G-bundle. Then there is a homeomorphism*

$$\mathcal{G}_0(M) \simeq \mathrm{Map}(M, G).$$

Proof. The given principle G-bundle $P_0 \longrightarrow M$ is the trivial bundle $M \times G \xrightarrow{\pi_1} M$ where π_1 is the projection. Let $f : M \times G \longrightarrow M \times G$ be a G-equivariant automorphism which fixes M, that is, $f \in \mathcal{G}_0(M)$. The action of G on $M \times G$ is given by its action on G, so as f is G-equivariant it is determined by how it acts on $M \times 1$, where 1 is the identity element of G. That is, f is determined by the composite $f' : M \xrightarrow{i_1} M \times G \xrightarrow{f} M \times G$ where i_1 is the inclusion of the first factor. Since f fixes M, f' is the identity when projected to M, so f' is determined by its projection onto G. Hence, f is determined by the composite $f'' : M \xrightarrow{i_1} M \times G \xrightarrow{f} M \times G \xrightarrow{\pi_2} G$ where π_2 is the projection onto the second factor. Therefore, we obtain a map $F : \mathcal{G}_0(M) \longrightarrow \mathrm{Map}(M, G)$. Each step in producing ϕ is continuous so ϕ is continuous.

On the other hand, given a map $h : M \longrightarrow G$ we obtain a G-equivariant automorphism $h' : M \times G \longrightarrow M \times G$ that fixes M by letting $h'(m, g) = (m, (h(m))^g)$, where $(h(m))^g$ is g acting on $h(m)$. Thus, we obtain a continuous map $H : \mathrm{Map}(M, G) \longrightarrow \mathcal{G}_0(M)$. Since $h'(m, 1) = (m, h(m))$, we have $F \circ H$ equal to the identity map, and by the constructions in the first paragraph we have $f(m, g) = (m, (f''(m))^g)$, so we also have $H \circ F$ equal to the identity map. Thus, $\mathcal{G}_0(M)$ is homeomorphic to $\mathrm{Map}(M, G)$. $\square$

PROPOSITION 5.2. *With hypotheses as in Lemma 5.1, there is a homotopy equivalence*

$$\mathcal{G}_0(M) \simeq G \times \mathcal{G}_0^*(M).$$

Proof. Consider the fibration $\mathrm{Map}^*(M, G) \xrightarrow{h} \mathrm{Map}(M, G) \xrightarrow{ev} G$, where ev evaluates a map at the basepoint of M. Restricting to the component containing the constant map, we obtain a fibration $\mathrm{Map}_0^*(M, G) \xrightarrow{h_0} \mathrm{Map}_0(M, G) \xrightarrow{ev} G$. In this fibration, the map ev has a section $s : G \longrightarrow \mathrm{Map}(M, G)$ defined by sending $g \in G$ to the constant map which takes every element of M to g. Since G is a group, there is an induced multiplication m on $\mathrm{Map}_0(M, G)$ given by taking two maps $f, g \in \mathrm{Map}_0(M, G)$ and defining $m(f, g)$ pointwise, $m(f, g)(x) = f(x) \cdot g(x)$. Thus the existence of a section for the evaluation fibration implies that the composite $G \times \mathrm{Map}_0^*(M, G) \xrightarrow{s \times h_0} \mathrm{Map}_0(M, G) \times \mathrm{Map}_0(M, G) \xrightarrow{m} \mathrm{Map}_0(M, G)$ is a homotopy equivalence.

Next, since G is a topological group it has a classifying space BG. Observe that the pointed exponential law implies that there are homotopy equivalences $\mathrm{Map}_0^*(M, G) \simeq \mathrm{Map}_0^*(\Sigma M, BG) \simeq \Omega \mathrm{Map}_0^*(M, BG)$. As well, we have $\mathrm{Map}_0^*(M, BG) = B\mathcal{G}_0^*$ so $\Omega \mathrm{Map}_0^*(M, BG) \simeq \mathcal{G}_0^*(M)$. Thus, $\mathrm{Map}_0(M, G) \simeq G \times \mathrm{Map}_0^*(M, G) \simeq G \times \mathcal{G}_0^*(M)$. But $\mathrm{Map}_0(M, G) \simeq \mathcal{G}_0(M)$ by Lemma 5.1, so we obtain the decomposition asserted by the proposition. $\square$

Combining the decompositions in Propositions 5.2 and 4.1, we obtain the following.

Proposition 5.3. *Let G be a connected, compact Lie group. Let $P_0 \longrightarrow M_m$ be the trivial principle G-bundle over M_m. Then there is a homotopy equivalence*

$$\mathcal{G}_0(M_m) \simeq G \times \Omega G\{2\} \times \left(\prod_{i=2}^{m} \Omega G\right).$$

6. The gauge group of the nontrivial $U(n)$-bundle

In this section, we consider the nontrivial principal $U(n)$-bundle $P \longrightarrow M_m$, with gauge group $\mathcal{G}_1(M_m)$. By Proposition 4.1, there is a homotopy equivalence $\mathcal{G}_1(M_m) \simeq \mathcal{G}_1(\mathbb{R}P^2) \times (\prod_{i=2}^{m} \Omega U(n))$. We will show that the homotopy type of $\mathcal{G}_1(M_m)$ is distinct from the gauge group $\mathcal{G}_0(M_m)$ of the trivial bundle by showing that the fibration $\Omega U(n)\{2\} \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$ in Lemma 4.2 does not split.

To set this up, observe that $\mathrm{Map}^*(S^2, BU(n)) \simeq \Omega U(n) \simeq \mathbb{Z} \times \Omega SU(n)$ has components indexed by the integers. Specifically, if $f : S^2 \longrightarrow BU(n)$ has degree k in homology, let $\mathrm{Map}^*_k(S^2, BU(n))$ be the component of $\mathrm{Map}^*(S^2, BU(n))$ containing f. For each $k \in \mathbb{Z}$, there is a fibration sequence

$$U(n) \xrightarrow{\partial_k} \mathrm{Map}^*_k\big(S^2, BU(n)\big) \longrightarrow \mathrm{Map}_k\big(S^2, BU(n)\big) \xrightarrow{ev} BU(n),$$

where ev evaluates a map at the basepoint, and the fibration defines the connecting map ∂_k. On the other hand, Lemmas 2.4 and 2.7 imply that $\mathrm{Map}^*(\mathbb{R}P^2, BU(n)) \simeq U(n)\{2\}$ has two components. Specifically, if $f : \mathbb{R}P^2 \longrightarrow BU(n)$ has degree k in homology then either $k = 0$ or $k = 1$. In either case, let $\mathrm{Map}^*_k(\mathbb{R}P^2, BU(n))$ be the component of $\mathrm{Map}^*(\mathbb{R}P^2, BU(n))$ containing f. Then we obtain a fibration sequence $U(n) \xrightarrow{\overline{\partial}_k} \mathrm{Map}^*_k(\mathbb{R}P^2, BU(n)) \longrightarrow \mathrm{Map}_k(\mathbb{R}P^2, BU(n)) \xrightarrow{ev} BU(n)$ which defines the map $\overline{\partial}_k$. Since the pinch map $q : \mathbb{R}P^2 \longrightarrow S^2$ to the top cell is degree one in $H_2(\cdot)$, if we restrict to the $k = 1$ components, we obtain a diagram of fibration sequences

(7)

$$\begin{tikzcd}
U(n) \arrow[r, "\partial_1"] \arrow[d, equal] & \mathrm{Map}^*_1\big(S^2, BU(n)\big) \arrow[r] \arrow[d, "q^*"] & \mathrm{Map}_1\big(S^2, BU(n)\big) \arrow[r, "ev"] \arrow[d] & BU(n) \arrow[d, equal] \\
U(n) \arrow[r, "\overline{\partial}_1"] & \mathrm{Map}^*_1\big(\mathbb{R}P^2, BU(n)\big) \arrow[r] & \mathrm{Map}_1\big(\mathbb{R}P^2, BU(n)\big) \arrow[r, "ev"] & BU(n).
\end{tikzcd}$$

We want to analyze the left square of (7) more closely.

The pinch map q is the connecting map for the cofibration $S^1 \xrightarrow{2} S^1 \longrightarrow \mathbb{R}P^2$. Thus, there is an induced fibration sequence $\mathrm{Map}^*(S^2, BU(n)) \xrightarrow{q^*} \mathrm{Map}^*(\mathbb{R}P^2, BU(n)) \longrightarrow \mathrm{Map}^*(S^1, BU(n)) \xrightarrow{2^*} \mathrm{Map}^*(S^1, BU(n))$. We have $\mathrm{Map}^*(S^1, BU(n)) \simeq U(n)$ and the map 2^* induces multiplication by 2 on the loop structure of $U(n) \simeq \Omega BU(n)$. Thus, this fibration sequence is equivalent to the fibration sequence $\Omega U(n) \xrightarrow{\rho} U(n)\{2\} \longrightarrow U(n) \xrightarrow{2} U(n)$. In particular, the map q^* in (7) can be identified with the restriction of ρ to the 1-components of $\Omega U(n)$ and $U(n)\{2\}$. To identify this more precisely, the inclusion of the subgroup $SU(n)$ into $U(n)$ induces a fibration diagram

$$\begin{tikzcd} \Omega SU(n) \arrow[r, "\overline{\rho}"] \arrow[d] & SU(n)\{2\} \arrow[r] \arrow[d] & SU(n) \arrow[r, "2"] \arrow[d] & SU(n) \arrow[d] \\ \Omega U(n) \arrow[r, "\rho"] & U(n)\{2\} \arrow[r] & U(n) \arrow[r, "2"] & U(n). \end{tikzcd} \tag{8}$$

By Lemma 2.7, the connected cover of $U(n)\{2\}$ is $SU(n)\{2\}$ and as $\Omega U(n) \simeq \mathbb{Z} \times \Omega SU(n)$, the connected cover of $\Omega U(n)$ is $\Omega SU(n)$. Thus, (8) implies that the restriction of ρ to the 1-components is $\overline{\rho}$. Summarizing, we have the following.

LEMMA 6.1. *There is a homotopy commutative diagram*

$$\begin{tikzcd} U(n) \arrow[r, "\partial_1"] \arrow[d, equal] & \Omega SU(n) \arrow[d, "\overline{\rho}"] \\ U(n) \arrow[r, "\overline{\partial}_1"] & SU(n)\{2\}. \end{tikzcd}$$

We aim to show that $\overline{\partial}_1$ is nontrivial. Let $\imath : \Sigma\mathbb{C}P^{n-1} \longrightarrow SU(n)$ be the canonical map which induces a projection onto the generating set in cohomology. Compose with the standard map $SU(n) \longrightarrow U(n)$ to obtain a map $\imath' : \Sigma\mathbb{C}P^{n-1} \longrightarrow U(n)$. We will show that $\overline{\partial}_1$ is nontrivial by showing in Corollary 6.6 that $\overline{\partial}_1 \circ \imath'$ is nontrivial. We begin with some preliminary lemmas.

LEMMA 6.2. *Let X be a CW-complex of dimension $\leq 2n-3$ with cells only in odd dimensions. Then any map $X \xrightarrow{g} \Omega SU(n)$ is null homotopic.*

Proof. The map $SU(n) \longrightarrow SU(\infty)$ is $(2n-1)$-connected. So by Bott periodicity $\pi_{2m}(SU(n)) \cong 0$ for $2m \leq 2n-2$. Equivalently, $\pi_{2m-1}(\Omega SU(n)) \cong 0$ for $2m-1 \leq 2n-3$. We induct on the number of odd dimensions in which the cells of X appear. If there is only one dimension, say $2m+1$, then $X \simeq \bigvee S^{2m+1}$. As X has dimension $\leq 2n-3$, the map $X \simeq \bigvee S^{2m+1} \xrightarrow{g} \Omega SU(n)$ is null homotopic.

Now suppose that the lemma holds for any space with cells only in $l-1$ distinct odd dimensions. Suppose the cells of X are in dimensions $\{2m_1+1, \ldots, 2m_l+1\}$ where $m_1 < \cdots < m_l$. Since the dimension of X is $\leq 2n-3$, we have $2m_1+1 \leq 2n-3$, so including the bottom cells into X, the composite $\bigvee S^{2m_1+1} \longrightarrow X \xrightarrow{g} \Omega SU(n)$ is null homotopic. Thus, g factors through a map $g' : Y \longrightarrow \Omega SU(n)$ where $Y = X/(\bigvee S^{2m_1+1})$. Since Y is a CW-complex of dimension $\leq 2n-3$ and has cells only in the $l-1$ odd dimensions $\{2m_2+1,\ldots,2m_l+1\}$, by inductive hypothesis the map g' is null homotopic. Since g factors through g', we have g null homotopic. Hence, the lemma holds by induction. $\square$

For example, let $X=\Sigma\mathbb{C}P^{n-2}$. This has dimension $2n-3$ and has cells only in odd dimensions. So Lemma 6.2 implies that $[\Sigma\mathbb{C}P^{n-2},\Omega SU(n)]\cong 0$. We now draw two consequences of this fact. Let $p\colon\Sigma\mathbb{C}P^{n-1}\longrightarrow S^{2n-1}$ be the pinch map to the top cell.

Corollary 6.3. *Any map $h:\Sigma\mathbb{C}P^{n-1}\longrightarrow\Omega SU(n)$ factors as a composite $\Sigma\mathbb{C}P^{n-1}\xrightarrow{p}S^{2n-1}\xrightarrow{h'}\Omega SU(n)$ for some map h'.*

Proof. Consider the homotopy cofibration $\Sigma\mathbb{C}P^{n-2}\longrightarrow\Sigma\mathbb{C}P^{n-1}\xrightarrow{p}S^{2n-1}$. The isomorphism $[\Sigma\mathbb{C}P^{2n-2},\Omega SU(n)]\cong 0$ implies that the composite

$$\Sigma\mathbb{C}P^{n-2}\longrightarrow\Sigma\mathbb{C}P^{n-1}\xrightarrow{h}\Omega SU(n)$$

is null homotopic. Thus h factors through p, as asserted. $\square$

Corollary 6.4. *The map $\Sigma\mathbb{C}P^{n-1}\xrightarrow{p}S^{2n-1}$ induces an epimorphism*

$$\left[S^{2n-1},\Omega SU(n)\right]\xrightarrow{p^*}\left[\Sigma\mathbb{C}P^{n-1},\Omega SU(n)\right].$$

So if $f:S^{2n-1}\longrightarrow\Omega SU(n)$ represents a generator of $\pi_{2n-1}(\Omega SU(n))\cong\mathbb{Z}/n!\mathbb{Z}$, then the composite $\Sigma\mathbb{C}P^{n-1}\xrightarrow{p}S^{2n-1}\xrightarrow{f}\Omega SU(n)$ represents a generator of $[\Sigma\mathbb{C}P^{n-1},\Omega SU(n)]$.

Proof. The homotopy cofibration $\Sigma\mathbb{C}P^{n-2}\longrightarrow\Sigma\mathbb{C}P^{n-1}\xrightarrow{p}S^{2n-1}$ induces an exact sequence

$$\left[S^{2n-1},\Omega SU(n)\right]\xrightarrow{p^*}\left[\Sigma\mathbb{C}P^{n-1},\Omega SU(n)\right]\longrightarrow\left[\Sigma\mathbb{C}P^{n-2},\Omega SU(n)\right].$$

Since $[\Sigma\mathbb{C}P^{n-2},\Omega SU(n)]\cong 0$ and $[S^{2n-1},\Omega SU(n)]\cong\pi_{2n-1}(\Omega SU(n))\cong\mathbb{Z}/n!\mathbb{Z}$, the assertions of the corollary follow. $\square$

Proposition 6.5. *The composite $\Sigma\mathbb{C}P^{n-1}\xrightarrow{\imath'}U(n)\xrightarrow{\partial_1}\Omega SU(n)\xrightarrow{\overline{\rho}}SU(n)\{2\}$ is nontrivial.*

Proof. In [T1], it was shown that there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma\mathbb{C}P^{n-1} \arrow[r, "\imath'"] \arrow[d, "q"] & U(n) \arrow[r, "\partial_1"] & \Omega SU(n) \arrow[d, equal] \\
S^{2n-1} \arrow[rr, "f"] & & \Omega SU(n),
\end{tikzcd} \tag{9}$$

where f represents the generator of $\pi_{2n}(SU(n)) \cong \mathbb{Z}/n!\mathbb{Z}$. Corollary 6.4 therefore implies that $\partial_1 \circ \imath'$ represents a generator of $[\Sigma\mathbb{C}P^{n-1}, \Omega SU(n)]$. In particular, $\partial_1 \circ \imath'$ is not divisible by 2. That is, there is no map $\Sigma\mathbb{C}P^{n-1} \xrightarrow{\lambda} \Omega SU(n)$ with the property that $\partial_1 \circ \imath' \simeq 2 \circ \lambda$. Hence, there is no lift

$$\begin{tikzcd} & & \Omega SU(n) \arrow[d, "2"] \\ \Sigma\mathbb{C}P^{n-1} \arrow[r, "\imath'"] \arrow[urr, "\lambda"] & U(n) \arrow[r, "\partial_1"] & \Omega SU(n). \end{tikzcd}$$

In other words, the composite $\Sigma\mathbb{C}P^{n-1} \xrightarrow{\imath'} U(n) \xrightarrow{\partial_1} \Omega SU(n) \xrightarrow{\overline{\rho}} SU(n)\{2\}$ is nontrivial. $\square$

COROLLARY 6.6. *The map* $\overline{\partial}_1 \simeq \overline{\rho} \circ \partial_1$ *is nontrivial.*

As $\overline{\partial}_1$ fits in the homotopy fibration sequence

$$\Omega SU(n)\{2\} \longrightarrow \mathcal{G}_1\big(\mathbb{R}P^2\big) \longrightarrow U(n) \xrightarrow{\overline{\partial}_1} SU(n)\{2\}$$

and Lemma 2.8 lets us identify $\Omega SU(n)\{2\}$ with $\Omega U(n)\{2\}$, the nontriviality of $\overline{\partial}_1$ immediately implies the following.

PROPOSITION 6.7. *The homotopy fibration* $\Omega U(n)\{2\} \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$ *is nontrivial.*

7. A partial decomposition for the gauge group of nontrivial bundles

In Proposition 6.7, we showed that the homotopy fibration $\Omega U(n)\{2\} \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$ does not split as $\mathcal{G}_1(\mathbb{R}P^2) \simeq U(n) \times \Omega U(n)\{2\}$. This makes determining the homotopy groups of $\mathcal{G}_1(\mathbb{R}P^2)$ more difficult. However, in this section we will show that there is a splitting through an appropriate dimensional range, and a splitting on the level of homotopy groups through a slightly higher range. Thus, even though the fibration does not split, we can still determine the homotopy groups of $\mathcal{G}_1(\mathbb{R}P^2)$ through a range.

We are considering the homotopy fibration sequence

$$\Omega SU(n)\{2\} \longrightarrow \mathcal{G}_1\big(\mathbb{R}P^2\big) \longrightarrow U(n) \xrightarrow{\overline{\partial}_1} SU(n)\{2\},$$

where Lemma 2.8 lets us identify $\Omega SU(n)\{2\}$ with $\Omega U(n)\{2\}$. By Lemma 6.1, the map $\overline{\partial}_1$ factors as the composite $U(n) \xrightarrow{\partial_1} \Omega SU(n) \xrightarrow{\rho} SU(n)\{2\}$. We first consider properties of ∂_1.

Recall that we regard $\Omega SU(n)$ as being homotopy equivalent to the component of $\Omega U(n)$ containing the continuous, pointed, degree one maps from the circle to $U(n)$, so ∂_1 can be written as $U(n) \xrightarrow{\partial_1} \Omega_1 U(n)$. Lang [L] showed that the adjoint of ∂_1, the map $S^1 \wedge U(n) \longrightarrow U(n)$, is homotopic to the

Samelson product $\langle i,1\rangle$ of the inclusion $i: S^1 \longrightarrow U(n)$ and the identity map $1: U(n) \longrightarrow U(n)$. Let $U = U(\infty)$ be the infinite unitary group and consider the homotopy fibration $\Omega(U/U(n)) \longrightarrow U(n) \longrightarrow U$, where the right map is the group inclusion. Since U is an infinite loop space, it is a homotopy associative, homotopy commutative H-space. So the Samelson product $\langle i,1\rangle$ on $U(n)$ composes trivially to U. Therefore, $\langle i,1\rangle$ lifts through the map $\Omega(U/U(n)) \longrightarrow U(n)$. Adjointing back, $\overline{\partial}_1$ lifts through the map $\Omega^2(U/U(n)) \longrightarrow \Omega_1 U(n)$. Rewriting this, as $\Omega(U/U(n)) \simeq \Omega(SU/SU(n))$, where $SU = SU(\infty)$ is the infinite special unitary group, and the fibration $\Omega(SU/SU(n)) \longrightarrow SU(n) \longrightarrow SU$ is the connected cover of the fibration $\Omega(U/U(n)) \longrightarrow U(n) \longrightarrow U$, when we adjoint back we obtain a lift

$$\begin{tikzcd} & \Omega^2\big(SU/SU(n)\big) \arrow[d] \\ U(n) \arrow[ur, "\lambda"] \arrow[r, "\partial_1"] & \Omega SU(n) \end{tikzcd} \tag{10}$$

for some map λ. Observe that as a CW-complex, $\Omega^2(SU/SU(n))$ is $(2n-2)$-connected, and its $(2n+1)$-skeleton is a two-cell complex $S^{2n-1} \cup e^{2n+1}$.

LEMMA 7.1. *The map* $U(n) \xrightarrow{\partial_1} \Omega SU(n)$ *is null homotopic in dimensions* $\leq 2n-2$.

Proof. Since $\Omega^2(SU/SU(n))$ is $(2n-2)$-connected, the map λ in (10) is null homotopic in dimensions $\leq 2n-2$. The homotopy commutativity of (10) therefore implies that ∂_1 is null homotopic in dimensions $\leq 2n-2$. □

Since $\overline{\partial}_1$ factors through ∂_1, Lemma 7.1 immediately implies the following.

COROLLARY 7.2. *The map* $U(n) \xrightarrow{\overline{\partial}_1} SU(n)\{2\}$ *is null homotopic in dimensions* $\leq 2n-2$.

This null homotopy through a dimensional range lets us decompose $\mathcal{G}_1(\mathbb{R}P^2)$ through a dimensional range.

PROPOSITION 7.3. *In dimensions* $\leq 2n-3$, *the fibration*

$$\Omega U(n)\{2\} \longrightarrow \mathcal{G}_1\big(\mathbb{R}P^2\big) \longrightarrow U(n)$$

splits to give a homotopy equivalence $\mathcal{G}_1(\mathbb{R}P^2) \simeq U(n) \times \Omega U(n)\{2\}$.

Proof. Consider the homotopy fibration sequence

$$\Omega SU(n)\{2\} \longrightarrow \mathcal{G}_1\big(\mathbb{R}P^2\big) \longrightarrow U(n) \xrightarrow{\overline{\partial}_1} SU(n)\{2\}.$$

By Corollary 7.2, $\overline{\partial}_1$ is null homotopic in dimensions $\leq 2n-2$. Therefore the

$(2n-2)$-skeleton $U(n)_{2n-2}$ of $U(n)$ lifts through the map $\mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$. This lets us define the composite

$$\psi : U(n)_{2n-2} \times \Omega SU(n)\{2\} \longrightarrow \mathcal{G}_1\big(\mathbb{R}P^2\big) \times \mathcal{G}_1\big(\mathbb{R}P^2\big) \xrightarrow{\mu} \mathcal{G}_1\big(\mathbb{R}P^2\big),$$

where μ is the loop multiplication on $\mathcal{G}_1(\mathbb{R}P^2) \simeq \Omega\mathrm{Map}(\mathbb{R}P^2, BU(n))$. Since the skeletal inclusion $U(n)_{2n-2} \longrightarrow U(n)$ induces an isomorphism on homotopy groups in dimensions $\leq 2n-3$, the map ψ induces an isomorphism on homotopy groups in dimensions $\leq 2n-3$. Thus, ψ is a (weak) homotopy equivalence in dimensions $\leq 2n-3$. Finally, in the composition defining ψ, we can use Lemma 2.8 to identify $\Omega SU(n)\{2\}$ as $\Omega U(n)\{2\}$. $\square$

The dimensional range in Proposition 7.3 is optimal. For by Proposition 6.5, the composite $\Sigma\mathbb{C}P^{n-1} \xrightarrow{\iota'} U(n) \xrightarrow{\partial_1} \Omega SU(n) \xrightarrow{\overline{\rho}} SU(n)\{2\}$ is nontrivial. Therefore, $\overline{\partial}_1 \simeq \overline{\rho} \circ \partial_1$ cannot be null homotopic in dimensions $\leq 2n-1$, implying that the fibration $\Omega U(n)\{2\} \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$ does not split in dimensions $\leq 2n-2$.

However, if we only care about a splitting on the level of homotopy groups rather than spaces, then we can show that $\overline{\partial}_1$ induces the zero map on homotopy groups in dimension $2n-1$ and does the same in dimension $2n$ provided n is odd. To see this, we first require some information about the homotopy groups of unitary groups. Let $q : U(n) \longrightarrow S^{2n-1}$ be the quotient map, where $S^{2n-1} \cong U(n)/U(n-1)$. Let $j : S^{2n-1} \longrightarrow \Omega(SU/SU(n))$ be the inclusion of the bottom cell. For $m \geq 3$, let $\eta : S^{m+1} \longrightarrow S^m$ represent the generator of the stable homotopy group $\pi_{m+1}(S^m) : \mathbb{Z}/2\mathbb{Z}$. The statements in Lemma 7.4 are a combination of results proved in [BH], [K].

LEMMA 7.4. *The following hold*:

(a) $\pi_{2n-1}(U(n)) \cong \mathbb{Z}$, *and a representative* $c : S^{2n-1} \longrightarrow U(n)$ *of the generator can be chosen so that the composite* $S^{2n-1} \xrightarrow{c} U(n) \xrightarrow{q} S^{2n-1}$ *has degree* $(n-1)!$;

(b) $\pi_{2n}(U(n)) \cong \mathbb{Z}/n!\mathbb{Z}$, *a representative of the generator is the composite* $d : S^{2n} \xrightarrow{j} \Omega(SU/SU(n)) \longrightarrow SU(n) \longrightarrow U(n)$, *and this generator has the property that the composite* $S^{2n} \xrightarrow{d} U(n) \xrightarrow{q} S^{2n-1}$ *is null homotopic if* n *is odd and is homotopic to* η *if* n *is even*;

(c) $\pi_{2n+1}(U(n)) \cong 0$ *if* n *is odd and is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$ *if* n *is even, and in the even case a representative of the generator is the composite* $S^{2n+1} \xrightarrow{\eta} S^{2n} \xrightarrow{d} U(n)$.

Next, we relate these homotopy groups to the map $U(n) \xrightarrow{\partial_1} \Omega SU(n)$. These arguments will hold integrally, that is, before any localization. First note that $U(n) \simeq S^1 \times SU(n)$ so for $m > 1$ we have $\pi_m(SU(n)) \cong \pi_m(U(n))$, implying that the homotopy group calculations in Lemma 7.4 are equally valid for $SU(n)$. Let $i : S^{2n-1} \longrightarrow \Omega^2(SU/SU(n))$ be the inclusion of the bottom cell, which is adjoint to the map j above. Consider the diagrams

(11)

$$\begin{tikzcd} & S^{2n-1} \arrow[r, "i"] & \Omega^2\big(SU/SU(n)\big) \arrow[d] \\ S^{2n-1} \arrow[ur, "a"] \arrow[r, "c"] & U(n) \arrow[ur, "\lambda"] \arrow[r, "\partial_1"] & \Omega SU(n) \end{tikzcd} \qquad \begin{tikzcd} & S^{2n-1} \arrow[r, "i"] & \Omega^2\big(SU/SU(n)\big) \arrow[d] \\ S^{2n} \arrow[ur, "b"] \arrow[r, "d"] & U(n) \arrow[ur, "\lambda"] \arrow[r, "\partial_1"] & \Omega SU(n), \end{tikzcd}$$

where the maps a and b will be defined momentarily. The right triangle in each diagram homotopy commutes by (10). By connectivity, the composites $\lambda \circ c$ and $\lambda \circ d$ factor through the $2n$-skeleton of $\Omega^2(SU/SU(n))$, which is homotopy equivalent to S^{2n-1}. That is, $\lambda \circ c$ factors as $S^{2n-1} \xrightarrow{a} S^{2n-1} \xrightarrow{i} \Omega^2(SU/SU(n))$ for some map a, and $\lambda \circ d$ factors as $S^{2n} \xrightarrow{b} S^{2n-1} \xrightarrow{i} \Omega^2(SU/SU(n))$ for some map b. Thus, both diagrams in (11) homotopy commute.

Observe in the left diagram in (11) that as the map $U(n) \xrightarrow{q} S^{2n-1}$ is degree 1 in integral homology, the fact that $q \circ c$ has degree $(n-1)!$ implies that the map c has degree $(n-1)!$ in integral homology. Thus $\lambda \circ c$ has degree dividing $(n-1)!$, so as i is degree 1, a has degree dividing $(n-1)!$. Observe in the right diagram in (11) that as $\pi_{2n}(S^{2n-1}) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by η, that $b \simeq t \cdot \eta$ for some $t \in \mathbb{Z}/2\mathbb{Z}$.

LEMMA 7.5. *If $n \geq 3$, then the map $U(n) \xrightarrow{\overline{\partial}_1} SU(n)\{2\}$ induces the zero map on π_{2n-1}.*

Proof. Since the map a in (11) has degree dividing $(n-1)!$, if $n \geq 3$ then the degree of a is divisible by 2. Thus, the homotopy commutativity of the left diagram in (11) implies that $\partial_1 \circ c$ is divisible by 2. That is, $\partial_1 \circ c \simeq g \circ \underline{2}$ for some map g, where $\underline{2}$ is the degree 2 map on S^{2n-1}.

In general, for spaces A and B the homotopy classes of maps $[\Sigma A, \Omega B]$ has two group structures, one induced from the co-H-structure on ΣA and the other from the loop structure on ΩB. It is well known that these two group structures coincide. Therefore, in $[S^{2n-1}, \Omega SU(n)]$ we have $g \circ \underline{2} \simeq 2 \circ g$, where 2 is the 2nd-power map on $\Omega SU(n)$. Thus $\partial_1 \circ c \simeq 2 \circ g$.

As $\overline{\partial}_1 \simeq \overline{\rho} \circ \partial_1$, we obtain $\overline{\partial}_1 \circ c \simeq \overline{\rho} \circ 2 \circ g$. But the composite $\Omega SU(n) \xrightarrow{2} \Omega SU(n) \xrightarrow{\overline{\rho}} SU(n)\{2\}$ is two consecutive maps in a homotopy fibration and so is null homotopic. Thus $\overline{\partial}_1 \circ c$ is null homotopic. As c represents the generator of $\pi_{2n-1}(U(n))$, we see that $\overline{\partial}_1$ induces ther zero map on π_{2n-1}. $\square$

LEMMA 7.6. *If $n \geq 3$, then the map $U(n) \xrightarrow{\partial_1} \Omega SU(n)$ induces the zero map on π_{2n} if n is odd and is nontrivial on π_{2n} if n is even.*

Proof. If n is odd then $\pi_{2n}(\Omega SU(n)) \cong \pi_{2n+1}(SU(n)) \cong \pi_{2n+1}(U(n))$, so by Lemma 7.4(c), $\pi_{2n}(\Omega SU(n)) \cong 0$. Thus in this case ∂_1 induces the zero map on π_{2n}.

If n is even, we will show that the map b in (11) is homotopic to η. Granting this, by Lemma 7.4(b) and the fact that $\pi_{2n}(U(n)) \cong \pi_{2n}(SU(n))$, the composite $S^{2n} \xrightarrow{j} \Omega(SU/SU(n)) \longrightarrow SU(n)$ represents the generator of $\pi_{2n}(SU(n))$. The adjoint of this generator is the composite $\varepsilon : S^{2n-1} \xrightarrow{i} \Omega^2(SU/SU(n)) \longrightarrow \Omega SU(n)$ appearing in (11). By Lemma 7.4(c), the composite $\varepsilon \circ \eta$ represents the generator of $\pi_{2n}(\Omega SU(n)) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore, as $\partial_1 \circ d \simeq \varepsilon \circ b$ by (11) and $b \simeq \eta$ by assumption, we see that $\partial_1 \circ d$ is nontrivial and the statement of the lemma follows.

To see that $b \simeq \eta$, consider the homotopy cofibration diagram

$$\begin{tikzcd}
S^{2n} \arrow[r, "d"] \arrow[d, equal] & U(n) \arrow[r] \arrow[d, "\lambda"] & C \arrow[d, "f"] \\
S^{2n} \arrow[r, "i \circ b"] & \Omega^2\big(SU/SU(n)\big) \arrow[r] & D,
\end{tikzcd}$$

where the left square homotopy commutes by (11), and the spaces C and D and the map f are defined by the homotopy pushout. Take cohomology with mod-2 coefficients. Observe that for dimensional reasons, the two horizontal arrows in the right square induce isomorphisms in H^{2n-1}. As well, by [T1], λ_* induces an isomorphism in H^{2n-1}. The homotopy commutativity of the right square above then implies that f^* induces an isomorphism in H^{2n-1}. Observe next that as d represents the generator of $\pi_{2n}(U(n))$, it attaches the cell to $U(n+1)$ that represents the ring generator $x \in H^{2n+1}(U(n+1))$. As n is even, $x = Sq^2(y)$, where y is the ring generator in $H^{2n-1}(U(n+1))$. Note that we may regard y as the ring generator of $H^{2n-1}(U(n))$ as well. This information about d implies that the cofibre C of d has the property that there is a generator $\bar{x} \in H^{2n+1}(C)$ such that $\bar{x} = Sq^2(\bar{y})$ for $\bar{y} \in H^{2n-1}(C)$, where the map $U(n) \longrightarrow C$ sends $\bar{y}$ to the ring generator $y \in H^{2n-1}(U(n))$. Now consider the map $C \xrightarrow{f} D$. We have already seen that f^* is an isomorphism in H^{2n-1}, so if $z \in H^{2n-1}(D)$ represents the generator, then $f^*(z) = \bar{x}$. The naturality of the Steenrod operation Sq^2 then implies that $f^*(Sq^2(z)) = Sq^2(f^*(z)) = Sq^2(\bar{x}) = \bar{y}$. Thus $Sq^2(z) \neq 0$.

Next, since D is the homotopy cofibre of $i \circ b$, if B is the homotopy cofibre of $S^{2n} \xrightarrow{b} S^{2n-1}$ then there is an induced map $B \longrightarrow D$ which induces isomorphisms in H^{2n-1} and H^{2n+1}. The fact that Sq^2 acts nontrivially on $H^{2n-1}(D)$ therefore implies that the same is true of $H^{2n-1}(B)$. Since Sq^2 detects the map η, we must therefore have $b \simeq \eta$. $\square$

Corollary 7.7. *If $n \geq 3$, then the map $U(n) \xrightarrow{\overline{\partial}_1} SU(n)\{2\}$ induces the zero map on π_{2n} if n is odd and is nontrivial on π_{2n} if n is even.*

Proof. We have $\overline{\partial}_1 \simeq \overline{\rho} \circ \partial_1$. If n is odd then Lemma 7.6 states that ∂_1 induces the zero map on π_{2n}, and therefore so does $\overline{\partial}_1$ since it factors through ∂_1. If n is even then Lemma 7.6 states that ∂_1 is nontrivial on π_{2n}. To be concrete, let $x \in \pi_{2n}(\Omega SU(n))$ be the image of $(\partial_1)_*$ applied to $\pi_{2n}(U(n))$. By Lemma 7.4(c), $\pi_{2n}(\Omega SU(n)) \cong \pi_{2n}(\Omega U(n)) \cong \mathbb{Z}/2\mathbb{Z}$, so the fact that x is nontrivial implies it is a generator of this homotopy group. In particular, x is not divisible by 2, so in the long exact sequence of homotopy groups induced by the fibration $\Omega SU(n) \xrightarrow{2} \Omega SU(n) \xrightarrow{\overline{\rho}} SU(n)\{2\}$, a representative for x cannot lift through 2. In other words, $\overline{\rho}_*(x)$ is nontrivial in $\pi_{2n}(SU(n)\{2\})$. □

The homotopy group results in Lemma 7.5 and Corollary 7.7 have the following consequences for the gauge group $\mathcal{G}_1(\mathbb{R}P^2)$.

Proposition 7.8. *The fibration $\Omega SU(n)\{2\} \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$ induces a splitting of homotopy groups*

$$\pi_m\big(\mathcal{G}_1\big(\mathbb{R}P^2\big)\big) \cong \pi_m\big(U(n)\big) \oplus \pi_m\big(\Omega SU(n)\{2\}\big)$$

for $m \leq 2n-1$ if n is odd and for $m \leq 2n-2$ if n is even.

Proof. If $m \leq 2n-3$, then the statement of the lemma is a consequence of Proposition 7.3. Otherwise, the homotopy fibration

$$\Omega SU(n)\{2\} \longrightarrow \mathcal{G}_1\big(\mathbb{R}P^2\big) \longrightarrow U(n)$$

induces a long exact sequence of homotopy groups with connecting map induced by $U(n) \xrightarrow{\overline{\partial}_1} SU(n)\{2\}$. By Proposition 7.3 and Lemma 7.5, $\overline{\partial}_1$ induces the zero map on π_{2n-2} and π_{2n-1}, respectively. Thus, there is a split short exact sequence of homotopy groups $0 \longrightarrow \pi_{2n-2}(\Omega SU(n)\{2\}) \longrightarrow \pi_{2n-2}(\mathcal{G}_1(\mathbb{R}P^2)) \longrightarrow \pi_{2n-1}(U(n)) \longrightarrow 0$. The same reasoning gives a splitting in π_{2n-1} if n is odd using Lemma 7.5 and Corollary 7.7. □

8. Localizing the gauge groups of nontrivial bundles

In this section, we consider the homotopy type of $\mathcal{G}_1(\mathbb{R}P^2)$ when $G = U(n)$ after localizing at a prime or rationally. We first dispense with the straightforward odd primary and rational cases.

Lemma 8.1. *Localize at an odd prime or the rationals. Then the evaluation map $B\mathcal{G}_1(\mathbb{R}P^2) \simeq \mathrm{Map}_1(\mathbb{R}P^2, BU(n)) \xrightarrow{ev} BU(n)$ is a homotopy equivalence.*

Proof. If localization at p is denoted by a subscript (p) (use (0) for the rationals), then by [HMR] there is a homotopy equivalence $\mathrm{Map}_1^*(\mathbb{R}P^2, BU(n))_{(p)} \simeq \mathrm{Map}_1^*(\mathbb{R}P^2_{(p)}, BU(n)_{(p)})$. But if p is odd or $p=0$ then $\mathbb{R}P^2_{(p)}$ is homotopy equivalent to a point. Thus, $\mathrm{Map}_1^*(\mathbb{R}P^2, BU(n))_{(p)}$ is contractible. Consequently, in the evaluation fibration

$$\mathrm{Map}_1^*\big(\mathbb{R}P^2, BU(n)\big) \longrightarrow \mathrm{Map}_1\big(\mathbb{R}P^2, BU(n)\big) \xrightarrow{ev} BU(n)$$

the map ev induces a homotopy equivalence when localized at an odd prime or rationally. □

Lemma 8.1 implies that the nontrivial homotopy fibration $\mathcal{G}_1^*(\mathbb{R}P^2) \longrightarrow \mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(n)$ in Proposition 6.7 is trivial after localization at an odd prime or rationally. Thus, the nontriviality of the fibration is a 2-primary phenomenon. In Propositions 7.3 and 7.8, we showed that the fibration splits in a certain dimensional range, and splits on the level of homotopy groups in a slightly higher range. In general, it is difficult to say more, but when $n=2$ we can give an explicit 2-local homotopy decomposition of $\mathcal{G}_1(\mathbb{R}P^2)$ after looping. To show this, we begin with some general arguments that do not require localization.

By Lemma 6.1 and (7), there is a homotopy fibration $\mathcal{G}_1(\mathbb{R}P^2) \longrightarrow U(2) \xrightarrow{\overline{\partial}_1} SU\{2\}$ where $\overline{\partial}_1$ factors as the composite $U(2) \xrightarrow{\partial_1} \Omega SU(2) \xrightarrow{\overline{\rho}} SU\{2\}$. Observe that $SU(2) \simeq S^3$, so $SU\{2\} \simeq S^3\{2\}$, and therefore we can rewrite $\overline{\partial}_1$ as $U(2) \xrightarrow{\partial_1} \Omega S^3 \xrightarrow{\overline{\rho}} S^3\{2\}$, and we have a homotopy fibration

$$\mathcal{G}_1\big(\mathbb{R}P^2\big) \longrightarrow U(2) \xrightarrow{\overline{\partial}_1} S^3\{2\}.$$

As well, consider the fibration $U(2) \xrightarrow{\partial_1} \Omega S^3 \simeq \mathrm{Map}_1^*(S^2, BU(2)) \longrightarrow \mathrm{Map}_1(S^2, BU(2))$ from (7). By Atiyah and Bott's result stated in Section 4, we have $\mathrm{Map}_1(S^2, BU(n)) \simeq B\mathcal{G}_1(S^2)$, where $\mathcal{G}_1(S^2)$ is the gauge group of the principal $U(2)$-bundle over S^2 classified by the degree one map $S^2 \longrightarrow BU(2)$. Thus, there is a homotopy fibration

$$\mathcal{G}_1\big(S^2\big) \longrightarrow U(2) \xrightarrow{\partial_1} \Omega S^3.$$

We aim to identify the homotopy type of $\Omega\mathcal{G}_1(\mathbb{R}P^2)$. This will involve first identifying the homotopy type of $\mathcal{G}_1(S^2)$, relating this to $\mathcal{G}_1(\mathbb{R}P^2)$, and then describing why looping yields more information.

The factorization $\overline{\partial}_1 \simeq \overline{\rho} \circ \partial_1$ in Lemma 6.1 induces a homotopy fibration diagram

$$
\begin{tikzcd}
\mathcal{G}_1(S^2) \arrow[r] \arrow[d, equal] & \mathcal{G}_1(\mathbb{R}P^2) \arrow[r] \arrow[d] & \Omega S^3 \arrow[d, "2"] \\
\mathcal{G}_1(S^2) \arrow[r] & U(2) \arrow[r, "\partial_1"] \arrow[d, "\overline{\partial}_1"] & \Omega S^3 \arrow[d, "\overline{\rho}"] \\
 & S^3\{2\} \arrow[r, equal] & S^3\{2\}.
\end{tikzcd} \tag{12}
$$

We have $U(2) \simeq S^1 \times SU(2) \simeq S^1 \times S^3$. Let $\pi : U(2) \longrightarrow S^3$ be the projection.

Lemma 8.2. *The map $U(2) \xrightarrow{\partial_1} \Omega S^3$ factors as the composite $U(2) \xrightarrow{\pi} S^3 \xrightarrow{\eta} \Omega S^3$ where η represents a generator of $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. Observe that by connectivity, the restriction of ∂_1 to S^1 is null homotopic. In general, for a fibration sequence $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$, there is a homotopy action $\theta : F \times \Omega B \longrightarrow F$ with the property that there is a homotopy commutative diagram

$$
\begin{tikzcd}
\Omega B \times \Omega B \arrow[r, "\mu"] \arrow[d, "\partial \times 1"] & \Omega B \arrow[d, "\partial"] \\
F \times \Omega B \arrow[r, "\theta"] & F,
\end{tikzcd}
$$

where μ is the loop multiplication on ΩB. In our case, combining the fact that ∂_1 is a fibration connecting map with the property that its restriction to S^1 is null homotopic, we obtain a homotopy commutative diagram

$$
\begin{tikzcd}
S^1 \times S^3 \arrow[r] \arrow[d, "* \times 1"] & U(2) \times U(2) \arrow[r, "\mu"] \arrow[d, "\partial_1 \times 1"] & U(2) \arrow[d, "\partial_1"] \\
* \times S^3 \arrow[r] & \Omega S^3 \times U(2) \arrow[r, "\theta"] & \Omega S^3.
\end{tikzcd}
$$

Observe that the top row in this diagram is a homotopy equivalence. Thus the diagram implies that $U(2) \xrightarrow{\partial_1} \Omega S^3$ factors as a composite $U(2) \xrightarrow{\pi} S^3 \xrightarrow{\eta} \Omega S^3$ for some map η. By [Su], ∂_1 is nontrivial, so η must be nontrivial. As $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, η must represent this generator. $\square$

Next, we use the factorization in Lemma 8.2 to identify the homotopy type of $\mathcal{G}_1(S^2)$. At this point, we need to localize all spaces and maps at 2. By [J], there is a 2-local homotopy fibration $S^2 \xrightarrow{E} \Omega S^3 \xrightarrow{H} \Omega S^5$, where H is the James–Hopf invariant and E is the suspension map. The map E sends the Hopf-invariant one map $S^3 \longrightarrow S^2$ to η. Thus, there is a homotopy fibration diagram

$$\begin{tikzcd}
S^1 \arrow[r, "s"] \arrow[d, equal] & X \arrow[r] \arrow[d] & \Omega^2 S^5 \arrow[d] \\
S^1 \arrow[r] & S^3 \arrow[r] \arrow[d, "\eta"] & S^2 \arrow[d, "E"] \\
& \Omega S^3 \arrow[r, equal] & \Omega S^3
\end{tikzcd} \tag{13}$$

which defines the space X and the map s. Let $\varphi : \Omega^2 S^3 \longrightarrow X$ be the connecting map for the fibration in the middle column.

Recall that $S^3\langle 3\rangle$ is the three-connected cover of S^3. Using the fact that $S^1 \simeq K(\mathbb{Z}, 1)$, let $r : \Omega^2 S^3 \longrightarrow S^1$ be the map representing $H^1(\Omega^2 S^3) \cong \mathbb{Z}$. Including the bottom cell into $\Omega^2 S^3$ gives a right homotopy inverse for r, so there is a homotopy equivalence $\Omega^2 S^3 \simeq S^1 \times \Omega^2 S^3\langle 3\rangle$.

LEMMA 8.3. *Localize at* 2. *The map* s *in* (13) *has a left homotopy inverse. Consequently, there is a homotopy decomposition* $X \simeq S^1 \times \Omega^2 S^5$. *Further, under this equivalence, the connecting map* φ *is homotopic to* $\Omega^2 S^3 \xrightarrow{r\times\Omega H} S^1 \times \Omega^2 S^5$.

Proof. Apply π_1 to the fibration along the top row of (13). Since $\Omega^2 S^5$ is 2-connected, we have $\pi_1(S^1) \cong \pi_1(X)$. Thus the map $g : X \longrightarrow K(\mathbb{Z}, 1)$, representing the generator of $H^1(X; \mathbb{Z})$ is a left homotopy inverse of s. The decomposition of X follows immediately.

Further, since the S^3 in the homotopy fibration $\Omega^2 S^3 \xrightarrow{\varphi} X \longrightarrow S^3$ is simply-connected, there is an isomorphism $\pi_1(\Omega^2 S^3) \cong \pi_1(X)$, so the composite $\Omega^2 S^3 \xrightarrow{\varphi} X \xrightarrow{g} K(\mathbb{Z}, 1)$ represents the generator of $H^1(X; \mathbb{Z})$. But this generator is also represented by r, so $g \circ \varphi \simeq r$. As well, by (13), the composite $\Omega^2 S^3 \xrightarrow{\varphi} X \longrightarrow \Omega^2 S^5$ is ΩH. Hence, using the homotopy equivalence in the first paragraph, the composite $\Omega^2 S^3 \xrightarrow{\varphi} X \xrightarrow{\simeq} S^1 \times \Omega^2 S^5$ is homotopic to $r \times \Omega H$. $\square$

By Lemma 8.2, $\partial_1 \simeq \eta \circ \pi$, so using the decomposition of X in Lemma 8.3 we obtain a homotopy fibration diagram

$$\begin{tikzcd}
S^1 \arrow[r, "t"] \arrow[d, equal] & \mathcal{G}_1\big(S^2\big) \arrow[r] \arrow[d] & S^1 \times \Omega^2 S^5 \arrow[d] \\
S^1 \arrow[r] & U(2) \arrow[r, "\pi"] \arrow[d, "\partial_1"] & S^3 \arrow[d, "\eta"] \\
 & \Omega S^3 \arrow[r, equal] & \Omega S^3
\end{tikzcd}$$

which defines the map t. Since the inclusion of S^1 into $U(2)$ has a left homotopy inverse, the homotopy commutativity of the upper left triangle in this diagram implies that t has a left homotopy inverse. Thus we immediately obtain the following, reproducing a result from [T2].

LEMMA 8.4. *There is a 2-local homotopy decomposition $\mathcal{G}_1(S^2) \simeq S^1 \times S^1 \times \Omega^2 S^5$.*

Now turn to $\mathcal{G}_1(\mathbb{R}P^2)$. Let γ be the composite $\gamma : S^3 \xrightarrow{\eta} \Omega S^3 \xrightarrow{\overline{\rho}} S^3\{2\}$. Define the space Y by the homotopy fibration diagram

$$\begin{tikzcd}
S^1 \times \Omega^2 S^5 \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & \Omega S^3 \arrow[d, "2"] \\
S^1 \times \Omega^2 S^5 \arrow[r] & S^3 \arrow[r, "\eta"] \arrow[d, "\gamma"] & \Omega S^3 \arrow[d, "\overline{\rho}"] \\
 & S^3\{2\} \arrow[r, equal] & S^3\{2\}.
\end{tikzcd} \tag{14}$$

LEMMA 8.5. *There is a 2-local homotopy decomposition $\mathcal{G}_1(\mathbb{R}P^2) \simeq S^1 \times Y$.*

Proof. The factorization of ∂_1 in Lemma 8.2 and the factorization of $\overline{\partial}_1$ in (12) implies that $\overline{\partial}_1 \simeq \gamma \circ \pi$. Thus, there is a homotopy fibration diagram

$$\begin{tikzcd}
S^1 \arrow[r, "u"] \arrow[d, equal] & \mathcal{G}_1\big(\mathbb{R}P^2\big) \arrow[r] \arrow[d] & Y \arrow[d] \\
S^1 \arrow[r] & U(2) \arrow[r, "\pi"] \arrow[d, "\overline{\partial}_1"] & S^3 \arrow[d, "\gamma"] \\
 & S^3\{2\} \arrow[r, equal] & S^3\{2\}
\end{tikzcd}$$

which defines the map u. Since the inclusion $S^1 \longrightarrow U(2)$ has a left homotopy inverse, the homotopy commutativity of the upper left square above implies that u also has a left homotopy inverse. Hence, $\mathcal{G}_1(\mathbb{R}P^2) \simeq S^1 \times Y$. $\square$

This is as far as we can go in identifying the homotopy type of $\mathcal{G}_1(\mathbb{R}P^2)$ at 2. However, after looping more can be said. Continuing the fibration sequences in (14) horizontally and using Lemma 8.3 to identify the connecting map in the lower fibration, we obtain a homotopy fibration diagram

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d] & \Omega^2 S^3 \arrow[r] \arrow[d, "2"] & S^1 \times \Omega^2 S^5 \arrow[d, equal] \\
\Omega S^3 \arrow[r, "\Omega\eta"] & \Omega^2 S^3 \arrow[r, "r \times \Omega H"] & S^1 \times \Omega^2 S^5.
\end{tikzcd}$$

By [C], the composite $\Omega^2 S^3 \xrightarrow{2} \Omega^2 S^3 \xrightarrow{\Omega H} \Omega^2 S^5$ is null homotopic. Thus $\Omega Y \simeq F \times \Omega^3 S^5$, where F is the homotopy fibre of the composite $\Omega^2 S^3 \xrightarrow{2} \Omega^2 S^3 \xrightarrow{r} S^1$. Since the 2nd-power map on $\Omega^2 S^3$ is degree 2 on the bottom cell (in dimension 1), the latter composite is homotopic to the composite $\Omega^2 S^3 \xrightarrow{r} S^1 \xrightarrow{2} S^1$. Thus, there is a homotopy fibration diagram

$$\begin{tikzcd}
\Omega^2 S^3\langle 3\rangle \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & S^1\{2\} \arrow[d] \\
\Omega^2 S^3\langle 3\rangle \arrow[r] & \Omega^2 S^3 \arrow[r, "r"] \arrow[d, "2 \circ r"] & S^1 \arrow[d, "2"] \\
& S^1 \arrow[r, equal] & S^1.
\end{tikzcd}$$

Since $\Omega^2 S^3\langle 3\rangle$ retracts off $\Omega^2 S^3$, the homotopy commutativity of the upper left square in this diagram implies that it retracts off F as well. Thus, from the fibration along the top row, we obtain a homotopy equivalence $F \simeq S^1\{2\} \times \Omega^2 S^3\langle 3\rangle$. Note that a check of homotopy groups shows that $S^1\{2\} \cong K(\mathbb{Z}/2\mathbb{Z}, 0)$, that is, $S^1\{2\} \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, we obtain the following.

LEMMA 8.6. *There is a 2-local homotopy decomposition* $\Omega Y \simeq \mathbb{Z}/2\mathbb{Z} \times \Omega^2 S^3\langle 3\rangle \times \Omega^3 S^5$.

REMARK 8.7. The decomposition of ΩY relied on the fact that $\Omega H \circ 2$ is null homotopic. The loop is necessary, as the composite $H \circ 2$ is nontrivial. It is the nontriviality of this composite that acts as an obstruction to a finer decomposition of Y, and hence of $\mathcal{G}_1(\mathbb{R}P^2) \simeq S^1 \times Y$.

Combining Lemmas 8.5 and 8.6, we obtain a precise description of the homotopy type of $\mathcal{G}_1(\mathbb{R}P^2)$.

Proposition 8.8. *For $G = U(2)$, there is a 2-local homotopy decomposition*

$$\Omega\mathcal{G}_1\left(\mathbb{R}P^2\right) \simeq \mathbb{Z}/2\mathbb{Z} \times S^1 \times \Omega^2 S^3\langle 3\rangle \times \Omega^3 S^5.$$

To conclude, we combine all our results to prove Theorem 1.1.

Proof of Theorem 1.1. Parts (a), (b), (c), (d) and (e) are the statements of Proposition 5.3, Proposition 6.7, Propositions 7.3 and 7.8, Lemma 8.1 and Proposition 8.8, respectively. □

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