A NORMAL FORM FOR ALL LEVI-NONDEGENERATE ALMOST CR STRUCTURES

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ABSTRACT. We propose a unified normal form for all Levi-nondegenerate hypersurface type almost CR structures.

1. Introduction

In the recent article [Z12], the author extended the Chern–Moser normal form theory [CM74] to *nonintegrable* (or *noninvolutive*) (hypersurface type) almost CR structures. The extension has been obtained for all *strongly pseudoconvex* almost CR structures. However, in the general case of *Levi-nondegenerate* almost CR structures a difficulty arose that lead to an additional condition called in [Z12] "*strong nondegeneracy*" involving both the Levi form and the Nijenhuis nonintegrability tensor.

The present paper removes that additional condition and yields a *unified normal form* for all Levi-nondegenerate (hypersurface type) almost CR structures. There is a price to pay: our normal form does not extend the Chern–Moser one but rather a modification involving higher order derivatives. The question of convergence of the normal form for real-analytic almost CR structures is not treated here and remains an interesting open problem.

Recall that an *almost CR structure* on a real manifold M consists of a subbundle $H = HM$ of the tangent bundle $T = TM$ and a vector bundle automorphism $J : H \to H$ satisfying $J^2 = -\,\mathsf{id}$. Thus, J makes every fiber H_p, $p \in M$, into a complex vector space. A special case of an almost CR structure corresponding to $H = T$ is the *almost complex structure*, whereas the *hypersurface type* corresponds to H being a corank 1 subbundle of T. Equivalently, an almost CR structure is given by the i-eigenspace subbundle $H^{1,0} \subset \mathbb{C} \otimes H \subset \mathbb{C} \otimes T$ of J, which can be chosen as an arbitrary complex

subbundle of $\mathbb{C} \otimes T$ satisfying $H^{1,0} \cap \overline{H^{1,0}} = 0$. A *CR structure* is an almost CR structure satisfying the integrability condition $[H^{1,0}, H^{1,0}] \subset H^{1,0}$. See [BER99] for more details.

We refer to [Z12] for the history of the problem. Here we recall the approach by S. S. Chern and J. Moser [CM74]. According to it, every real-analytic Levi-nondegenerate hypersurface type CR structure near a fixed point p can be realized by an embedded real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ of the form

$$(1.1) \qquad \operatorname{Im} w = F(z, \bar{z}, \operatorname{Re} w), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

with p corresponding to the origin and F having near the origin the expansion

$$(1.2) \qquad F(z, \bar{z}, u) = \sum_{j=1}^{n} \varepsilon_j |z_j|^2 + \sum_{k,l \geq 2} F_{kl}(z, \bar{z}, u),$$

where $\varepsilon_j = \pm 1$ and the summands $F_{kl}(z, \bar{z}, u)$ are bihomogeneous of bidegree (k, l) in $(z, \bar{z})$ (i.e. $F_{kl}(tz, s\bar{z}, u) = t^k s^l F_{kl}(z, \bar{z}, u)$ for $t, s \in \mathbb{R}$) satisfying the normalization conditions

$$(1.3) \qquad \operatorname{tr} F_{22} = 0, \qquad \operatorname{tr}^2 F_{23} = 0, \qquad \operatorname{tr}^3 F_{33} = 0,$$

where $\operatorname{tr} := \sum_{j=1}^{n} \varepsilon_j \frac{\partial^2}{\partial z_j \, \partial \bar{z}_j}$. The CR structure here is induced by the embedding of M in $\mathbb{C}^{n+1}$ in the sense that $HM = TM \cap iTM$ and J is the restriction to HM of the multiplication by i in $\mathbb{C}^{n+1}$. If the given CR structure is merely smooth, its Taylor series expansion at p can still be realized by a *formal normal form* (1.2) satisfying (1.3), that is, as a smooth hypersurface (1.1) such that the right-hand side of (1.2) is a formal power series expansion satisfying (1.3) and such that the CR structure induced by the embedding coincides with the given one at the infinite order at 0.

Our goal here is to obtain a normal form for more general almost CR structures that may not necessarily satisfy the integrability condition. Such structures arise naturally, for instance, when one is deforming or glueing CR structures. The first problem one faces here is that the nonintegrable almost CR structures do not admit any realization as real submanifolds in $\mathbb{C}^{n+1}$ even at the formal level. Speaking informally, the normal form (1.2), (1.3) is *extrinsic* whereas almost CR structures are *intrinsic*.

In [Z12], we suggested a way of overcoming this difficulty based on an intrinsic analogue of the Chern–Moser normal form. The main idea is to impose normalizing conditions on the almost CR structure itself rather than on the defining equation in the extrinsic setting. However, almost CR structures are given by objects of different nature (complex subbundles of $\mathbb{C} \otimes T$) than functions. In order to relate with the setting of [CM74] we constructed a new function F associated with a given almost CR structure in given coordinates. Roughly speaking, F is obtained by "restricting" the almost CR structure to the Euler vector field. More precisely, consider any intrinsic coordinates on M that we group as $(z, u) \in \mathbb{C}^n \times \mathbb{R}$, where the subspace $\mathbb{C}^n \times \{0\}$ with

its complex structure corresponds to the given almost CR structure only at the origin. Using the standard complex structure, we write the complexification of $\mathbb{C}^n$ as the direct sum $\mathbb{C}^n_z \oplus \mathbb{C}^n_{\bar{z}}$ of the spaces of $(1,0)$ and $(0,1)$ vectors ($\pm i$-eigenspaces of J). Then $\mathbb{C} \otimes T$ can be identified at every point with $\mathbb{C}^n_z \oplus \mathbb{C}^n_{\bar{z}} \oplus \mathbb{C}_w$ and the almost CR structure corresponds to a complex subbundle $H^{1,0} \subset \mathbb{C} \otimes T$, which, at each point $p = (z,u)$ near 0, is a graph of a uniquely determined complex-linear map $L(z,\bar{z},u) : \mathbb{C}^n_z \to \mathbb{C}^n_{\bar{z}} \oplus \mathbb{C}_w$. We now consider the Euler (or radial) vector field $e(z) = \sum_j z_j \frac{\partial}{\partial z_j}$ on $\mathbb{C}^n_z$ and set

$$(1.4) \qquad \widetilde{L}(z,\bar{z},u) := L(z,\bar{z},u)\big(e(z)\big), \qquad \widetilde{L} = \big(\widetilde{L}^{\bar{z}}, \widetilde{L}^w\big) \in \mathbb{C}^n_{\bar{z}} \oplus \mathbb{C}_w.$$

Recall of the main results of [Z12]:

THEOREM 1.1 ([Z12]). *Any smooth strictly pseudoconvex hypersurface type almost CR structure admits a formal normal form given by $\widetilde{L}^{\bar{z}} = 0$, $\mathrm{Re}\,\widetilde{L}^w = 0$ and $F = \mathrm{Im}\,\widetilde{L}^w$ satisfying the Chern–Moser normalization* (1.2), (1.3). *This normal form is determined as uniquely as the Chern–Moser normal form, that is, up to the isotropy group of the hyperquadric associated with the Levi form.*

When attempting to extend the statement to all Levi-nondegenerate almost CR structures, we have encountered the difficulty that the normalization procedure breaks down at the crucial step, where the ODEs defining chains are being set up. Here, instead of the nondegeneracy of the Levi form, one needs it for a certain linear combination of the Levi form and the transversal component of the nonintegrability tensor. More precisely, given the function $L = (L^{\bar{z}}, L^w) : \mathbb{C}^n_z \to \mathbb{C}^n_{\bar{z}} \oplus \mathbb{C}_w$ as above defining the almost CR structure, the Levi form at 0 corresponds, up to an imaginary multiple, to the antihermitian part $\mathcal{L}(\bar{\xi},\eta)$ of the derivative of L^w in $\bar{z}$, whereas the nonintegrability tensor corresponds to the antisymmetric part $\mathcal{N}(\xi,\eta)$ of the derivative of L in z, and its transversal component $\mathcal{N}^w(\xi,\eta)$ to that of L^w. (In fact, $\mathcal{L}$ and $\mathcal{N}$ are the only 2nd order obstructions to the flatness as immediately follows from our normal form.) We then called the almost CR structure *strongly nondegenerate* at 0 if it is Levi-nondegenerate and in addition, the bilinear form $3\mathcal{L} + \mathcal{N}^w$ is nondegenerate in the sense that

$$(1.5) \qquad 3\mathcal{L}(\bar{\xi},\eta) + \mathcal{N}^w(\xi,\eta) = 0 \quad \text{for all } \eta \quad \Longrightarrow \quad \xi = 0.$$

Obviously, for (integrable) CR structures, strong nondegeneracy means the same as Levi-nondegeneracy. Furthermore, a strongly pseudoconvex almost CR structure is automatically strongly nondegenerate. Indeed, since $\mathcal{N}$ is antisymmetric, substituting $\eta = \xi$ into the left-hand side of (1.5) leads to $\mathcal{L}(\bar{\xi},\xi) = 0$, which in the case $\mathrm{Im}\,\mathcal{L}$ is positive definite, implies $\xi = 0$. However, if the Levi form has mixed signature, strong nondegeneracy is a stronger property than Levi-nondegeneracy. We then obtained the following extensions of Theorems 1.1 covering, in particular, all Levi-nondegenerate (integrable) CR structures.

THEOREM 1.2 ([Z12]). *The conclusion of Theorem 1.1 holds for all strongly nondegenerate hypersurface type almost CR structures.*

In this paper, we obtain the following normal form result that holds for all Levi-nondegenerate almost CR structures, that are not necessarily strongly nondegenerate.

THEOREM 1.3. *For every formal power series $L(z, \bar{z}, u) : \mathbb{C}^n_z \to \mathbb{C}^n_{\bar{z}} \times \mathbb{C}_w$ without constant terms corresponding to any Levi-nondegenerate almost CR structure and every vector $v \in \mathbb{C}^n \times \mathbb{R}$, there exist unique formal power series $f(z, \bar{z}, u) \in \mathbb{C}^n$ and $g(z, \bar{z}, u) \in \mathbb{R}$ without constant and linear terms such that $(f_{u^2}, g_{u^2})(0) = v$ and the map $h = \mathsf{id} + (f, g)$ transforms L' into L satisfying the normalization*

$$\widetilde{L}^{\bar{z}}_{z^a \bar{z}^b u^c} = 0,$$

(1.6)
$$\operatorname{Re} \widetilde{L}^{w}_{z^a \bar{z}^b u^c} = 0, \qquad \widetilde{L}^{w}_{z\bar{z}u^{c+1}} = 0, \qquad \widetilde{L}^{w}_{z^{a+1}\bar{z}u^c} = 0,$$
$$\operatorname{tr}\big(\widetilde{L}^{w}_{z^2 \bar{z}^2 u^c}\big) = 0, \qquad \operatorname{tr}^3\big(\widetilde{L}^{w}_{z^3 \bar{z}^3 u^c}\big) = 0,$$
$$\operatorname{tr}^3\big(48 \widetilde{L}^{w}_{z^3 \bar{z}^2 u^{s-1}} \widetilde{L}^{w}_{z\bar{z}} - 7 \widetilde{L}^{w}_{z^4 \bar{z}^3 u^{s-2}}\big) = 0,$$

for all $a, b, c \geq 0$.

Theorem 1.3 is proved in Section 4. The "mysterious" coefficients 48 and -7 appear naturally in the normal form calculation and are determined up to a common factor.

2. Almost CR structures

For reader's convenience, we recall basic definitions and constructions. Let M be a real manifold with almost CR structure given by H and J or, equivalently by a complex subbundle $H^{1,0} \subset \mathbb{C} \otimes T$ satisfying $H^{1,0} \cap \overline{H^{1,0}} = 0$. The complex dimension of the fiber H_p, $p \in M$, is called the *CR dimension* $\dim_{\mathrm{CR}} M$ of an almost CR manifold M and the real codimension of H_p in T_p the *CR codimension* $\operatorname{codim}_{\mathrm{CR}} M$ of M. The pair $(\dim_{\mathrm{CR}} M, \operatorname{codim}_{\mathrm{CR}} M)$ is sometimes called the *type* of M and in case $\operatorname{codim}_{\mathrm{CR}} M = 1$, M is said to be of *hypersurface type*.

Many formulas and calculations become simpler when working with the complexified tangent bundle $\mathbb{C}T := \mathbb{C} \otimes T$. Here J extends to a complex bundle automorphism of $\mathbb{C}H =: \mathbb{C} \otimes H$, which splits into direct sum of its $(\pm i)$-eigenspaces

$$H^{1,0} := \{\xi \in \mathbb{C}H : J\xi = i\xi\}, \qquad H^{0,1} := \{\xi \in \mathbb{C}H : J\xi = -i\xi\}.$$

These eigenspaces form complex subbundles of $\mathbb{C}H$ satisfying

$$H^{0,1} = \overline{H^{1,0}}, \qquad \mathbb{C}H = H^{1,0} \oplus H^{0,1}$$

and each of them uniquely determines the almost CR structure.

3. Coordinate setting

We briefly outline the construction of [Z12] that we shall used here. For a given almost CR structure (H, J) on M of CR dimension n and CR codimension d and a reference point $p_0 \in M$, we consider a system of coordinates $(x + iy, u) \in \mathbb{C}^n_{x+iy} \times \mathbb{R}^d_u$ on M such that

$$(3.1) \qquad p_0 = (0, 0, 0), \qquad H_0 = \mathbb{C}^n \times \{0\}, \qquad J_0(\xi, 0) = (i\xi, 0).$$

3.1. Complex coordinates. Instead of using the real coordinates (x, y, u), we adopt the complexified point of view and switch to the coordinates

$$(3.2) \qquad z = x + iy, \qquad \bar{z} = x - iy \in \mathbb{C}^n, \qquad w \in \mathbb{C}^d \quad \text{with } u = \operatorname{Re} w.$$

Then our preliminary normalization (3.1) is expressed by

$$(3.3) \qquad\qquad H_0^{1,0} = \{d\bar{z} = 0, dw = 0\}.$$

Now consider a general point $p \in M$. If p is sufficiently close to 0, the subspace $H_p^{1,0} \subset \mathbb{C}T_p$ is the graph of a uniquely determined complex-linear map

$$(3.4) \qquad L(p) : \mathbb{C}^n_z \to \mathbb{C}^n_{\bar{z}} \times \mathbb{C}^d_w, \qquad H_p^{1,0} = \big\{(d\bar{z}, dw) = L(p)dz\big\}.$$

We shall also distinguish the components of L:

$$(3.5) \qquad L(p) = \big(L^{\bar{z}}(p), L^w(p)\big), \quad L^{\bar{z}}(p) : \mathbb{C}^n_z \to \mathbb{C}^n_{\bar{z}}, L^w(p) : \mathbb{C}^n_z \to \mathbb{C}^d_w.$$

Then (3.3) can be rewritten as

$$(3.6) \qquad\qquad L(0) = \big(L^{\bar{z}}(0), L^w(0)\big) = 0.$$

3.2. Evaluation along the Euler vector field. One basic idea of [Z12] was to write normalization conditions not for L directly but for its *evaluation along the Euler vector field*.

Consider the following Euler (or radial) type vector field:

$$(3.7) \qquad\qquad e = e(z) := z\frac{\partial}{\partial z} = \sum_j z_j \frac{\partial}{\partial z_j} \in T_z^{1,0}\mathbb{C}^n.$$

Given a (formal) map L as in (3.4), we evaluate it along e to obtain a $\mathbb{C}^n_{\bar{z}} \times \mathbb{C}^d_w$-valued formal power series

$$(3.8) \qquad\qquad \widetilde{L}(z, \bar{z}, u) := L(z, \bar{z}, u)e(z) = L(z, \bar{z}, u)z\frac{\partial}{\partial z}.$$

We write $L_{z^a \bar{z}^b u^c}$ for the derivative at 0 regarded as a multihomogeneous polynomial of degree a in z, b in $\bar{z}$ and c in u, which is given by

$$(3.9) \quad L_{z^a \bar{z}^b u^c}(z, \bar{z}, u)$$
$$:= \sum L_{z_{i_1} \cdots z_{i_a} \bar{z}_{j_1} \cdots \bar{z}_{j_b} u_{k_1} \cdots z_{k_c}}(0) z_{i_1} \cdots z_{i_a} \bar{z}_{j_1} \cdots \bar{z}_{j_b} u_{k_1} \cdots u_{k_c}$$

in terms of the partial derivatives of L, where the summation is taken over all collections of indices $i_1, \ldots, i_a,\ j_1, \ldots, j_b \in \{1, \ldots, n\},\ k_1, \ldots, k_c \in \{1, \ldots, d\}$. Then

$$(3.10) \qquad \widetilde{L}_{z^a \bar{z}^b u^c}(z, \bar{z}, u) = a L_{z^{a-1} \bar{z}^b u^c}(z, \bar{z}, u) e(z)$$

for all integers $a, b, c \geq 0$.

3.3. Partial normalization. We recall from [Z12] the following partial normalization that holds for almost CR structures for any codimension without any nondegeneracy assumption.

PROPOSITION 3.1 ([Z12], Proposition 5.2). *For every formal power series* $L(z, \bar{z}, u) : \mathbb{C}_z^n \to \mathbb{C}_{\bar{z}}^n \times \mathbb{C}_w^d$ *without constant terms and every formal power series* $f_0(z, u)$ *and* $g_0(u)$ *without constant and linear terms, there exist unique formal power series* $f(z, \bar{z}, u)$ *and* $g(z, \bar{z}, u)$ *without constant and linear terms such that* $f(z, 0, u) = f_0(z, u)$ *and* $g(0, 0, u) = g_0(u)$ *and the map* $\mathsf{id} + (f, g)$ *transforms* L *into* L' *satisfying the normalization*

$$(3.11) \qquad \widetilde{L}'^{\bar{z}'}\left(z', \bar{z}', u'\right) = 0, \qquad \mathsf{Re}\, \widetilde{L}'^{w'}\left(z', \bar{z}', u'\right) = 0.$$

4. A normal form for all Levi-nondegenerate almost CR structures

Our goal here is to modify our previous construction in [Z12] to obtain a different normal form that is valid for *all* Levi-nondegenerate almost CR structures (that are not necessarily strongly nondegenerate). For this, note that strong nondegeneracy has only been used in [Z12, (7.35)] in order to determine $f_{u^{s+1}}$ for $s \geq 1$, whereas only Levi-nondegeneracy has been used in the preceding normalizations [Z12, (7.22), (7.26), (7.29), (7.32)].

If the given almost CR structure is not strongly nondegenerate, we may not be able to obtain the normalization [Z12, (7.35)]. Instead we consider the identities [Z12, (7.3)] corresponding to (a, b, c) equal to $(4, 3, s-2)$ and $(3, 4, s-2)$ for $s \geq 2$, where we keep the notation and induction hypotheses of Section 7 of [Z12]. Using [Z12, (7.5)] to eliminate the nonpure terms as before, we obtain:

$$
\begin{aligned}
& 4 g_{z^4 \bar{z}^3 u^{s-2}} + 12 g_{z^3 \bar{z}^2 u^{s-1}} \widetilde{L}_{z\bar{z}}^w + \widetilde{L}_{z^4 \bar{z}^3 u^{s-2}}^w \\
& \quad = \widetilde{L}'^{w'}_{z'^4 \bar{z}'^3 u'^{s-2}} + 216 L'^{w'}_{\bar{z}'}(\bar{z}; f_{z^2 u^s})\left(\widetilde{L}_{z\bar{z}}^w\right)^2 + \left(120 L'^{w'}_{z'}(z; f_{u^{s+1}})\right. \\
(4.1) & \qquad \left. - 24 L'^{w'}_{\bar{z}'}(\bar{f}_{u^{s+1}}; z)\right)\left(\widetilde{L}_{z\bar{z}}^w\right)^3 + \cdots, \\
& 3 g_{z^3 \bar{z}^4 u^{s-2}} + 12 g_{z^2 \bar{z}^3 u^{s-1}} \widetilde{L}_{z\bar{z}}^w + \widetilde{L}_{z^3 \bar{z}^4 u^{s-2}}^w \\
& \quad = \widetilde{L}'^{w'}_{z'^3 \bar{z}'^4 u'^{s-2}} + 36 L'^{w'}_{\bar{z}'}(\bar{f}_{\bar{z}^2 u^s}; z)\left(\widetilde{L}_{z\bar{z}}^w\right)^2 + 144 L'^{w'}_{\bar{z}'}(\bar{z}; f_{u^{s+1}})\left(\widetilde{L}_{z\bar{z}}^w\right)^3 + \cdots.
\end{aligned}
$$

As in Section 7.3 of [Z12], wee keep the normalization [Z12, (7.36)], in particular, we assume

$$(4.2) \qquad \widetilde{L}'^{w'}_{z'^4 \bar{z}'^3 u'^{s-2}} + \overline{\widetilde{L}'^{w'}_{z'^3 \bar{z}'^4 u'^{s-2}}} = 0.$$

Furthermore, subtracting 4 times the conjugate of the second identity in (4.1) from 3 times the first identity, we obtain

$$(4.3) \qquad 7 \cdot 12 g_{z^3 \bar{z}^2 u^{s-1}} \widetilde{L}^w_{z\bar{z}} + \left(3\widetilde{L}^w_{z^4 \bar{z}^3 u^{s-2}} - 4\overline{\widetilde{L}^w_{z^3 \bar{z}^4 u^{s-2}}} \right)$$
$$= \left(3\widetilde{L}'^{w'}_{z'^4 \bar{z}'^3 u'^{s-2}} - 4\overline{\widetilde{L}'^{w'}_{z'^3 \bar{z}'^4 u'^{s-2}}} \right) + 792 L'^{w'}_{\bar{z}'}(\bar{z}; f_{z^2 u^s}) \left(\widetilde{L}^w_{z\bar{z}} \right)^2$$
$$+ \left(360 L'^{w'}_{z'}(z; f_{u^{s+1}}) - 648 L'^{w'}_{\bar{z}'}(\bar{f}_{u^{s+1}}; z) \right) \left(\widetilde{L}^w_{z\bar{z}} \right)^3 + \cdots.$$

We use [Z12, (7.25)] to solve for $g_{zu^{s+1}}$, substitute into the second identity in [Z12, (7.30)] and solve it for $L'^{w'}_{\bar{z}'}(\bar{z}; f_{z^2 u^s})$. Then substitute it in [Z12, (7.33)] and solve for $g_{z^3 \bar{z}^2 u^{s-1}}$ and finally substitute everything into (4.3):

$$(4.4) \qquad -\frac{96 \cdot 12}{5} \left(7 L'^{w'}_{\bar{z}'}(\bar{f}_{u^{s+1}}; z) + 4 L'^{w'}_{z'}(f_{u^{s+1}}; z) \right) \left(\widetilde{L}^w_{z\bar{z}} \right)^3$$
$$+ \left(3\widetilde{L}^w_{z^4 \bar{z}^3 u^{s-2}} - 4\overline{\widetilde{L}^w_{z^3 \bar{z}^4 u^{s-2}}} \right)$$
$$= \left(3\widetilde{L}'^{w'}_{z'^4 \bar{z}'^3 u'^{s-2}} - 4\overline{\widetilde{L}'^{w'}_{z'^3 \bar{z}'^4 u'^{s-2}}} \right) + \cdots.$$

Multiplying [Z12, (7.34)] by $-12 \cdot 4\widetilde{L}^w_{z\bar{z}}$, (4.4) by 5 and adding them together, we obtain

$$(4.5) \qquad 96 \cdot 60 L'^{w'}_{\bar{z}'}(\bar{f}_{u^{s+1}}; z) \left(\widetilde{L}^w_{z\bar{z}} \right)^3$$
$$- 48 \left(2\widetilde{L}^w_{z^3 \bar{z}^2 u^{s-1}} - 3\overline{\widetilde{L}^w_{z^2 \bar{z}^3 u^{s-1}}} \right) \widetilde{L}^w_{z\bar{z}}$$
$$+ 5 \left(3\widetilde{L}^w_{z^4 \bar{z}^3 u^{s-2}} - 4\overline{\widetilde{L}^w_{z^3 \bar{z}^4 u^{s-2}}} \right)$$
$$= -48 \left(2\widetilde{L}'^{w'}_{z'^3 \bar{z}'^2 u'^{s-1}} - 3\overline{\widetilde{L}'^{w'}_{z'^2 \bar{z}'^3 u'^{s-1}}} \right) \widetilde{L}^w_{z\bar{z}}$$
$$+ 5 \left(3\widetilde{L}'^{w'}_{z'^4 \bar{z}'^3 u'^{s-2}} - 4\overline{\widetilde{L}'^{w'}_{z'^3 \bar{z}'^4 u'^{s-2}}} \right).$$

Using [Z12, (7.32)] and (4.2), we conclude that the expression

$$(4.6) \qquad 48\widetilde{L}'^{w'}_{z'^3 \bar{z}'^2 u'^{s-1}} \widetilde{L}^w_{z\bar{z}} - 7\widetilde{L}'^{w'}_{z'^4 \bar{z}'^3 u'^{s-2}}$$

is determined up to a multiple of $L'^{w'}_{\bar{z}'}(\bar{f}_{u^{s+1}}; z)(\widetilde{L}^w_{z\bar{z}})^3$. Hence, $\bar{f}_{u^{s+1}}$ for $s \geq 2$ can be uniquely determined by the normalization condition

$$(4.7) \qquad \mathrm{tr}^3 \left(48\widetilde{L}'^{w'}_{z'^3 \bar{z}'^2 u'^{s-1}} \widetilde{L}^w_{z\bar{z}} - 7\widetilde{L}'^{w'}_{z'^4 \bar{z}'^3 u'^{s-2}} \right) = 0.$$

Note that in contrast to the previous section, the remaining derivative f_{u^2} is not determined and is to be treated as free parameter along with g_{u^2}. Summarizing we obtain Theorem 1.3.

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