Illinois Journal of Mathematics
Volume 56, Number 1, Spring 2012, Pages 101–125
S 0019-2082

FORMAL THEORY OF SEGRE VARIETIES

GIUSEPPE DELLA SALA, ROBERT JUHLIN AND BERNHARD LAMEL

ABSTRACT. We define a category of formal CR manifolds in a purely algebraic way, study their basic properties, and prove the Baouendi–Ebenfelt–Rothschild minimality criterion in this setting. The objects of our category include deformations of classical CR manifolds (real-analytic or formal).

1. Introduction and statement of results

In this paper, we develop the basic theory of Segre maps from a purely formal point of view. Segre maps and Segre sets appear naturally when one studies the orbit structure of the CR equations on a real-analytic CR manifold and have turned out to be a basic tool in the field; they were introduced in Baouendi, Ebenfelt and Rothschild's study of algebraicity [1] and subsequently used in many results in the theory of mapping problems. There have been several studies of the Segre maps undertaken in the past, see, for example, [3], [4] as well as the book [2], so why a new one? While the cited papers are actually applicable to the formal situation, it is not quite clear how one can use these approaches in a situation where the orbit structure is not locally uniform, that is, the dimension of the orbits changes. It is not even clear how to properly state this in the formal context, as one can really speak only about the formal orbit of *one point* in the known framework!

This situation is unsatisfactory, since in many instances, one wants to "deform" a given geometric object, for example, by choosing a new basepoint for a construction. This point of view has turned up repeatedly in newer studies of mapping problems, and we think it is thus necessary and interesting to present a general framework which can be used to treat these deformations in a unified manner.

Received December 1, 2011; received in final form May 3, 2012.

The authors were supported by the START Prize Y377 of the Austrian Federal Ministry of Science and Research bmwf.

2010 *Mathematics Subject Classification.* 32V05, 32V35, 32H02.

Our approach is to consider instead of formal CR manifolds, that is, objects defined by real ideals in a power series ring $\mathbb{C}[[Z,\zeta]]$, formal CR manifolds which are defined by ideals in a power series ring $\mathcal{A}[[Z,\zeta]]$, where now $\mathcal{A}$ is some *ring* (commutative with unit) which has an involution σ, and we consider ideals which are invariant under the extension $\tilde{\sigma}$ of σ to $\mathcal{A}[[Z,\zeta]]$ defined by $\tilde{\sigma}|_{\mathcal{A}} = \sigma$ and $\tilde{\sigma}(Z) = \zeta$.

One can for example use the (complex) coordinate ring $\mathbb{C}[[\mathcal{M}]] = \mathbb{C}[[p,q]]/I$ of a formal (or real-analytic) submanifold defined by the ideal I for $\mathcal{A}$, and tautologically induce from I, considered as $I \subset \mathbb{C}[[Z,\zeta]]$ an ideal $\tilde{I} \subset \mathbb{C}[[\mathcal{M}]][[Z,\zeta]]$ generated by $\varphi(p+Z, q+\zeta) - \varphi(p,q)$ for $\varphi \in I$; one can also use the (smooth complex) coordinate ring of a smooth CR submanifold $M \subset \mathbb{C}^N$ in a similar manner. We will call these objects *formal CR structures*.

Each of these uses encapsulates some *nonlocal* behaviour of a manifold in a localised way. We will thus have to be careful with "straightforward" generalizations of well-known notions: In our framework, there is a big difference between some independence relation being satisfied with respect to $\mathcal{A}$ or with respect to its quotient field $\mathbb{K}$. In the case of induced CR structures, the former corresponds to a locally satisfied condition, while the latter corresponds to a condition which is only satisfied generically.

Our main result in this paper is that the minimality criterion of Baouendi, Ebenfelt and Rothschild holds in the setting of these formal CR structures. Minimality is defined here by a Lie-algebra condition on the family of $(1,0)$- and $(0,1)$-vector fields of a formal CR structure (to be defined in detail below), and the formal analogues of the Segre maps are also defined below; we refer the reader to Section 2 or our basic definitions of a formal CR structure, and to Section 6 for the definition of the associated Segre maps.

THEOREM 1. *Assume that $\mathcal{M}$ is a formal CR structure over a ring $\mathcal{A}$ with involution σ. Then there exists an integer k such that the Segre map of order k is generically of full rank if and only $\mathcal{M}$ is generic and of finite type. Furthermore, if $\mathcal{M}$ is generic, of finite type, and of CR codimension d, then the Segre map of order $d+1$ is already of full rank. In that case, there exists a formal manifold Σ such that the Segre map of order $2d-1$ maps Σ onto 0 and is generically of full rank on Σ.*

More generally, we find that the image of the Segre maps of high order coincides in a certain sense with the CR orbit of a formal CR structure. We also build some of the basic theory of formal CR structures as defined here (as we need to use it).

Let us discuss a bit the technique we use in this paper. Our framework *never* allows us to evaluate our power series sensibly at any point besides of 0. We shall overcome this technical obstacle by building algebraically on formal flows of vector fields. The main technical ingredient is that any formal family of vector fields fulfills a finite type condition if and only if their iterated flows are of full rank (over a field of fractions). This result requires us to restrict ourselves to rings which contain the rational numbers as a subfield (in all applications we have in mind, this is the case). Although doubtlessy this fact is known, we deduce it for sake of completeness (and because we haven't been able to find a suitable reference in the literature) in Section 3.

Finally, let us note that the "formal CR structures" introduced here allow for an elegant way to state (and prove) some results known from the literature, for example, how complexifications vary with the basepoint, as one can deduce these results by applying generalizations of the local arguments to these more general structures.

The paper is organized as follows. In Section 2, we define the objects and maps we are going to study, and state some basic properties. We present the material on flows and their composition, as already mentioned, in Section 3. In the special case of multiple smooth foliations we present another way of looking at the composition in Section 4; in the setting of CR manifolds, this ties together iterated complexifications of CR manifolds with the images of iterated Segre maps. That compositions of flows increase their rank is proved in Section 5. Finally, in Section 6, we define the iterated Segre maps of the formal CR structures and prove Theorem 1.

2. Formal manifolds and CR structures

2.1. Formal manifold ideals. Let $\mathcal{A}$ be a ring (commutative, with unit); we shall assume that $\mathcal{A}$ has no zero divisors and that it contains the field of rational numbers as a subring (i.e., all multiples of 1 are units in $\mathcal{A}$). We say that an ideal $I\subset\mathcal{A}[[x]]$ in the formal power series ring in the unknowns $x=(x_1,\dots,x_n)$ with coefficients in $\mathcal{A}$ is a *manifold ideal* if there exist generators $\rho=(\rho_1,\dots,\rho_d)$ of I whose differentials have the property that the matrix

$$d\rho(0)=\begin{pmatrix}\rho_{1,x_1}(0) & \cdots & \rho_{1,x_n}(0)\\ \vdots & & \vdots\\ \rho_{d,x_1}(0) & \cdots & \rho_{d,x_n}(0)\end{pmatrix}$$

has a $d\times d$-minor which is a unit in $\mathcal{A}$. By the implicit function theorem, this is equivalent to the fact that after renumbering the x_j if necessary, there exist formal power series $\varphi_j(x_1,\dots,x_{n-d})\in\mathcal{A}[[x_1,\dots,x_{n-d}]]$, $j=1,\dots,d$, such that I is generated by $x_{n-d+j}-\varphi_j(x_1,\dots,x_{n-d})$, $j=1,\dots,d$; in other words, if we write $x=(x',x'')$ where $x'=(x_1,\dots,x_{n-d})$, and $\varphi(x')=(\varphi_1(x'),\dots\varphi_d(x'))$, then $\mathcal{A}[[x]]/I$ is isomorphic to $\mathcal{A}[[x']]$ by the substitution homomorphism $\Phi:\mathcal{A}[[x]]\to\mathcal{A}[[x']]$, $\Phi(a(x))=a(x',\varphi(x'))$.

The $\mathcal{A}$-linear derivations $\mathcal{D}$ of $\mathcal{A}[[x]]$ are identified with the formal vector fields

$$X=\sum_{j=1}^{n}a_j(x)\frac{\partial}{\partial x_j},\quad a_j\in\mathcal{A}[[x]];$$

the condition that I is a manifold ideal can be restated once more by saying that the $\mathcal{A}$-linear derivations of $\mathcal{A}[[x]]$ which map I into itself, the space of which we denote by $\mathcal{D}_I = \mathrm{Der}_{\mathcal{A}}(\mathcal{A}[[x]], I)$, form a free module of rank $n-d$ over $\mathcal{A}[[x]]$ with the additional property that the vector space $\mathcal{D}_I(0)$ over the quotient field $\mathbb{K}$ of $\mathcal{A}$ is of dimension $n-d$. Restating this once more, I is a manifold ideal if and only if for every $a \in \mathcal{A}^n$ with $df(0)a = 0$ for all $f \in I$ there exists a formal vector field $X \in \mathrm{Der}$ with $XI \subset I$ which satisfies $X(0) = a$.

We shall find it convenient to think about the formal manifold $\mathcal{M} \subset \mathcal{A}^n$ associated to I when speaking about geometric concepts associated to I, even though we will not give a precise meaning to this object. In particular, we refer to $\mathcal{D}_I(0)$ as the tangent space $T_0\mathcal{M}$ of $\mathcal{M}$.

2.2. Rank of maps and homomorphisms. Given a power series map $H(x) = (f_1(x), \ldots, f_m(x))$ with $f_j(x) \in \mathcal{A}[[x_1, \ldots, x_n]]$, we define the generic rank $\mathrm{rk}\, H$ of H as the maximum integer r such that there exists an $r \times r$-minor of the matrix $\frac{\partial H}{\partial x}$ which is a nonvanishing formal power series with coefficients in $\mathcal{A}$. In other words, $\mathrm{rk}\, H$ is the dimension of the vector space spanned by the columns (or rows) of $\frac{\partial H}{\partial x}$ over the field of fractions $\mathbb{K}$ of $\mathcal{A}$. Given a homomorphism of power series rings $\Psi : \mathcal{A}[[y_1, \ldots, y_m]] \to \mathcal{A}[[x]]$, we define the rank $\mathrm{rk}\, \Psi$ of Ψ as $\mathrm{rk}(\Psi(y_1), \ldots, \Psi(y_m))$. H is *generically of full rank* if $\mathrm{rk}\, H = \min(m, n)$. In what follows, we will abuse notation by identifying power series maps with their induced homomorphisms; this will cause no difficulty, since it will be clear from the context what is meant, and we shall be careful to distinguish in places where confusion might arise.

2.3. Formal CR structures. Now let us assume that $\mathcal{A}$ has an involution, which we denote by $a \mapsto \sigma a$. We extend σ to an involution $\tilde{\sigma}$ of the formal power series ring $\mathcal{A}[[Z, \zeta]]$, where $Z = (Z_1, \ldots, Z_N)$ and $\zeta = (\zeta_1, \ldots, \zeta_N)$ are independent variables, by setting $\tilde{\sigma} Z_j = \zeta_j$. Explicitly, this involution is given by

$$\tilde{\sigma}\big(\rho(Z, \zeta)\big) = (\sigma\rho)(\zeta, Z),$$

where for $\rho(Z, \zeta) = \sum_{\alpha,\beta} a_{\alpha,\beta} Z^\alpha \zeta^\beta$, we define $\sigma\rho(Z, \zeta) = \sum_{\alpha,\beta} (\sigma a_{\alpha,\beta}) Z^\alpha \zeta^\beta$.

The space of $\mathcal{A}$-linear derivations of $\mathcal{A}[[Z, \zeta]]$ splits into a direct sum,

$$\mathcal{D} = \mathcal{D}^{(1,0)} \oplus \mathcal{D}^{(0,1)},$$

where $\mathcal{D}^{(1,0)} = \mathrm{ann}(\mathcal{A}[[\zeta]])$ and $\mathcal{D}^{(0,1)} = \mathrm{ann}(\mathcal{A}[[Z]])$; here $\mathrm{ann}\, S = \{X : Xm = 0 \text{ for all } m \in S\}$.

We are mainly interested in the properties of $\tilde{\sigma}$-invariant ideals I, that is, ideals which satisfy $\tilde{\sigma} I = I$; in the case $\mathcal{A} = \mathbb{C}$ with σ denoting complex conjugation, the standard terminology is to refer to $I \subset \mathcal{A}[[Z, \zeta]]$ as *real* if $\bar{I} = I$.

A formal $\tilde{\sigma}$-invariant manifold $\mathcal{M}$ of codimension d is given by a $\tilde{\sigma}$-invariant manifold ideal $I = I(\mathcal{M}) \subset \mathcal{A}[[Z, \zeta]]$. We note that given a $\tilde{\sigma}$-invariant ideal, it can always be generated by elements ρ which either satisfy $\tilde{\sigma}\rho = \rho$ or $\tilde{\sigma}\rho = -\rho$. By definition,

$$d\rho(0) := \begin{pmatrix} \rho_{1,Z_1}(0) & \cdots & \rho_{1,Z_N}(0) & \rho_{1,\zeta_1}(0) & \cdots & \rho_{1,\zeta_N}(0) \\ \vdots & & \vdots & \vdots & & \vdots \\ \rho_{d,Z_1}(0) & \cdots & \rho_{d,Z_N}(0) & \rho_{d,\zeta_1}(0) & \cdots & \rho_{d,\zeta_N}(0) \end{pmatrix}$$

has a $d \times d$ minor which is a unit in $\mathcal{A}$ and the tangent space of $\mathcal{M}$ at 0 is a free submodule of $\mathcal{A}^{2N}$ of rank $2N - d$; it is given by the $(A, B) \in \mathcal{A}^{2N}$ which satisfy

$$\sum_j \rho_{\ell,Z_j}(0) A_j + \sum_k \rho_{\ell,\zeta_j}(0) B_j = 0, \quad \ell = 1, \dots, d.$$

We define the space of $(1,0)$-tangent vectors $T_0^{(1,0)}\mathcal{M} \subset \mathcal{A}^N$ to consist of all $A = (A_1, \dots, A_n)$ which satisfy that $\sum_j f_{Z_j}(0) A_j = 0$ for all $f \in I$, and similarly $T_0^{(0,1)}\mathcal{M}$ to consist of all $B = (B_1, \dots, B_n)$ with $\sum_j f_{\zeta_j}(0) A_j = 0$ for all $f \in I$. We note that σ defines an isomorphism $T_0^{(1,0)}\mathcal{M} \to T_0^{(0,1)}\mathcal{M}$.

We define the spaces $\mathcal{D}_I^{(1,0)} = \mathcal{D}_I \cap \mathcal{D}^{(1,0)}$ and $\mathcal{D}_I^{(0,1)} = \mathcal{D}_I \cap \mathcal{D}^{(0,1)}$. We shall say that $\mathcal{M}$ is *CR* or *Cauchy–Riemann* if $\mathcal{D}_I^{(1,0)}(0) = T_0^{(1,0)}\mathcal{M}$; if we want to emphasize the ring $\mathcal{A}$ and the involution σ, we will refer to $\mathcal{M}$ as a formal CR structure, defined by I, over $(\mathcal{A}, \sigma)$. If the matrix

$$\rho_Z(0) := \begin{pmatrix} \rho_{1,Z_1}(0) & \cdots & \rho_{1,Z_N}(0) \\ \vdots & & \vdots \\ \rho_{d,Z_1}(0) & \cdots & \rho_{d,Z_N}(0) \end{pmatrix}$$

has a $d \times d$ minor which is a unit in $\mathcal{A}$, then $\mathcal{M}$ is said to be *generic.*

Our first lemma analyzes the structure of general formal CR manifolds; the corresponding statement for the case $\mathcal{A} = \mathbb{C}$ is well-known (see, e.g., [2]). It allows us to restrict ourselves most of the time to generic manifolds.

LEMMA 1. *Let $\mathcal{M}$ be a formal CR manifold of codimension d, with associated manifold ideal I. Then there exists integers d_1, d_2, and d_3 with $N = d_1 + d_2 + d_3$, $d = 2d_1 + d_3$ such that after renumbering Z and ζ if necessary, we can write $Z = (Z^1, Z^2, Z^3)$, $\zeta = (\zeta^1, \zeta^2, \zeta^3)$, where $Z^j = (Z^j_1, \dots, Z^j_{d^j})$, $\zeta^j = (\zeta^j_1, \dots, \zeta^j_{d^j})$, and we can choose a set of generators of I given by $Z^1 - \varphi(Z^2, Z^3)$, $\zeta^1 - \sigma\varphi(\zeta^2, \zeta^3)$, and $\zeta^2 - R(Z^2, Z^3, \zeta^3)$ for some $\varphi \in \mathcal{A}[[Z^2, Z^3]]^{d_1}$ and $R \in \mathcal{A}[[Z^2, Z^3, \zeta^3]]^{d_2}$.*

Proof. Assume that $\mathcal{M}$ is of codimension d, and let $\rho = (\rho_1, \dots, \rho_d)$ be generators of I. Let $d' > 0$ be the biggest number such that $\rho' = (\rho_1, \dots, \rho_{d'})$ satisfies that the matrix ρ'_ζ has a $d' \times d'$ minor which is a unit in $\mathcal{A}$. By the implicit function theorem, after renumbering the ζ if necessary, we can write $\zeta = (\zeta', \zeta'')$ such that with an invertible $d' \times d'$-matrix A we can write $\rho'(Z, \zeta) = A(Z, \zeta)(\zeta' - R(\zeta'', Z))$; we can thus assume that $\rho_j = \zeta_j - R_j(\zeta'', Z)$

for $j = 1, \ldots, d'$. We write $\eta = (\rho_{d'+1}(Z, R(\zeta'', Z), \zeta''), \ldots, \rho_d(Z, R(\zeta'', Z), \zeta'')) = (\eta_1, \ldots, \eta_e)$; note that (ρ', η) are again generators for I. Since I is a manifold ideal, the matrix $\eta_Z(0)$ necessarily has an $e \times e$-minor which is a unit in $\mathcal{A}$. Thus, using the implicit function theorem, we can choose $k_1, \ldots, k_e$ and that $\eta_j = \hat{Z}_{k_j} - \varphi_j(\tilde{Z}, \zeta'')$ (where the $\tilde{Z}$ contains all the Z_k for $k \neq k_j$).

By assumption (since $\mathcal{M}$ is CR), each $\eta_{j,\zeta}$ can be written as

$$\eta_{j,\zeta} = \sum_{k=1}^{d'} a_j^k \rho_{k,\zeta} \tag{1}$$

modulo I. Since η_j does not depend on ζ', this implies that $a_j^k \in I$. (1) implies that we can actually write $\eta_{j,\zeta} = \sum_k b_k \rho_k + \sum_\ell c_\ell \eta_\ell$. But ρ_k contains a ζ_k which does not appear anywhere else. Hence, $b_k = 0$; also, η_ℓ does contain a Z_ℓ which does not appear anywhere else, and so $c_\ell = 0$. We conclude that actually $\eta_\ell = Z_\ell - \varphi_\ell(\tilde{Z})$. Since $\tilde{\sigma}\eta_\ell \in I$, we can arrange by renumbering again that $k_j = j$ and replace ρ_j by $\bar{\eta}_j$ for $j = 1, \ldots, e$. We set $d_1 = e$, $d_2 = d' - e$, and arrive at the conclusion of the lemma. ☐

In particular, $\mathcal{A}[[Z, \zeta]]/I$ is isomorphic to $\mathcal{A}[[Z^2, Z^3, \zeta^2, \zeta^3]]/\tilde{I}$ where $\tilde{I}$ is generated by $\zeta^2 - R(Z^2, Z^3, \zeta^3)$. In this sense, we can reduce to the study of generic submanifolds.

Example 1. We start by considering a formal manifold M over $\mathbb{C}$, given by $\rho(p, q) = 0$. To it, we attach a formal manifold $\tilde{M}$ over the coordinate ring of M (which we denote by $\mathbb{C}[[M]]$), its ideal being generated by $\tilde{\rho}(Z, \zeta) = \rho(Z + p, \zeta + q) - \rho(p, q)$. Clearly $\tilde{\rho}_Z(0) = \rho_p(p, q)$, so $\tilde{M}$ is generic if M is.

2.4. Generic and $\mathcal{A}$-submanifolds. Given a manifold ideal $J \subset \mathcal{A}[[Z]]$, there exists a unique minimal $\tilde{\sigma}$-invariant ideal $I \subset \mathcal{A}[[Z, \zeta]]$ such that $I \cap \mathcal{A}[[Z]] = J$. We shall say that such a manifold ideal is associated to an $\mathcal{A}$-submanifold or say that I is an $\mathcal{A}[[Z]]$-ideal. The following lemma summarizes ways to define $\mathcal{A}$-submanifolds; we leave its proof to the reader.

Lemma 2. *If $I \subset \mathcal{A}[[Z, \zeta]]$ is a $\tilde{\sigma}$-invariant manifold ideal, the following are equivalent*:

(i) *I is an $\mathcal{A}[[Z]]$-ideal*;
(ii) $I = (I \cap \mathcal{A}[[Z]]) + (I \cap \mathcal{A}[[\zeta]])$;
(iii) $\mathcal{D}_I = \mathcal{D}_I^{(1,0)} \oplus \mathcal{D}_I^{(0,1)}$;
(iv) *There exists a splitting $Z = (Z^1, Z^2)$, $Z^j = (Z_1^j, \ldots, Z_{N^j}^j)$ and a map $\varphi(Z^1) \in \mathcal{A}[[Z^1]]^{N^2}$ such that the ring homomorphism induced by $Z^2 = \varphi(Z^1)$, $\zeta^2 = \sigma\varphi(\zeta^1)$ is an isomorphism $\mathcal{A}[[Z^1, \zeta^1]] \cong \mathcal{A}[[Z, \zeta]]/I$.*

A generic submanifold has characterizations which starkly contrast these.

Lemma 3. *If* $I \subset \mathcal{A}[[Z,\zeta]]$ *is a* $\tilde{\sigma}$*-invariant manifold ideal, the following are equivalent*:

(i) I *is generic*;
(ii) $I \cap \mathcal{A}[[Z]] = \{0\}$;
(iii) $\mathcal{D}_I + \tilde{\sigma}\mathcal{D}_I = \mathcal{D}$, *or more specifically, with* $\mathcal{T} = \mathcal{D}_I/(\mathcal{D}_I^{(1,0)} \oplus \mathcal{D}_I^{(0,1)})$, $\mathcal{D} = \mathcal{D}_I \oplus \mathcal{T}$;
(iv) *There exists a splitting* $Z = (Z^1, Z^2)$, $Z^j = (Z^j_1, \dots, Z^j_{N^j})$ *and a map* $R(Z^1, Z^2, \zeta^2) \in \mathcal{A}[[Z^1, Z^2, \zeta^2]]^{N^1}$ *such that the ring homomorphism induced by* $\zeta^2 = \varphi(Z^1, Z^2, \zeta^1)$ *is an isomorphism of* $\mathcal{A}[[Z^1, Z^2, \zeta^1]] \cong \mathcal{A}[[Z,\zeta]]/I$.

We can now restate Lemma 1 by saying that for a formal CR structure in $\mathcal{A}[[Z,\zeta]]$, there exists a unique $\mathcal{A}$-submanifold in which it is generic.

2.5. CR mappings. We now consider two rings $\mathcal{A}$ and $\mathcal{B}$, with involutions σ and ω, respectively. We say that a ring homomorphism $\Phi : \mathcal{A}[[Z',\zeta']] \to \mathcal{B}[[Z,\zeta]]$ is compatible with (σ,ω) if $\omega \circ \Phi = \Phi \circ \sigma$. As usual, a ring homomorphism is identified with a power series map defined by $Z'_j = F_j(Z,\zeta)$, $\zeta'_j = G_j(Z,\zeta)$, where $F_j = \Phi(Z'_j)$, $G_j = \Phi(\zeta'_j)$, so Φ is compatible if it is given by a map which satisfies $G_j = \omega F_j$. We say that a ring homomorphism is holomorphic if Φ^* maps $(1,0)$-vector fields to $(1,0)$-vector fields; equivalently, for a $\mathcal{B}$-linear derivation X which annihilates $\mathcal{B}[[\zeta]]$, $X' = X \circ \Phi$ is an $\mathcal{A}$-linear derivation X' of $\mathcal{A}[[Z',\zeta']]$ which annihilates $\mathcal{A}[[\zeta']]$.

Lemma 4. *In the setting above, a holomorphic homomorphism* Φ *is given by a power series map of the form* $Z'_j = F_j(Z)$, $\zeta'_j = \omega F_j(\zeta)$; *furthermore,* Φ *maps constants to constants, so if we denote* $\Phi|_{\mathcal{A}} = \varphi$, *then*

$$\Phi\left(\sum_{\alpha,\beta} a_{\alpha,\beta} Z'^{\alpha} \zeta'^{\beta}\right) = \sum_{\alpha,\beta} \varphi(a_{\alpha,\beta}) F(Z)^{\alpha} \big(\omega F(\zeta)\big)^{\beta}.$$

Proof. We only need to check that Φ maps constants to constants, the rest of the lemma is then an easy consequence. We note that $a \in \mathcal{A}$ if and only if $X'a = 0$ for all $\mathcal{A}$-linear derivation X' of $\mathcal{A}[[Z',\zeta']]$, similarly for $b \in \mathcal{B}$. But by assumption, $X' \circ \Phi$ is a $\mathcal{B}$-linear derivation, and thus $X' \circ \Phi(a) = 0$ for all X' and a, and thus $\Phi(a)$ is a constant. □

A CR mapping of a formal CR structure over $(\mathcal{A},\sigma)$ defined by an ideal I' and a formal CR structure over $(\mathcal{B},\omega)$ defined by an ideal I is a holomorphic ring homomorphism Φ, compatible with (σ,ω) (or, equivalently, a formal power series mapping), which in addition satisfies $\Phi(I')\subset I$. Two formal CR structures are equivalent if there exists a CR mapping between them which is invertible and whose inverse is also CR. Thus, an equivalence between two formal CR structures is given by an equivalence of the CR structure defined by the trivial ideal which also respects the ideals.

Lemma 5. *A CR mapping Φ between $\mathcal{A}[[Z',\zeta']]$ and $\mathcal{B}[[Z,\zeta]]$ is an equivalence if and only if the induced mapping on the constants $\varphi:\mathcal{A}\to\mathcal{B}$ is an isomorphism and it is given by a power series map $Z'=H(Z)$ with the property that* $\det H'(0)$ *is a unit in $\mathcal{B}$.*

Proof. We only need to show sufficiency of the conditions. Since $\det H'(0)$ is a unit in $\mathcal{B}$, we can find a power series map $G(Z)\in\mathcal{B}[[Z]]$ with $G(H(Z))=H(G(Z))=Z$. We define the map $\Psi:\mathcal{B}[[Z,\zeta]]\to\mathcal{A}[[Z',\zeta']]$ by $\Psi|_{\mathcal{B}}=\varphi^{-1}$ and by the power series map $(\varphi^{-1}G)(Z')$ (where φ^{-1} acts on the coefficients of G). □

In particular, we shall refer to a CR equivalence of $\mathcal{A}[[Z,\zeta]]$ with itself as a choice of coordinates.

2.6. Coordinate choices—normal coordinates. Our aim in this section is to show that given a generic $\tilde\sigma$-invariant manifold ideal $I\subset\mathcal{A}[[Z,\zeta]]$ of codimension d, we can choose *normal coordinates* $Z=H(z,w)$, $\zeta=\sigma H(\chi,\tau)$, $z=(z_1,\dots,z_n)$, $\chi=(\chi_1,\dots\chi_n)$, $w=(w_1,\dots,w_d)$, $\tau=(\tau_1,\dots,\tau_d)$, $N=n+d$. $H\in\mathcal{A}[[Z]]^N$ satisfies that $\det H'(0)$ is a unit in $\mathcal{A}$ and there are generators of I of the form $w_j-Q_j(z,\chi,\tau)$, $j=1,\dots,d$, satisfying

$$Q_j(z,0,\tau)=Q_j(0,\chi,\tau)=\tau,\qquad Q_j\big(z,\chi,\sigma Q(\chi,z,w)\big)=w. \tag{2}$$

In particular, the induced homomorphism of this coordinate change on the constants is the identity. As before, it is convenient to write the generators in vector notation, i.e. as $w-Q(z,\chi,\tau)$.

Our first step is to choose linear coordinates $\tilde Z=(\tilde z,\tilde w)$ (that is, we choose an invertible $N\times N$ matrix A with entries in $\mathcal{A}$ and set $Z=A\tilde Z$) such that the set of generators $\tilde\rho(\tilde Z,\tilde\zeta)=\rho(A\tilde Z,\sigma A\tilde\zeta)$ satisfy $\tilde\rho_{\tilde z}(0)=0$ and $\tilde\rho_{\tilde w}(0)=I$. For this, we just need to find A such that $\rho_Z(0)A=(0I)$, which is possible since by assumption $\rho_Z(0)$ has a $d\times d$-minor which is a unit in $\mathcal{A}$.

We can now apply the implicit function theorem to write generators of I in the form $\tilde w-R(\tilde z,\tilde\chi,\tilde\tau)$, with $R_{\tilde\tau}(0,0,0)=I$, and $R(\tilde z,\tilde\chi,\bar R(\tilde\chi,\tilde z,\tilde w))=\tilde w$. We claim that after another change of coordinates of the form $w+g(z,w)=\tilde w$, where we choose g with the property that $\sigma g(0,w)=-g(0,w)$, we arrive at the form we wanted. The transformed generators are then $w+g(z,w)-R(z,\chi,\tau-g(\chi,\tau))=\rho(z,w,\chi,\tau)$. We need to choose g such that $\rho(z,w,0,w)=0$. Now consider the equation $w+Y-R(0,0,w-Y)=0$. By the implicit function theorem, this equation has a unique solution $Y=g(0,w)$. We have already noted that $R(\tilde{z},\tilde{\chi},\bar{R}(\tilde{\chi},\tilde{z},w))=w$, $R(z,\chi,\sigma R(\chi,z,w))=w$, so that we have $w+\sigma g(0,w)=\sigma R(0,0,R(0,0,w+\sigma g(0,w)))$; from the uniqueness of g, it follows that $R(0,0,w+\sigma g(0,w))=w-\sigma g(0,w)$, and hence (again by uniqueness) $g(0,w)=-\sigma g(0,w)$ as required. The change of coordinates is now defined by $g(z,w)=R(z,0,w-g(0,w))-w$.

2.7. Finite type. We say that a formal CR structure defined by an ideal I is of finite type if the Lie algebra generated by $\mathcal{D}_I^{(1,0)}\oplus\mathcal{D}_I^{(0,1)}$ has the property that its evaluation at 0 spans $\mathcal{D}_I(0)$ over $\mathbb{K}$. Note that we do *not* require that it spans over $\mathcal{A}$.

3. Homomorphisms, flows and their iterations

In this section, we consider a formal power series ring $\mathcal{A}[[x]]$, where $x=(x_1,\dots,x_n)$. A homomorphism $\Psi:\mathcal{A}[[x]]\to\mathcal{A}[[x,t]]$, where $t=(t_1,\dots,t_d)$ is given by substitution with a formal map $\psi(x,t)=(\psi_1(x,t),\dots,\psi_n(x,t))$, where $\Psi(x_j)=\psi_j(x,t)$. Using the notation $t^{[k]}=(t^1,\dots,t^k)$, where $t^j=(t^j_1,\dots,t^j_d)$, we define the kth iteration $\Psi^{[k]}:\mathcal{A}[[x]]\to\mathcal{A}[[x,t^{[k]}]]$ inductively by

$$\Psi^{[1]}=\Psi,\qquad \Psi^{[k]}f(x)=f\big(\psi^k\big(x,t^{[k]}\big)\big)=f\big(\psi\big(\psi^{k-1}\big(x,t^{[k-1]}\big),t^k\big)\big).\tag{3}$$

We also define the kth restricted iteration $\Psi_0^{[k]}:\mathcal{A}[[x,t^{[k]}]]\to\mathcal{A}[[t^{[k]}]]$ by the composition of $\Psi^{[k]}$ with the projection of $\mathcal{A}[[x,t^{[k]}]]$ onto $\mathcal{A}[[t^{[k]}]]$.

Definition 6. We say that Ψ is of finite type if there exists an integer k such that the map $t^{[k]}\mapsto\psi(0,t^{[k]})$ is generically of full rank.

The meaning of the preceding definition is that the matrix $\frac{\partial\psi}{\partial t^{[k]}}(0,t^{[k]})$ has an $n\times n$ minor which is not identically vanishing; this condition is sometimes referred to as the geometric rank of $\Psi_0^{[k]}$ is full or that $\Psi_0^{[k]}$ is strongly injective. More generally, recall that we define the generic rank $\operatorname{rk}h$ of a power series map $h(y)\in\mathcal{A}[[y]]$ as the largest integer r for which the matrix $\frac{\partial h}{\partial y}$ has a nonvanishing $r\times r$ minor, and the generic rank of a homomorphism as the generic rank of its associated formal map.

Next, observe that $\operatorname{rk}\Psi_0^{[1]}\le\operatorname{rk}\Psi_0^{[2]}\le\cdots\le\operatorname{rk}\Psi_0^{[k]}\le n$; thus there exist numbers k and s such that $\operatorname{rk}\Psi_0^{[\ell]}=s$ for all $\ell\ge k$. (We will observe in Section 5 that the increase of the rank is strict; that is, if $\operatorname{rk}\Psi_0^{[j]}=\operatorname{rk}\Psi_0^{[j+1]}$, then $\operatorname{rk}\Psi_0^{[j]}=\operatorname{rk}\Psi_0^{[j+s]}$ for all $s\in\mathbb{N}$.)

DEFINITION 7. The *substitution rank* $\operatorname{sk}\Psi$ is $\max_k \operatorname{rk}\Psi_0^{[k]}$.

Given a family of vector fields $X = (X_1, \dots, X_d)$, $X_j \in \operatorname{Der}_{\mathcal{A}}(\mathcal{A}[[x]])$, we define their formal flow by the power series map $\psi_X(x,t) = e^{t_1 X_1} \cdots e^{t_d X_d} x$, and the associated homomorphism by Ψ_X.

DEFINITION 8. We say that the family $X = (X_1, \dots, X_d)$ is of finite type if the Lie algebra $\operatorname{Lie}(X) = \operatorname{Lie}(X_1, \dots, X_d) \subset \mathcal{A}[[x]]^n$ generated by the X_j has the property that its evaluation $\operatorname{Lie}(X)(0)$ at 0 spans $\mathbb{K}^n$. The rank $\operatorname{rk} X$ of X is the dimension of the $\mathbb{K}$-vector space $\operatorname{Lie}(X)(0)$.

REMARK 1. The meaning of the preceding definition of finite type is that if we identify formal vector fields $Y = (a_1(x), \dots, a_n(x))$ with their coefficients, then we can collect $Y^1, \dots, Y^n \in \operatorname{Lie}(X)$, $Y^j = (a_1^j(x), \dots, a_n^j(x))$ such that the matrix $(a_k^j(0))$ has nonzero determinant (it need *not* be a unit). More generally, $\operatorname{rk} X$ is the maximum number r such that we can choose $Y_1, \dots, Y_r \in \operatorname{Lie}(X)$ such that the matrix $(a_k^j(0))$ has a nonzero $r \times r$ minor.

The main result of this section is that finite type of a family of vector fields and finite type of their flows are equivalent.

THEOREM 2. *Let X be a finite family of vector fields on $\mathcal{A}[[x]]$. Then $\operatorname{sk}\Psi_X = \operatorname{rk} X$. In particular, the homomorphism Ψ_X is of finite type if and only if X is of finite type.*

The proof of Theorem 2 is based on the following well-known lemma (we include a proof valid in our setting).

LEMMA 9. *Let $X = \sum_j a_j(x) \frac{\partial}{\partial x_j}$ and $Y = \sum_j b_j(x) \frac{\partial}{\partial x_j}$ be formal vector fields on $\mathcal{A}[[x]]$, and let $\varphi(x,t)$ denote the (formal) flow of X. Define a formal power series map $W(x,t) = (w_1(x,t), \dots, w_n(x,t))$ by*

$$W(x,t) = \varphi_x\big(\varphi(x,t), -t\big) Y\big(\varphi(x,t)\big). \tag{4}$$

Then $W(x,t)$ solves the differential equation $\frac{dW}{dt} = \varphi_x(\varphi(x,t), -t)[X,Y](\varphi(x, t))$ with $W(x,0) = Y(x)$; in particular,

$$\frac{d}{dt} W(x,0) = [X,Y](x). \tag{5}$$

Proof. By the flow property, $\varphi(\varphi(x,t), -t) = x$, so $\varphi_x(x,t) = (\varphi_x(\varphi(x,t), -t))^{-1}$. We differentiate the equation

$$\varphi_x(x,t) W(x,t) = Y\big(\varphi(x,t)\big)$$

with respect to t. On the right-hand side, we obtain the vector $(Xb_1, \dots, Xb_n)$; on the left-hand side, we need to compute $\varphi_{x,t}(x,t) W(x,t)$. Since $\varphi_t(x,t) =$

$X(\varphi(x,t))$, we have that $\varphi_{x,t} = X_x(\varphi(x,t))\varphi_x(x,t)$ (where we consider X as a map given by $(a_1,\dots,a_n)$), so $\varphi_{x,t}(x,t)W(x,t) = X_x(\varphi(x,t))Y(\varphi(x,t))$, which is exactly the vector $(Ya_1,\dots,Ya_n)$. Recalling that $[X,Y] = \sum_j c_j \frac{\partial}{\partial x_j}$ with $c_j = (Xb_j - Ya_j)$ gives the claimed result. $\square$

We need the following consequence of Lemma 9.

LEMMA 10. *Let $X^1,\dots,X^k$ and Y be formal vector fields on $\mathcal{A}[[x]]$; denote the flow of X^j by $\varphi^j(x,t_j)$, and let $\varphi^{[j]}(x,t^{[j]})$ be defined inductively by $\varphi^{[1]} = \varphi^1$, $\varphi^{[j]}(x,t^{[j]}) = \varphi^j(\varphi^{[j-1]}(x,t^{[j-1]}),t_j)$. Define $W(t^{[k]})$ by*

$$\begin{aligned}(6)\qquad & W(x,t_1,\dots,t_k)\\ &= \varphi_x^1\bigl(\varphi^1(x,t_1),-t_1\bigr)\cdots\varphi_x^k\bigl(\varphi^{[k]}(x,t^{[k]}),-t_k\bigr)Y\bigl(\varphi^{[k]}(x,t^{[k]})\bigr).\end{aligned}$$

Then

$$\frac{\partial^k W}{\partial t_1\dots\partial t_k}(x,0) = \bigl[X_1,\bigl[\dots\bigl[X_{k-1},[X_k,Y]\bigr]\dots\bigr]\bigr](x). \qquad (7)$$

Proof. The case $k=1$ is just Lemma 9. In order to finish the inductive step, use the inductive hypothesis for the family of vector fields $X_2,\dots,X_k$ and replace x by $\varphi^1(x,t_1)$ in the resulting equation. $\square$

Proof of Theorem 2: $\operatorname{rk} X \le \operatorname{sk}\Psi_X$. Assume that $X = (X_1,\dots,X_d)$ is a family of vector fields of finite type, and let $\varphi^j(x,t) = \varphi^{j,t}(x)$ be the flow of X_j. For any sequence of integers $J = (j_1,\dots,j_{|J|})$, we write $t_J = (t_1,\dots,t_{|J|})$, and define $\varphi_J^{t_J} = \varphi_J(x,t_1,\dots,t_{|J|}) = \varphi^{j_1,-t_1}\circ\varphi^{j_2,-t_2}\circ\cdots\circ\varphi^{j_{|J|-1},-t_{|J|-1}}\circ\varphi^{j_{|J|},t_{|J|}}\circ\varphi^{j_{|J|-1},t_{|J|-1}}\circ\cdots\circ\varphi^{j_1,t_1}$.

Now choose r sequences of integers $J^1,\dots,J^r$ with the property that

$$Z_\ell = \bigl[X_{j_1^\ell},\bigl[X_{j_2^\ell},\bigl[\dots,[X_{j^\ell_{|J^\ell|-1}},X_{j^\ell_{|J^\ell|}}]\dots\bigr]\bigr]\bigr](0)$$

are linearly independent over $\mathbb{K}$; without loss of generality, assume that if we write $Z_\ell = \sum_j a_\ell^j(x)\frac{\partial}{\partial x_j}$, then the $r\times r$ matrix $(a_\ell^j(0))_{1\le j,\ell\le r}$ has nonvanishing determinant. We claim that the map $\psi(t_j^\ell) = \varphi_{J^1}^{t_{J^1}}\circ\cdots\circ\varphi_{J^n}^{t_{J^n}}(0)$ is of generic rank at least r. To show this, we compute the $r\times r$ minor $D(t)$ of ψ_t comprised of the columns corresponding to $t^\ell_{|J^\ell|}$ and the first r rows, and claim that we have

$$\frac{\partial^{|J^1|+\dots+|J^n|-n}D}{\partial t_1^1\cdots\partial t^n_{|J^n|-1}}(0)\neq 0,$$

where the derivative is with respect to all t_j^ℓ with $j\neq j_{|J^\ell|}$. Indeed, when we compute the derivative of ψ with respect to $t^\ell_{|J^\ell|}$ and set all $t^\ell_{|J^\ell|}=0$, $1\le\ell\le n$, it is given by the derivative of $\varphi^{t_J}_{J^\ell}$ with respect to $t^\ell_{|J^\ell|}$, which turns out to depend only on t^ℓ_j, $1 \le j \le |J^\ell|$ and is given by

$$\varphi_x^{j_1}\big(\varphi^{j_1}\big(0,t_1^\ell\big),-t_1^\ell\big)\cdots\varphi_x^{j_k}\big(\varphi^{[k]}\big(0,t^{[k]}\big),-t_k^\ell\big)X_{j_{k+1}}\big(\varphi^{[k]}\big(0,t^{[k]}\big)\big),$$

where we set $k=|J^\ell|-1$ and define $\varphi^{[j]}$ in the obvious inductive manner. Thus, we can apply Lemma 10 to compute the derivative of this vector with respect to $t_1^\ell,\dots,t_k^\ell$, which evaluated at 0 is just Z_ℓ. Thus,

$$\frac{\partial^{|J^1|+\cdots+|J^n|-n}D}{\partial t_1^1\cdots\partial t^n_{|J^n|-1}}(0)=\det\big[\big(a_\ell^j(0)\big)_{1\le j,\ell\le r}\big]\neq 0,$$

as claimed. $\square$

In order to prove the opposite inequality, we need to employ a version of the Baker–Campbell–Hausdorff formula. We recall that for vector fields X,Y, the adjoint map ad is defined by $(\operatorname{ad}X)(Y)=[X,Y]$. We now record the following extension of Lemma 9, which is obtained by induction:

LEMMA 11. *Let X,Y be formal vector fields on $\mathcal{A}[[x]]$, and denote the flow of X by $\varphi(x,t)$. Then we have*

$$\begin{aligned}\varphi_x\big(\varphi(x,t),-t\big)Y\big(\varphi(x,t)\big)&=\sum_{j=0}^\infty\frac{t^j}{j!}\big((\operatorname{ad}X)^jY\big)(x),\\ \varphi_x(x,t)Y(x)&=\sum_{j=0}^\infty\frac{(-1)^jt^j}{j!}\big((\operatorname{ad}X)^jY\big)\big(\varphi(x,t)\big).\end{aligned}\tag{8}$$

To study iterated flows along a family of vector fields, we also need a result analogous to Lemma 10:

LEMMA 12. *Let $X^1,\dots,X^k$ and Y be formal vector fields on $\mathcal{A}[[x]]$, and denote the flow of X^j by $\varphi^j(x,t_j)=\varphi^{j,t_j}(x)$. Let $\Phi(x,t)=\varphi^{k,t_k}\circ\cdots\circ\varphi^{1,t_1}$ and $\Phi^{-1}(x,t)=\varphi^{1,-t_1}\circ\cdots\circ\varphi^{k,-t_k}$. Then*

$$\begin{aligned}&\Phi_x(x,t)Y(x)\\ &\quad=\sum_{\alpha\in\mathbb{N}^k}\frac{(-1)^{|\alpha|}t^\alpha}{\alpha!}\big(\operatorname{ad}X^k\big)^{\alpha_k}\circ\cdots\circ\big(\operatorname{ad}X^1\big)^{\alpha_1}(Y)\big(\Phi(x,t)\big),\\ &\Phi_x^{-1}\big(\Phi(x,t),t\big)Y\big(\Phi(x,t)\big)\\ &\quad=\sum_{\alpha\in\mathbb{N}^k}\frac{t^\alpha}{\alpha!}\big(\operatorname{ad}X^1\big)^{\alpha_1}\circ\cdots\circ\big(\operatorname{ad}X^k\big)^{\alpha_k}(Y)(x).\end{aligned}\tag{9}$$

Proof. Note that the second formula follows from the first one, the proof of which we now turn to. If $k=1$, this is the second formula in Lemma 11.

For $k>1$ we use induction, and write $\Phi=\varphi^{k,t^k}\circ\tilde\Phi$, $t=(t_k,\tilde t)$. We then have

$$\Phi_x(x,t)Y(x)=\varphi_x^k\big(\tilde\Phi(x,\tilde t),t_k\big)\tilde\Phi_x(x,\tilde t)Y(x)=\varphi_x^k\big(\tilde\Phi(x,\tilde t),t_k\big)Z\big(\tilde\Phi(x,\tilde t),\tilde t\big).$$

Now note that $y=\tilde\Phi(x,\tilde t)$ is a well-defined change of variable in the ring $\mathcal{A}'[[x]]$, where $\mathcal{A}'=\mathcal{A}[[\tilde t]]$. We can thus use Lemma 11 to compute

$$\varphi_x^k(y,t_k)Z(y,\tilde t)=\sum_{j=0}^{\infty}\frac{(-1)^j {t_k}^j}{j!}\big((\operatorname{ad}X_k)^jZ\big)\big(\varphi^k(y,t_k)\big).$$

The result follows by applying the induction hypothesis and replacing y by $\tilde\Phi(x,\tilde t)$. □

Our next observation is that the rank of a Lie algebra of vector fields is constant along its flows; stated formally, we mean the following.

PROPOSITION 13. *Let $\mathcal{L}\subset\mathcal{A}[[x]]^n$ be a Lie algebra of formal vector fields on $\mathcal{A}[[x]]$. For any collection $X_1,\dots,X_k\in\mathcal{L}$, denote the flow of X^j by $\varphi^j(x,t_j)=\varphi^{j,t_j}(x)$. Let $\Phi(x,t)=\varphi^{k,t_k}\circ\cdots\circ\varphi^{1,t_1}$, and consider the set $\mathcal{L}(\Phi(0,t))\subset\mathcal{A}[[t]]^n$; let $\mathbb{K}((t))$ be the quotient field of $\mathcal{A}[[t]]$. Then* $\dim_{\mathbb{K}((t))}\mathcal{L}(\Phi(0,t))=\dim_{\mathbb{K}}\mathcal{L}(0)$.

Proof. The inequality $\dim_{\mathbb{K}_t}\mathcal{L}(\Phi(0,t))\ge\dim_{\mathbb{K}}\mathcal{L}(0)$ is immediate. In order to prove the opposite inequality, choose any vector fields $V_1,\dots,V_n\in\mathcal{L}$, and consider the matrix $V=(V_1,\dots V_n)$ with columns V_j. Since $\Phi_x^{-1}(\Phi(x,t),t)|_{x=t=0}=I$, the rank of $V'=(V_1',\dots V_n')$, where $V_j'=\Phi_x^{-1}(\Phi(0,t),t)V_j(\Phi(0,t))$, over $\mathbb{K}((t))$ coincides with the rank of $V(\Phi(0,t))$ over $\mathbb{K}((t))$. But Lemma 12 implies that $V_j'(t)=\sum_\alpha C_\alpha^j t^\alpha$, where $C_\alpha^j\in\mathcal{L}(0)$. Thus, if all minors of size r of $n\times n$ matrices with columns in $\mathcal{L}(0)$ vanish identically, so do all minors of size r of $V(\Phi(0,t))$. We conclude that $\dim_{\mathbb{K}((t))}\mathcal{L}(\Phi(0,t))=\dim_{\mathbb{K}}\mathcal{L}(0)$. □

Proof of Theorem 2: $\operatorname{sk}\Psi_X\le\operatorname{rk}X$. Consider an iterated flow of the vector fields $X_1,\dots X_d$; i.e. we have a sequence of integers $(j_1,\dots,j_\ell)$ with $1\le j_k\le d$, and we consider the map $\Phi:(x,t_1,\dots,t_\ell)=(x,t)\mapsto\varphi^{j_\ell,t_\ell}\circ\cdots\circ\varphi^{j_1,t_1}(x)$. We first claim that for any k, the vector $\frac{\partial\Phi}{\partial t_k}$ can be written as a series of Lie brackets of the X_j evaluated along $\Phi(x,t)$. This is obvious for $\ell=1$, and we proceed by induction on ℓ. Note that it is enough to consider the case $k=1$, as all other cases are automatically covered by the induction assumption. We thus compute

$$\frac{\partial}{\partial t_1}\Phi(x,t)=\frac{\partial}{\partial t_1}\varphi^{j_k,t_k}\circ\tilde\Phi(x,\tilde t)=\varphi_x^{j_k,t_k}\big(t_k,\tilde\Phi(x,t)\big)\frac{\partial}{\partial t_1}\tilde\Phi(x,\tilde t).$$

Assuming that $\frac{\partial}{\partial t_1}\tilde\Phi(x,\tilde t)=\sum_\alpha\tilde t^\alpha Z_\alpha(\tilde\Phi(x,\tilde t))$ with $Z_\alpha\in\operatorname{Lie}(X)$ we see that we can proceed as in the proof of Lemma 12 and finish the induction by applying Lemma 11.

It follows that the columns of the matrix $\frac{\partial\Phi}{\partial t}(0,t)$ can be written as formal series of elements of $\operatorname{Lie}(X)$, evaluated at $\Phi(0,t)$. Thus, by Proposition 13, we conclude that $\operatorname{sk}\Psi_X \leq \dim \operatorname{Lie}(X)(\Phi(0,t)) = \dim \operatorname{Lie}(X)(0) = \operatorname{rk} X$. $\square$

4. Strongly independent vector fields and strictly regular maps

Now assume that $X = (X_1, \dots, X_d)$ is a family of formal vector fields on $\mathcal{A}[[x_1, \dots, x_n]]$; as before, we shall identify each X_j with its coefficients. We shall furthermore assume that $(X_1, \dots, X_d)$ have the property that there exists a $d \times d$ minor of the $r \times d$ matrix $(X_1(0), \dots, X_d(0))$ which is a unit in $\mathcal{A}$. We say that X is strongly independent if it satisfies this property. In particular, note that in this case $d = \dim_{\mathbb{K}} \langle \{X_1, \dots X_d\} \rangle$. Now if X is strictly regular, the composition of their flows $\varphi(t,x) = \varphi^{d,t_d} \circ \cdots \circ \varphi^{1,t_1}(x)$ has the property that $\varphi_t(0,x) = X(x)$ has a $d \times d$ minor which is a unit in $\mathcal{A}[[x]]$. We shall be interested in substitution maps with this particular property; we thus give the following definition.

Definition 14. A homomorphism $\Psi : \mathcal{A}[[x]] \to \mathcal{A}[[x,t]]$, or equivalently a power series map $\psi(x,t) \in \mathcal{A}[[x,t]]^n$, is strictly regular if $\psi_t(x,0)$ has a minor which is invertible in $\mathcal{A}[[x]]$.

Our goal is to understand the iterations $\Psi^{[k]}$ in a bit of a different way. The crucial observation is the following lemma.

Lemma 15. *Let $\psi(x,t) = (\psi^1(x,t), \dots, \psi^n(x,t))$ be a strictly regular map, $t = (t_1, \dots, t_d)$. Then there exists a formal power series map $f(x,y) \in \mathcal{A}[[x, y]]^{n-d}$ such that $f_y(0,0)$ has an $(n-d) \times (n-d)$ minor which is a unit in $\mathcal{A}$ and which satisfies*

$$f\big(x, \psi(x,t)\big) = 0.$$

Proof. Without loss of generality, we assume that the $d \times d$ matrix $M(x) = (\psi^j_{t_k}(x))_{1 \leq j,k \leq d}$ has the property that $\det M(x)$ is a unit in $\mathcal{A}[[x]]$. To construct $f = (f^1, \dots, f^{n-d})$ (which is highly nonunique), take the derivative of $f(x, \psi(x,t))$ with respect to t and set $t = 0$ to obtain

$$f_y(x,0)\psi_t(x,0) = 0.$$

Since we would like that $f_y(0,0)$ has a minor of size $n-d$ which is a unit in $\mathcal{A}$, we make the the choice $f^j_{y_{d+k}}(x,0) = \delta^k_j$ for $j,k = 1, \dots, n-d$. Now write $y = (z,w)$ where z denotes the first d entries of y, and write $\psi_t(x) = (M(x) \;\; N(x))^t$. We thus have $f_z(x,0)M(x) = -f_w(x,0)N(x) = -N(x)$; multiplying by the classical adjoint $M^T(x)$ from the right gives $(\det M(x)) f_z(x, 0) = -N(x)M^T(x)$, which determines $f_z(x,0)$.

The construction of higher order derivatives proceeds by induction on the order: If we have determined all derivatives of f up to order $k-1$, we take the derivatives of $f(x,\varphi(t,x))$ with respect to t of order k, substitute for all derivatives $f_{y^\alpha}(x,0)$ for $|\alpha|<k$, decide to leave all $f_{z^\alpha w^\beta}(x,0)$ where $|\alpha|+|\beta| = k$ and $\beta \neq 0$ undetermined (or set them equal to 0), and find that we can solve the equations for the remaining $f_{z^\alpha}(x,0)$ where $|\alpha| = k$. $\square$

We thus give the following definition.

Definition 16. An ideal $I \subset \mathcal{A}[[x,y]]$ is strictly regular of order e (with respect to y) if there exist e generators $f_1(x,y),\dots,f_e(x,y)$ with the property that $f_y(x,0)$ has a minor of size e which is a unit in $\mathcal{A}[[x]]$.

Consider, now, a sequence $I_1,\dots,I_\ell$ of strictly regular ideals, with I_j of order e_j. From the sequence I_j, we define an operation of iterated substitution as follows:

$$\begin{aligned}
&I^{[k]} \subset \mathcal{A}[[x,y^1,\dots,y^k]] = \mathcal{A}[[x,y^{[k]}]], \qquad I^{[1]} = \big(g\big(x,y^1\big)\big)_{g\in I_1},\\
&I^{[k]} = I^{[k-1]} + \big(g\big(y^{k-1},y^k\big)\big)_{g\in I_k}, \quad k = 2,\dots,\ell,\\
&I_0^{[k]} \subset \mathcal{A}[[y^1,\dots,y^k]] = \mathcal{A}[[y^{[k]}]], \qquad I_0^{[k]} = I^{[k]}|_{x=0}.
\end{aligned}$$

We refer to the sequence $I_0^{[k]}$ as the iterated substitution "starting from the origin."

Definition 17. A regular parametrization of a strictly regular ideal $I \subset \mathcal{A}[[x,y]]$ is any strictly regular map $\psi(x,t)$ which satisfies $f(x,\psi(x,t)) = 0$ for all $f \in I$.

The following lemma summarizes some easily proved facts.

Lemma 18. *Let $I_1,\dots,I_\ell \subset \mathcal{A}[[x,y]]$ be strictly regular ideals as above, and let $\psi^j(x,t^j)$ be a strictly regular parametrization for I_j; we write $\Psi = (\psi^1,\dots,\psi^\ell)$. Then for any $1 \le k \le \ell$, $I^{[k]}$ is a manifold ideal of dimension $(k+1)n - \sum_{j=1}^k e_j$, and a parametrization of $I^{[k]}$ is given by $(\psi^{[1]}(x,t^{[1]}),\dots, \psi^{[k]}(x,t^{[k]}))$, where*

$$\begin{aligned}
&\psi^{[1]}\big(x,t^{[1]}\big) = \psi^1\big(x,t^1\big),\\
&\psi^{[k]}\big(x,t^{[k]}\big) = \psi^k\big(\psi^{[k-1]}\big(x,t^{[k-1]}\big),t^k\big), \quad k = 2,\dots,\ell,
\end{aligned}$$

and for any $1 \le k \le \ell$, $I_0^{[k]}$ is a manifold ideal of dimension $kn - \sum_{j=1}^k e_j$, and a parametrization of $I_0^{[k]}$ is given by $(\psi^{[1]}(0,t^{[1]}),\dots,\psi^{[k]}(0,t^{[k]}))$. In particular, for any two regular parametrizations Ψ_1 and Ψ_2 for $(I_1\dots,I_\ell)$, we have $\operatorname{rk}\psi_1^{[k]}(0,t^{[k]}) = \operatorname{rk}\psi_2^{[k]}(0,t^{[k]})$.

Proof. Let $f_1^j,\dots,f_{e_j}^j$ be generators for I_j as in Definition 16; then the differential of the (f_i^j) with respect to $y^{[l]}$ is a block triangular matrix:

$$d_{y^{[\ell]}}(f_i^j)(y^{[\ell]}) = \begin{pmatrix} d_{y^1}(f_i^1)(0,y^1) & 0 & \cdots & 0 & 0 \\ d_{y^1}(f_i^2)(y^1,y^2) & d_{y^2}(f_i^2)(y^1,y^2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{y^{\ell-1}}(f_i^{\ell-1})(y^{\ell-2},y^{\ell-1}) & 0 \\ 0 & 0 & \cdots & d_{y^{\ell-1}}(f_i^{\ell})(y^{\ell-1},y^{\ell}) & d_{y^{\ell}}(f_i^{\ell})(y^{\ell-1},y^{\ell}) \end{pmatrix}$$

from which it follows immediately that $I^{[k]}$ is a manifold ideal of dimension $kn-\sum_{j=1}^k e_j$. Next, we observe that since $f_i^j(x,\psi^j(x,t))\equiv 0$ by definition, also

$$\begin{aligned} &f_i^j\big(\psi^{[j-1]}\big(0,t^{[j-1]}\big),\psi^{[j]}\big(0,t^{[j]}\big)\big) \\ &\quad = f_i^j\big(\psi^{[j-1]}\big(0,t^{[j-1]}\big),\psi^j\big(\psi^{[j-1]}\big(0,t^{[j-1]}\big),t^j\big)\big)\equiv 0. \end{aligned}$$

Finally, we notice that the differential of the map $(\psi^{[1]}(0,t^{[1]}),\dots,\psi^{[k]}(0,t^{[k]}))$ with respect to $t^{[k]}$ is a triangular matrix; analogously as before, this implies that its rank is $\sum_j e_j$, and thus that it represents a parametrization for $I^{[k]}$. $\square$

5. The rank increase property

In this section, we study compositions of maps of the kind $\varphi(x,t)$, corresponding to homomorphisms $\mathcal{A}[[x]]\to\mathcal{A}[[x,t]]$, satisfying the property $\varphi(x,0)\equiv x$. We fix a set of such formal maps $\{\varphi^1(x,t),\dots,\varphi^d(x,t)\}$. For all $k\in\mathbb{N}$, we denote by I_k the set of all the maps $\mathfrak{i}:\{1,\dots,k\}\to\{1,\dots,d\}$; moreover, for any such $\mathfrak{i}$ we consider the (ordered) list $\varphi^{\mathfrak{i}(1)},\dots,\varphi^{\mathfrak{i}(k)}$, and we define the substitution maps $\Phi^{\mathfrak{i},[l]}(x,t^{[l]})$, where $l\le k$ and $t^{[l]}=(t_1,\dots,t_l)$, inductively as follows:

$$\Phi^{\mathfrak{i},[1]}(x,t_1)=\varphi^{\mathfrak{i}(1)}(x,t_1),\qquad \Phi^{\mathfrak{i},[j+1]}\big(x,t^{[j+1]}\big)=\varphi^{\mathfrak{i}(j+1)}\big(\Phi^{\mathfrak{i},[j]}\big(x,t^{[j]}\big),t_{j+1}\big)$$

(cf. equation (3)); we refer to l as the *length* of $\Phi^{\mathfrak{i},[l]}$. We define $\operatorname{rk}_j\Phi$, the *generic rank at step* j of the set $\Phi=\{\varphi^1,\dots,\varphi^d\}$, to be

$$\operatorname{rk}_j\Phi=\max_{\mathfrak{i}\in I_j}\big\{\operatorname{rk}\Phi_0^{\mathfrak{i},[j]}\big\}$$

with $\Phi_0^{\mathfrak{i},[j]}$ as in Section 3. In analogy with Section 3, we define the substitution rank of Φ as $\operatorname{sk}\Phi=\max_j\operatorname{rk}_j\Phi$. We want to show that rk_j increases strictly before stabilizing.

Lemma 19. *Let $\Phi=\{\varphi^1,\dots,\varphi^d\}$ be a set of homomorphisms $\mathcal{A}[[x]]\to\mathcal{A}[[x,t]]$ as before, and let $\operatorname{sk}\Phi$ be the substitution rank of Φ. Then $\operatorname{rk}_j\Phi=\min\{j,\operatorname{sk}\Phi\}$.*

Proof. It is clear by definition that $\mathrm{rk}_l \Phi \le \mathrm{rk}_{l+1} \Phi \le \mathrm{rk}_l \Phi + 1$ for all $l \in \mathbb{N}$. Now, fix $j \in \mathbb{N}$ and $\mathfrak{i} \in I_j$, and consider the maps $\Phi^{\mathfrak{i},[l]}$, $l \le j$, as above. Let $J \subset \{1, \dots, j\}$ be defined as $J = \{l \le j : \mathrm{rk}\, \Phi_0^{\mathfrak{i},[l]} = \mathrm{rk}\, \Phi_0^{\mathfrak{i},[l+1]}\}$, and let $j_0 = \min J$. An inspection of the proof of Proposition 3.1 in [3] shows that it also works in our context, i.e. with $\mathbb{C}$ replaced by $\mathcal{A}$; we employ that proposition with $A(t^{[j_0]}, t_{j_0+1}) = \Phi_0^{\mathfrak{i},[j_0+1]}(t^{[j_0+1]})$, $B(t^{[j_0]}) = A(t^{[j_0]}, 0) = \Phi_0^{\mathfrak{i},[j_0]}(t^{[j_0]})$ (the last equality holds because $\varphi^{j_0}(x, 0) \equiv x$) and $F(x, t^{[k \ge j_0+2]}) = \Phi(x, t^{[k \ge j_0+2]})$ being the substitution map associated to the list $\varphi^{\mathfrak{i}(j_0+2)}, \dots, \varphi^{\mathfrak{i}(j)}$, obtaining that $\mathrm{rk}\, \Phi_0^{\mathfrak{i},[j - \{j_0\}]} = \mathrm{rk}\, \Phi_0^{\mathfrak{i},[j]}$. Iterating this elimination argument, we find a substitution map of length precisely $r = \mathrm{rk}\, \Phi_0^{\mathfrak{i},[j]}$ whose generic rank is also r. Choosing $\mathfrak{i} \in I_j$ such that $\mathrm{rk}\, \Phi_0^{\mathfrak{i},[j]} = \mathrm{rk}_j \Phi$, we then have that $\mathrm{rk}_{\mathrm{rk}_j \Phi} \Phi = \mathrm{rk}_j \Phi$ for all $j \in \mathbb{N}$; in particular $\mathrm{rk}_m \Phi = \mathrm{rk}_{m+1} \Phi$ implies that $\mathrm{rk}_{m+1} \Phi = \mathrm{rk}_{m+2} \Phi$ (in fact, otherwise we would have $\mathrm{rk}_{\mathrm{rk}_{m+2} \Phi} \Phi = \mathrm{rk}_{m+2} \Phi > \mathrm{rk}_m \Phi$, while $\mathrm{rk}_{m+2} \Phi \le \mathrm{rk}_{m+1} \Phi + 1 = \mathrm{rk}_m \Phi + 1 \le m + 1$ and hence $\mathrm{rk}_{\mathrm{rk}_{m+2} \Phi} \Phi \le \mathrm{rk}_{m+1} \Phi = \mathrm{rk}_m \Phi$, a contradiction). This, together with the definition of $\mathrm{sk}\, \Phi$, immediately implies the claim. □

Now we turn back to the composition of formal maps φ obtained by the integration of a formal vector field X. As in Section 3, we consider a set $X_1, \dots, X_h$ of formal vector fields and we denote by $\varphi^j(x, t_j) = \varphi^{j,t_j}(x)$ the flow of X_j. We also consider $\Phi(x, t) = \varphi^{h,t_h} \circ \cdots \circ \varphi^{1,t_1}$.

We remark that, if $x = (x_1, \dots x_n)$, the components $\Phi_1(0, t), \dots, \Phi_n(0, t)$ of $\Phi(0, t)$ define an ideal $I(\Phi)$ of $\mathcal{A}[[t]]$. If J is any other ideal of $\mathcal{A}[[t]]$ for which $I(\Phi) \subset J$, we say that J consists of *formal closed paths* at $x = 0$ for Φ. If, furthermore, J is the ideal of a formal manifold Σ which is parametrized by a ring homomorphism $\Psi : \mathcal{A}[[t]] \to \mathcal{A}[[s]]$ (where $s = (s_1, \dots, s_k)$) associated to a mapping of the form $\psi(s) = (\psi_1(s), \dots, \psi_h(s))$, the condition is equivalent to $\Phi(0, \psi(s)) \equiv 0$. Geometrically, this means that “for any fixed s”the flow Φ at the time $\psi(s)$ maps the origin (in the x-space) back to itself.

We say that *the image of* $\Phi(0, t)$ *has rank (at least)* r *at* $x = 0$ if there exists an ideal consisting of closed paths, paramaterized by $\psi(s)$ as above, for Φ such that the rank over the quotient field of $\mathcal{A}[[s]]$ of the matrix $\Phi_t(0, \psi(s))$ is r. The following lemma can be seen as a refinement of the construction carried out in the proof of Theorem 2.

LEMMA 20. *Let* $X = \{X_1, \dots, X_k\}$ *be a family of formal vector fields of rank* r*. Then there exists a composition of* $2r - 1$ *of their flows whose image has rank* r *at* $x = 0$.

Proof. Let Ψ_X be the set of the flows of the fields X_j; by Theorem 2 we have that $\mathrm{sk}\, \Psi_X = r$, and by Lemma 19 we can choose $j_1, \dots, j_r \in \{1, \dots, k\}$ such that $\Phi(0, t) = \varphi^{j_r,t_r} \circ \cdots \circ \varphi^{j_1,t_1}(0)$ has generic rank r. Let $t = (t_1, \dots, t_r) =$

(t', t_r) and $\Phi'(0, t') = \Phi(0, (t', 0))$: we remark that the matrix representing $\Phi_t(0, t)$ has rank r over $\mathcal{A}[[t]]$ if and only if r is also the rank of $\Phi_t(0, (t', 0))$ over $\mathcal{A}[[t']]$. In fact, the columns of $\Phi_t(0, (t', 0))$ are given by the vectors $W_l(\Phi(0, t')) = \frac{\partial \Phi}{\partial t_l}(\Phi'(0, t'))$ together with the vector $X_{j_r}(\Phi'(0, t'))$, while those of $\Phi_t(0, t)$ are given by the $\varphi_x^{j_r}(\Phi'(0, t'), t_r) W_l(\Phi(0, t))$ and $X_{j_r}(\Phi(0, t))$: the claim then follows from the fact that $\varphi_x^{j_r}(\Phi(0, t), -t_r) X_{j_r}(\Phi(0, t)) = X_{j_r}(\Phi'(0, t'))$.

Let, now, $s = (s_1, \ldots, s_{2r-1})$ and $\Phi''(x, s) = \varphi^{j_1, s_{2r-1}} \circ \cdots \circ \varphi^{j_{r-1}, s_{r+1}} \circ \varphi^{j_r, s_r} \circ \varphi^{j_{r-1}, s_{r-1}} \circ \cdots \circ \varphi^{j_1, s_1}(x)$; then $\psi(t') = (t_1, \ldots, t_{r-1}, 0, -t_{r-1}, \ldots, -t_1)$ is a closed path for Φ'' and the matrix of $\Phi''_s(0, \psi(t'))$ is given by the composition of $\Phi_t(0, t')$ and $\varphi_x^{j_{r-1}, t_{r-1}} \circ \cdots \circ \varphi_x^{j_1, t_1}$, hence it has rank r. □

6. Formal Segre maps

Given a $\tilde{\sigma}$-invariant manifold ideal in $\mathcal{A}[[Z, \zeta]]$, we define its kth iteration ideal by

$$\begin{aligned} I^{[k]} &= \big(\{f(Z^1, 0), f(Z^j, \zeta^j), f(Z^j, \zeta^{j-1}) : 1 \le j \le \ell, f \in I\}\big) \\ &\quad \subset \mathcal{A}[[Z^1, \zeta^1, \ldots, Z^\ell, \zeta^\ell]], \quad k = 2\ell, \\ I^{[k]} &= \big(\{f(Z^1, 0), f(Z^j, \zeta^j), f(Z^{j+1}, \zeta^j) : 1 \le j \le \ell - 1, f \in I\}\big) \\ &\quad \subset \mathcal{A}[[Z^1, \zeta^1, \ldots, \zeta^{\ell-1}, Z^\ell]], \quad k = 2\ell - 1. \end{aligned}$$

Each ideal $I^{[k]}$ is a manifold ideal; indeed, a parametrization $\Psi_{2\ell} : \mathcal{A}[[Z^1, \zeta^1, \ldots, Z^\ell, \zeta^\ell]] \to \mathcal{A}[[z^1, \chi^1, \ldots, z^\ell, \chi^\ell]]$ of $I^{[2\ell]}$ is given by

$$\begin{aligned} Z^1 &= (z^1, 0), \\ \zeta^1 &= \big(\chi^1, \sigma Q(\chi^1, Z^1)\big) = \big(\chi^1, \sigma Q(\chi^1, z^1, 0)\big), \\ &\;\;\vdots \\ Z^\ell &= \big(z^\ell, Q(z^\ell, \zeta^{\ell-1})\big) = \big(z^\ell, Q(z^\ell, \chi^{\ell-1}, \sigma Q(\chi^{\ell-1}, z^{\ell-1}, \ldots))\big), \\ \zeta^\ell &= \big(\chi^\ell, \sigma Q(\chi^\ell, Z^\ell)\big) \\ &= \big(\chi^\ell, \sigma Q(\chi^\ell, z^\ell, Q(z^\ell, \chi^{\ell-1}, \sigma Q(\chi^{\ell-1}, z^{\ell-1}, \ldots)))\big), \end{aligned} \tag{10}$$

where we have chosen normal coordinates $Z = (z, w)$, $\zeta = (\chi, \tau)$, and assume that I is generated by $w - Q(z, \chi, \tau)$ as in Section 2.6.

Similarly, $I^{[2\ell+1]}$ is parametrized by $\Psi_{2\ell+1} : \mathcal{A}[[Z^1, \zeta^1, \ldots, \zeta^\ell, Z^{\ell+1}]] \to \mathcal{A}[[z^1, \chi^1, \ldots, z^\ell, \chi^\ell, z^{\ell+1}]]$ by combining $\psi^{2\ell}$ from (10) with

$$\begin{aligned} Z^{\ell+1} &= \big(z^{\ell+1}, Q(z^{\ell+1}, \zeta^\ell)\big) \\ &= \big(z^{\ell+1}, \chi^\ell, Q(\chi^\ell, \sigma Q(\chi^\ell, z^\ell, Q(z^\ell, \chi^{\ell-1}, \sigma Q(\chi^{\ell-1}, z^{\ell-1}, \ldots))))\big). \end{aligned}$$

The *Segre maps* are obtained by composing the parametrizations above with suitable projections: more precisely, for $k = 2\ell$ the Segre map $S^k = S^{2\ell}$ is defined as $\pi_{\zeta^\ell} \circ \psi_{2\ell}$, and for $k = 2\ell + 1$ it is defined by $\pi_{Z^{\ell+1}} \circ \psi_{2\ell+1}$, where π_{ζ^ℓ} and $\pi_{Z^{\ell+1}}$ are the projections on the last coordinate; we denote the corresponding homomorphism as usual by $\Pi_{\zeta^\ell} : \mathcal{A}[[\zeta^\ell]] \to \mathcal{A}[[Z^1, \zeta^1, \dots, Z^\ell, \zeta^\ell]]$ and $\Pi_{Z^{\ell+1}} : \mathcal{A}[[Z^{\ell+1}]] \to \mathcal{A}[[Z^1, \zeta^1, \dots, \zeta^\ell, Z^{\ell+1}]]$ respectively. With these definitions, one can verify that the Segre maps satisfy the following recurrence relation:

$$S^{2\ell+1}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell, z^{\ell+1}\big) = \big(z^{\ell+1}, Q\big(z^{\ell+1}, \sigma S^{2\ell}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell\big)\big)\big)$$

and a similar one for $k = 2\ell$. We note that this relation uniquely determines S^k, and may in fact taken as an alternative definition of the Segre maps.

We are now going to show how the Segre maps defined above fall into the framework described in Section 4. We start by considering the flows the formal vector fields associated to the generators $w_j - Q^j(z, \chi, \tau)$, that is,

$$V_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^{d} Q^k_{z_j} \frac{\partial}{\partial w_k} \in \operatorname{Der}_{\mathcal{A}}\big(\mathcal{A}[[z, w, \chi, \tau]], I\big), \quad j = 1, \dots, n$$

which form a basis of $\mathcal{D}_I^{(1,0)}$. Since $\tilde\sigma$ induces an isomorphism between $\mathcal{D}_I^{(1,0)}$ and $\mathcal{D}_I^{(0,1)}$,

$$\tilde\sigma V_j = \frac{\partial}{\partial \chi_j} + \sum_{k=1}^{d} \tilde\sigma Q^k_{\chi_j} \frac{\partial}{\partial \tau_k}, \quad j = 1, \dots, n$$

form a basis of $\mathcal{D}_I^{(0,1)}$. Moreover, $[V_{j_1}, V_{j_2}] = [\tilde\sigma V_{j_1}, \tilde\sigma V_{j_2}] = 0$ for all $1 \le j_1, j_2 \le n$. We define the combined flow of the vector fields V_j at the time $z = (z_1, \dots, z_n)$ as follows:

$$\varphi_V^z = \varphi_{V_1}^{z_1} \circ \cdots \circ \varphi_{V_n}^{z_n}$$

and analogously for the $\tilde\sigma V_j$,

$$\varphi_{\tilde\sigma V}^{\chi} = \varphi_{\tilde\sigma V_1}^{\chi_1} \circ \cdots \circ \varphi_{\tilde\sigma V_n}^{\chi_n};$$

note that because the V_j are pairwise commuting, the previous definitions do not depend on the order in which the flows are composed.

Remark 2. Let $\Psi : \mathcal{A}[[Z, \zeta]] \to \mathcal{A}[[x]]$ be a homomorphism, induced by a power series map $x \to \psi(x) = (\psi^Z(x), \psi^\zeta(x))$. Then one has

$$\operatorname{rk}\big(\varphi_V\big(\psi(x), t\big)\big) > \operatorname{rk}\psi(x) \quad \Leftrightarrow \quad \operatorname{rk}\big(\pi_Z\big(\varphi_V\big(\psi(x), t\big)\big)\big) > \operatorname{rk}\big(\pi_Z\big(\psi(x)\big)\big).$$

This is a consequence of the fact that $\pi_\zeta(\varphi_V^t(Z, \zeta)) = \pi_\zeta(Z, \zeta) = \zeta$ (which follows immediately from the fact that V_j is a combination of $\partial/\partial z_j$ and the fields $\partial/\partial w_k$), so that $\varphi_V(\psi(x), t)$ has a bigger (generic) rank than $\psi(x)$ if and only if their projections to the Z-coordinates have this property. A similar statement holds of course for $\varphi^s_{\sigma V}$ and the projection π_ζ.

In the ring $\mathcal{A}[[Z,\zeta,Z_0,\zeta_0]]$, we define two ideals J_1, J_2 as follows:

$$\begin{aligned} J_1 &= \big(w_j - Q_j(z,\chi,\tau) - (w_0)_j + Q_j(z_0,\chi_0,\tau_0),\\ &\qquad \chi_k - (\chi_0)_k, \tau_j - (\tau_0)_j\big)_{1\le j\le d, 1\le k\le n},\\ J_2 &= \big(\tau_j - \sigma Q_j(\chi,z,w) - (\tau_0)_j + \sigma Q_j(\chi_0,z_0,w_0),\\ &\qquad z_k - (z_0)_k, w_j - (w_0)_j\big)_{1\le j\le d, 1\le k\le n}. \end{aligned}$$

LEMMA 21. *J_1 (resp. J_2) is a strictly regular ideal of order $n+2d$ with respect to the variables (Z,ζ) (resp. (Z_0,ζ_0)), for which $\varphi_V^t(Z_0,\zeta_0)$ (resp. $\varphi_{\sigma V}^s(Z,\zeta)$) constitutes a regular parametrization.*

Proof. It suffices to prove the claim for J_1 (the case of J_2 is similar). One computes that the differential $d_{(Z,\zeta)}$ of the generators $w_j - Q_j(z,\chi,\tau) - (w_0)_j + Q_j(z_0,\chi_0,\tau_0), \chi_k - (\chi_0)_k, \tau_j - (\tau_0)_j$ at $(Z,\zeta,Z_0,\zeta_0) = (0,0,0,0)$ has rank $n+2d$. One also sees that the flow $\varphi_V^t(Z_0,\zeta_0)$ is a strictly regular map (as explained in the beginning of Section 4) so what we need to verify is that if $f(Z,\zeta,Z_0,\zeta_0)$ is one of those generators we have $f(\varphi_V^t(Z_0,\zeta_0),Z_0,\zeta_0)\equiv 0$. We use the explicit computation of the flow:

$$\varphi_V^t(z_0,w_0,\chi_0,\tau_0) = \big(z_0+t, w_0 - Q(z_0,\chi_0,\tau_0) + Q(z_0+t,\chi_0,\tau_0),\chi_0,\tau_0\big)$$

to obtain, if $f = \chi_k - (\chi_0)_k$ or $\tau_j - (\tau_0)_j$,

$$f\big(\varphi_V^t(Z_0,\zeta_0),Z_0,\zeta_0\big) = (\chi_0)_k - (\chi_0)_k \equiv 0 \quad\text{or}\quad (\tau_0)_j - (\tau_0)_j \equiv 0,$$

and in the case when $f = w_j - Q_j(z,\chi,\tau) - (w_0)_j + Q_j(z_0,\chi_0,\tau_0)$

$$\begin{aligned} f\big(\varphi_V^t(Z_0,\zeta_0),Z_0,\zeta_0\big) &= \big((w_0)_j - Q_j(z_0,\chi_0,\tau_0) + Q_j(z_0+t,\chi_0,\tau_0)\big)\\ &\qquad - Q_j(z_0+t,\chi_0,\tau_0) - (w_0)_j + Q_j(z_0,\chi_0,\tau_0) \equiv 0. \quad \square \end{aligned}$$

REMARK 3. An alternative proof of the previous lemma can be obtained by observing that the vector fields V_j are not only elements of $\mathcal{D}_I$, but actually annihilate the defining functions: $V_j(w - Q(z,\chi,\tau)) \equiv 0$. The claim above, then, corresponds to the fact that the orbit of the flow $\varphi_V^t(Z_0,\zeta_0)$ is contained in the level set of $w - Q(z,\chi,\tau)$ through (Z_0,ζ_0).

Now we define, for $k = 2\ell$

$$J^{[k]} = J^{[2\ell]} \subset \mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']]$$

and for $k = 2\ell - 1$

$$J^{[k]} = J^{[2\ell-1]} \subset \mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_{\ell-1}',\zeta_{\ell-1}',Z_{\ell-1}'',\zeta_{\ell-1}'',Z_\ell',\zeta_\ell']]$$

as the iteration from the origin, according to the scheme of Section 4, of a sequence of 2ℓ (resp. $2\ell-1$) ideals of $\mathcal{A}[[Z,\zeta,Z_0,\zeta_0]]$ alternating between J_1 and J_2, starting with J_1; notice that, varying slightly from the formalism employed in that section, the role of the variables y is alternately assumed by (Z,ζ) and (Z_0,ζ_0), starting with (Z,ζ). Define, for $k = 2\ell$, the ring homomorphism corresponding to the immersion $(Z_1,\zeta_1,\dots,Z_\ell,\zeta_\ell)\mapsto(Z_1,0,Z_1,\zeta_1,Z_2,\zeta_1,Z_2,\zeta_2,\dots,\zeta_\ell)$

$$\mathfrak{i}_k=\mathfrak{i}_{2\ell}:\mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']]\to\mathcal{A}[[Z^1,\zeta^1,\dots,Z^\ell,\zeta^\ell]]$$

as

$$\begin{aligned}&\mathfrak{i}_k(Z_1')=Z^1, &&\mathfrak{i}_k(\zeta_1')=0, &&\mathfrak{i}_k(\zeta_1'')=\zeta^1, &&\mathfrak{i}_k(Z_1'')=Z^1, &&\dots,\\ &\mathfrak{i}_k(Z_\ell')=Z^\ell, &&\mathfrak{i}_k(\zeta_\ell')=\zeta^{\ell-1}, &&\mathfrak{i}_k(\zeta_\ell'')=\zeta^\ell, &&\mathfrak{i}_k(Z_\ell'')=Z^\ell;\end{aligned}$$

we give an analogous definition for $k=2\ell-1$. Moreover, we define the ring homomorphism corresponding to the submersion $(Z_1',\zeta_1',Z_1'',\zeta_1''\dots,Z_\ell'',\zeta_\ell'')\mapsto(Z_1',\zeta_1'',Z_2',\zeta_2'',\dots)$

$$\mathfrak{p}_k=\mathfrak{p}_{2\ell}:\mathcal{A}[[Z^1,\zeta^1,\dots,Z^\ell,\zeta^\ell]]\to\mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']]$$

by

$$\mathfrak{p}_k(Z^1)=Z_1',\qquad \mathfrak{p}_k(\zeta^1)=\zeta_1'',\qquad \dots,\qquad \mathfrak{p}_k(Z^\ell)=Z_\ell',\qquad \mathfrak{p}_k(\zeta^\ell)=\zeta_\ell''$$

and similarly for $k=2\ell-1$.

Remark 4. From now on, we will abuse notation in the following way: Even for $k=2\ell-1$ odd, we write down the variables $(Z_1,\zeta_1,\dots,Z_\ell,\zeta_\ell)$ with the understanding that for $k=2\ell-1$, one has to disregard the last ζ_ℓ in the corresponding equations or replace it by Z_ℓ where appropriate.

Lemma 22. *We have* $\mathfrak{i}_k(J^{[k]})=I^{[k]}$ *and* $\mathfrak{p}_k(I^{[k]})\subset J^{[k]}$; *moreover, the induced homomorphisms*

$$\widetilde{\mathfrak{i}}_k:\mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']]/J^{[k]}\to\mathcal{A}[[Z^1,\zeta^1,\dots,Z^\ell,\zeta^\ell]]/I^{[k]}$$

and

$$\widetilde{\mathfrak{p}}_k:\mathcal{A}[[Z^1,\zeta^1,\dots,Z^\ell,\zeta^\ell]]/I^{[k]}\to\mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']]/J^{[k]}$$

are ring isomorphisms and $\widetilde{\mathfrak{i}}_k=\widetilde{\mathfrak{p}}_k^{-1}$. *In other words, the restriction of* $\mathfrak{i}_k$ *to the formal manifold* $\mathcal{M}^{[k]}$ *defined by* $I^{[k]}$ *induces an isomorphism between* $\mathcal{M}^{[k]}$ *and the formal manifold* $\mathcal{N}^{[k]}$ *defined by* $J^{[k]}$, *whose inverse is given by the restriction of* $\mathfrak{p}_k$ *to* $\mathcal{N}^{[k]}$.

Proof. We verify the first claim inductively. For $k=1$, we have

$$J^{[1]}=\left((w_1')_j-Q_j(z_1',\chi_1',\tau_1'),(\chi_1')_k,(\tau_1')_j\right)_{1\le j\le d,1\le k\le n},$$

so that

$$\mathfrak{i}_1(J^{[1]})=\left(w_j^1-Q_j(z^1,0,0)\right)_{1\le j\le d}=I^{[1]}.$$

For the inductive step, we restrict to the case $k=2\ell+1$ (the other is similar). We have

$$\begin{aligned}J^{[2\ell+1]}=J^{[2\ell]}+\big(&(w_{\ell+1}')_j-Q_j(z_{\ell+1}',\chi_{\ell+1}',\tau_{\ell+1}')-(w_\ell'')_j+Q_j(z_\ell'',\chi_\ell'',\tau_\ell''),\\&(\chi_{\ell+1}')_k-(\chi_\ell'')_k,(\tau_{\ell+1}')_j-(\tau_\ell'')_j\big)_{1\le j\le d,1\le k\le n}\end{aligned}$$

so that

$$\begin{aligned}\mathfrak{i}_{2\ell+1}\big(J^{[2\ell+1]}\big) &= \mathfrak{i}_{2\ell+1}\big(J^{[2\ell]}\big) + \Big(\big(w^{\ell+1}\big)_j - Q_j\big(z^{\ell+1},\chi^\ell,\tau^\ell\big) - \big(w^\ell\big)_j \\ &\qquad + Q_j\big(z^\ell,\chi^\ell,\tau^\ell\big), \big(\chi^\ell\big)_k - \big(\chi^\ell\big)_k, \big(\tau^\ell\big)_j - \big(\tau^\ell\big)_j\Big)_{1\le j\le d, 1\le k\le n} \\ &= I^{[2\ell]} + \Big(\big(w^{\ell+1}\big)_j - Q_j\big(z^{\ell+1},\chi^\ell,\tau^\ell\big)\Big)_{1\le j\le d} = I^{[2\ell+1]},\end{aligned}$$

where we used the facts that $\mathfrak{i}_{2\ell+1}(J^{[2\ell]}) = \mathfrak{i}_{2\ell}(J^{[2\ell]}) = I^{[2\ell]}$ and that $-(w^\ell)_j + Q_j(z^\ell,\chi^\ell,\tau^\ell) \in I^{[2\ell]}$. The verification that $\mathfrak{p}_k(I^{[k]}) \subset J^{(k)}$ is very similar and we shall omit it.

To prove the second claim, we will show that $\widetilde{\mathfrak{i}}_k$ and $\widetilde{\mathfrak{p}}_k$ invert each other. Since $\mathfrak{i}_k \circ \mathfrak{p}_k$ is the identity in $\mathcal{A}[[Z^1,\zeta^1,\dots,Z^\ell,\zeta^\ell]]$, it follows immediately that $\widetilde{\mathfrak{i}}_k \circ \widetilde{\mathfrak{p}_k} = id$. For $\mathfrak{p}_k \circ \mathfrak{i}_k$, we have

$$\begin{gathered}\mathfrak{p}_k \circ \mathfrak{i}_k\big(Z_1'\big) = Z_1', \qquad \mathfrak{p}_k \circ \mathfrak{i}_k\big(\zeta_1'\big) = 0, \qquad \mathfrak{p}_k \circ \mathfrak{i}_k\big(Z_1''\big) = Z_1', \\ \mathfrak{p}_k \circ \mathfrak{i}_k\big(\zeta_1''\big) = \zeta_1'', \qquad \dots, \qquad \mathfrak{p}_k \circ \mathfrak{i}_k\big(Z_\ell'\big) = Z_\ell', \\ \mathfrak{p}_k \circ \mathfrak{i}_k\big(\zeta_\ell'\big) = \zeta_{\ell-1}'', \qquad \mathfrak{p}_k \circ \mathfrak{i}_k\big(Z_\ell''\big) = Z_\ell', \qquad \mathfrak{p}_k \circ \mathfrak{i}_k\big(\zeta_\ell''\big) = \zeta_\ell'';\end{gathered}$$

this homomorphism induces the identity on $\mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']]/J^{[k]}$, since

$$J^{[k]} \supset \big(\zeta_1', Z_1''-Z_1',\dots,Z_\ell''-Z_\ell',\zeta_\ell'-\zeta_{\ell-1}''\big). \qquad \square$$

REMARK 5. The homomorphism $\mathfrak{p}_k = \mathfrak{p}_{2\ell}$ is also commuting with the relevant projections, in the sense that the diagram

$$\begin{array}{ccc}\mathcal{A}[[Z^1,\zeta^1,\dots,Z^\ell,\zeta^\ell]] & \xrightarrow{\mathfrak{p}_k} & \mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']] \\ \big\uparrow \Pi_{\zeta^\ell} & & \big\uparrow \Pi_{(Z_\ell'',\zeta_\ell'')} \\ \mathcal{A}[[\zeta^\ell]] & \xrightarrow{\Pi''_{\zeta^\ell}} & \mathcal{A}[[Z_\ell'',\zeta_\ell'']]\end{array}$$

is commutative, where Π''_{ζ^ℓ} is defined by $\Pi''_{\zeta^\ell}(\zeta^\ell) = \zeta_\ell''$.

Proof of Theorem 1: We consider the combined flow

$$\varphi^{[2\ell]}\big(Z,\zeta,t^1,s^1,\dots,t^\ell,s^\ell\big) = \varphi_{\tilde\sigma V}^{s^\ell} \circ \varphi_V^{t^\ell} \circ \cdots \circ \varphi_{\tilde\sigma V}^{s^1} \circ \varphi_V^{t^1}(Z,\zeta),$$

and the homomorphism $\Phi_k : \mathcal{A}[[Z_1',\zeta_1',Z_1'',\zeta_1'',\dots,Z_\ell',\zeta_\ell',Z_\ell'',\zeta_\ell'']] \to \mathcal{A}[[t^1, s^1,\dots,t^\ell,s^\ell]]$ induced by the map

$$\begin{cases}(t^1,s^1,\dots,t^\ell,s^\ell) \longrightarrow_{\varphi_{2\ell}} (\varphi^{[1]}(0,0,t^1),\dots,\varphi^{[2\ell]}(0,0,t^1,\dots,s^\ell)), \\ \quad k = 2\ell, \\ (t^1,s^1,\dots,t^\ell,s^\ell,t^{\ell+1}) \longrightarrow_{\varphi_{2\ell+1}} (\varphi^{[1]}(0,0,t^1),\dots,\varphi^{[2\ell+1]}(0,0,t^1,\dots,s^\ell,t^{\ell+1})), \\ \quad k = 2\ell+1.\end{cases}$$

According to Lemmata 21 and 18, Φ_k gives a parametrization of the ideal $J^{[k]}$. By Lemma 22 we have that $\Phi_k \circ \mathfrak{p}_k$ is a parametrization of $I^{[k]}$. Since, then, $\Phi_k \circ \mathfrak{p}_k$ and Ψ_k are both parametrizations of $I^{[k]}$, we have that $\Phi_k \circ \mathfrak{p}_k \circ \Pi_{\zeta^\ell}$

has the same rank as $\pi_{\zeta^\ell} \circ \psi_k = S^k$, the Segre map of order $k = 2\ell$. The case of odd k is treated analogously.

Denote by X the family of vector fields $(V_1, \dots, V_n, \tilde{\sigma} V_1, \dots, \tilde{\sigma} V_n)$. By Theorem 2 (applied to the pull-back of X by any parametrization of $\mathcal{M}$) we have that $\operatorname{sk} \Psi_X = 2n + d$ if and only if $\operatorname{rk} X = 2n + d$, i.e. if $\mathcal{M}$ is of finite type. Next, we observe that an arbitrary composition of flows of the V_j and $\tilde{\sigma} V_j$ can be reduced to one of the form $\varphi^{[k]}$; indeed, if for some i_0 the flows $\varphi^t_{V_{i_0}}$ and $\varphi^s_{V_{i_0}}$ are listed without any $\varphi_{\tilde{\sigma} V_j}$ in between them, then they can be brought by commutation to a single flow $\varphi^{t+s}_{V_{i_0}}$ (which contributes to the rank in the same way as $\varphi^t_{V_{i_0}}$). It follows that the generic rank of $\varphi^{[k]}$ is equal to $2n + d$ for big enough k if and only if $\mathcal{M}$ is of finite type.

Notice now, that $\Phi^{[k]} = \Phi_k \circ \Pi_{(Z''_\ell, \zeta''_\ell)}$, hence

$$\Phi^{[k]} \circ \Pi''_{\zeta^\ell} = \Phi_k \circ \Pi_{(Z''_\ell, \zeta''_\ell)} \circ \Pi''_{\zeta^\ell} = \Phi_k \circ \mathfrak{p}_k \circ \Pi_{\zeta^\ell}$$

(where we have used Remark 5), which as observed above has the same (generic) rank as S^k. The proof is finished by observing that the restrictions of π''_{ζ^ℓ} to $\mathcal{M}$ is a submersion. The fact about the order of the Segre map now follows immediately from Lemma 19 (taking in account Remark 2 after the definition of the flows φ^z_V and $\varphi^\chi_{\tilde{\sigma} V}$). The last claim of the theorem is obtained as a straightforward application of Lemma 20.

The previous construction shows that the flows of the vector fields V_j, $\tilde{\sigma} V_j$, after suitable projections, give rise to maps with the same rank as the Segre maps. We conclude the section by showing that it is possible to use the mentioned flows to obtain *precisely* the Segre maps. In order to achieve this, one has to correctly reparametrize the "time" variables, as in the following lemma.

LEMMA 23. *Define a flow Υ^k in the following way: for $k = 2\ell$*

$$\Upsilon^{2\ell}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell\big)(Z, \zeta) = \varphi_{\tilde{\sigma} V}^{\chi^\ell - \chi^{\ell-1}} \circ \varphi_V^{z^\ell - z^{\ell-1}} \circ \cdots \circ \varphi_{\tilde{\sigma} V}^{\chi^1} \circ \varphi_V^{z^1}(Z, \zeta)$$

and for $k = 2\ell + 1$

$$\begin{aligned} &\Upsilon^{2\ell+1}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell, z^{\ell+1}\big)(Z, \zeta) \\ &\quad = \varphi_V^{z^{\ell+1} - z^\ell} \circ \varphi_{\tilde{\sigma} V}^{\chi^\ell - \chi^{\ell-1}} \circ \cdots \circ \varphi_{\tilde{\sigma} V}^{\chi^1} \circ \varphi_V^{z^1}(Z, \zeta). \end{aligned}$$

Then

$$S^{2\ell}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell\big) = \pi_\zeta \circ \sigma \Upsilon^{2\ell}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell\big)(0, 0)$$

and

$$S^{2\ell+1}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell, z^{\ell+1}\big) = \pi_Z \circ \Upsilon^{2\ell+1}\big(z^1, \chi^1, \dots, z^\ell, \chi^\ell, z^{\ell+1}\big)(0, 0).$$

Proof. The (composed) flow Υ^t_V is computed explicitly as follows:

$$\Upsilon^t_V(z, w, \chi, \tau) = \big(z + t, w - Q(z, \chi, \tau) + Q(z + t, \chi, \tau), \chi, \tau\big)$$

and the flow $\Upsilon^t_{\tilde\sigma V}$ is given by

$$\Upsilon^t_{\tilde\sigma V}(z,w,\chi,\tau) = \big(z,w,\chi+t,\tau-\sigma Q(\chi,z,w)+\sigma Q(\chi+t,z,w)\big).$$

It follows that $\Upsilon^1(z_1)(0,0) = \Upsilon^{z_1}_V(0,0) = (z^1,0,0,0)$; projecting on the $Z = (z,w)$-space we see that the conclusion holds for $k=1$. Arguing by induction, for odd k we have

$$\begin{aligned}
&\Upsilon^{2\ell+1}\big(z^1,\chi^1,\dots,z^\ell,\chi^\ell,z^{\ell+1}\big)(0,0)\\
&\quad= \varphi_V^{z^{\ell+1}-z^\ell}\circ\Upsilon^{2\ell}\big(z^1,\chi^1,\dots,z^\ell,\chi^\ell\big)(0,0)\\
&\quad= \big(z^\ell+\big(z^{\ell+1}-z^\ell\big),\pi_w\big(\Upsilon^{2\ell}\big)-Q\big(z^\ell,\pi_\zeta\big(\Upsilon^{2\ell}\big)\big)\\
&\qquad+Q\big(z^\ell+\big(z^{\ell+1}-z^\ell\big),\pi_\zeta\big(\Upsilon^{2\ell}\big)\big),\pi_\zeta\big(\Upsilon^{2\ell}\big)\big)\\
&\quad= \big(z^{\ell+1},Q\big(z^{\ell+1},\pi_\zeta\big(\Upsilon^{2\ell}\big)\big),\pi_\zeta\big(\Upsilon^{2\ell}\big)\big),
\end{aligned}$$

where we used the fact that $\pi_w(\Upsilon^{2\ell})-Q(z^\ell,\pi_\zeta(\Upsilon^{2\ell}))\equiv 0$ since the vector fields involved in the flow all belong to $\mathcal{D}_I$. Now, from the previous computation we have

$$\pi_Z\big(\Upsilon^{2\ell+1}\big) = \big(z^{\ell+1},Q\big(z^{\ell+1},\pi_\zeta\big(\Upsilon^{2\ell}\big)\big)\big) = \big(z^{\ell+1},Q\big(z^{\ell+1},\sigma\pi_\zeta\big(\sigma\Upsilon^{2\ell}\big)\big)\big)$$

which means that the maps $\pi_Z(\Upsilon^{2\ell+1})$ and $\pi_\zeta(\sigma\Upsilon^{2\ell})$ satisfy the same recurrence relation defining the Segre maps (the verification for k even is completely analogous); the claim is then obtained by induction. □

REMARK 6. The combined flow which we had previously considered,

$$\varphi^{[2\ell]}\big(z^1,\chi^1,\dots,z^\ell,\chi^\ell\big)(Z,\zeta) = \varphi^{\chi^\ell}_{\tilde\sigma V}\circ\varphi^{z^\ell}_V\circ\cdots\circ\varphi^{\chi^1}_{\tilde\sigma V}\circ\varphi^{z^1}_V(Z,\zeta);$$

has the same generic rank as $\Upsilon^{2\ell}$; the two flows are obtained one from the other by an invertible (and in fact linear) transformation of the formal parameters $(z^1,\chi^1,\dots,z^\ell,\chi^\ell)$. The same holds for $k=2\ell+1$.

References

[1] M. S. Baouendi, P. Ebenfelt and L. P. Rothschild, *Algebraicity of holomorphic mappings between real algebraic sets in* $\mathbf{C}^n$, Acta Math. **177** (1996), 225–273. MR 1440933

[2] M. S. Baouendi, P. Ebenfelt and L. P. Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, vol. **47**, Princeton University Press, Princeton, NJ, 1999. MR 1668103

[3] M. S. Baouendi, P. Ebenfelt and L. P. Rothschild, *Dynamics of the Segre varieties of a real submanifold in complex space*, J. Algebraic Geom. **12** (2003), 81–106. MR 1948686

[4] J. Merker, *Vector field construction of Segre sets*; available at arXiv:math/9901010 [math.CV].

GIUSEPPE DELLA SALA, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, A-1090 WIEN, AUSTRIA

E-mail address: giuseppe.dellasala@univie.ac.at

Robert Juhlin, Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: robert.juhlin@univie.ac.at

Bernhard Lamel, Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: bernhard.lamel@univie.ac.at