Illinois Journal of Mathematics
Volume 54, Number 1, Spring 2010, Pages 129–135
S 0019-2082

COMPUTING EQUATIONS FOR RESIDUALLY FREE GROUPS

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ABSTRACT. We show that there is no algorithm deciding whether the maximal residually free quotient of a given finitely presented group is finitely presentable or not.

Given a finitely generated subgroup G of a finite product of limit groups, we discuss the possibility of finding an explicit set of defining equations (i.e., of expressing G as the maximal residually free quotient of an explicit finitely presented group).

1. Introduction

Any countable group G has a largest residually free quotient $\mathrm{RF}(G)$, equal to $G/\bigcap_{f\in\mathcal{H}}\ker f$ where $\mathcal{H}$ is the set of all homomorphisms from G to a non-Abelian free group $\mathbb{F}$. Since any two countably generated non-Abelian free groups can be embedded in each other, this notion does not depend on the rank of the free group $\mathbb{F}$ considered.

In the language of [BMR99], if R is a finite set of group equations on a finite set of variables S, then $G = \mathrm{RF}(\langle S \mid R\rangle)$ is the *coordinate group* of the variety defined by the system of equations R. We say that R is a *set of defining equations* of G over S. Equational noetherianness of free groups implies that any finitely generated residually free group G has a (finite) set of defining equations [BMR99].

On the other hand, any finitely generated residually free group embeds into a finite product of limit groups (also known as finitely generated fully residually free groups), which correspond to the *irreducible components* of the variety defined by R [BMR99, KM98, Sel01]. Conversely, any subgroup of a finite product of limit groups is residually free.

This gives three possibilities to define a finitely generated residually free group G in an explicit way:

Received June 12, 2009; received in final form October 23, 2009.
2000 *Mathematics Subject Classification.* 20F65, 20F10, 20E26, 20F67.

(1) give a finite presentation of G (if G is finitely presented);
(2) give a set of defining equations of G: write $G = \mathrm{RF}(\langle S \mid R\rangle)$, with S and R finite;
(3) write G as the subgroup of $L_1 \times \cdots \times L_n$ generated by a finite subset S, where $L_1, \dots, L_n$ are limit groups given by some finite presentations.

We investigate the algorithmic possibility to go back and forth between these ways of defining G.

One can go from 2 to 3: given a set of defining equations of G, one can find an explicit embedding into some product of limit groups [KM98, KM05, BHMS09, GW09].

Conversely, if G is given as a subgroup of a product of limit groups, *and if one knows that G is finitely presented*, one can compute a presentation of G [BHMS09]. Obviously, a finite presentation is a set of defining equations.

Since residually free groups are not always finitely presented, we investigate the following question:

Question. *Let $L = L_1 \times \cdots \times L_n$ be a product of limit groups. Let G be the subgroup generated by a finite subset $S \subset L$. Can one algorithmically find a finite set of defining equations for G, that is, find a finite presentation $\langle S \mid R\rangle$ such that $G = \mathrm{RF}(\langle S \mid R\rangle)$?*

We will prove that this question has a negative answer. On the other hand, we introduce a closely related notion which has better algorithmic properties.

Let $\mathrm{RF}_{na}(G)$ be the quotient $G/\bigcap_{f\in\mathcal{H}_{na}} \ker f$ where $\mathcal{H}_{na}$ is the set of all homomorphisms from G to $\mathbb{F}$ with *non-Abelian image*. Of course, $\mathrm{RF}_{na}(G)$ is a quotient of $\mathrm{RF}(G)$, which forgets the information about morphisms to $\mathbb{Z}$. In fact (Lemma 2.2), it is the quotient of $\mathrm{RF}(G)$ by its center.

We say that G is a *residually non-Abelian free* group if $G = \mathrm{RF}_{na}(G)$, i.e., if every non-trivial element of G survives in a non-Abelian free quotient of G; equivalently, G is residually non-Abelian free if and only if G is residually free and has trivial center. Given a residually non-Abelian free group G, we say that R is a *set of na-equations* of G over S if $G = \mathrm{RF}_{na}(\langle S \mid R\rangle)$.

We write $Z(G)$ for the center of G, and $b_1(G)$ for the torsion-free rank of $H_1(G, \mathbb{Z})$.

Theorem 1.

- *There is an algorithm which takes as input presentations of limit groups $L_1, \dots, L_n$, and a finite subset $S \subset L_1 \times \cdots \times L_n$, and which computes a finite set of na-equations for $G/Z(G) = \mathrm{RF}_{na}(G)$, where $G = \langle S\rangle$.*
- *One can compute a finite set of defining equations for $G = \langle S\rangle$ if and only if one can compute $b_1(G)$.*

Since there is no algorithm computing $b_1(\langle S\rangle)$ from $S \subset \mathbb{F}_2 \times \mathbb{F}_2$ [BM09], we deduce the following corollary.

Corollary 1. *There is no algorithm which takes as an input a finite subset* $S \subset \mathbb{F}_2 \times \mathbb{F}_2$ *and computes a finite set of equations for* $\langle S \rangle$.

We also investigate the possibility to decide whether a residually free quotient is finitely presented. Using Theorem 1 and [Gru78], we prove the following theorem.

Theorem 2. *There is no algorithm which takes as an input a finite group presentation* $\langle S \mid R \rangle$, *and which decides whether* $\mathrm{RF}(\langle S \mid R \rangle)$ *is finitely presented.*

2. The residually non-Abelian free quotient RF_{na}

We always denote by G a finitely generated group, and by $\mathbb{F}$ a non-Abelian free group.

Definition 2.1. $\mathrm{RF}(G)$ is the quotient of G by the intersection of the kernels of all morphisms $G \to \mathbb{F}$.

$\mathrm{RF}_{na}(G)$ is the quotient of G by the intersection of the kernels of all morphisms $G \to \mathbb{F}$ with non-Abelian image.

One may view $\mathrm{RF}(G)$ as the image of G in $\mathbb{F}^{\mathcal{H}}$, where $\mathcal{H}$ is the set of all morphisms $G \to \mathbb{F}$, and $\mathrm{RF}_{na}(G)$ as the image in $\mathbb{F}^{\mathcal{H}_{na}}$, where $\mathcal{H}_{na}$ is the set of all morphisms with non-Abelian image.

Every homomorphism $G \to \mathbb{F}$ factors through $\mathrm{RF}(G)$ (through $\mathrm{RF}_{na}(G)$ if its image is not Abelian). By definition, G is residually free if and only if $G = \mathrm{RF}(G)$, residually non-Abelian free if and only if $G = \mathrm{RF}_{na}(G)$.

Lemma 2.2. *There is an exact sequence*

$$1 \to Z(\mathrm{RF}(G)) \to \mathrm{RF}(G) \to \mathrm{RF}_{na}(G) \to 1.$$

In particular, G is residually non-Abelian free if and only if G is residually free and $Z(G) = 1$. If G is a non-Abelian limit group, it has trivial center and $\mathrm{RF}_{na}(G) = \mathrm{RF}(G) = G$.

Proof of Lemma 2.2. Recall that $\mathbb{F}$ is *commutative transitive*, that is, that centralizers of nontrivial elements are Abelian (i.e., cyclic) [LS01]. Let $H = \mathrm{RF}(G)$. Consider $a \in Z(H)$ and $f : H \to \mathbb{F}$ with $f(a) \neq 1$. The image of f centralizes $f(a)$, so is Abelian by commutative transitivity of $\mathbb{F}$. Thus, a has trivial image in $\mathrm{RF}_{na}(H) = \mathrm{RF}_{na}(G)$.

Conversely, consider $a \in H \setminus Z(H)$, and $b \in H$ with $[a,b] \neq 1$. There exists $f : H \to \mathbb{F}$ such that $f([a,b]) \neq 1$. Then $f(H)$ is non-Abelian, and $f(a) \neq 1$. This means that the image of a in $\mathrm{RF}_{na}(G)$ is nontrivial. □

Any epimorphism $f : G \to H$ induces epimorphisms $f_{\mathrm{RF}} : \mathrm{RF}(G) \to \mathrm{RF}(H)$ and $f_{na} : \mathrm{RF}_{na}(G) \to \mathrm{RF}_{na}(H)$.

Lemma 2.3. *Let $f : G \to H$ be an epimorphism. Then $f_{\mathrm{RF}} : \mathrm{RF}(G) \to \mathrm{RF}(H)$ is an isomorphism if and only if $f_{na} : \mathrm{RF}_{na}(G) \to \mathrm{RF}_{na}(H)$ is an isomorphism and $b_1(G) = b_1(H)$.*

Proof. Note that f_{RF} (resp., f_{na}) is an isomorphism if and only if any morphism $G \to \mathbb{F}$ (resp., any such morphism with non-Abelian image) factors through f. The lemma then follows from the fact that the embedding $\mathrm{Hom}(H, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(G, \mathbb{Z})$ induced by f is onto if and only if $b_1(G) = b_1(H)$. □

Given a product $L_1 \times \cdots \times L_n$, we denote by p_i the projection onto L_i.

Lemma 2.4. *Let $G \subset L = L_1 \times \cdots \times L_n$ with L_i a limit group. Let $I \subset \{1, \ldots, n\}$ be the set of indices such that $p_i(G)$ is Abelian. Then $\mathrm{RF}_{na}(G)$ is the image of G in $L' = \prod_{i \notin I} L_i$ (viewed as a quotient of $L_1 \times \cdots \times L_n$).*

Proof. Note that $G = \mathrm{RF}(G)$. An element $(x_1, \ldots, x_n) \in G$ is in $Z(G)$ if and only if x_i is central in $p_i(G)$ for every i. Since $p_i(G)$ is Abelian or has trivial center, $Z(G)$ is the kernel of the natural projection $L \to L'$. The result follows from Lemma 2.2. □

Lemma 2.5. $\mathrm{RF}(G)$ *is finitely presented if and only if* $\mathrm{RF}_{na}(G)$ *is.*

Proof. If H is any residually free group, the abelianization map $H \to H_{ab}$ is injective on $Z(H)$ since any element of $Z(H)$ survives in some free quotient of H, which has to be cyclic (see [BHMS09, Lemma 6.2]). In particular, $Z(H)$ is finitely generated if H is. Applying this to $H = R(G)$, the exact sequence of Lemma 2.2 gives the required result. □

3. Proof of the theorems

Let S be a finite set of elements in a group. We define $S_0 = S \cup \{1\}$. If R, R' are sets of words on $S \cup S^{-1}$, then R^{S_0} is the set of all words obtained by conjugating elements of R by elements of S_0, and $[R^{S_0}, R']$ is the set of all words obtained as commutators of words in R^{S_0} and words in R'.

Proposition 3.1. *Let $A_1, \ldots, A_n$ be arbitrary groups, with $n \geq 2$. Let $G \subset A_1 \times \cdots \times A_n$ be generated by $S = \{s_1, \ldots, s_k\}$. Let $p_i : G \to A_i$ be the projection. Assume that $p_i(G) = \mathrm{RF}_{na}(\langle S \mid R_i \rangle)$ for some finite set of relators R_i.*

Then the set

$$\tilde{R} = [R_n^{S_0}, [R_{n-1}^{S_0}, \ldots [R_3^{S_0}, [R_2^{S_0}, R_1]] \ldots]]$$

is a finite set of na-equations of $\mathrm{RF}_{na}(G)$ *over S, i.e.,* $\mathrm{RF}_{na}(G) = \mathrm{RF}_{na}(\langle S \mid \tilde{R} \rangle)$.

An equality such as $p_i(G) = \mathrm{RF}_{na}(\langle S \mid R_i\rangle)$ means that there is an isomorphism commuting with the natural projections $F(S) \to p_i(G)$ and $F(S) \to \mathrm{RF}_{na}(\langle S \mid R_i\rangle)$, where $F(S)$ denotes the free group on S.

Proof of Proposition 3.1. Recall that a free group $\mathbb{F}$ is CSA: commutation is transitive on $\mathbb{F} \setminus \{1\}$, and maximal Abelian subgroups are malnormal [MR96]. In particular, if two nontrivial subgroups commute, then both are Abelian. If A, B are nontrivial subgroups of $\mathbb{F}$, and if A commutes with $B, B^{x_1}, \dots, B^{x_p}$ for elements $x_1, \dots, x_p \in \mathbb{F}$, then $\langle A, B, x_1, \dots, x_p\rangle$ is Abelian.

We write

$$\tilde{G} = \langle S \mid \tilde{R}\rangle = \langle S \mid [R_n^{S_0}, [R_{n-1}^{S_0}, \dots [R_2^{S_0}, R_1] \dots]]\rangle.$$

We always denote by $\varphi : F(S) \to \mathbb{F}$ a morphism with non-Abelian image. We shall show that *such a φ factors through G if and only if it factors through $\tilde{G}$.* This implies the desired result $\mathrm{RF}_{na}(G) = \mathrm{RF}_{na}(\tilde{G})$: both groups are equal to the image of $F(S)$ in $\mathbb{F}^{\mathcal{H}_{na}}$, where $\mathcal{H}_{na}$ is the set of all φ's which factor through G and $\tilde{G}$.

We proceed by induction on n. We first claim that φ is trivial on $\tilde{R}$ if and only if it is trivial on some R_i. The if direction is clear. For the only if direction, observe that the image of $[R_{n-1}^{S_0}, \dots [R_2^{S_0}, R_1] \dots]$ commutes with all conjugates of $\varphi(R_n)$ by elements of $\varphi(F(S))$, so R_n or $[R_{n-1}^{S_0}, \dots [R_2^{S_0}, R_1] \dots]$ has trivial image. The claim follows by induction.

Now suppose that φ factors through $\tilde{G}$. Then φ kills $\tilde{R}$, hence some R_i. It follows that φ factors through $p_i(G)$, hence through G.

Conversely, suppose that φ factors through $f : G \to \mathbb{F}$. Consider the intersection of G with the kernel of $p_n : G \to A_n$ and the kernel of $p_{1,\dots,n-1} : G \to A_1 \times \cdots \times A_{n-1}$. These are commuting normal subgroups of G. If both have nontrivial image in $\mathbb{F}$, the CSA property implies that the image of f is Abelian, a contradiction. We deduce that f factors through p_n or through $p_{1,\dots,n-1}$, and by induction that it factors through some p_i. Thus, φ kills R_i, hence $\tilde{R}$ as required. $\square$

Proof of Theorem 1. Given a finite subset $S \subset L_1 \times \cdots \times L_n$, where each L_i is a limit group, we want to find a finite set of na-equations for $G/Z(G) = \mathrm{RF}_{na}(G)$, where $G = \langle S\rangle$.

Using a solution of the word problem in a limit group, one can find the indices i for which $p_i(G) \subset L_i$ is Abelian (this amounts to checking whether the elements of $p_i(S)$ commute).

First, assume that no $p_i(G)$ is Abelian. As pointed out in [GW09] or [BHMS09, Lemma 7.5], one deduces from [Wil08] an algorithm yielding a finite presentation $\langle S \mid R_i\rangle$ of $p_i(G)$. Since $p_i(G)$ is not Abelian, one has $p_i(G) = \mathrm{RF}_{na}(\langle S \mid R_i\rangle)$, and Proposition 3.1 yields a finite set of na-equations for $\mathrm{RF}_{na}(G)$ over S (if $n = 1$, then $\mathrm{RF}_{na}(G) = p_1(G)$). If some $p_i(G)$'s are

Abelian, we simply replace G by its image in L' as in Lemma 2.4. This proves the first assertion of the theorem.

We now prove that one can find a finite set of defining equations if and only if one can compute $b_1(G)$. Suppose that $b_1(G)$ is known. We want a finite set R such that $\mathrm{RF}(G)=\mathrm{RF}(\langle S \mid R\rangle)$. If $n=1$, then G is a subgroup of the limit group L_1, and one can find a finite presentation of G as explained above. So assume $n\geq 2$. Consider the finite presentation $\tilde G=\langle S \mid \tilde R\rangle$ given by Proposition 3.1, so that $\mathrm{RF}_{na}(\tilde G)=\mathrm{RF}_{na}(G)$.

We claim that G is a quotient of $\tilde G$. To see this, we consider an $x\in F(S)$ which is trivial in $\tilde G$ and we prove that it is trivial in G. If not, residual freeness of G implies that x survives under a morphism $\varphi: F(S)\to\mathbb{F}$ which factors through G. If φ has non-Abelian image, it factors through $\mathrm{RF}_{na}(G)=\mathrm{RF}_{na}(\tilde G)$, hence through $\tilde G$, contradicting the triviality of x in $\tilde G$. If the image is Abelian, φ also factors through $\tilde G$ because all relators in $\tilde R$ are commutators.

Since $\tilde R$ is finite, we can compute $b_1(\tilde G)$. If $b_1(\tilde G)=b_1(G)$, we are done by Lemma 2.3 since G is a quotient of $\tilde G$. If $b_1(\tilde G)>b_1(G)$, we enumerate all trivial words of G (using an enumeration of trivial words in each $p_i(G)$), and we add them to the presentation of $\tilde G$ one by one. We compute b_1 after each addition, and we stop when we reach the known value $b_1(G)$.

Conversely, if we have a finite set of defining equations for G, so that $G=\mathrm{RF}(\langle S \mid R\rangle)$, we can compute $b_1(\langle S\mid R\rangle)$, which equals $b_1(G)$ by Lemma 2.3. □

Theorem 3. *There is no algorithm which takes as input a finite group presentation $\langle S \mid \tilde R\rangle$, and which decides whether $\mathrm{RF}(\langle S\mid \tilde R\rangle)$ is finitely presented.*

Proof. Given a finite set $S\subset \mathbb{F}_2\times\mathbb{F}_2$, Theorem 1 provides a finite set $\tilde R$ such that $\mathrm{RF}_{na}(\langle S\rangle)=\mathrm{RF}_{na}(\langle S\mid\tilde R\rangle)$. Using Lemma 2.5, we see that finite presentability of $\mathrm{RF}(\langle S\mid\tilde R\rangle)$ is equivalent to that of $\mathrm{RF}_{na}(\langle S\mid \tilde R\rangle)$, hence to that of $\mathrm{RF}(\langle S\rangle)=\langle S\rangle$. But it follows from [Gru78] that there is no algorithm which decides, given a finite set $S\subset\mathbb{F}_2\times\mathbb{F}_2$, whether $\langle S\rangle$ is finitely presented. □

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