Illinois Journal of Mathematics
Volume 54, Number 1, Spring 2010, Pages 89–108
S 0019-2082

DOMAINS OF PROPER DISCONTINUITY ON THE BOUNDARY OF OUTER SPACE

ILYA KAPOVICH AND MARTIN LUSTIG

The article is dedicated to the mathematical contributions of Paul E. Schupp.

Abstract. Motivated by the work of McCarthy and Papadopoulos for subgroups of mapping class groups, we construct domains of proper discontinuity in the compactified Outer space and in the projectivized space of geodesic currents for any "dynamically large" subgroup of $\mathrm{Out}(F_N)$ (that is, a subgroup containing an atoroidal iwip).

As a corollary, we prove that for $N \geq 3$ the action of $\mathrm{Out}(F_N)$ on the subset of $\mathbb{P}\mathrm{Curr}(F_N)$ consisting of all projectivized currents with full support is properly discontinuous.

1. Introduction

One of several important recent events in the study of mapping class groups and Teichmüller spaces is the introduction and development of the theory of convex-cocompact subgroups of mapping class groups through the work of Farb and Mosher [14], Hamenstadt [18], by Kent and Leininger [29]–[31] and others. This theory is inspired by the classical notion of convex-cocompactness for Kleinian groups and is motivated, in part, by looking for new examples of word-hyperbolic extensions of surface groups by nonelementary word-hyperbolic groups. A key component of the theory of convex-cocompactness in the mapping class group context is the construction of domains of discontinuity for subgroups of mapping class groups in the Thurston boundary of the Teichmüller space. This construction of domains of discontinuity was first put forward by Masur for the handlebody group [34] and for arbitrary subgroups of mapping class groups by McCarthy and Papadopoulos [35], the main case

Received February 25, 2009; received in final form November 15, 2009.
The first author was supported by the NSF grants DMS-0603921 and DMS-0904200.
2000 *Mathematics Subject Classification.* Primary 20F. Secondary 57M, 37B, 37D.

being that of "sufficiently large" subgroups of mapping class groups. More detail about the work of McCarthy and Papadopoulos will be given below.

For automorphisms of free groups, many concepts and results proved for mapping classes have been successfully "translated" in the past 20 years, sometimes giving rise to new interesting variations of the mapping class group ideas, and occasionally even to a deeper understanding of them. In some rare cases, the innovative impulse has even gone in the converse direction.

A satisfying translation of the subgroup theory into the $\mathrm{Out}(F_N)$ world, however, is still far from being achieved. In particular, it would be much desirable if some analogues of the above quoted results for convex-cocompact subgroups could be established. The aim of this paper is to provide a first step into this direction. In order to state our result, we first provide some terminology; more detail will be given later.

Teichmüller space and its Thurston boundary admit two independent translations into the $\mathrm{Out}(F_N)$ world: One candidate is Culler–Vogtmann's (compactified) Outer space $\overline{\mathrm{CV}}_N$, and the other one is the projectivized space of currents on F_N, which is also compact but, unlike $\overline{\mathrm{CV}}_N$, infinite dimensional. The group $\mathrm{Out}(F_N)$ acts on both spaces, and it is known that *atoroidal iwip* automorphisms (which are strong analogues of pseudo-Anosov mapping classes, see Definition 3.1 and Remark 3.2 below) act on both spaces with North–South dynamics. For precise references and more details, see Sections 2 and 3.

THEOREM 1.1. *Let* $G \subseteq \mathrm{Out}(F_N)$ *be a subgroup which contains at least one atoroidal iwip. Then there exist canonical nonempty open* G*-invariant subsets* $\widehat{\Delta}_G^{\mathrm{cv}} \subseteq \overline{\mathrm{CV}}_N$ *and* $\widehat{\Delta}_G^{\mathrm{curr}} \subseteq P\,\mathrm{Curr}(F_N)$ *on which the action of* G *is properly discontinuous.*

The precise definitions of the sets $\widehat{\Delta}_G^{\mathrm{cv}} \subset \overline{\mathrm{CV}}_N$ and $\widehat{\Delta}_G^{\mathrm{curr}} \subset P\,\mathrm{Curr}(F_N)$ are given in Section 4, and we also revisit their construction in the discussion below.

In order to motivate and explain Theorem 1.1 properly, we need to recall some details of the above mentioned construction of McCarthy and Papadopoulos [35]. Let S be a closed hyperbolic surface, let $\mathrm{Mod}(S)$ be the mapping class group of S and let $G \subseteq \mathrm{Mod}(S)$ be a "sufficiently large subgroup", that is, G contains two independent pseudo-Anosov elements. For such G, there is a well-defined *limit set* $\Lambda_G \subseteq \mathcal{PML}(S)$ which is the unique smallest G-invariant closed subset of $\mathcal{PML}(S)$. Here, $\mathcal{PML}(S)$ is the space of projective measured laminations on S. One then defines the *zero locus* Z_G of G as the set of all $[\lambda] \in \mathcal{PML}(S)$ such that there exists $[\lambda'] \in \Lambda_G$ satisfying $i(\lambda, \lambda') = 0$. Then Z_G is a closed G-invariant set containing Λ_G. Put $\Delta_G = \mathcal{PML}(S) \setminus Z_G$. McCarthy and Papadopoulos prove [35] that G acts properly discontinuously on Δ_G. Moreover, G acts properly discontinuously on $\widehat{\Delta}_G := \mathcal{T}(S) \cup \Delta_G$, where $\mathcal{T}(S)$ is the Teichmüller space of S [30].

In the free group context, the most frequently used analogue of Teichmüller space is given by the above mentioned Outer space. Let F_N be a free group of finite rank $N \geq 2$. The (unprojectivized) Outer space cv_N consists of all minimal free and discrete isometric actions of F_N on $\mathbb{R}$-trees. The projectivized Outer space $\mathrm{CV}_N = \mathbb{P}\,\mathrm{cv}_N$ consists of equivalence classes of points from cv_N up to homothety. The closure $\overline{\mathrm{cv}}_N$ of cv_N in the equivariant Gromov–Hausdorff convergence topology consists precisely of all the *very small* minimal isometric actions of F_N on $\mathbb{R}$-trees, considered up to F_N-equivariant isometry. The projectivization $\overline{\mathrm{CV}}_N = \mathbb{P}\overline{\mathrm{cv}}_N$ of $\overline{\mathrm{cv}}_N$ is compact and contains CV_N as a dense subset. The space $\overline{\mathrm{CV}}_N$ is an analogue of the Thurston compactification of the Teichmüller space. The difference $\partial\,\mathrm{CV}_N := \overline{\mathrm{CV}}_N \setminus \mathrm{CV}_N$ is called the *boundary of the Outer space* CV_N. All of the above spaces come equipped with natural $\mathrm{Out}(F_N)$-actions, see Section 2 for more details and further references.

There is a companion space for cv_N consisting of *geodesic currents* on F_N. A geodesic current on F_N is a positive Radon measure μ on $\partial^2 F_N$ (i.e., the complement of the diagonal in $\partial F_N \times \partial F_N$) that is invariant under the natural F_N-translation action and under the "flip" map interchanging the two coordinates of $\partial^2 F_N$. Motivated by the work of Bonahon about geodesic currents on hyperbolic surfaces [5]–[7], geodesic currents on free groups have been introduced in a 1995 dissertation of Reiner Martin [33]. The notion was recently reintroduced in the work of Kapovich [20, 21] and the theory of geodesic currents on free groups has been developed in the work of Kapovich and Lustig [23]–[26], Kapovich [22], Kapovich and Nagnibeda [28], Francaviglia [15], Coulbois, Hilion and Lustig [12] and others (in particular see [4], [19] for recent applications). The space $\mathrm{Curr}(F_N)$ of all geodesic currents is locally compact and comes equipped with a natural continuous action of $\mathrm{Out}(F_N)$ by linear transformations. There is a projectivization $\mathbb{P}\,\mathrm{Curr}(F_N)$ of $\mathrm{Curr}(F_N)$ that consists of projective classes $[\mu]$ of nonzero geodesic currents μ, where two such currents are in the same projective class if they are positive scalar multiples of each other. The space $\mathbb{P}\,\mathrm{Curr}(F_N)$ is compact and inherits a natural $\mathrm{Out}(F_N)$-action. A crucial tool in this theory is the notion of a continuous *geometric intersection form* $\langle\cdot,\cdot\rangle : \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to \mathbb{R}$ that was constructed by the authors in [24]. This intersection form has some key properties in common with Bonahon's notion of a geometric intersection number between two geodesic currents on a surface, see Proposition 2.1 below for a precise formulation.

In order to construct domains of discontinuity for subgroups of $\mathrm{Out}(F_N)$ in $\partial\,\mathrm{CV}_N$, it turns out to be necessary to "undualize" the picture and to play the spaces $\overline{\mathrm{CV}}_N$ and $\mathbb{P}\,\mathrm{Curr}(F_N)$ off each other. We say that a subgroup $G \subseteq \mathrm{Out}(F_N)$ is *dynamically large* if it contains an atoroidal iwip (irreducible with irreducible powers) element, see Definition 3.1 for a precise definition of iwips. Being an atoroidal iwip element of $\mathrm{Out}(F_N)$ is the strongest free group analogue of being a pseudo-Anosov mapping class. Atoroidal iwips act with

“North–South” dynamics both on $\overline{\mathrm{CV}}_N$ and $\mathbb{P}\mathrm{Curr}(F_N)$ (see Section 3 for precise statements). This fact allows us to define in Section 4, for a dynamically large $G \subseteq \mathrm{Out}(F_N)$, its *limit sets* $\Lambda_G^{\mathrm{cv}} \subseteq \overline{\mathrm{CV}}_N$ and $\Lambda_G^{\mathrm{curr}} \subseteq \mathbb{P}\mathrm{Curr}(F_N)$. When G is not virtually cyclic, these limit sets are exactly the unique minimal closed G-invariant subsets of $\overline{\mathrm{CV}}_N$ and $\mathbb{P}\mathrm{Curr}(F_N)$ accordingly. We then define the *zero sets* of G:

$$Z_G^{\mathrm{cv}} = \{[T] \in \overline{\mathrm{CV}}_N \mid \langle T, \mu\rangle = 0 \text{ for some } [\mu] \in \Lambda_G^{\mathrm{curr}}\} \subseteq \overline{\mathrm{CV}}_N$$

and

$$Z_G^{\mathrm{curr}} = \{[\mu] \in \mathbb{P}\mathrm{Curr}(F_N) \mid \langle T, \mu\rangle = 0 \text{ for some } [T] \in \Lambda_G^{\mathrm{cv}}\} \subseteq \overline{\mathrm{CV}}_N.$$

The zero sets are closed, G-invariant and contain the corresponding limit sets. We put $\widehat{\Delta}_G^{\mathrm{cv}} = \overline{\mathrm{CV}}_N \setminus Z_G^{\mathrm{cv}}$ and $\widehat{\Delta}_G^{\mathrm{curr}} = \mathbb{P}\mathrm{Curr}(F_N) \setminus Z_G^{\mathrm{curr}}$, so that $\widehat{\Delta}_G^{\mathrm{cv}}$ and $\widehat{\Delta}_G^{\mathrm{curr}}$ are open G-invariant sets.

Thus, Theorem 1.1 can be viewed as a free group analogue of the result of McCarthy and Papadopoulos [35] for subgroups of $\mathrm{Mod}(S)$, mentioned above.

For a dynamically large subgroup $G \subseteq \mathrm{Out}(F_N)$, it is in general rather difficult to decide which points of $\partial \mathrm{CV}_N$ belong to the limit set Λ_G^{cv}, and which points of $\mathbb{P}\mathrm{Curr}(F_N)$ belong to $\Lambda_G^{\mathrm{curr}}$. For the larger zero sets Z_G^{cv} and Z_G^{curr}, this question is a little easier, as shown by the following result.

Proposition 1.2 (Proposition 5.14 below). *Let $G \subseteq \mathrm{Out}(F_N)$ be a dynamically large subgroup. Then the following hold*:

(1) *Let $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$ and let $[T_\infty]$ be a accumulation point of $[T]G$. Then $[T_\infty] \in Z_G^{\mathrm{cv}}$.*
(2) *Let $[\mu] \in \widehat{\Delta}_G^{\mathrm{curr}}$ and let $[\mu_\infty]$ be a accumulation point of $G[\mu]$. Then $[\mu_\infty] \in Z_G^{\mathrm{curr}}$.*

It is easy to see that for a dynamically large subgroup $G \subseteq \mathrm{Out}(F_N)$ we always have $\mathrm{CV}_N \subseteq \widehat{\Delta}_G^{\mathrm{cv}}$ because for any $[T] \in \mathrm{CV}_N$ and any $[\mu] \in \mathbb{P}\mathrm{Curr}(F_N)$ we have $\langle T, \mu\rangle > 0$. The main result of [25] implies a similar property for currents with full support: if $\mu \in \mathrm{Curr}(F_N)$ is such a current then for any $[T] \in \overline{\mathrm{CV}}_N$ we have $\langle T, \mu\rangle > 0$. As a consequence, we obtain the following application of Theorem 1.1, proved below as Corollary 6.7.

Corollary 1.3. *Let $N \geq 3$ and denote by $\mathbb{P}\mathrm{Curr}_+(F_N)$ the set of all $[\mu] \in \mathbb{P}\mathrm{Curr}(F_N)$ such that μ has full support. Then the action of* $\mathrm{Out}(F_N)$ *on* $\mathbb{P}\mathrm{Curr}_+(F_N)$ *is properly discontinuous.*

Examples of currents with full support include the *Patterson–Sullivan currents* corresponding to points of cv_N, see [28] for details.

The domains of discontinuity provided by Theorem 1.1 are not, in general, maximal possible. In [17], Guirardel constructed a nonempty open $\mathrm{Out}(F_N)$-invariant subset $\mathcal{O}_n \subseteq \partial \mathrm{CV}_N$ such that $\mathrm{Out}(F_N)$ acts properly discontinuously on $\mathcal{O}_n$. As explained in Remark 6.6 below, there are points in $\mathcal{O}_n$ which do not belong to our discontinuity domain $\widehat{\Delta}_{\mathrm{Out}(F_N)}^{\mathrm{cv}}$.

On the other hand, there is a natural class of examples where our domains of discontinuity are likely to be maximal possible. Namely, suppose $\varphi, \psi \in \mathrm{Out}(F_N)$ are atoroidal iwips such that the subgroup $\langle \varphi, \psi \rangle$ is not virtually cyclic. Then, as we proved in [26], there exist $n, m \geq 1$ such that $G = \langle \varphi^n, \psi^m \rangle$ is free of rank two and such that every nontrivial element of G is again an atoroidal iwip and such that $F_N \rtimes \langle \Phi^n, \Psi^m \rangle$ is word-hyperbolic (where $\Phi, \Psi \in \mathrm{Aut}(F_N)$ are representatives of φ, ψ). Based on the mapping class group analogy (where similar statements are known as a general part of the theory of convex-cocompact subgroups of mapping class groups, see [14]), we believe that in this case it should be true that $\Lambda_G^{\mathrm{cv}} = Z_G^{\mathrm{cv}}$. This would imply that $\widehat{\Delta}_G^{\mathrm{cv}} \subset \overline{\mathrm{CV}}_N$ is the maximal domain of discontinuity for G in this case. Moreover, again by analogy with the known mapping class group results for convex-cocompact subgroups [14], we expect that in this situation there is a natural homeomorphism between the hyperbolic boundary of the free group $\langle \varphi^n, \psi^m \rangle$ and the limit set Λ_G^{cv}. Furthermore, every $[T] \in \Lambda_G^{\mathrm{cv}}$ (or rather, the underlying topological dynamical system, that is, the "dual algebraic lamination" $L^2(T)$, see [11]) should be uniquely ergodic, in the following strong double meaning:

(a) Every tree $T' \in \partial\,\mathrm{cv}_N$ which arises from T by an equivariant change of the metric, without changing the topology of any finite subtree, defines the same point $[T'] = [T] \in \partial\,\mathrm{CV}_N$ ("T is uniquely ergometric", see [10]).

(b) There is a unique $[\mu] \in \mathbb{P}\,\mathrm{Curr}(F_N)$ such that $\langle T, \mu \rangle = 0$ ("T is dually uniquely ergodic", see [12], Section 5). If true, these statements would indicate that the "Schottky type" groups $G = \langle \varphi^n, \psi^m \rangle$ as above are good candidates for being examples of "convex-cocompact" subgroups of $\mathrm{Out}(F_N)$.

We are very grateful to Chris Leininger for explaining to us the proof of McCarthy–Papadopoulos and for many helpful conversations.

2. Outer space and the space of geodesic currents

We give here only a brief overview of basic facts related to Outer space and the space of geodesic currents. We refer the reader to [1, 2, 13, 16, 17, 21, 37] for more detailed background information.

2.1. Outer space. Let $N \geq 2$. The *unprojectivized Outer space* cv_N consists of all minimal free and discrete isometric actions on F_N on $\mathbb{R}$-trees (where two such actions are considered equal if there exists an F_N-equivariant isometry between the corresponding trees). There are several different topologies on cv_N that are known to coincide, in particular the equivariant Gromov–Hausdorff convergence topology and the so-called *length function* topology, [36]. Every $T \in \mathrm{cv}_N$ is uniquely determined by its *translation length function* $\|\cdot\|_T : F_N \to \mathbb{R}$, where $\|g\|_T$ is the translation length of g on T. Two trees $T_1, T_2 \in \mathrm{cv}_N$ are close if the functions $\|\cdot\|_{T_1}$ and $\|\cdot\|_{T_1}$ are close pointwise on a large ball in F_N. The closure $\overline{\mathrm{cv}}_N$ of cv_N in either of these two topologies is well-understood and known to consists precisely of all the so-called *very small* minimal isometric actions of F_N on $\mathbb{R}$-trees, see [2] and [9]. The Outer automorphism group $\mathrm{Out}(F_N)$ has a natural continuous right action on $\overline{\mathrm{cv}}_N$ (that leaves cv_N invariant) given at the level of length functions as follows: for $T \in \mathrm{cv}_N$ and $\varphi \in \mathrm{Out}(F_N)$ we have $\|g\|_{T\varphi} = \|\varphi(g)\|_T$, where $g \in F_N$. In terms of tree actions, $T\varphi$ is equal to T as a metric space, but the action of F_N is modified as: $g \underset{T\varphi}{\cdot} x = \Phi(g) \underset{T}{\cdot} x$ where $x \in T$, $g \in F_N$ are arbitrary and where $\Phi \in \mathrm{Aut}(F_N)$ is some fixed representative of the outer automorphism φ is $\mathrm{Aut}(F_N)$. The *projectivized Outer space* $\mathrm{CV}_N = \mathbb{P}\mathrm{cv}_N$ is defined as the quotient $\mathrm{cv}_N / \sim$ where for $T_1 \sim T_2$ whenever $T_2 = cT_1$ for some $c > 0$. One similarly defines the projectivization $\overline{\mathrm{CV}}_N = \mathbb{P}\overline{\mathrm{cv}}_N$ of $\overline{\mathrm{cv}}_N$ as $\overline{\mathrm{cv}}_N/\sim$ where $\sim$ is the same as above. The space $\overline{\mathrm{CV}}_N$ is compact and contains CV_N as a dense $\mathrm{Out}(F_N)$-invariant subset. The compactification $\overline{\mathrm{CV}}_N$ of CV_N is a free group analogue of the Thurston compactification of Teichmüller space. For $T \in \overline{\mathrm{cv}}_N$ its $\sim$-equivalence class is denoted by $[T]$, so that $[T]$ is the image of T in $\overline{\mathrm{CV}}_N$.

2.2. Geodesic currents. Let $\partial^2 F_N := \{(\xi, \zeta) \mid \xi, \zeta \in \partial F_N, \xi \neq \zeta\}$. The action of F_N by translations on its hyperbolic boundary ∂F_N defines a natural diagonal action of F_N on $\partial^2 F_N$. A *geodesic current* on F_N is a positive Radon measure on $\partial^2 F_N$ that is F_N-invariant and is also invariant under the "flip" map $\partial^2 F_N \to \partial^2 F_N$, $(\xi, \zeta) \mapsto (\zeta, \xi)$. The space $\mathrm{Curr}(F_N)$ of all geodesic currents on F_N has a natural $\mathbb{R}_{\geq 0}$-linear structure and is equipped with the weak*-topology of pointwise convergence on continuous functions.

Every point $T \in \mathrm{cv}_N$ defines a *simplicial chart* on $\mathrm{Curr}(F_N)$ which allows one to think about geodesic currents as systems of nonnegative weights satisfying certain Kirchhoff-type equations; see [21] for details. We briefly recall the simplicial chart construction for the case where $T_A \in \mathrm{cv}_N$ is the Cayley tree corresponding to a free basis A of F_N. For a nondegenerate geodesic segment $\gamma = [p, q]$ in T_A the *two-sided cylinder* $\mathrm{Cyl}_A(\gamma) \subseteq \partial^2 F_N$ consists of all $(\xi, \zeta) \in \partial^2 F_N$ such that the geodesic from ξ to ζ in T_A passes through $\gamma = [p, q]$. Given a nontrivial freely reduced word $v \in F(A) = F_N$ and a current $\mu \in \mathrm{Curr}(F_N)$, the "weight" $\langle v, \mu \rangle_A$ is defined as $\mu(\mathrm{Cyl}_A(\gamma))$ where γ is any segment in the Cayley graph T_A labelled by v (the fact that the measure μ is F_N-invariant implies that a particular choice of γ does not matter). A current μ is uniquely determined by a family of weights $(\langle v, \mu \rangle_A)_{v \in F_N \setminus \{1\}}$. The weak*-topology on $\mathrm{Curr}(F_N)$ corresponds to pointwise convergence of the weights for every $v \in F_N, v \neq 1$.

There is a natural left action of $\mathrm{Out}(F_N)$ on $\mathrm{Curr}(F_N)$ by continuous linear transformations. Specifically, let $\mu \in \mathrm{Curr}(F_N)$, $\varphi \in \mathrm{Out}(F_N)$ and let $\Phi \in \mathrm{Aut}(F_N)$ be a representative of φ in $\mathrm{Aut}(F_N)$. Since Φ is a quasi-isometry of F_N, it extends to a homeomorphism of ∂F_N and, diagonally, defines a homeomorphism of $\partial^2 F_N$. The measure $\varphi\mu$ on $\partial^2 F_N$ is defined as follows. For a Borel subset $S \subseteq \partial^2 F_N$, we have $(\varphi\mu)(S) := \mu(\Phi^{-1}(S))$. One then checks that $\varphi\mu$ is a current and that it does not depend on the choice of a representative Φ of φ. Note, however, that in general one has $\langle v, \varphi\mu\rangle_A \neq \langle \Phi(v), \mu\rangle_A$.

A current μ is said to have *of full support* if the Radon measure μ on $\partial^2 F_N$ has all of $\partial^2 F_N$ as support. This is equivalent to stating that for any basis A of F_N one has $\langle v, \mu\rangle_A > 0$ for all $v \in F_N \setminus \{1\}$. We denote by $\mathrm{Curr}_+(F_N)$ the set of all currents with full support. Then $\mathrm{Curr}_+(F_N)$ is clearly an $\mathrm{Out}(F_N)$-invariant open subspace (a subcone) of $\mathrm{Curr}(F_N)$.

For every $g \in F_N, g \neq 1$, there is an associated *counting current* $\eta_g \in \mathrm{Curr}(F_N)$. If A is a free basis of F_N and the conjugacy class $[g]$ of g is realized by a "cyclic word" W (that is a cyclically reduced word in $F(A)$ written on a circle with no specified base-vertex), then for every nontrivial freely reduced word $v \in F(A) = F_N$ the weight $\langle v, \eta_g\rangle_A$ is equal to the total number of occurrences of $v^{\pm 1}$ in W (where an occurrence of v in W is a vertex on W such that we can read v in W clockwise without going off the circle). We refer the reader to [21] for a detailed exposition on the topic. By construction, the counting current η_g depends only on the conjugacy class $[g]$ of g and it also satisfies $\eta_g = \eta_{g^{-1}}$. One can check [21] that for $\Phi \in \mathrm{Aut}(F_N)$ representing $\varphi \in \mathrm{Out}(F_N)$ and any $g \in F_N \setminus \{1\}$ we have $\varphi\eta_g = \eta_{\Phi(g)}$. Scalar multiples $c\eta_g \in \mathrm{Curr}(F_N)$, where $c \geq 0$, $g \in F_N, g \neq 1$, are called *rational currents.* A key fact about $\mathrm{Curr}(F_N)$ states that the set $\mathrm{Curr}_{rat}(F_N)$ of all rational currents is dense in $\mathrm{Curr}(F_N)$ (see, for example, [21]).

The *space of projectivized geodesic currents* is defined as $\mathbb{P}\mathrm{Curr}(F_N) = (\mathrm{Curr}(F_N) \setminus \{0\})/\sim$ where $\mu_1 \sim \mu_2$ whenever there exists $c > 0$ such that $\mu_2 = c\mu_1$. The $\sim$-equivalence class of $\mu \in \mathrm{Curr}(F_N) \setminus \{0\}$ is denoted by $[\mu]$. The action of $\mathrm{Out}(F_N)$ on $\mathrm{Curr}(F_N)$ descends to a continuous action of $\mathrm{Out}(F_N)$ on $\mathbb{P}\mathrm{Curr}(F_N)$. The space $\mathbb{P}\mathrm{Curr}(F_N)$ is compact and the set $\{[\eta_g] \mid g \in F_N, g \neq 1\}$ is a dense subset of it.

2.3. Intersection form. In [24], we constructed a natural geometric *intersection form* which pairs trees and currents.

Proposition 2.1. [24] *Let $N \geq 2$. There exists a unique continuous map*

$$\langle \cdot, \cdot \rangle : \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to \mathbb{R}_{\geq 0}$$

with the following properties:

(1) $\langle T, c_1\mu_1 + c_2\mu_2\rangle = c_1\langle T, \mu_1\rangle + c_2\langle T, \mu_2\rangle$ *for any* $T \in \overline{\mathrm{cv}}_N$, $\mu_1, \mu_2 \in \mathrm{Curr}(F_N)$, $c_1, c_2 \geq 0$.
(2) $\langle cT, \mu\rangle = c\langle T, \mu\rangle$ *for any* $T \in \overline{\mathrm{cv}}_N$, $\mu \in \mathrm{Curr}(F_N)$ *and* $c \geq 0$.
(3) $\langle T\varphi, \mu\rangle = \langle T, \varphi\mu\rangle$ *for any* $T \in \overline{\mathrm{cv}}_N$, $\mu \in \mathrm{Curr}(F_N)$ *and* $\varphi \in \mathrm{Out}(F_N)$.
(4) $\langle T, \eta_g\rangle = \|g\|_T$ *for any* $T \in \overline{cv}_N$ *and* $g \in F_N, g \neq 1$.
(5) $\langle T_A, \mu\rangle = \sum_{a_i \in A} \langle a_i, \mu\rangle_A$ *for any basis* A *of* F_N *and* $\mu \in \mathrm{Curr}(F_N)$.

3. North–South dynamics for atoroidal iwips

DEFINITION 3.1. An element $\varphi \in \mathrm{Out}(F_N)$ is *reducible* if there exists a free product decomposition $F_N = C_1 * \cdots * C_k * F'$, where $k \geq 1$ and $C_i \neq \{1\}$, such that φ permutes the conjugacy classes of subgroups $C_1, \ldots, C_k$ in F_N. An element $\varphi \in \mathrm{Out}(F_N)$ is called *irreducible* if it is not reducible. An element $\varphi \in \mathrm{Out}(F_N)$ is said to be *irreducible with irreducible powers* or an *iwip* for short, if for every $n \geq 1$ φ^n is irreducible (sometimes such automorphisms are also called *fully irreducible*).

REMARK 3.2. It is easy to see that $\varphi \in \mathrm{Out}(F_N)$ is an iwip if and only if no positive power of φ preserves the conjugacy class of a proper free factor of F_N. Recall also that $\varphi \in \mathrm{Out}(F_N)$ is called *atoroidal* if it has no nontrivial periodic conjugacy classes, that is, if there do not exist $n \geq 1$ and $g \in F_N \setminus \{1\}$ such that φ^n fixes the conjugacy class $[g]$ of g in F_N.

By a result of Brinkmann [8], for $\varphi \in \mathrm{Out}(F_N)$ the condition of being atoroidal is equivalent to being *hyperbolic*, that is such that the mapping torus group $F_N \rtimes_\Phi \mathbb{Z}$ is word-hyperbolic, where $\Phi \in \mathrm{Aut}(F_N)$ is some (equivalently, any) representative of φ. Further, in the case where $\varphi \in \mathrm{Out}(F_N)$ is an iwip, being atoroidal is also equivalent to "not induced by any surface homeomorphism" (see [3]).

The following result is due to Reiner Martin [33].

PROPOSITION 3.3. *Let $\varphi \in \mathrm{Out}(F_N)$ be a atoroidal iwip. Then there exist unique $[\mu_+], [\mu_-] \in \mathbb{P}\mathrm{Curr}(F_N)$ with the following properties*:

(1) *The elements $[\mu_+], [\mu_-] \in \mathbb{P}\mathrm{Curr}(F_N)$ are the only fixed points of φ in $\mathbb{P}\mathrm{Curr}(F_N)$.*
(2) *For any $[\mu] \neq [\mu_-]$ we have $\lim_{n\to\infty} \varphi^n[\mu] = [\mu_+]$ and for any $[\mu] \neq [\mu_+]$ we have $\lim_{n\to\infty} \varphi^{-n}[\mu] = [\mu_-]$.*
(3) *We have $\varphi\mu_+ = \lambda_+\mu_+$ and $\varphi^{-1}\mu_- = \lambda_-\mu_-$ where $\lambda_+ > 1$ and $\lambda_- > 1$. Moreover λ_+ is the Perron–Frobenius eigenvalue of any train-track representative of φ and λ_- is the Perron–Frobenius eigenvalue of any train-track representative of φ^{-1}.*

A similar statement is known for $\overline{\mathrm{CV}}_N$ by a result of Levitt and Lustig [32].

PROPOSITION 3.4. *Let $\varphi \in \mathrm{Out}(F_N)$ be an iwip. Then there exist unique $[T_+], [T_-] \in \overline{\mathrm{CV}}_N$ with the following properties*:

(1) *The elements $[T_+], [T_-] \in \overline{\mathrm{CV}}_N$ are the only fixed points of φ in $\overline{\mathrm{CV}}_N$.*
(2) *For any $[T] \in \overline{\mathrm{CV}}_N$, $[T] \neq [T_-]$ we have $\lim_{n\to\infty}[T\varphi^n] = [T_+]$ and for any $[T] \in \overline{\mathrm{CV}}_N$, $[T] \neq [T_+]$ we have $\lim_{n\to\infty}[T\varphi^{-n}] = [T_-]$.*
(3) *We have $T_+\varphi = \lambda_+ T$ and $T_-\varphi^{-1} = \lambda_- T_-$ where $\lambda_+ > 1$ and $\lambda_- > 1$. Moreover λ_+ is the Perron–Frobenius eigenvalue of any train-track representative of φ and λ_- is the Perron–Frobenius eigenvalue of any train-track representative of φ^{-1}.*

The following statements have been proved in [25, 26].

PROPOSITION 3.5. *Let* $\varphi \in \mathrm{Out}(F_N)$ *be an iwip and let* $[T_+],[T_-] \in \overline{\mathrm{CV}}_N$ *and* $[\mu_+],[\mu_-] \in \mathbb{P}\mathrm{Curr}(F_N)$ *be as in Proposition* 3.4 *and Proposition* 3.3 *accordingly. Then the following hold*:

(1) *For* $T \in \overline{\mathrm{cv}}_N$ *we have* $\langle T,\mu_+\rangle = 0$ *if and only if* $[T] = [T_-]$ *and we have* $\langle T,\mu_-\rangle = 0$ *if and only if* $[T] = [T_+]$.
(2) *For* $\mu \in \mathrm{Curr}(F_N)$, $\mu \neq 0$ *we have* $\langle T_+,\mu\rangle = 0$ *if and only if* $[\mu] = [\mu_-]$ *and we have* $\langle T_-,\mu\rangle = 0$ *if and only if* $[\mu] = [\mu_+]$.
(3) *We have* $\langle T_+,\mu_+\rangle > 0$ *and* $\langle T_-,\mu_-\rangle > 0$.
(4) *We have*

$$\begin{aligned}\mathrm{Stab}_{\mathrm{Out}(F_N)}([T_+]) &= \mathrm{Stab}_{\mathrm{Out}(F_N)}([T_-])\\ &= \mathrm{Stab}_{\mathrm{Out}(F_N)}([\mu_+]) = \mathrm{Stab}_{\mathrm{Out}(F_N)}([\mu_-])\end{aligned}$$

and this stabilizer is virtually cyclic.

We also need the following fact [26, 27].

PROPOSITION 3.6. *Let* $G \subseteq \mathrm{Out}(F_N)$ *be a subgroup and such that there exist an atoroidal iwip* $\varphi \in G$. *Let* $[T_+(\varphi)],[T_-(\varphi)] \in \overline{\mathrm{CV}}_N$, $[\mu_+(\varphi)],[\mu_-(\varphi)] \in \mathbb{P}\mathrm{Curr}(F_N)$ *be the attracting and repelling fixed points of* φ *in* $\overline{\mathrm{CV}}_N$ *and* $\mathbb{P}\mathrm{Curr}(F_N)$ *accordingly. Then exactly one of the following occurs*:

(1) *The group* G *is virtually cyclic and preserves the sets* $\{[T_+(\varphi)],[T_-(\varphi)]\} \subseteq \overline{\mathrm{CV}}_N$, $\{[\mu_+(\varphi)],[\mu_-(\varphi)]\} \subseteq \mathbb{P}\mathrm{Curr}(F_N)$.
(2) *The group* G *contains an atoroidal iwip* $\psi = g\varphi g^{-1}$ *for some* $g \in G$ *such that* $\{[T_+(\varphi)],[T_-(\varphi)]\} \cap \{[T_+(\psi)],[T_-(\psi)]\} = \emptyset$ *and* $\{[\mu_+(\varphi)],[\mu_-(\varphi)]\} \cap \{[\mu_+(\psi)],[\mu_-(\psi)]\} = \emptyset$. *In this case there are some* $m,n \geq 1$ *such that* $\langle \varphi^m,\psi^n\rangle \leq \mathrm{Out}(F_N)$ *is free of rank two and*, *moreover*, *every nontrivial element of* $\langle \varphi^m,\psi^n\rangle$ *is an atoroidal iwip.*

4. Dynamically large subgroups and their associated invariant sets

DEFINITION 4.1 (*Dynamically large*). We say that a subgroup $G \subseteq \mathrm{Out}(F_N)$ is *dynamically large* if there exist atoroidal iwip $\varphi \in G$. We will say that a dynamically large subgroup $G \subseteq \mathrm{Out}(F_N)$ is *elementary* if it is virtually cyclic and that it is *nonelementary* otherwise. Thus, by Proposition 3.6, a nonelementary dynamically large subgroup contains a free subgroup of rank two.

PROPOSITION–DEFINITION 4.2 (Limit set for a nonelementary dynamically large subgroup). *Let* $G \subseteq \mathrm{Out}(F_N)$ *be a nonelementary dynamically large subgroup. Then the following hold*:

(1) *There exists a unique minimal nonempty closed* G*-invariant subset* Λ_G^{cv} *of* $\overline{\mathrm{CV}}_N$. *Moreover*, $\Lambda_G^{\mathrm{cv}} \subseteq \partial\,\mathrm{CV}_N$ *and every* G*-orbit of a point of* Λ_G^{cv} *is dense in* Λ_G^{cv}. *We call* Λ_G^{cv} *the* limit set *of* G *in* $\overline{\mathrm{CV}}_N$.

(2) *There exists a unique minimal nonempty closed G-invariant subset $\Lambda_G^{\mathrm{curr}}$ of $\mathbb{P}\,\mathrm{Curr}(F_N)$. Moreover, every G-orbit of a point of $\Lambda_G^{\mathrm{curr}}$ is dense in $\Lambda_G^{\mathrm{curr}}$. We call $\Lambda_G^{\mathrm{curr}}$ the* limit set *of G in $\mathbb{P}\,\mathrm{Curr}(F_N)$.*

Proof. We will only prove part (1) of the proposition as the proof of (2) is completely analogous.

By Propositions 3.5 and 3.6, there exist atoriodal iwips $\varphi, \psi \in G$, such that $[T_\pm(\varphi)], [T_\pm(\psi)]$ are four distinct points and $[\mu_\pm(\varphi)], [\mu_\pm(\psi)]$ are four distinct points.

Put Λ_G^{cv} to be the closure in $\overline{\mathrm{CV}}_N$ of the orbit $[T_+(\varphi)]G$. Thus, Λ_G^{cv} is a closed G-invariant subset. Let $X \subseteq \overline{\mathrm{CV}}_N$ be a nonempty closed invariant subset. Note that X must contain a point $[T] \neq [T_\pm(\varphi)]$ since G is nonelementary and $\psi \in G$ does not leave invariant a nonempty subset of $\{[T_\pm(\varphi)]\}$. Then $\lim_{n\to\infty} \varphi^n[T] = [T_+(\varphi)]$ and hence $[T_+\varphi] \in X$. Since X is closed and G-invariant, it follows that $\Lambda_G^{\mathrm{cv}} = \overline{[T_+(\varphi)]G} \subseteq X$. Thus, $\Lambda_G^{\mathrm{cv}} = \overline{[T_+(\varphi)]G}$ is the unique minimal nonempty closed G-invariant subset of $\overline{\mathrm{CV}}_N$. It follows that the G-orbit of every point of Λ_G^{cv} is dense in Λ_G^{cv}, since the closure of such an orbit is a closed G-invariant set and thus must contain Λ_G^{cv}. $\square$

Definition 4.3 (*Limit set of an elementary dynamically large subgroup*). Let $G \subseteq \mathrm{Out}(F_N)$ be an elementary dynamically large subgroup and let $\varphi \in G$ be an atoroidal iwip. We put $\Lambda_G^{\mathrm{cv}} := \{[T_+(\varphi)], [T_-(\varphi)]\}$ and call it the *limit set* of G in $\overline{\mathrm{CV}}_N$. Similarly, we put $\Lambda_G^{\mathrm{curr}} := \{[\mu_+(\varphi)], [\mu_-(\varphi)]\}$ and call it the *limit set* of G in $\mathbb{P}\,\mathrm{Curr}(F_N)$.

Definition 4.4 (*Zero sets*). Let $G \subseteq \mathrm{Out}(F_N)$ be a dynamically large subgroup.

(1) Put

$$Z_G^{\mathrm{cv}} = \{[T] \in \overline{\mathrm{CV}}_N : \langle T, \mu\rangle = 0 \text{ for some } [\mu] \in \Lambda_G^{\mathrm{curr}}\}.$$

(2) Put

$$Z_G^{\mathrm{curr}} = \{[\mu] \in \mathbb{P}\,\mathrm{Curr}(F_N) : \langle T, \mu\rangle = 0 \text{ for some } [T] \in \Lambda_G^{\mathrm{cv}}\}.$$

The following is an immediate corollary of the definitions and of Proposition 3.5.

Proposition 4.5. *Let $G \subseteq \mathrm{Out}(F_N)$ be an elementary dynamically large subgroup and let $\varphi \in G$ be an atoroidal iwip. Then $Z_G^{\mathrm{cv}} = \Lambda_G^{\mathrm{cv}} = \{[T_\pm(\varphi)]\}$ and $Z_G^{\mathrm{curr}} = \Lambda_G^{\mathrm{curr}} = \{[\mu_\pm(\varphi)]\}$.*

Remark 4.6. Let $G \subseteq \mathrm{Out}(F_N)$ and let $\varphi \in G$ be an atoroidal iwip. Then $[T_\pm(\varphi)] \in \Lambda_G^{\mathrm{cv}}$ and $[\mu_\pm(\varphi)] \in \Lambda_G^{\mathrm{curr}}$. Indeed, if G is elementary, this follows from the definitions. If G is nonelementary, then the proof of Proposition–Definition 4.2 shows that there exists $[T] \in \Lambda_G^{\mathrm{cv}}$ such that $[T] \neq [T_\pm(\varphi)]$. Then $\lim_{n\to\pm\infty} [T]\varphi^n = [T_\pm] \in \Lambda_G^{\mathrm{cv}}$. The argument for $[\mu_\pm]$ is the same.

From Remark 4.6, from the fact [Proposition 3.5(1) or (2)] that $\langle T_+(\varphi), \mu_-(\varphi)\rangle = \langle T_-(\varphi), \mu_+(\varphi)\rangle = 0$, and from the continuity of the intersection form we obtain directly the following preposition.

Proposition 4.7. *Let G be a dynamically large subgroup of* $\mathrm{Out}(F_N)$. *Then $Z_G^{\mathrm{cv}} \subseteq \overline{\mathrm{CV}}_N$ and $Z_G^{\mathrm{curr}} \subseteq \mathbb{P}\,\mathrm{Curr}(F_N)$ are closed G-invariant subsets and $\Lambda_G^{\mathrm{cv}} \subseteq Z_G^{\mathrm{cv}}$ and $\Lambda_G^{\mathrm{curr}} \subseteq Z_G^{\mathrm{curr}}$.*

Definition 4.8. Let $G \subseteq \mathrm{Out}(F_N)$ be a dynamically large subgroup. Put

$$\widehat{\Delta}_G^{\mathrm{cv}} := \overline{\mathrm{CV}}_N \setminus Z_G^{\mathrm{cv}}$$

and

$$\widehat{\Delta}_G^{\mathrm{curr}} := \mathbb{P}\,\mathrm{Curr}(F_N) \setminus Z_G^{\mathrm{curr}}.$$

Note that by construction $\widehat{\Delta}_G^{\mathrm{cv}}$ and $\widehat{\Delta}_G^{\mathrm{curr}}$ are open G-invariant subsets of $\overline{\mathrm{CV}}_N$ and of $\mathbb{P}\,\mathrm{Curr}(F_N)$ accordingly.

5. Dichotomy

Convention 5.1. Through this section, unless specified otherwise, let $G \subseteq \mathrm{Out}(F_N)$, where $N \geq 3$, be a dynamically large subgroup. Let $g \in G$ be an atoroidal iwip. Let $[T_+], [T_-] \in \overline{\mathrm{CV}}_N$, $[\mu_+], [\mu_-] \in \mathbb{P}\,\mathrm{Curr}(F_N)$ be the attracting and repelling fixed points of g in $\overline{\mathrm{CV}}_N$ and $\mathbb{P}\,\mathrm{Curr}(F_N)$ accordingly.

5.1. Basic dichotomy for trees.

Lemma 5.2. *Let $T \in \mathrm{cv}(F)$ and let $g_n \in \mathrm{Out}(F_N)$ be an infinite sequence of distinct elements. Let $T_\infty \in \overline{\mathrm{cv}}_N$, $c_n \geq 0$ be such that* $\lim_{n\to\infty} c_n T g_n \to T_\infty$ *in $\overline{\mathrm{cv}}_N$. Then* $\lim_{n\to\infty} c_n = 0$.

Proof. Recall that $\mathrm{CV}_N = \mathbb{P}\,\mathrm{cv}_N$ and $\overline{\mathrm{CV}}_N = \mathbb{P}\overline{\mathrm{cv}}_N$. Since $\lim_{n\to\infty}[T]g_n = [T_\infty]$ in $\overline{\mathrm{CV}}_N$ and since the action of $\mathrm{Out}(F_N)$ on CV_N is properly discontinuous, it follows that $[T_\infty] \in \partial\,\mathrm{CV}_N = \overline{\mathrm{CV}}_N \setminus \mathrm{CV}_N$. Therefore, F_N has nontrivial elements of arbitrarily small translation length with respect to the action on T_∞. On the other hand, the action of F_N on T is free and discrete and therefore there exists $C > 0$ such that for every $w \in F_N, w \neq 1$ and for every $n \geq 1$ we have $\|w\|_{Tg_n} = \|g_n(w)\|_T \geq C$. The statement of the lemma now follows from point-wise convergence of the translation length functions of Tg_n to that of T_∞. □

Corollary 5.3. *Let $g_n \in \mathrm{Out}(F_N)$ be an infinite sequence of distinct elements. Then there exists a conjugacy class $[w]$ in F_N such that the set of conjugacy classes $g_n[w]$ is infinite.*

Proof. Let $T \in \mathrm{cv}_N$ be arbitrary. Choose a limit point $[T_\infty]$ of $[T]g_n$ in $\overline{\mathrm{CV}}_N$. Then, after passing to a subsequence, we have $\lim_{n\to\infty} c_n T g_n = T_\infty$. Choose $w \in F_N$ such that $\|w\|_{T_\infty} > 0$. Thus $\lim_{n\to\infty} c_n \|g_n(w)\|_T = \|w\|_{T_\infty} > 0$. Since by Lemma 5.2, we have $\lim_{n\to\infty} c_n = 0$, it follows that $\lim_{n\to\infty} \|g_n(w)\|_T = \infty$, so that the sequence of conjugacy classes $g_n[w]$ contains infinitely many distinct elements. $\square$

LEMMA 5.4. *Let $T \in \mathrm{cv}_N$ and $\mu \in \mathrm{Curr}(F_N)$. Let $g_n \in \mathrm{Out}(F_N)$ be an infinite sequence of distinct elements. Then one of the following holds*:

(1) *The sequence $\langle T, g_n\mu\rangle$ is unbounded.*
(2) *For every accumulation point $[T_\infty]$ of the $[T]g_n$, we have $\langle T_\infty, \mu\rangle = 0$.*

Proof. Suppose that (2) fails and there exists a limit point $[T_\infty]$ of $[T]g_n$ such that $\langle T_\infty, \mu\rangle > 0$. After passing to a subsequence g_{n_i} of g_n, we have $T_\infty = \lim_{i\to\infty} c_i T g_{n_i}$ for some $c_i \ge 0$. Then

$$\langle T_\infty, \mu\rangle = \lim_{i\to\infty} \langle c_i T g_{n_i}, \mu\rangle = \lim_{i\to\infty} c_i \langle T, g_{n_i}\mu\rangle.$$

Since by Lemma 5.2 we have $\lim_{i\to\infty} c_i = 0$ and since $\langle T_\infty, \mu\rangle > 0$, it follows that $\lim_{i\to\infty} \langle T, g_{n_i}\mu\rangle = \infty$. Therefore, the sequence $\langle T, g_n\mu\rangle$ is unbounded, as required. $\square$

Recall that according to Convention 5.1 the element $g \in G$ is an atoroidal iwip and that $[T_\pm]$, $[\mu_\pm]$ are its fixed points in $\overline{\mathrm{CV}}_N$ and $\mathbb{P}\mathrm{Curr}(F_N)$.

The following fact is established in [25].

PROPOSITION 5.5. *Let $T_0 \in \mathrm{cv}_N$ be arbitrary. Then the functions $\|\cdot\|_{T_0}$ and $\|\cdot\|_{T_+} + \|\cdot\|_{T_-}$ on F_N are bi-Lipschitz equivalent.*

The following is an analogue of Lemma 5.2.

LEMMA 5.6. *Let $g_n \in \mathrm{Out}(F_N)$ be an infinite sequence of distinct elements, and assume that there exist coefficients $c_n^+, c_n^- \ge 0$ and trees $T_\infty^+, T_\infty^- \in \overline{\mathrm{cv}}_N$ such that $\lim_{n\to\infty} c_n^+ T_+ g_n = T_\infty^+$ and $\lim_{n\to\infty} c_n^- T_- g_n = T_\infty^-$.*

Then there exists an infinite subsequence of indices $n_i \in \mathbb{N}$ and some $* \in \{+,-\}$ which satisfies $\lim_{i\to\infty} c_{n_i}^* = 0$.

Proof. By Proposition 5.5, for any $T_0 \in \mathrm{cv}_N$ there exists $C > 0$ such that

$$\|\cdot\|_{T_+} + \|\cdot\|_{T_-} \ge C\|\cdot\|_{T_0} \quad \text{on } F_N.$$

Corollary 5.3 implies that there exists a nontrivial conjugacy class $[w]$ in F_N such that the set of conjugacy classes $g_n[w]$ is infinite, so that $\lim_{n\to\infty} \|g_n(w)\|_{T_0} = \infty$. Hence, from the above inequality, we conclude that there is an infinite subsequence g_{n_i} of the g_n such that for some $* \in \{+,-\}$ we have $\lim_{i\to\infty} \|g_{n_i}(w)\|_{T_*} = \infty$.

Since there exists a finite limit

$$\|w\|_{T_\infty^*} = \lim_{i\to\infty} c_{n_i}^* \|w\|_{T_* g_{n_i}} = \lim_{i\to\infty} c_{n_i}^* \|g_{n_i}(w)\|_{T_*},$$

it follows that $\lim_{i\to\infty} c^*_{n_i} = 0$. □

CONVENTION 5.7. For the remainder of this section, unless specified otherwise, we assume that $g_n \in G$ is a (fixed) infinite sequence of distinct elements of a dynamically large subgroup $G \subseteq \mathrm{Out}(F_N)$.

COROLLARY 5.8. *For any* $[\mu] \in \widehat{\Delta}^{\mathrm{curr}}_G$ *there is a tree* $T_* \in \{T_+, T_-\}$ *and an infinite subsequence* g_{n_i} *of the* g_n *such that* $\lim_{i\to\infty}\langle T_*, g_{n_i}\mu\rangle = \infty$. *In particular, the sequence* $\langle T_*, g_n\mu\rangle$ *is unbounded.*

Proof. By compactness of $\overline{\mathrm{CV}}_N$, we can assume that after passing to a subsequence the g_n satisfy the hypotheses of Lemma 5.6. Let $n_i \in \mathbb{N}$ be the infinite subsequence and let $* \in \{+,-\}$ be as provided by that lemma.

Suppose now, by contradiction, that for some $[\mu] \in \widehat{\Delta}^{\mathrm{curr}}_G$, we have $\lim_{n\to\infty}\langle T_*, g_{n_i}\mu\rangle \neq \infty$. Then there exists an infinite subsequence $g_{n_i'}$ of g_{n_i} such that the sequence $\langle T_*, g_{n_i'}\mu\rangle$ is bounded. By Lemma 5.6, we have $\lim_{i\to\infty} c^*_{n_i'} = 0$. Therefore, as in the proof of Lemma 5.4, either the sequence $\langle T_*, g_{n_i'}\mu\rangle$ is unbounded or $\langle T^*_\infty, \mu\rangle = 0$. The former is impossible since by assumption the sequence $\langle T_*, g_{n_i'}\mu\rangle$ is bounded. Thus, $\langle T^*_\infty, \mu\rangle = 0$. However, $[T_\pm] \in \Lambda^{\mathrm{cv}}_G$ by Remark 4.6 and hence $[T^*_\infty] \in \Lambda^{\mathrm{cv}}_G$. Therefore, the possibility that $\langle T^*_\infty, \mu\rangle = 0$ is ruled out by the assumption that $[\mu] \in \widehat{\Delta}^{\mathrm{curr}}_G$, yielding a contradiction. □

5.2. Basic dichotomy for currents. We now want to dualize the previous arguments for the case of geodesic currents. The difficulty here is that currents analogues of Lemmas 5.2 and 5.6 are not readily available. We get around this problem by using the results of the previous steps to obtain such analogues below.

LEMMA 5.9. *Let* $[\mu] \in \widehat{\Delta}^{\mathrm{curr}}_G$. *Suppose that there are* $c_n \geq 0$ *and* $\mu_\infty \in \mathrm{Curr}(F_N)$ *be such that* $\lim_{n\to\infty} c_n g_n \mu = \mu_\infty$. *Then* $\lim_{n\to\infty} c_n = 0$.

Proof. By contradiction, suppose that there is an infinite subsequence c_{n_k} of the c_n which are bounded below by some $c_0 > 0$. We apply Corollary 5.8 to the subsequence g_{n_k} of the g_n, to obtain a further infinite subsequence $g_{n_{k(i)}}$ of the g_{n_k} as well as a tree $T_* \in \{T_+, T_-\}$, which satisfy $\lim_{i\to\infty}\langle T_*, g_{n_{k(i)}}\mu\rangle = \infty$.

Now, the finiteness of

$$\langle T_*, \mu_\infty\rangle = \lim_{i\to\infty}\big\langle T_*, c_{n_{k(i)}} g_{n_{k(i)}}\mu\big\rangle = \lim_{i\to\infty} c_{n_{k(i)}}\big\langle T_*, g_{n_{k(i)}}\mu\big\rangle,$$

together with the equation $\lim_{i\to\infty}\langle T_*, g_{n_{k(i)}}\mu\rangle = \infty$ implies $\lim_{i\to\infty} c_{n_{k(i)}} = 0$, which contradicts our initial assumption on the c_{n_k}. □

We now obtain an analogue of Lemma 5.4 for currents, with the important difference however, that, as specified in Convention 5.7, here we do need to assume that $g_n \in G$, while the proof of Lemma 5.4 works for an arbitrary infinite sequence $g_n \in \mathrm{Out}(F_N)$. Similarly, we had to assume that $g_n \in G$ in

Lemma 5.9 while this assumption was not needed in Lemma 5.2. On the other hand, in Lemmas 5.2 and 5.4, we need the assumption $T \in \mathrm{cv}_N$, which is stronger than just demanding $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$.

LEMMA 5.10. *Let $T \in \overline{\mathrm{cv}}_N$ be arbitrary and let $[\mu] \in \widehat{\Delta}_G^{\mathrm{curr}}$. Then one of the following holds*:

(1) *The sequence $\langle Tg_n, \mu\rangle$ is unbounded.*
(2) *We have $\langle T, \mu_\infty\rangle = 0$ for any accumulation point $[\mu_\infty] \in \mathbb{P}\mathrm{Curr}(F_N)$ of the sequence $[g_n\mu]$.*

Proof. Suppose that (2) fails, so that there exist $c_i \geq 0$ and $\mu_\infty \in \mathrm{Curr}(F_N)$ such that, after possibly passing to a subsequence g_{n_i} of g_n, we have $\lim_{i\to\infty} c_i g_{n_i}\mu = \mu_\infty$ and $\langle T, \mu_\infty\rangle > 0$. Then

$$\langle T, \mu_\infty\rangle = \lim_{i\to\infty}\langle T, c_i g_{n_i}\mu\rangle = \lim_{i\to\infty} c_i\langle Tg_{n_i}, \mu\rangle.$$

Since by Lemma 5.9 we have $\lim_{i\to\infty} c_i = 0$, and since by assumption $\langle T, \mu_\infty\rangle > 0$, it follows that the sequence $\langle Tg_{n_i}, \mu\rangle$ is unbounded. Thus, alternative (1) holds. □

LEMMA 5.11. *Let $g_n \in G$ be an infinite sequence of distinct elements, and assume that there exist coefficients $c_n^+, c_n^- \geq 0$ and currents $\mu_\infty^+, \mu_\infty^- \in \mathrm{Curr}(F_N)$ such that $\lim_{n\to\infty} c_n^+ g_n\mu_+ = \mu_\infty^+$ and $\lim_{n\to\infty} c_n^- g_n\mu_- = \mu_\infty^-$.*

Then there exists an infinite subsequence of indices $n_i \in \mathbb{N}$ and some $* \in \{+,-\}$ which satisfies $\lim_{i\to\infty} c_{n_i}^* = 0$.

Proof. By way of contradiction, we assume that there is a constant $c_0 > 0$ which is a lower bound $c_0 < c_n^+$ and $c_0 < c_n^-$ for all indices $n \in \mathbb{N}$.

Let $T \in \mathrm{cv}_N$, and choose an accumulation point $[T_\infty]$ of the sequence of $[Tg_n]$. By Proposition 3.5, we cannot have both, $\langle T_\infty, \mu_+\rangle = 0$ and $\langle T_\infty, \mu_-\rangle = 0$. Thus, there is $* \in \{+,-\}$ with $\langle T_\infty, \mu_*\rangle > 0$.

Then by Lemma 5.4 the sequence $\langle T, g_n\mu_*\rangle$ is unbounded. Thus, for some infinite subsequence of g_{n_i} we have $\lim_{i\to\infty}\langle T, g_{n_i}\mu_*\rangle = \infty$. Now the finiteness of

$$\langle T, \mu_\infty^*\rangle = \lim_{i\to\infty}\langle T, c_{n_i}^* g_{n_i}\mu_*\rangle = \lim_{i\to\infty} c_{n_i}^*\langle T, g_{n_i}\mu_*\rangle$$

yields a contradiction, since $\lim_{i\to\infty}\langle T, g_{n_i}\mu_*\rangle = \infty$ and $c_{n_i}^* \geq c_0 > 0$ for every $i \geq 1$. □

COROLLARY 5.12. *For any $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$, there is a current $\mu_* \in \{\mu_+, \mu_-\}$ and an infinite subsequence g_{n_i} of the g_n such that $\lim_{i\to\infty}\langle Tg_{n_i}, \mu_*\rangle = \infty$. In particular, the sequence $\langle Tg_n, \mu_*\rangle$ is unbounded.*

Proof. The proof is a word-by-word dualization of the proof of Corollary 5.8, where Lemma 5.6 is replaced by Lemma 5.11 and Lemma 5.4 by Lemma 5.10. Since this scheme of proof will be used below more often, we carry it through in detail once more. By compactness of $\mathbb{P}\mathrm{Curr}(F_N)$ [replaces

$\overline{\mathrm{CV}}_N$], we can assume that after passing to a subsequence the g_n satisfy the hypotheses of Lemma 5.11 [replaces Lemma 5.6]. Let $n_i \in \mathbb{N}$ be the infinite subsequence and let $* \in \{+,-\}$ be as provided by that lemma.

Suppose now, by contradiction, that for some $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$ [replaces $[\mu] \in \widehat{\Delta}_G^{\mathrm{curr}}$], we have $\lim_{n\to\infty}\langle Tg_{n_i}, \mu_*\rangle \neq \infty$ [replaces $\lim_{n\to\infty}\langle T_*, g_{n_i}\mu\rangle \neq \infty$]. Then there exists an infinite subsequence $g_{n_i'}$ of g_{n_i} such that the sequence $\langle Tg_{n_i'}, \mu_*\rangle$ [replaces $\langle T_*, g_{n_i'}\mu\rangle$] is bounded. By Lemma 5.11 [replaces Lemma 5.6] we have $\lim_{i\to\infty} c^*_{n_i'} = 0$. Therefore, as in the proof of Lemma 5.10 [replaces Lemma 5.4], either the sequence $\langle Tg_{n_i'}, \mu_*\rangle$ [replaces $\langle T_*, g_{n_i'}\mu\rangle$] is unbounded or $\langle T, \mu^*_\infty\rangle = 0$ [replaces $\langle T^*_\infty, \mu\rangle = 0$]. The former is impossible since by assumption the sequence $\langle Tg_{n_i'}, \mu_*\rangle$ [replaces $\langle T_*, g_{n_i'}\mu\rangle$] is bounded. Thus, $\langle T, \mu^*_\infty\rangle = 0$ [replaces $\langle T^*_\infty, \mu\rangle = 0$]. However, $[\mu_\pm] \in \Lambda_G^{\mathrm{curr}}$ [replaces $[T_\pm] \in \Lambda_G^{\mathrm{cv}}$] by Remark 4.6 and hence $[\mu^*_\infty] \in \Lambda_G^{\mathrm{curr}}$ [replaces $[T^*_\infty] \in \Lambda_G^{\mathrm{cv}}$]. Therefore, the possibility that $\langle T, \mu^*_\infty\rangle = 0$ [replaces $\langle T^*_\infty, \mu\rangle = 0$] is ruled out by the assumption that $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$ [replaces $[\mu] \in \widehat{\Delta}_G^{\mathrm{curr}}$], yielding a contradiction. □

COROLLARY 5.13. *Let $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$. Suppose that there are $c_n \geq 0$ and $T_\infty \in \overline{\mathrm{cv}}_N$ be such that $\lim_{n\to\infty} c_n g_n T = T_\infty$. Then $\lim_{n\to\infty} c_n = 0$.*

Proof. Again the proof is a word-by-word dualization, of the proof of Lemma 5.9, with Corollary 5.8 replaced by Corollary 5.12. □

PROPOSITION 5.14. *Let $G \subseteq \mathrm{Out}(F_N)$ be a dynamically large subgroup. Then the following hold*:

(1) *Let $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$ and let $[T_\infty]$ be a accumulation point of $[T]G$. Then $[T_\infty] \in Z_G^{\mathrm{cv}}$.*
(2) *Let $[\mu] \in \widehat{\Delta}_G^{\mathrm{curr}}$ and let $[\mu_\infty]$ be a accumulation point of $G[\mu]$. Then $[\mu_\infty] \in Z_G^{\mathrm{curr}}$.*

Proof. (1) Let $[T] \in \widehat{\Delta}_G^{\mathrm{cv}}$ and let $[T_\infty]$ be a limit point of $[T]G$. Then there is an infinite sequence of distinct elements $g_n \in G$ as well as coefficients $c_n \geq 0$ such that $\lim_{n\to\infty} c_n T g_n = T_\infty$. By Corollary 5.13, we have $\lim_{i\to\infty} c_i = 0$.

We now consider the sequence $h_n = g_n^{-1}$, and by compactness of $\mathbb{P}\mathrm{Curr}(F_N)$ we can extract an infinite subsequence h_{n_i} with the property that for some coefficients $c^+_{n_i}, c^-_{n_i} > 0$ and some currents $\mu^+_\infty, \mu^-_\infty \in \mathrm{Curr}(F_N)$ one has both, $\lim_{i\to\infty} c^+_{n_i} h_{n_i}\mu_+ = \mu^+_\infty$ and $\lim_{n\to\infty} c^-_{n_i} h_{n_i}\mu_- = \mu^-_\infty$.

We now apply Lemma 5.11 and conclude that, after passing to a further subsequence, for some $* \in \{+,-\}$ we have $\lim_{i\to\infty} c^*_{n_i} = 0$.

Therefore,

$$\langle T_\infty, \mu_\infty\rangle = \lim_{i\to\infty} c_{n_i} c^*_{n_i}\langle Tg_{n_i}, g_{n_i}^{-1}\mu_*\rangle = \lim_{i\to\infty} c_{n_i} c^*_{n_i}\langle T, \mu_*\rangle = 0.$$

By construction, we have $[\mu_*] \in \{[\mu_+], [\mu_-]\} \subseteq \Lambda_G^{\mathrm{curr}}$ and hence $[\mu_\infty] = \lim_{i\to\infty} g_{n_i}^{-1}[\mu_*] \in \Lambda_G^{\mathrm{curr}}$. Therefore, by definition of the zero-set, we have $[T_\infty] \in Z_G^{\mathrm{cv}}$, as required.

(2) The proof of the second part of the proposition is again a word-by-word dualization of the proof of part (1), where the use of Corollary 5.13 and Lemma 5.11 has to be replaced by the use of Lemma 5.9 and Lemma 5.6, respectively. □

6. Domains of discontinuity

The presentation in this section follow rather closely the arguments given in Section 3.2 of the paper [30] by Kent and Leininger.

Convention 6.1. For the remainder of this section, let $G \subseteq \mathrm{Out}(F_N)$ be a dynamically large subgroup, let $\varphi \in G$ be an atoroidal iwip and let $[T_\pm]$, $[\mu_\pm]$ be the fixed points of φ in $\overline{\mathrm{CV}}_N$ and in $\mathbb{P}\mathrm{Curr}(F_N)$ accordingly.

For any tree $T \in \overline{\mathrm{cv}}_N$, we denote $\langle T, \mu_\pm\rangle = \langle T, \mu_+\rangle + \langle T, \mu_-\rangle$. Similarly, for $\mu \in \mathrm{Curr}(F_N)$, we denote $\langle T_\pm, \mu\rangle = \langle T_+, \mu\rangle + \langle T_-, \mu\rangle$.

Notation 6.2. Denote

$$D_G^{\mathrm{curr}} = \{[\mu] \in \widehat{\Delta}_G^{\mathrm{curr}} : \langle T_\pm, \mu\rangle \le \langle T_\pm, g\mu\rangle \text{ for every } g \in G\}$$

and

$$D_G^{\mathrm{cv}} = \{[T] \in \widehat{\Delta}_G^{\mathrm{cv}} : \langle T, \mu_\pm\rangle \le \langle Tg, \mu_\pm\rangle \text{ for every } g \in G\}.$$

Lemma 6.3. *We have $GD_G^{\mathrm{curr}} = \widehat{\Delta}_G^{\mathrm{curr}}$ and $D_G^{\mathrm{cv}} G = \widehat{\Delta}_G^{\mathrm{cv}}$.*

Proof. Let $[\mu] \in \widehat{\Delta}_G^{\mathrm{curr}}$. Then for any $C \ge 1$ the set

$$\{g \in G : \langle g\mu, T_\pm\rangle \le C\}$$

is finite by Corollary 5.8. Hence, there exists $g_0 \in G$ with $\langle g_0\mu, T_\pm\rangle = \min_{g\in G}\langle g\mu, T_\pm\rangle$, that is, $g_0[\mu] \in D_G^{\mathrm{curr}}$. Therefore, $GD_G^{\mathrm{curr}} = \widehat{\Delta}_G^{\mathrm{curr}}$, as required. The proof that $GD_G^{\mathrm{cv}} = \widehat{\Delta}_G^{\mathrm{cv}}$ is precisely the same, except that it has to be dualized (i.e., the role of trees and of currents are interchanged, as in the previous section), with the use of Corollary 5.8 replaced by Corollary 5.12. □

Lemma 6.4. *For any compact set $K \subseteq \widehat{\Delta}_G^{\mathrm{cv}}$ and for any compact set $K' \subseteq \widehat{\Delta}_G^{\mathrm{curr}}$, we have*

(1) *$\{g \in G : K \cap D_G^{\mathrm{cv}} g \ne \emptyset\}$ is finite, and*
(2) *$\{g \in G : K' \cap gD_G^{\mathrm{curr}} \ne \emptyset\}$ is finite.*

Proof. (1) Suppose that there exists an infinite sequence of distinct elements $g_n \in G$ such that $K \cap D_G^{\mathrm{cv}} g_n^{-1} \ne \emptyset$ for all $n \ge 1$. Then there is also a sequence $[T_n] \in K \cap D_G^{\mathrm{cv}} g_n^{-1}$. After passing to a subsequence, we may assume that $[T_n] \to [T_\infty]$ in $\overline{\mathrm{CV}}_N$. Moreover, after choosing the projective representatives of $[T_n]$ appropriately, we may assume that $T_n \to T_\infty$ as $n \to \infty$. Note that $[T_\infty] \in K$ since K is compact.

By Lemma 5.11, after passing to a further subsequence, there exist $\mu \in \{\mu_\pm\}$, $c_n \geq 0$ and $0 \neq \mu_\infty \in \mathrm{Curr}(F_N)$ such that $\lim_{n\to\infty} c_n g_n \mu = \mu_\infty$ and such that $\lim_{n\to\infty} c_n = 0$. We have

$$\begin{aligned}\langle T_\infty, \mu_\infty\rangle &= \lim_{n\to\infty} c_n \langle T_n, g_n\mu\rangle \leq \lim_{n\to\infty} c_n \langle T_n, g_n\mu_\pm\rangle = \lim_{n\to\infty} c_n \langle T_n g_n, \mu_\pm\rangle \\ &\overset{(*)}{\leq} \lim_{n\to\infty} c_n \langle T_n, \mu_\pm\rangle = \lim_{n\to\infty} c_n \lim_{n\to\infty} \langle T_n, \mu_\pm\rangle = 0 \cdot \langle T_\infty, \mu_\pm\rangle = 0,\end{aligned}$$

where the inequality marked by $(*)$ holds because $[T_n] \in D_G^{\mathrm{cv}} g_n^{-1}$ and thus $[T_n]g_n \in D_G^{\mathrm{cv}}$.

Thus, $\langle T_\infty, \mu_\infty\rangle = 0$. Recall that $[T_\infty] \in K \subseteq \widehat{\Delta}_G^{\mathrm{cv}}$. Also, $[\mu] \in \{[\mu_\pm]\}$ and hence $[\mu] \in \Lambda_G^{\mathrm{curr}}$ and therefore $[\mu_\infty] \in \Lambda_G^{\mathrm{curr}}$. The fact that $\langle T_\infty, \mu_\infty\rangle = 0$ now gives a contradiction with the definition of $\widehat{\Delta}_G^{\mathrm{cv}}$.

(2) The proof that $\{g \in G : K' \cap gD_G^{\mathrm{curr}} \neq \emptyset\}$ is finite is again an exact analog of part (1), where the use of Lemma 5.11 has to be replaced by the use of Lemma 5.6. □

In this paper, we use as definition of a "properly discontinuous group action" what is termed in the French literature "a proper group action", namely that any compact set meets only finitely many of its translates. For locally compact spaces, this is equivalent to the older definition from low-dimensional topology, namely that every point has a neighborhood which meets only finitely many of its translates.

Theorem 6.5. *Let $G \subseteq \mathrm{Out}(F_N)$ be a dynamically large subgroup. Then the actions of G on $\widehat{\Delta}_G^{\mathrm{cv}}$ and $\widehat{\Delta}_G^{\mathrm{curr}}$ are properly discontinuous.*

Proof. We will show that the action of G on $\widehat{\Delta}_G^{\mathrm{curr}}$ is properly discontinuous. The argument for $\widehat{\Delta}_G^{\mathrm{cv}}$ is obtained through word-by-word dualization.

Let $K' \subseteq \widehat{\Delta}_G^{\mathrm{curr}}$ be a compact subset and suppose that $\{g \in G \mid K' \cap gK' \neq \emptyset\}$ is infinite. Let $g_n \in G$ be an infinite sequence of distinct elements such that $K' \cap g_n K' \neq \emptyset$ for every $n \geq 1$.

By Lemma 6.4, there exists a finite collection $h_1, \ldots, h_t \in G$ such that

$$\{h_1, \ldots, h_t\} = \{g \in G \mid K' \cap gD_G^{\mathrm{curr}} \neq \emptyset\}.$$

Moreover, since by Lemma 6.3 we have $GD_G^{\mathrm{curr}} = \widehat{\Delta}_G^{\mathrm{curr}}$, it follows that $K' \subseteq \bigcup_{i=1,\ldots,t} h_i D_G^{\mathrm{curr}}$.

For every $n \geq 1$, there exists $[\mu_n] \in K' \cap g_n K'$ and hence $[\mu_n] \in K' \cap \bigcup_{i=1,\ldots,t} h_i D_G^{\mathrm{curr}}$. Hence, for every $n \geq 1$ there exists an index i_n with $1 \leq i_n \leq t$, such that $K' \cap g_n h_{i_n} D_G^{\mathrm{curr}} \neq \emptyset$. It follows that

$$\{g_n h_{i_n} \mid n \geq 1\} \subseteq \{h_1, \ldots, h_t\}$$

which yields a contradiction, since by assumption the sequence $(g_n)_{n\geq 1}$ consists of infinitely many distinct elements of G. □

REMARK 6.6. The domains of discontinuity constructed in Theorem 6.5 are not necessarily maximal. For instance, in the case of $G = \mathrm{Out}(F_N)$ Guirardel [17] exhibited an open G-invariant subset $\mathcal{O}_N \subset \overline{\mathrm{CV}}_N$ on which $\mathrm{Out}(F_N)$ acts properly discontinuously. In Guirardel's construction, one has $\mathrm{CV}_N \subseteq \mathcal{O}_N$ but $\mathrm{CV}_N \neq \mathcal{O}_N$. One can see directly that for every $[T] \in \mathcal{O}_N \setminus \mathrm{CV}_N$ (a so-called "Martian") we have $[T] \in Z^{\mathrm{cv}}_{\mathrm{Out}(F_N)}$ and hence $[T] \notin \widehat{\Delta}^{\mathrm{cv}}_G$: the reason is that every such T corresponds to a (nonfree) simplicial action of F_N with trivial edge stabilizers (plus some additional conditions on the quotient graph that are not relevant here), and therefore there exists a primitive element a of F_N such that a fixes a vertex in T. For every such a, we have $[\eta_a] \in \Lambda^{\mathrm{curr}}_{\mathrm{Out}(F_N)}$ (see [23]) and, by Proposition 2.1(4), $\langle T, \eta_a\rangle = \|a\|_T = 0$, so that indeed $[T] \in Z^{\mathrm{cv}}_{\mathrm{Out}(F_N)}$, as claimed.

Recall from Section 2.2 that $\mathrm{Curr}_+(F_N)$ denotes the set of all $\mu \in \mathrm{Curr}(F_N)$ with full support and let $\mathbb{P}\mathrm{Curr}_+(F_N) = \{[\mu] \mid \mu \in \mathrm{Curr}_+(F_N)\}$. Note that $\mathbb{P}\mathrm{Curr}_+(F_N)$ is an open $\mathrm{Out}(F_N)$-invariant subset of $\mathbb{P}\mathrm{Curr}(F_N)$.

COROLLARY 6.7. *Let $N \geq 3$. Then the action of* $\mathrm{Out}(F_N)$ *on* $\mathbb{P}\mathrm{Curr}_+(F_N)$ *is properly discontinuous.*

Proof. Note that for $N \geq 3$ the group $G = \mathrm{Out}(F_N)$ is dynamically large. By the main result of [25], we have $\langle T, \mu\rangle > 0$ for any $T \in \overline{\mathrm{cv}}_N$ and any $\mu \in \mathrm{Curr}_+(F_N)$. Therefore, by Definition 4.8, we have $\mathbb{P}\mathrm{Curr}_+(F_N) \subseteq \widehat{\Delta}^{\mathrm{curr}}_{\mathrm{Out}(F_N)}$. Hence, the action of $\mathrm{Out}(F_N)$ on $\widehat{\Delta}^{\mathrm{curr}}_{\mathrm{Out}(F_N)}$ is properly discontinuous by Theorem 6.5 and therefore the action of $\mathrm{Out}(F_N)$ on $\mathbb{P}\mathrm{Curr}_+(F_N)$ is properly discontinuous as well. $\square$

Note, however, that in the proof of Corollary 6.7 the containment $\mathbb{P}\mathrm{Curr}_+(F_N) \subseteq \widehat{\Delta}^{\mathrm{curr}}_{\mathrm{Out}(F_N)}$ is proper. Recall that a current $\mu \in \mathrm{Curr}(F_N)$ is called *filling* if for every $T \in \overline{\mathrm{cv}}_N$ we have $\langle T, \mu\rangle > 0$. The same argument as in the proof of Corollary 6.7 shows that if $\mu \in \mathrm{Curr}(F_N)$ is filling then $[\mu] \in \widehat{\Delta}^{\mathrm{curr}}_{\mathrm{Out}(F_N)}$. It was proved in [25] that there exist filling rational currents $\eta_g \in \mathrm{Curr}(F_N)$ and in fact, the property of being filling for a rational current is, in a sense, "generic". However, a rational current never has full support, so if η_g is filling, then $\eta_g \in \widehat{\Delta}^{\mathrm{curr}}_{\mathrm{Out}(F_N)} \setminus \mathbb{P}\mathrm{Curr}_+(F_N)$.

As a final comment, note that $\mathbb{P}\mathrm{Curr}_+(F_N)$ is dense in $\mathbb{P}\mathrm{Curr}(F_N)$: For any $\mu_+ \in \mathbb{P}\mathrm{Curr}_+(F_N)$ and any $\mu \in \mathbb{P}\mathrm{Curr}(F_N)$ the line segment $[\mu_+, \mu] := \{[\mu_t] = [t\mu + (1-t)\mu_+] \mid t \in [0,1]\}$ is entirely contained in $\mathbb{P}\mathrm{Curr}_+(F_N)$ except for possibly its endpoint μ. This picture, together with the result of the Corollary 6.7, suggests that perhaps one should view $\mathbb{P}\mathrm{Curr}_+(F_N)$ as a kind of "interior" of the current space $\mathbb{P}\mathrm{Curr}(F_N)$, in analogy to CV_N being the interior of $\overline{\mathrm{CV}}_N$.

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Ilya Kapovich, Department of Mathematics, University of Illinois at Urbana–Champaign, 1409 West Green Street, Urbana, IL 61801, USA

URL: http://www.math.uiuc.edu/~kapovich/

E-mail address: kapovich@math.uiuc.edu

Martin Lustig, Mathématiques (LATP), Université Paul Cézanne—Aix Marseille III, 52 av. Escadrille Normandie-Niémen, 13397 Marseille 20, France

E-mail address: Martin.Lustig@univ-cezanne.fr