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THE MAXIMAL PURE SPECTRUM OF AN ABELIAN GROUP

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Abstract. This paper introduces the notion of the maximal pure spectrum of an Abelian group—this is the set of isomorphism classes of maximal proper pure subgroups—and focuses on the situation in which this spectrum is small. The converse situation is also examined i.e., given a collection of isomorphism classes of groups, can one find an Abelian group having precisely this collection as its maximal pure spectrum. Finally, it is shown that in some familiar situations, the answers to these questions may be undecidable.

1. Introduction

An arbitrary nonzero Abelian group G has the property that it contains maximal pure proper subgroups—see Proposition 2.1 below. In general, one would expect that for an arbitrary group G, the set of such maximal pure subgroups would be large and varied (up to isomorphism): recall that Boyer showed [2], that an uncountable group G has $2^{|G|}$ pure subgroups, each having cardinality equal to $|G|$. Nevertheless, there are situations when, up to isomorphism, the collection of maximal pure subgroups has a particularly simple form, and this is one focus of the present work. We formalize this notion by defining the *maximal pure spectrum* of a group G as the set of isomorphism classes of maximal pure subgroups; we denote this set by $\mathrm{MPSpec}(G)$. (The notion clearly has a family resemblance to the notion of torsion-free Crawley groups introduced by the authors and A. L. S. Corner recently [4].) We shall also be interested in the converse situation of when a given collection of isomorphism classes of groups can occur as the maximal pure spectrum of

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some group; we shall say that a collection of isomorphism classes of groups $\{H_i\}$ *supports* a maximal pure spectrum if there exists some group G such that $\mathrm{MPSpec}(G) = \{H_i\}$.

Throughout all groups are additively written Abelian groups and standard concepts and notation for Abelian groups may be found in the books [8], [9], while set-theoretic concepts may be found in [7]; note that here mappings are written on the right.

2. Basic results

We begin by establishing the existence of a maximal pure spectrum for any nonzero group. Recall that a group is said to be pure simple if its only proper pure subgroup is the zero group $\{0\}$; it is well known that such a group is isomorphic to a subgroup of $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$, for some prime p—see [8, Exercise 7, p. 119].

Proposition 2.1. *If G is a nonzero group, then* $\mathrm{MPSpec}(G) \neq \emptyset$. *Moreover, if G is uncountable and X is any subgroup of G with $|X| < |G|$, then there is a proper maximal pure subgroup of G containing X.*

Proof. If G is not torsion-free, then it has a direct summand which is pure simple and then any complement of this summand is a maximal pure subgroup of G. If G is torsion-free, then it can be embedded as an essential subgroup of a direct sum of copies of $\mathbb{Q}$ and the kernel of any projection onto a single factor, when restricted to G, will be a maximal pure subgroup of G since the image is pure simple. To establish the second statement, note that by Szele's result—see, for example [8, Proposition 26.2]—there is a pure subgroup Y of G containing X and $|Y| = \max\{|X|, \aleph_0\} < |G|$. So G/Y is a nonzero group and by the first part of our result, there is a proper maximal pure subgroup M/Y of G/Y. Clearly, M is a maximal pure subgroup of G and contains X. □

Not surprisingly, the situation for divisible groups is straightforward and the elementary proof of the next proposition is left to the reader.

Proposition 2.2. *If $G = \bigoplus_{p\in\mathbb{P}'} \bigoplus_{\alpha_p} \mathbb{Z}(p^\infty) \oplus \bigoplus_\beta \mathbb{Q}$ is divisible, then* $\mathrm{MPSpec}(G)$ *consists of the collection of groups of the form*

$$\bigoplus_{p\in\mathbb{P}'} \bigoplus_{\alpha_p^*} \mathbb{Z}(p^\infty) \oplus \bigoplus_{\beta^*} \mathbb{Q},$$

where precisely one of α_p^*, β^* differs from the α_p, β and, in that case it is either $\alpha_p - 1$ or $\beta - 1$, with the usual conventions applying if the α_p or β are infinite. In particular, if G is torsion-free divisible of rank β, then $\mathrm{MPSpec}(G)$ *consists of a singleton $\bigoplus_{\beta-1} \mathbb{Q}$.*

The converse situation is equally clear, at least when all the invariants are infinite.

PROPOSITION 2.3. *If* $H = \bigoplus_{p\in\mathbb{P}'} \bigoplus_{\alpha_p} \mathbb{Z}(p^\infty) \oplus \bigoplus_\beta \mathbb{Q}$ *is a divisible group and each invariant* α_p $(p \in \mathbb{P}'), \beta$ *is infinite, then* H *supports the maximal pure spectrum of only one group viz.* H *itself.*

Proof. If G is a group with $\mathrm{MPSpec}(G) = \{H\}$, then if M is any maximal pure subgroup of G, G/M is pure simple and G splits as $G = M \oplus X$, where $X \leq \mathbb{Q}$, or $X \leq \mathbb{Z}(p^\infty)$, for some prime p. Let M_1 be maximal pure in M and note that $M_1 \oplus X$ is then, by hypothesis, isomorphic to H. It follows immediately that $X \cong \mathbb{Q}$ or $X \cong \mathbb{Z}(p^\infty)$ for some $p \in \mathbb{P}'$. Since each invariant is infinite, G is then isomorphic to H. Finally, it follows from Proposition 2.2 that the maximal pure spectrum is, in this case, actually equal to $\{H\}$. □

It is not, of course, true that given a group H, one can find a group G having $\{H\}$ as maximal pure spectrum. For example, if $H = J_p$, the group of p-adic integers, then H does not support any maximal pure spectrum; in fact, we have the more general in the following.

PROPOSITION 2.4. *If* H *is a reduced torsion-free algebraically compact group, then there does not exist any group* G *with* $\mathrm{MPSpec}(G) = \{H\}$.

Proof. Suppose such a group G exists, then G must be an extension of H by a rank one group, R, say. Since H is algebraically compact such an extension must split: $G = H \oplus R$. Let K be a maximal pure subgroup of H, so that $K \oplus R$ is maximal pure in G. Thus, $K \oplus R$ must, by hypothesis, be isomorphic to H and this is impossible since any direct summand of a reduced algebraically compact group, and thus in particular, R, must again be reduced algebraically compact. This contradiction establishes the result. □

The situation for free groups is also straightforward, although in this case there is an interesting *twist.*

LEMMA 2.5. *If* G *is an arbitrary uncountable group having a free subgroup* M *of infinite rank, with countable quotient* G/M, *then* $G = M_1 \oplus N$ *where* M_1 *is a summand of* M *of uncountable rank, and* N *is countable.*

Proof. If A is the subgroup generated by the preimages of the coset representatives of G/M, then $G = A + M$. Moreover, since $A \cap M$ is countable, there is a decomposition $M = M_0 \oplus M_1$ with $A \cap M \leq M_0$ and $M_1 \neq 0$. Set $N = A + M_0$, a countable group and note that $G = N + M_1$ while $N \cap M_1 = 0$. Thus, $G = N \oplus M_1$ as required. □

PROPOSITION 2.6. *If* G *is a torsion-free group of infinite rank having the property that every maximal pure subgroup is free, then* G *is free. Conversely, if* G *is free of infinite rank, then all maximal pure subgroups are isomorphic and free.*

Proof. We consider first the case where G has uncountable rank. Let M be a maximal pure subgroup of G, so that M is, by assumption, free of uncountable rank. However, since M is maximal pure in G, the quotient G/M is torsion-free of rank 1; in particular G/M is countable. By Lemma 2.5, $G = M_1 \oplus N$, where M_1 is a summand of M of uncountable rank, and so is free. Now choose a maximal pure subgroup H of M_1 and consider the subgroup $H \oplus N$ of G. Clearly, it is a maximal pure subgroup of G and so by hypothesis, it too is free. Consequently, N is free and so also is $G = M_1 \oplus N$.

Suppose now that G is countably infinite. It will suffice, by Pontryagin's Theorem [8, Theorem 19.1], to show that every finite rank subgroup of G is free. Embed G in its divisible hull D, a countable direct sum of copies of $\mathbb{Q}$. Now if F is a finite rank subgroup of G, there is a finite rank summand D_1 of D such that $F \leq D_1 : D = D_1 \oplus D_2$, where D_2 is divisible of countable rank. Let π be the projection of D onto a rank one summand of D_2, so that $F \leq \operatorname{Ker} \pi$. Now if $H = \operatorname{Ker} \pi \restriction G = G \cap \operatorname{Ker} \pi$, then $F \leq H$. However, $G/H \cong \operatorname{Im} \pi \restriction G \leq \mathbb{Q}$ and so H is a maximal pure subgroup of G. By hypothesis, H is free and hence, so also is F.

The converse is essentially immediate: subgroups of free groups are free and maximal pure subgroups are of the same rank as the whole group. □

Corollary 2.7. *If H is a free group of infinite rank, then $\{H\}$ supports the maximal pure spectrum of a single group G, and $G \cong H$.*

The twist here is that the proposition above fails for all finite ranks $n \geq 2$: Corner has exhibited a group of finite rank n with all rank $(n-1)$ subgroups free, but the group is indecomposable—see e.g., [9, Exercise 8, Section 88].

Recall that a group A is said to be $\aleph_1$-separable if every countable subset of A is contained in a countable free summand of A and that A is said to be hereditarily (or totally) $\aleph_1$-separable, if every subgroup of A is $\aleph_1$-separable. Finally, slightly modifying a standard definition, we say that A is of *strong quotient type* $\mathbb{Q}$ if A has a filtration $A = \bigcup_{\alpha<\lambda} A_\alpha$ such that $A_{\alpha+1}/A_\alpha \cong \mathbb{Q}$ for all limit ordinals $\alpha < \lambda$. Note that any other filtration of A will agree with the given one on a cub; see Chapter VIII in [7] for the above concepts. We note that an $\aleph_1$-separable group A of cardinality $\aleph_1$ may be defined via a filtration $A = \bigcup_{\alpha<\lambda} A_\alpha$ in which $A_{\alpha+1}$ is a free direct summand of A for all α. Hence, if A is also of strong quotient type $\mathbb{Q}$ and $B = \bigcup_{\alpha<\lambda} B_\alpha$ is an $\aleph_1$-filtration, then the existence of an isomorphism $\phi : A \to B$ implies that there is a cub C such that $B_\alpha = A_\alpha \phi$ and B_α is not closed in B for all $\alpha \in C$: the existence of a cub with $B_\alpha = A_\alpha \phi$ for all $\alpha \in C$ follows from a standard back and forth argument. Since $A_{\alpha+1}$ is a summand of A, it is the closure of A_α, and thus $A_{\alpha+1}/A_\alpha$ is mapped isomorphically onto $\overline{B}_\alpha/B_\alpha$, implying that B_α is not closed, as required.

Proposition 2.8. *If G is a group such that every maximal pure subgroup is (hereditarily) $\aleph_1$-separable, then all the maximal pure subgroups of G are* *isomorphic and are isomorphic to G. In particular, G is again (hereditarily) $\aleph_1$-separable.*

Proof. If the maximal pure subgroups are of countable rank, then they are free and so the result reduces to the corresponding result, Proposition 2.6 above. Assume then that the maximal pure subgroups, and hence G, are uncountable. Then, by an identical argument to that used in Lemma 2.5, if M is maximal pure in G, $G = N \oplus M_1$ and N is countable, while M_1 is an uncountable direct summand of M. Since M is assumed to be $\aleph_1$-separable, so also is M_1—see e.g., [7, Exercise 1, Chapter VIII]. However, as M_1 is $\aleph_1$-separable, it has a free summand of countable rank and so $M_1 \cong M_1 \oplus N = G$ and $M_1 \cong M$. Thus, all maximal pure subgroups are isomorphic to G. The case where the maximal pure subgroups are assumed to be hereditarily $\aleph_1$-separable, follows immediately. □

We are now in a position to establish the following theorem.

THEOREM 2.9. *Let $\{H_i : i \in I\}$ be a collection of isomorphism classes of hereditarily $\aleph_1$-separable groups. Then, there exists a group G with* $\mathrm{MPSpec}(G) = \{H_i : i \in I\}$ *if, and only if, $|I| = 1$. Moreover, in this case* $\mathrm{MPSpec}(G) = G$ *itself.*

Proof. If any H_i is countable, we are immediately reduced to the situation where all the groups are free, so we may suppose that each H_i is uncountable. So, if there is a group G with $\mathrm{MPSpec}(G) = \{H_i : i \in I\}$, it follows from Proposition 2.8 above that G is hereditarily $\aleph_1$-separable and every maximal pure subgroup of G is isomorphic to G itself. Hence, $|I| = 1$, as required.

Conversely, given a hereditarily $\aleph_1$-separable group G, if M is a maximal pure subgroup of G, then M is a subgroup of countable index in G and it then follows from [11, Theorem 201] that $M \cong G$, i.e., $\mathrm{MPSpec}(G) = \{G\}$. □

If we weaken the assumption in the previous theorem to being simply that the collection $\{H_i : i \in I\}$ consists of $\aleph_1$-separable groups, the situation changes dramatically. Note that if G is of strong quotient type $\mathbb{Q}$, then G is not free. Recall that a group G is said to be a *Griffith group* if G is a (nonfree) $\aleph_1$-separable subgroup of $\mathbb{Z}^{\aleph_1}$ of cardinality $\aleph_1$ defined by a filtration $G = \bigcup_{\alpha<\omega_1} G_\alpha$ such that for each limit ordinal α, $G_{\alpha+1}/G_\alpha \cong \mathbb{Q}$—see [11, Theorem 147]. Clearly, Griffith groups are examples of groups of strong quotient type $\mathbb{Q}$.

Under the set-theoretic assumption $V = L$, we obtain the following theorem.

THEOREM 2.10 ($V = L$). *If $\{H_i : i \in I\}$ is collection of isomorphism classes of $\aleph_1$-separable groups of cardinality $\aleph_1$ and strong quotient type $\mathbb{Q}$, then there does not exist a group G with* $\mathrm{MPSpec}(G) = \{H_i : i \in I\}$.

Remark: The hypothesis of strong quotient type $\mathbb{Q}$ in Theorem 2.10 can also be weakened: it suffices to have that each of the groups H has the property that for each limit ordinal α, $H_{\alpha+1}/H_\alpha \cong R$, where R is a subgroup of $\mathbb{Q}$ containing a subring of $\mathbb{Q}$ different from $\mathbb{Z}$.

The theorem follows immediately from the next result which may be of some independent interest. We remark that Megibben [12] has produced a similar result in the category of Abelian p-groups, but our proof is necessarily somewhat more complicated than his. This is because, in a p-group, a closed countable pure subgroup of an $\aleph_1$-separable p-group is necessarily a direct summand, and so one has a convenient class of *test* subgroups from which one can demonstrate failure of the $\aleph_1$-separability property. No such convenient class seems to exist in the torsion-free case.

THEOREM 2.11 ($V = L$). *If G is an $\aleph_1$-separable group of cardinality $\aleph_1$, which is of strong quotient type $\mathbb{Q}$, then G has a maximal pure dense subgroup which is not $\aleph_1$-separable.*

Proof. Since G is an $\aleph_1$-separable of cardinality $\aleph_1$ of strong quotient type $\mathbb{Q}$, we can choose a filtration of G by countable free subgroups: $G = \bigcup_{\alpha<\omega_1} G_\alpha$ with $G_{\alpha+1}/G_\alpha \cong \mathbb{Q}$ for all limit ordinals $\alpha < \omega_1$. Next, we shift the labels of the filtration and begin with $G_{\omega+1}$ which now becomes G_0 or say just A. If B is the old G_ω, then $A/B \cong \mathbb{Q}$ and $E := \{\alpha < \omega_1 \mid G_{\alpha+1}/G_\alpha \cong \mathbb{Q}\} = \mathring{\omega}_1$, the set of limit ordinals $< \omega_1$, is a stationary subset of ω_1. Now partition E into disjoint stationary sets $E = E' \dot\cup E''$. Note that the set E' will be reserved solely to ensure that the group H to be constructed will be maximal pure in G. From $V = L$ follows that $\diamondsuit_{\omega_1}(E'')$ holds and so there are Jensen functions $f_\alpha : G_\alpha \to G_\alpha$ ($\alpha \in E''$) such that for each endomorphism $f : G \to G$, the set $\{\alpha \in E'' \mid f_\alpha = f \restriction G_\alpha\}$ is stationary in ω_1.

The subgroup B is maximal pure dense in A and so we may choose an element $z \in A$ such that $A = \langle B, z\rangle_*$, the pure subgroup of G generated by B and z. *Fix one such z for the remainder of our discussion.* Let $\mathcal{X} = \{X \mid B \le X \le G, |X| = \aleph_0,\ z \notin X\}$.

We now construct a group H as an $\aleph_1$-filtration $H = \bigcup_{\alpha<\omega_1} H_\alpha$ with $H_\alpha \le_* G$ as follows:

(0) $z \notin H_\alpha$ for any $\alpha < \omega_1$,
(1) $H_0 = B$,
(2) if α is a limit ordinal, $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$,
(3) assuming H_α has been constructed, choose a maximal linearly independent subset of

$$(G_{\alpha+1} + H_\alpha)/\langle H_\alpha, z\rangle_*,$$

lift this back to elements $\{x_j\}_{(j\in J)} \in G_{\alpha+1}$ and set $H_{\alpha+1} = \langle H_\alpha, x_j \mid j \in J\rangle_* \le G$, (3a) *UNLESS* $\alpha \in E''$ is a limit ordinal, $f_\alpha = \rho \restriction G_\alpha$ for some projection ρ of G with kernel $\ker \rho = \overline{X}^G$, $X \in \mathcal{X}$, $H_\alpha f_\alpha \subseteq H_\alpha$ and $H_\alpha f_\alpha$ *not* closed in G. In these circumstances, we choose $x_\alpha \in \overline{H_\alpha f_\alpha} \setminus H_\alpha f_\alpha$ and set $H_{\alpha+1} = \langle H_\alpha, x_\alpha - z \rangle_* \leq G$.

The only condition needing to be checked for consistency of this construction, is that $z \notin H_\alpha$ for any α. Clearly, $z \notin H_0$ and limit ordinals will present no difficulty if we have handled successors.

We show first that $z \notin H_{\alpha+1}$ when $H_{\alpha+1}$ was constructed as in (3a). Note that $x_\alpha \in \overline{H_\alpha f_\alpha}$ implies that $x_\alpha = \lim_{n\to\infty} h_{\alpha_n} f_\alpha$ for some $h_{\alpha_n} \in H_\alpha$ and, since f_α in this situation acts as the projection ρ, it extends uniquely to a homomorphism on the closure of H_α; there will be no confusion if we continue to call the extended map f_α. Now $x_\alpha f_\alpha = \lim_{n\to\infty}(h_{\alpha_n} f_\alpha) f_\alpha = \lim_{n\to\infty} h_{\alpha_n} f_\alpha = x_\alpha$. Now suppose for a contradiction that $z \in H_{\alpha+1}$; then there are integers r, s such that $sz = h_\alpha + r(x_\alpha - z)$ for some $h_\alpha \in H_\alpha$. Applying the map f_α to this equation, we recall that $z \in \overline{B} \leq \ker f_\alpha$ and thus $z f_\alpha = 0$, we get $0 = h_\alpha f_\alpha + r x_\alpha f_\alpha$, so that $r(x_\alpha f_\alpha) \in H_\alpha f_\alpha \leq H_\alpha$, whence $x_\alpha f_\alpha \in H_\alpha$ by purity. But $x_\alpha = x_\alpha f_\alpha \in H_\alpha f_\alpha$—contrary to the choice of $x_\alpha \notin H_\alpha f_\alpha$.

Finally, if the construction of $H_{\alpha+1}$ is as in (3) and if $z \in H_{\alpha+1} = \langle H_\alpha, x_j \mid j \in J \rangle_*$, then for some $s \neq 0, r_j \in \mathbb{Z}$ and x_j from (3) and $h_\alpha \in H_\alpha$ it follows $sz = \sum_{j\in J} x_j r_j + h_\alpha$. Hence, $h_\alpha = sz - \sum_{j \in J} x_j r_j$ and looking at the equation mod $\langle H_\alpha, z \rangle_*$ we have $r_j = 0$ for all $j \in J$ by the choice of the x_j's. Hence, $sz = h_\alpha$ and $z \in H_\alpha$ by purity of H_α is a contradiction. Thus, in either case is $z \notin H_{\alpha+1}$; the construction can go on and $H = \bigcup_{\alpha<\omega_1} H_\alpha \leq_* G$.

By the choice of E' it follows that case (3) appears on an unbounded sequence of steps. This implies that all independent elements of $G_{\alpha+1}$ mod $\langle H_\alpha, z \rangle_*$ are absorbed into H or equivalently at the end $G = \langle H, z \rangle_*$ which is to say that $G/H \cong \langle z + H \rangle_* \cong \mathbb{Q}$.

We claim that H is not $\aleph_1$-separable; specifically we show that B cannot be embedded in a countable summand of H. Suppose for a contradiction that $B \leq X$ and $H = X \oplus K$, where X is countable. Since $z \notin H$, we have $z \notin X$ and so $X \in \mathcal{X}$. Observe firstly that this implies that $G = \langle X, z \rangle_* \oplus K$: an argument on rank show that $G = \langle X, z \rangle_* + K$, while if $k \in \langle X, z \rangle_* \cap K$, then $tk = x + sz$ for some integers s, t and $x \in X, k \in K$. Thus, both $k, x \in H$ and $sz = tk - x \in H$ as well, a contradiction.

Now let ρ be the projection of H onto K with kernel X. Notice that there can be only one extension of ρ to a projection of G onto K with kernel $\langle X, z \rangle_*$: for if π, π_1 were such extensions, then the difference $\pi - \pi_1$ would induce a map $\overline{\pi - \pi_1}$ from G/H into K. However, $G/H \cong \mathbb{Q}$ and K is reduced, so the induced map must be the zero map i.e., $\pi = \pi_1$. Thus, there shall be no confusion if we refer to the projection of G onto K with kernel $\overline{X}$ as ρ also. Note that K is again an $\aleph_1$-separable group since it is a summand of G.

Let $Y_\alpha = H_\alpha\rho$ and note that $\bigcup_{\alpha<\omega_1} Y_\alpha$ is an $\aleph_1$-filtration of K. However, the subgroup $\langle X, z\rangle_*$ of G is countable and hence free, and so one can deduce that $G \cong K$ as K is $\aleph_1$-separable. Since the group G is of strong quotient type $\mathbb{Q}$, at each limit ordinal α one has that $G_{\alpha+1}/G_\alpha \cong \mathbb{Q}$ and so, as observed earlier, there exists a cub C^* such that $Y_\alpha = H_\alpha\rho$ is not closed in K for $\alpha \in C^*$. Since $\rho \in \operatorname{End}(G)$ and $\rho \restriction H \in \operatorname{End}(H)$, we can find a cub C in ω_1 such that for all $\alpha \in C$:

(1) $H_\alpha\rho \le H_\alpha$ and
(2) $G_\alpha\rho \le G_\alpha$.

So there is an ordinal $\beta \in E'' \cap (C \cap C^*)$. Hence, $f_\beta = \rho \restriction G_\beta$, $H_\beta\rho \le H_\beta$ so that $H_\beta f_\beta \le H_\beta$. Moreover, $H_\beta f_\beta = H_\beta\rho$ is not closed in K and hence in G. We conclude that by the above the construction of $H_{\beta+1}$ at step β must have proceeded according to (3a), so that $H_{\beta+1} = \langle H_\beta, x_\beta - z\rangle_*$ for some $x_\beta \in \overline{H_\beta f_\beta} \setminus H_\beta f_\beta$. This, however, is impossible: Since $z = x_\beta - (x_\beta - z)$ and f_β acts like the projection ρ, thereby fixing x_β and killing z, we have $x_\beta = (x_\beta - z)\rho \in H$. Since $x_\beta - z \in H_{\beta+1} \le H$, we would then be forced to conclude that $z \in H$, a contradiction. This contradiction shows that B cannot be embedded in a countable summand of H, so that H is not $\aleph_1$-separable, as required. □

We also remark that a special case of the above theorem is the statement that, under the assumption of $V = L$, there is an $\aleph_1$-separable group of cardinality $\aleph_1$, which is not hereditarily $\aleph_1$-separable. Such a group G has been previously obtained by Eklof [6, Theorem 4.5], however his proof does not establish the existence of a maximal pure dense subgroup of G which fails to be $\aleph_1$-separable. We also note that it is not possible to dispense with the set-theoretic hypothesis here, since Eklof also established in [6], that under the assumption $(MA + \neg CH)$, every $\aleph_1$-separable group of cardinality $\aleph_1$ is hereditarily $\aleph_1$-separable.

3. Whitehead and $\aleph_1$-coseparable groups

In the previous section, we exploited the 'separability-like' properties of some classes of groups to investigate their maximal pure spectra. Here, we shall make use of a more homological approach which, perhaps inevitably, requires some further cardinality restrictions. Recall that a group G is said to be a *Whitehead group* or W-group if $\operatorname{Ext}(G, \mathbb{Z}) = 0$.

PROPOSITION 3.1. *If G is a group such that every maximal pure subgroup of G is a W-group of rank λ, then if $2^\lambda > 2^{\aleph_0}$, G is a W-group; in particular if every maximal pure subgroup of G is a W-group of cardinality $2^{\aleph_0}$, then G is a W-group.*

Proof. Let M be a maximal pure subgroup of G and first observe that $\operatorname{Hom}(G, \mathbb{Z}) \neq 0$: consider the short exact sequence $0 \to M \to G \to X \to 0$, where X is torsion-free of rank 1 and assume, for a contradiction, that $\mathrm{Hom}(G,\mathbb{Z})=0$. Note that $X\not\cong\mathbb{Z}$ then follows, and so $\mathrm{Hom}(X,\mathbb{Z})=0$. Applying the contravariant functor $\mathrm{Hom}(-,\mathbb{Z})$ and using the assumption, gives

$$0\to\mathrm{Hom}(M,\mathbb{Z})\to\mathrm{Ext}(X,\mathbb{Z})\to\mathrm{Ext}(G,\mathbb{Z})\to\mathrm{Ext}(M,\mathbb{Z})=0.$$

Since $X\leq\mathbb{Q}$, it follows easily that $\mathrm{Ext}(X,\mathbb{Z})$ is an epimorphic image of $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$ and, since the latter is isomorphic to $\mathbb{Q}^{\aleph_0}$, it follows that $\mathrm{Ext}(X,\mathbb{Z})$ has cardinality at most $2^{\aleph_0}$. On the other hand, it has a subgroup $\mathrm{Hom}(M,\mathbb{Z})$ which, by a well-known result of Chase (see e.g., [9, Proposition 99.4]), has cardinality $2^\lambda>2^{\aleph_0}$—contradiction. Thus, it follows that $G=H\oplus\mathbb{Z}$, for some subgroup H. However, H is clearly a maximal pure subgroup of G and hence is a W-group. It is then immediate that G is also a W-group. □

The converse of the above proposition is not immediately clear. Under the assumption $(V=L)$, it is, of course, immediate and merely reduces to Proposition 2.6, since W-groups are free. Assuming $(MA+\neg CH)$, W-groups of power $\aleph_1$ are $\aleph_1$-coseparable, and so it is natural to investigate this class of groups; recall—see e.g., [11, Chapter VIII]—that a group G is $\aleph_1$-coseparable precisely if $\mathrm{Ext}(G,S)=0$, where S is the free group of rank $\aleph_0$ and that G is hereditarily $\aleph_1$-separable if, and only if, it is both $\aleph_1$-separable and $\aleph_1$-coseparable.

Proposition 3.2. *If G is a group such that every maximal pure subgroup is an $\aleph_1$-coseparable group of rank λ, then if $2^\lambda>2^{\aleph_0}$, G is an $\aleph_1$-coseparable group.*

Proof. The proof is essentially identical to that of Proposition 3.1 once one notes the following: (i) if $\mathrm{Hom}(G,\mathbb{Z})=0$, then $\mathrm{Hom}(G,S)=0$; (ii) replace the functor $\mathrm{Hom}(-,\mathbb{Z})$ with the functor $\mathrm{Hom}(-,S)$ and note that $\mathrm{Ext}(X,S)$ is a homomorph of $\mathrm{Ext}(\mathbb{Q},S)$ and that this latter still has cardinality $2^{\aleph_0}$; (iii) since the maximal pure subgroup M is $\aleph_1$-coseparable, and hence a W-group, and $\mathrm{Hom}(M,\mathbb{Z})\leq\mathrm{Hom}(M,S)\leq\mathrm{Ext}(X,S)$, we can reach the desired result by applying Chase's result. □

However, in this situation, we have a converse and so we can establish the following theorem.

Theorem 3.3. *If H is an $\aleph_1$-coseparable group of rank λ and $2^\lambda>2^{\aleph_0}$, then $\{H\}$ supports the maximal pure spectrum of only one group, namely H itself.*

Proof. If M is any maximal pure subgroup of H, then it follows from [11, Theorem 201] that $M\cong H$. Thus, H itself has maximal pure spectrum $\{H\}$. Uniqueness follows from the cardinality hypothesis and the previous proposition. □

4. The Baer–Specker group

In the following, we will denote as usually by $P = \mathbb{Z}^{\aleph_0}$ the Baer–Specker group, where $\mathbb{Z}^\lambda = \prod_{\alpha<l} \mathbb{Z}e_\alpha$ is the cartesian product of λ copies of $\mathbb{Z}$. The first work on this type of problem is, to the best of our knowledge, due to A. L. S. Corner in an undated manuscript [3] from the late fifties or early sixties. The main result of that paper is the following theorem.

THEOREM 4.1 ([3]). *Let G be a group such that every maximal pure subgroup of G is isomorphic to $\mathbb{Z}^\lambda$, where λ is a given infinite cardinal. Then $G \cong \mathbb{Z}^\lambda$.*

Proof. Suppose, for a contradiction, that G satisfies the hypotheses but not the conclusion of the theorem. It is immediate that $\operatorname{Hom}(G,\mathbb{Z}) = 0$; otherwise $G = \mathbb{Z} \oplus M$ for some subgroup M which is clearly a maximal pure subgroup of G and so, by hypothesis, is isomorphic to $\mathbb{Z}^\lambda$. This would lead immediately to the contradiction that G is isomorphic to $\mathbb{Z}^\lambda$. Claim that every maximal pure subgroup of G is dense in the $\mathbb{Z}$-adic topology of G. Consider, a maximal pure subgroup H of G. The quotient G/H is visibly torsion-free and of rank one. Suppose, for a contradiction, that G/H is reduced. Then, as a countable reduced torsion-free group, it is slender. Moreover, since G/H is countable, while G itself is of cardinality at least the continuum, there is a maximal pure subgroup M of G that contains a representative of each coset of H in G, so that the composition $M \to G \to G/H$ is an epimorphism. Since G/H is slender while $M \cong \mathbb{Z}^\lambda$, the epimorphism must, in fact, vanish on a direct complement of some free direct summand of finite rank in M; hence G/H is finitely generated. But this then implies that $G/H \cong \mathbb{Z}$, which contradicts $\operatorname{Hom}(G,\mathbb{Z}) = 0$. Consequently, $G/H \cong \mathbb{Q}$ and the claim is established.

We continue to consider $H \leq G$ and note by the above that there is an element $g_0 \in \overline{H} \setminus H$, the $\mathbb{Z}$-adic closure of H in G such that $G = \langle g_0, H\rangle_*$ which, for convenience, can be viewed in the $\mathbb{Z}$-adic completion $\widehat{H}$ of H and clearly $\widehat{G} = \widehat{H}$.

By assumption of the theorem, we can write $H = \prod_{\delta\in\Delta} \mathbb{Z}e_\delta$, where Δ is an indexing set of cardinality λ. Now we can write $g_0 = \sum^*_{\delta\in\Delta} \pi^\delta e_\delta$, where $\pi^\delta \in \widehat{\mathbb{Z}}$.

Let M be a maximal pure subgroup of G that contains g_0 and the e_δ $(\delta \in \Delta)$; such an M exists by Proposition 2.1, since $|\langle g_0, e_\delta\ (\delta\in\Delta)\rangle| = \lambda < 2^\lambda = |G|$. Since M is a product of copies of $\mathbb{Z}$, there is a nonzero homomorphism $\phi : M \to \mathbb{Z}$. Now by the claim above, M is dense in G and ϕ extends to a $\widehat{\mathbb{Z}}$-homomorphism $\widehat{\phi} : \widehat{G} \to \widehat{\mathbb{Z}}$. Since the kernel of ϕ is of corank 1 in M and therefore of corank ≤ 2 in G, the image $G\widehat{\phi}$ is a subgroup of rank at most 2 in $\widehat{\mathbb{Z}}$; hence $G\widehat{\phi}$ is slender, see [8]. But $\widehat{\phi} \restriction H : H \to G\widehat{\phi}$, so that almost all the $e_\delta\phi$ must vanish. It now follows that $G\widehat{\phi} \leq \mathbb{Z}$: for if $g \in G$, then for some positive integer q we have

$$qg = \sum_{\delta \in \Delta}^{*} \xi^{\delta} e_{\delta} + \xi g_0$$

for suitable $\xi, \xi^{\delta} \in \mathbb{Z}$; hence

$$q(g\widehat{\phi}) = \sum_{\delta \in \Delta}^{*} \xi^{\delta}(e_{\delta}\phi) + \xi(g_0\phi) \in \mathbb{Z}$$

and therefore $g\widehat{\phi} \in \mathbb{Z}$, by the purity of $\mathbb{Z}$ in $\widehat{\mathbb{Z}}$. Since $\mathrm{Hom}(G,\mathbb{Z}) = 0$, we have $G\widehat{\phi} = 0$; and it follows that $\phi = 0$, contrary to the choice of ϕ. This contradiction establishes the theorem. □

Proposition 4.2. *If λ is an infinite cardinal, then there is a maximal pure dense subgroup H of $\mathbb{Z}^{\lambda}$ which is not isomorphic to $\mathbb{Z}^{\lambda}$.*

Assuming that we have established the proposition above, we deduce the following.

Corollary 4.3. *A group of the form $\mathbb{Z}^{\lambda}$, with λ infinite, does not support the maximal pure spectrum of any group G.*

Proof. Suppose that $\mathbb{Z}^{\lambda}$ did support the maximal pure spectrum of a group G, then by the previous theorem G itself would be of the form $\mathbb{Z}^{\lambda}$, which would force all maximal pure subgroups to be isomorphic to $\mathbb{Z}^{\lambda}$, contrary to Proposition 4.2. □

Proof of Proposition 4.2. First, we consider the case $\lambda = \aleph_0$. Take a countable free resolution $0 \to F_0 \to F_1 \to \mathbb{Q} \to 0$ of $\mathbb{Q}$ in which F_0, F_1 are of countable rank and set $P_1 = F_1^*, P_0 = F_0^*$, so that

$$0 \to P_1 \to P_0 \to \mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \cong \mathbb{Q}^{\aleph_0} \to 0.$$

Choose G containing P_1 so that P_0/G is a single copy of $\mathbb{Q}$; G is certainly pure of countable corank in P_0 and is not a summand of P_0.

Suppose $G \cong P_0$. Then we may apply Nunke [14, p. 69, Lemma 3]: We have $P_0/G \cong \mathbb{Q}$, thus $(P_0/G)^* = 0$ and from $G^* \cong P_0^* \cong F_0$, it follows that $U := G^*/P_0^*$ (with the canonical embedding $P_0^* \leq G^*$) is countable. From Nunke [14], it now follows that $U^* = 0$ and $\mathbb{Q} \cong P_0/G \cong \mathrm{Ext}(U,\mathbb{Z})$. Thus, $\mathrm{Ext}(U,\mathbb{Z}) \neq 0$ and since $U \neq 0$ is countable, $\mathrm{Ext}(U,\mathbb{Z})$ is well-known and must be uncountable (see, for example [7, Chapter XII, 4]) which contradicts the last equation. Thus, $G \ncong P_0$.

Now let λ be any infinite cardinal and set $H := G \oplus \mathbb{Z}^{\lambda} \leq P_0 \oplus \mathbb{Z}^{\lambda} = P$, so that H is a maximal pure dense subgroup of P and $P \cong \mathbb{Z}^{\lambda}$. We claim that H is not isomorphic to P. Otherwise, G would be isomorphic to a direct summand of $\mathbb{Z}^{\lambda}$. If λ is less than the first measurable cardinal, then another result of Nunke [13, p. 69, Theorem 5] applies, showing that G is a product P_2

of copies of $\mathbb{Z}$. Thus, $G^* \cong P_2^*$ must be free of countable rank, which implies $G \cong P_0$, contradicting our choice of G. If λ is the first measurable cardinal or even larger, then a parallel result by Eda [5] applies, and again G would be isomorphic to P_0, a contradiction. Since H is maximal pure dense in P, the proof of the proposition is complete. □

5. Torsion groups

The situation for separable p-groups is easily handled since any p-group has a cyclic direct summand whose complements are maximal pure subgroups.

PROPOSITION 5.1. *If G is a p-group such that all its maximal pure subgroups are direct sums of cyclic groups, then G is a direct sum of cyclic groups.*

Note: In the case of separable p-groups, one actually only needs the existence of a single maximal pure subgroup which is a direct sum of cyclic groups.

PROPOSITION 5.2. *If G is a separable p-group which has a maximal pure subgroup which is a direct sums of cyclic groups, then G is a direct sum of cyclic groups.*

Proof. Suppose that M is a maximal pure subgroup which is a direct sum of cyclic groups, then either $G/M \cong \mathbb{Z}(p^n)$ for some finite n, or $G/M \cong \mathbb{Z}(p^\infty)$. In the first case the group splits as $G \cong M \oplus \mathbb{Z}(p^n)$ which is clearly a direct sum of cyclic groups. In the second situation, M is a basic subgroup of G with countable quotient and so, since G is assumed separable, G is a direct sum of cyclic groups by [9, Proposition 68.3]. □

Separability is, of course, necessary for this proposition: the Prüfer group $H_{\omega+1}$ has an upper basic subgroup such that its quotient by this upper basic subgroup is a single copy of $\mathbb{Z}(p^\infty)$, however $H_{\omega+1}$ is not a direct sum of cyclic groups.

The converse situation also holds provided all Ulm invariants are infinite. For if G is a direct sum of cyclic groups with $f_n(G)$ infinite for all n, and M is any maximal pure subgroup, then M is also a direct sum of cyclic groups and, as above, either $G \cong M \oplus \mathbb{Z}(p^n)$ for some n or M is basic in G. In either case, G and M have the same Ulm invariants and hence are isomorphic. Thus, we have established the following proposition.

PROPOSITION 5.3. *If H is a direct sum of cyclic p-groups and each Ulm invariant of H is infinite, then there exists a group G with* $\mathrm{MPSpec}(G) = H$; *in fact $G \cong H$.*

The restriction on the Ulm invariants is necessary for suppose that H is a direct sum of cyclic groups and an Ulm invariant, say $f_N(H)$, is finite and nonzero, but $\{H\} = \mathrm{MPSpec}(G)$ for some separable p-group G. Then as in

Proposition 5.2, G will again be a direct sum of cyclic groups. If the quotient G/H is $\mathbb{Z}(p^{N+1})$, then $G \cong \bigoplus_\alpha \mathbb{Z}(p^{N+1}) \oplus H_1$, where $\alpha = f_N(H)+1$; note $f_N(H_1)=0$. Now choose M to be a direct summand complementing a single cyclic group in H_1. Then $X = \bigoplus_\alpha \mathbb{Z}(p^{N+1}) \oplus M$ is maximal pure in G and so, by assumption, must be isomorphic to H. This forces $f_N(X) = f_N(H)$ which in turn requires that $\alpha = f_N(H)+1 = f_N(H)$, a contradiction since $f_N(H)$ is finite. If $G/H \cong \mathbb{Z}(p^{k+1})$ for some $k \neq N$, choose a decomposition $H = M \oplus \mathbb{Z}(p^{N+1})$ so that $Y = M \oplus \mathbb{Z}(p^{k+1})$ is maximal pure in G and is isomorphic to H. A similar argument shows that this too violates the restriction that $f_N(H)$ is finite. Finally, if $G/H \cong \mathbb{Z}(p^\infty)$ then H is basic in G and so, as observed above, G is again a direct sum of cyclic groups and is actually isomorphic to H. Set $G = \bigoplus_\alpha \mathbb{Z}(p^{N+1}) \oplus G_1$ where $f_N(G_1)=0$. Then $X = \bigoplus_{\alpha-1} \mathbb{Z}(p^{N+1}) \oplus G_1$ is maximal pure but is not isomorphic to H since their Ulm invariants $f_N(X)$ and $f_N(H)$ are different.

The situation for torsion-complete groups is, however, completely different and reminiscent of the situation for the Baer–Specker group; observe that the proof of our next result depends only on the existence of a single maximal pure subgroup which is torsion-complete.

Proposition 5.4. *If G is a p-group such that every maximal pure subgroup is a torsion-complete p-group, then G itself is torsion-complete.*

Proof. Suppose that H is maximal pure in G so that G/H is isomorphic to a subgroup of $\mathbb{Z}(p^\infty)$. If the subgroup is proper, then the quotient is a finite cyclic group and thus G splits as $G = H \oplus X$ where X is finite cyclic. In this case, G is clearly torsion-complete being the direct sum of two such groups. Thus, it remains only to handle the situation where $0 \to H \to G \to Z(p^\infty) \to 0$ is a short pure-exact sequence. Applying the functor $\mathrm{Hom}(\mathbb{Z}(p^\infty), -)$ to this sequence and noting that a group K is torsion-complete if and only if $\mathrm{Pext}(\mathbb{Z}(p^\infty), K) = 0$ (see e.g., [9, Corollary 68.5]), one obtains the sequence $0 = \mathrm{Pext}(\mathbb{Z}(p^\infty), H) \to \mathrm{Pext}(\mathbb{Z}(p^\infty), G) \to \mathrm{Pext}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \to 0$. However, the final term is also zero since $\mathbb{Z}(p^\infty)$ is divisible, and so $\mathrm{Pext}(\mathbb{Z}(p^\infty), G) = 0$, which yields the desired result by applying [9, Corollary 68.5] again. □

We can now show that it is impossible to find a group G with maximal pure spectrum a fixed unbounded torsion-complete group. By the previous Proposition 5.4 such a group would, itself, be torsion-complete and so would have a direct summand which is maximal pure and unbounded torsion-complete. Hence, it will suffice to exhibit a maximal pure subgroup of an unbounded torsion-complete group which is not torsion-complete. Since an unbounded torsion-complete group can be expressed as a direct sum of two torsion-complete groups, one of which is unbounded with a basic subgroup which is countable, it will suffice to show that the latter has a maximal pure subgroup which is not torsion-complete. So suppose that $\overline{B}$ is unbounded torsion-complete with countable basic subgroup B. Then, by adopting the argument of Beaumont and Pierce [1]—see also [10]—one may exhibit a maximal pure subgroup of $\overline{B}$ with endomorphism ring a split extension of the ring of p-adic integers, J_p, by the ideal of small endomorphisms. Such a group is not torsion-complete. Thus we have proved the following.

Proposition 5.5. *There does not exist a p-group G having* $\mathrm{MPSpec}(G) = \{\overline{B}\}$, *where $\overline{B}$ is a fixed unbounded torsion-complete group.*

A considerable amount is known about p-groups which are $\aleph_1$-separable i.e., separable p-groups which have the property that any countable subset may be embedded in a countable direct summand. This is particularly the case when one makes the additional set-theoretic hypothesis $(MA + \neg CH)$—see e.g., the fundamental paper of Megibben [12]. Surprisingly, we do not need any additional set theory to derive our main result on such groups, but unsurprisingly the question of whether a single $\aleph_1$-separable p-group can support a maximal pure spectrum is undecidable. First, we derive a simple and standard proposition; note that as in Proposition 5.4, the existence of a single suitable maximal pure subgroup suffices.

Proposition 5.6. *If G is an uncountable p-group with all its maximal pure subgroups $\aleph_1$-separable, then G is $\aleph_1$-separable.*

Proof. Suppose that M is any maximal pure subgroup of G, so that G/M is countable. Then, as in Lemma 2.5, $G = M + A$ for some countable subgroup A. Now $M \cap A$ is a countable subset of M and so there is a direct decomposition $M = M_0 \oplus M_1$ where $M \cap A \leq M_0$. Set $N = A + M_0$ so that $G = N + M_1$. However, a routine check shows that $N \cap M_1 = \{0\}$, so that $G = N \oplus M_1$. Note that N is countable and so is a direct sum of cyclic groups, and hence certainly $\aleph_1$-separable. However, it is easy to show that the direct sum of two $\aleph_1$-separable p-groups is again $\aleph_1$-separable, so G is $\aleph_1$-separable, as required. □

Theorem 5.7. *If H is an uncountable $\aleph_1$-separable p-group with all its Ulm-invariants infinite, then any group G with maximal pure spectrum* $\mathrm{MPSpec}(G) = \{H\}$, *is isomorphic to H itself.*

Proof. Suppose that G is a p-group such that $\mathrm{MPSpec}(G) = \{H\}$, then by Lemma 5.6, G is again $\aleph_1$-separable. Moreover, G/H is then either a finite cyclic group or the quasi-cyclic group $\mathbb{Z}(p^\infty)$. In the former case, G splits and as the Ulm-invariants of H are all infinite, G is then isomorphic to H. In the latter case, we argue as in Lemma 5.6 that $G = H + C$ for some countable subgroup C, and furthermore $H = D \oplus K$ for a some summand D which contains $H \cap C$. Setting $A = D + C$, one obtains as before $G = A \oplus K$; note that A is still countable. A routine check shows that $A \cap H = D$ and so

$$A/D = A/A \cap H \cong A + H/H = G/H \cong \mathbb{Z}(p^\infty).$$

Now D is Σ-cyclic and pure in G, and hence in A, so D is a basic subgroup of A. Since A itself is countable, it is a direct sum of cyclic groups and thus is basic in itself. It follows immediately that $D \cong A$ and thus $H = D \oplus K \cong A \oplus K = G$, as required. □

The converse question is much trickier and leads to an independence result.

THEOREM 5.8. *If G is an $\aleph_1$-separable p-group of cardinality $\aleph_1$ which is not a direct sum of cyclic groups and $f_n(G)$ is infinite for all n, then*:

(i) $(MA + \neg CH)$ *all maximal pure subgroups of G are isomorphic to G itself i.e., G supports the maximal pure spectrum of a single group which is G itself.*

(ii) $(2^{\aleph_0} < 2^{\aleph_1})$ *G has a maximal pure subgroup which is not $\aleph_1$-separable and hence G does not support the maximal pure spectrum of any group.*

Proof. As noted in Theorem 5.7 above, nondense maximal pure subgroups split and the condition on the Ulm invariants ensures that for any such maximal pure subgroup, M say, its Ulm invariants are also all infinite, and hence $M \cong M \oplus \mathbb{Z}(p^n) \cong G$ for any n. If H is maximal pure dense in G, then the result follows from [12, Theorem 2.6].

However, the assumption $(2^{\aleph_0} < 2^{\aleph}_1)$ allows one to construct an $\aleph_1$-separable p-group G_0 having a maximal pure subgroup H_0 which is not $\aleph_1$-separable—see e.g., [12, Theorem 3.3]. By adding on, if necessary, a direct summand X which is a direct sum of cyclic groups with infinite Ulm-invariants, one obtains a group $G = G_0 \oplus X$ which is $\aleph_1$-separable and has all its Ulm-invariants infinite. However, the subgroup $H = H_0 \oplus X$ is maximal pure in G but not $\aleph_1$-separable since its summand H_0 is not. The final conclusion follows immediately from Theorem 5.7. □

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