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SOLID SEQUENCE F-SPACES OF L_0-TYPE OVER SUBMEASURES ON $\mathbb{N}$

LECH DREWNOWSKI AND IWO LABUDA

Abstract. We study solid sequence F-spaces $\lambda_0(\eta)$, nonseparable in general, and their closed separable subspaces $\lambda_{00}(\eta)$. The space $\lambda_0(\eta)$ is associated with a strictly positive submeasure η on $\mathbb{N}$ and equipped with the topology of convergence in submeasure. While $\lambda_0(\eta)$'s may be viewed as analogs of usual L_0-spaces, the relation between $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$ often resembles that between c_0 and l_∞. For many η's, the weak topology of these spaces coincides with that of coordinate-wise convergence, they are not locally pseudoconvex and yet have the Bounded Multiplier Property. Further, in agreement with the analogy to $L_0 = L_0[0,1]$, they possess copies of l_p for $0 < p \leq 2$, and yet in contrast to L_0 they contain a lot of well-located copies of c_0 and l_∞; also, the quotient $\lambda_0(\eta)/\lambda_{00}(\eta)$ contains a copy of L_0. All of this happens already for the spaces $\lambda_0 = \lambda_0(\bar{d})$ and $\lambda_{00} = \lambda_{00}(\bar{d})$ with $\bar{d}$ being a submeasure closely related to the standard density d, in which case, moreover: (1) There is a series in λ_{00} all of whose subseries of density zero are convergent, and yet its partial sums are unbounded. (2) The Orlicz–Pettis theorem fails in λ_0. (3) λ_{00} can be used to show that some earlier constructed normed barrelled spaces are not ultrabarrelled.

0. Introduction

Our object of investigation are solid nonseparable F-spaces $\lambda_0(\eta)$ of scalar sequences that are curious in many respects. On the one hand, they are analogs of F-spaces $\tilde{L}_0(\mu)$ ($\tilde{L}_0(\mu)$ consists of those measurable functions that

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are bounded outside a set of finite measure; see Section 1.G) over *infinite* positive measure spaces (S,Σ,μ), equipped with the topology of convergence in measure μ on all of S. In our setting, however, the basis for the definition of $\lambda_0(\eta)$ and its topology is a strictly positive submeasure η on $\mathbb{N}$. On the other hand, we distinguish in each of the spaces $\lambda_0(\eta)$ a closed separable subspace $\lambda_{00}(\eta)$, and observe that the relation between $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$ is very much like that between c_0 and l_∞.

In the very beginning, we studied a particular pair of spaces, λ_0 and λ_{00} in the present notation. Actually, we dealt with two variants of these spaces. The first variant of λ_0 was defined as the space of all scalar sequences $x=(\xi_j)$ such that $|||tx|||\to 0$ as $t\to 0$, where

$$|||x|||:=\sup_n |||x|||_n \quad \text{and} \quad |||x|||_n:=\frac{1}{2^n}\sum_{j\in D_n}\min(1,|\xi_j|),$$

and equipped with the F-norm $|||\cdot|||$; λ_{00} was understood as the space of all those x with $|||x|||_n\to 0$; $D_n:=\{j\in\mathbb{N}:2^n\le j<2^{n+1}\}$ for $n=0,1,2,\dots$.

For the second, the definitions were similar using

$$\|x\|:=\sup_n \|x\|_n \quad \text{and} \quad \|x\|_n:=\frac{1}{n}\sum_{j=1}^{n}\min(1,|\xi_j|).$$

These two definitions turned out to be equivalent, yielding the same linear spaces with equivalent F-norms.

We obtained the first variant of λ_0 as a sequential version of a naturally-looking example of an F-space of measurable functions over the half-line $[0,\infty)$ that was constructed while working on [13]. One of the objectives of that paper was to study topological vector spaces that have the Zero-density Convergence Property (ZCP), that is, those in which a series must be subseries convergent provided it is known that all its subseries of density zero are convergent. ('Subseries of density zero' means subseries over the sets of indices of density zero.) The example just mentioned showed that, in general, such a series need not even have bounded partial sums (while it has to in any locally pseudoconvex space or in 'usual' L_0-spaces). By construction, it remained to be so in the sequence spaces λ_0 and λ_{00}.

The second variant of λ_0 came up as a result of an effort to make its definition visibly related to the definition of the standard density d in $\mathbb{N}$,

$$d(A):=\limsup_{n\to\infty}\frac{1}{n}|A\cap\{1,\dots,n\}|.$$

(Density zero sets are just those on which d vanishes.) At this point, we were also able to identify the duals of these spaces as consisting of all finite linear combinations of coordinate functionals, and by means of a very simple series, $\sum_n e_n$, show that the Orlicz–Pettis theorem fails in λ_0. However, in many situations the first variant of λ_0 and λ_{00} has still an important advantage over the second one; one sees at once that these spaces are the l_∞- and c_0-sums of finite-dimensional spaces $L_0(D_n)$ $(n \geq 0)$, respectively.

Since the motivating example, as well as its sequential counterparts, disappeared from the final version of [13], we describe it briefly now. For every measurable function f on $[0,\infty)$ with Lebesgue measure λ, set

$$\|f\|_k = \int_{I_k} \min(1,|f|)\,d\lambda \quad \text{and} \quad \|f\| = \sup_k \|f\|_k,$$

where $I_k = [k,k+1)$ $(k = 0,1,2,\dots)$. Next, define Y to be the space of all f with $\lim_{t\to 0}\|tf\| = 0$. Then $Y = (Y,\|\cdot\|)$ is an F-space, and it is isometric to the l_∞-sum of the spaces $L_0(I_k) \simeq L_0([0,1])$. If $n \in \mathbb{N}$ and $n = 2^k + j$ for some $k \geq 0$ and $0 \leq j < 2^k$, let f_n be the characteristic function of the interval $I_{k,j} = [k + j2^{-k}, k + (j+1)2^{-k}) \subset I_k$. Next, for every $x = (\xi_n) \in \omega$, the space of all sequences, let

$$h_x = \sum_{n=1}^{\infty} \xi_n f_n$$

(the pointwise 'disjoint' sum). Now, and this is what was important, it turns out that in Y the series $\sum_{n=1}^{\infty} \xi_n f_n$ has all its subseries of density zero convergent, but if $|\xi_n| \to \infty$, the sequence of partial sums of the series is unbounded. Note that all this involves only countably simple functions of the type h_x. For such functions, the formula giving $\|h_x\|_k$ can be written as

$$\|h_x\|_k = \frac{1}{2^k} \sum_{n\in D_k} \min(1,|\xi_n|).$$

Therefore, replacing h_x by $x = (\xi_n)$, we may define $\|\cdot\|_k$ and $\|\cdot\|$ in the space ω and proceed to obtain a solid sequence F-space λ_0.

As the investigation of λ_0 and λ_{00} went on, we realized that these spaces are related to the standard density d in a much closer way than expected. In fact, they are *determined* by d because

$$\lambda_0 = \Big\{x \in \omega : \lim_{r\to\infty} d(s(x,r)) = 0\Big\}$$

and

$$\lambda_{00} = \{x \in \omega : s(x,r) \in \mathcal{Z},\ \forall r > 0\}$$

where $s(x,r) = \{j : |\xi_j| > r\}$, and $\mathcal{Z} = \{A \subset \mathbb{N} : d(A) = 0\}$. Moreover, their topology is simply the topology of convergence in submeasure $\bar{d}$, that is defined on $\mathbb{N}$ by

$$\bar{d}(A) := \sup_n \frac{1}{n}|A \cap \{1,\dots,n\}|,$$

and d can be derived from $\bar{d}$ by means of the formula

$$d(A) = \inf\{\bar{d}(A\setminus F) : F \subset A,\ |F| < \infty\}.$$

This prompted us to consider the truly general case. As the starting point, we take a strictly positive submeasure η on $\mathbb{N}$, like $\bar{d}$ above, and next define the core $\eta^\bullet$ of η so that $\eta^\bullet$ is related to η as d to $\bar{d}$ above. By using equalities like those above, we define the spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$ endowing them with the topology τ_η of convergence in submeasure η. Actually, an additional condition of lower semicontinuity (lsc) is imposed on η, and the topology τ_η is defined in all of ω; then $\lambda_0(\eta)$ is introduced as the largest topological vector subspace of (ω, τ_η), and it is shown that $\lambda_0(\eta) = \{x \in \omega : \lim_{r\to\infty} \eta(s(x,r)) = 0\}$.

We now describe the contents of the paper.

Section 1 is preliminary. The terminology and notation as well as a few less known concepts are explained. Some information concerning spaces $L_0(\mu)$ is provided including a few technical facts used later in Section 5.

In Section 2, L_0-like and quasi-L_0-like topological vector spaces are introduced together with a few basic facts about them.

Section 3 is devoted to submeasures on a set S. Here the core $\eta^\bullet$ of a submeasure η is defined and discussed, as are the notions of (strongly) nonatomic and core-nonatomic submeasures.

Section 4 is concerned with submeasures of special types, namely $\bar{d}_{\boldsymbol{\mu}}$ and $d_{\boldsymbol{\mu}}$, that are sup's and lim sup's of sequences $\boldsymbol{\mu} = (\mu_n)$ of finite measures on $\mathbb{N}$. This class includes, in particular, $\bar{d}$ and d (the standard density). Under a mild condition (A) on $\boldsymbol{\mu}$, one has $d_{\boldsymbol{\mu}} = (\bar{d}_{\boldsymbol{\mu}})^\bullet$.

In Section 5, for an *admissible* sequence $\boldsymbol{\mu} = (\mu_n)$, a related sequence $(\|\cdot\|^0_{\mu_n})$ of $L_0(\mu_n)$-F-seminorms, and the FG-norm $\|\cdot\|^0_{\boldsymbol{\mu}} = \sup_n \|\cdot\|^0_{\mu_n}$ are defined in ω and briefly discussed. In particular, it is shown that if the sequence $\boldsymbol{\mu}$ is uniformly bounded, then the FG-norm convergence and the convergence in submeasure $\bar{d}_{\boldsymbol{\mu}}$ coincide (Corollary 5.3).

In Section 6, the F-space $\lambda_0(\boldsymbol{\mu})$ is introduced as the largest vector subspace of ω on which $\|\cdot\|^0_{\boldsymbol{\mu}}$ is an F-norm, together with its closed subspace $\lambda_{00}(\boldsymbol{\mu})$ defined to consist of all $x \in \omega$ for which $\|x\|^0_{\mu_n} \to 0$. Then, for $\boldsymbol{\mu}$ uniformly bounded, a characterization of $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ is given in terms of the submeasure $d_{\boldsymbol{\mu}}$ (Proposition 6.4), and the uniform boundedness is shown to be necessary for that (Proposition 6.5).

In Section 7, inspired by Corollary 5.3 and Proposition 6.4 just mentioned, we finally reach the general setting. For an *arbitrary* strictly positive submeasure η on $\mathbb{N}$ that is lower semi-continuous (i.e., $\eta(A_n) \to \eta(A)$ whenever $A_n \uparrow A$), we introduce in ω the topology τ_η of convergence in submeasure η, together with an FG-norm $\|\cdot\|_\eta$ defining it, and then distinguish $\lambda_0(\eta)$ as the largest vector subspace of ω on which τ_η is a vector topology (and $\|\cdot\|_\eta$ is an F-norm). We go on to characterize $\lambda_0(\eta)$, and introduce its closed subspace $\lambda_{00}(\eta)$, in full agreement with what we had previously in Proposition 6.4.

A systematic investigation of the F-spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$ follows. Proposition 7.13 shows that the dual of $\lambda_{00}(\eta)$ consists merely of finite linear combinations of coordinate functionals iff $\eta(n) \to 0$. In the particular case of the space $\lambda_{00} = \lambda_{00}(\bar{d})$, Corollary 7.12 shows that there exists a series in λ_{00} whose subseries of density zero converge, and yet the sequence of partial sums of the series is unbounded. Theorem 7.16 describes how our spaces depend on whether or not $\eta^\bullet \neq 0$, and on the behavior of the sequence $(\eta(n))$. For instance, if $\eta^\bullet \neq 0$ and $\eta(n) \to 0$, $\lambda_{00}(\eta)$ is not locally pseudoconvex. At the end of the section, we use the space λ_{00} to prove that certain barrelled normed sequence spaces are not ultrabarrelled (Proposition 7.18). Also, in Remark 7.19, we clarify the relations between the two types of spaces: those based on sequences $\boldsymbol{\mu}$ vs. those based on submeasures η.

Section 8 deals with what we call simple embeddings into $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$. It is shown, for instance, that if $\eta^\bullet \neq 0$, then there exist isomorphic embeddings $T : l_\infty \to \lambda_0(\eta)$ such that $\lambda_{00}(\eta) \cap T(l_\infty) = T(c_0)$, as well as such that $\lambda_{00}(\eta) \cap T(l_\infty) = \{0\}$ (Corollary 8.3). In consequence, $\lambda_{00}(\eta)$ is then uncomplemented in $\lambda_0(\eta)$ (Corollary 8.4).

In Section 9, the quotient $\lambda_0(\eta)/\lambda_{00}(\eta)$ is considered. It is shown to be L_0-like (and hence have a zero dual) if η is core-nonatomic (Theorem 9.2), and to contain an isomorphic copy of l_∞/c_0 if $\eta^\bullet \neq 0$ (Theorem 9.3).

In Section 10, in the core-nonatomic case, the duals of $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$ are identified as consisting of finite linear combinations of the coordinate functionals (Theorem 10.1). As a consequence, the Orlicz–Pettis theorem is shown to fail in $\lambda_0(\eta)$ provided $\eta^\bullet \neq 0$ (Corollary 10.4).

In Section 11, again for η core-nonatomic, the results of the preceding section are considerably strengthened by proving that $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$ are quasi-L_0-like (Theorem 11.1). In consequence, their complemented locally pseudoconvex subspaces are precisely those that are of finite dimension or isomorphic to ω (Corollary 11.5).

Section 12 concerns (topologically or metrically) bounded subsets of $\lambda_0(\eta)$. For instance, for any bounded set in $\lambda_0(\eta)$, its closure in ω is again a bounded set in $\lambda_0(\eta)$ (Proposition 12.1).

In Section 13, we return to spaces $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$, but mostly in the special case of μ_n's having pairwise disjoint supports. In this case, the spaces are isometrically isomorphic to the l_∞- and c_0-sums of the spaces $L_0(\mu_n)$ ($n \in \mathbb{N}$), respectively. Here, we first verify rigorously that the two variants of λ_0 and λ_{00} which we discussed earlier are indeed identical with equivalent F-norms. Then we prove that in any c_0-sum of F-spaces which have no nontrivial locally bounded subspaces, any closed locally bounded subspace of infinite dimension contains a copy of c_0 that is 'extendable' to a copy of l_∞ in the respective l_∞-sum (Theorem 13.3). In particular, this result is applicable to $\lambda_{00}(\boldsymbol{\mu})$ and $\lambda_0(\boldsymbol{\mu})$ for $\boldsymbol{\mu}$ as specified above, but we also find a way to extend it to general $\boldsymbol{\mu}$'s (Theorem 13.8).

From this point on, we consider a still more special case of the sequence $\boldsymbol{\mu}$ of probability measures with pairwise disjoint supports. The three main results are Theorems 13.12, 13.13, and 13.15. In the first, it is shown that, for each $0 < p \leq 2$, $\lambda_{00}(\boldsymbol{\mu})$ contains an isomorphic copy of the c_0-sum $(\sum_n l_p^n)_0$ which, in addition, is 'extendable' to an isomorphic copy of the l_∞-sum $(\sum_n l_p^n)_\infty$ in $\lambda_0(\boldsymbol{\mu})$; in particular, $\lambda_0(\boldsymbol{\mu})$ contains a copy of l_p. In the second, $\lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$ is proved to contain a copy of $L_0[0,1]$. In the third, a Schwartz locally convex subspace with a Schauder basis that is not isomorphic to ω is exhibited in $\lambda_{00}(\mathbf{S}) = \lambda_{00}(\boldsymbol{\mu})$ for $\boldsymbol{\mu}$ consisting of uniform probability measures determined by a sequence $\mathbf{S} = (S_n)$ of disjoint finite sets with union $\mathbb{N}$ and $|S_n| \to \infty$.

In Section 14, we show that all the spaces $\lambda_0(\boldsymbol{\mu})$ as well as their quotients $\lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$ have the Bounded Multiplier Property. That is, whenever a series $\sum_n z_n$ in a space is unconditionally convergent, so is the series $\sum_n t_n z_n$ for all bounded sequences (t_n) of scalars.

In Section 15, we gather our main results in the case of spaces λ_0 and λ_{00}.

1. Some terminology, notation and facts

Our terminology and facts used are mostly standard: see [19] for topological vector spaces; [27] and [21] for F-spaces and p-Banach spaces, (Schauder) bases, basic sequences and their block sequences (also see [6]); and [1] for locally solid Riesz spaces. For the reader's convenience, we recall some frequently used notions and facts, and fix some notation.

1.A. General notation. Given a set S, we let $\mathcal{F}(S)$ stand for the family of all finite subsets of S, and $\mathcal{P}(S)$ for the power set of S. For a finite set A, we denote by $|A|$ the number of its elements. We write $\mathcal{F}$ and $\mathcal{P}$ when $S = \mathbb{N} = \{1, 2, \dots\}$. If $m, n \in \mathbb{N}$, then $[m,n] := \{k \in \mathbb{N} : m \leq k \leq n\}$ and $[n] = [1,n]$; other types of intervals in $\mathbb{N}$, like $[m,n)$ or $[m,\infty)$ are understood likewise.

Whenever we speak about a set function, say μ, on a set S, it is defined on the σ-algebra $\mathcal{P}(S)$, and we write $\mu(s)$ instead of $\mu(\{s\})$ for each $s \in S$.

For a scalar function f defined on a set S, we denote its *support* by $s(f)$, and its *support at level* $r > 0$ by $s(f,r)$:

$$s(f) := \{s \in S : f(s) \neq 0\} \quad \text{and} \quad s(f,r) := \{s \in S : |f(s)| > r\}.$$

Likewise, if μ is a positive measure on S, then its *support* $s(\mu)$ is defined as

$$s(\mu) = \{s \in S : \mu(s) > 0\}.$$

1.B. Topological vector spaces. TVS stands for *topological vector space*. An *F-space* is a complete metrizable TVS, i.e., one whose topology can be defined by a complete F-norm. A *p-Banach space* is a complete p-normed space. To avoid any ambiguity, we include some additional explanations.

Let X be a vector space and $q : X \to [0,\infty)$. Then q is an *F-seminorm* [*F-norm*] if $q(x+y) \leq q(x) + q(y)$, $q(tx) \leq q(x)$ whenever $|t| \leq 1$, $q(tx) \to 0$ as $t \to 0$ [and $q(x) = 0$ only when $x = 0$]. If $0 < \alpha \leq 1$ and, instead of the second and third conditions, q satisfies a stronger condition $q(tx) = |t|^\alpha q(x)$, it is an *α-seminorm* [*α-norm*]; we say simply *seminorm* [*norm*] when $\alpha = 1$. In each of these cases (including that of FG-norms below), $(x, y) \to q(x - y)$ is the translation invariant semimetric [metric] associated with q.

Given a family Q of F-seminorms (α-seminorms; seminorms), it converts X into a TVS (resp., locally α-convex space; locally convex space), see e.g., [19, Sections 2.7 and 6.5]. If each $q \in Q$ is an α-seminorm for some $\alpha = \alpha(q)$, then the resulting space is *locally pseudoconvex*, see [19, p. 109] or [27, Sec. 3.1]. For the converse results, see [19, Prop. 2.7.3 and Th. 6.5.1] and [27, Th. 3.1.4].

We shall constantly deal with sequence F-spaces continuously included in the space ω of all scalar sequences. The following consequence of the closed graph theorem is therefore worth keeping in mind.

FACT 1.1. *If X and Y are F-spaces that are continuously contained in a Hausdorff* TVS *Z, and $X \subset Y$, then the inclusion is continuous. In particular, if $X = Y$, then their F-norms are equivalent.*

An *F-lattice* is a complete metrizable topological Riesz space (or vector lattice); thus its topology can be defined by a complete monotone F-norm. In an F-lattice, every *disjoint* sequence (u_n) of nonzero elements is an unconditional Schauder basic sequence. In fact, it is even a *bounded multiplier* basic sequence: if a series $\sum_n \xi_n u_n$ converges, so does $\sum_n \alpha_n \xi_n u_n$ for all $(\alpha_n) \in l_\infty$.

1.C. The space ω. We denote by ω the locally convex F-lattice of all scalar sequences equipped with the product (or coordinate-wise convergence) topology. If $A \subset \mathbb{N}$, its characteristic function e_A is viewed as a sequence of 0's and 1's (so that $e_A \in \omega$). In particular, $e_n := e_{\{n\}}$ for $n \in \mathbb{N}$ are the *unit vectors* in ω. For each $x = (\xi_j) \in \omega$, its *support* $s(x)$, and its *support* $s(x, r)$ *at level* $r > 0$ are understood as explained earlier, and its nth section x^n $(n \in \mathbb{N})$ is defined as

$$x^n := \sum_{j=1}^{n} \xi_j e_j = x e_{[n]}.$$

Here, for $x \in \omega$ and $A \subset \mathbb{N}$, xe_A is the coordinate-wise product of x and e_A.

A *solid sequence space* is a solid vector subspace (or an ideal) in ω equipped with a locally solid Riesz topology. This class of spaces includes, in particular, solid sequence F-spaces and Banach spaces. Standard examples are c_0, l_∞, and l_p for $0 < p < \infty$, with the norms denoted $\|\cdot\|_\infty$ and $\|\cdot\|_p$, respectively.

1.D. $F\omega$-spaces and spaces with poor duals. Let $X = (X, \tau)$ be an infinite-dimensional metrizable TVS (F-normed space) with separating (or total) dual X'. Its Mackey topology $\mu(X, X')$ coincides with the strongest locally convex topology $\tau^c \leq \tau$. Hence, it is metrizable, because the absolutely convex hulls of τ-neighborhoods of zero form a base for the τ^c-neighborhoods at zero. Further, $\dim X' = \aleph_0$ iff the weak topology $\sigma(X, X')$ is metrizable iff $\sigma(X,X')=\tau^c$ iff (X,τ^c) is isomorphic to a (dense) subspace of ω. If this happens and X is an F-space, we call it an *$F\omega$-space* (see [7]).

We shall say that a TVS X has *poor* dual, or that its dual X' is *poor*, if X' is separating and $\dim X'=\aleph_0$.

FACT 1.2. *Let X be a Hausdorff solid sequence space containing all the unit vectors e_n, and let X_0 be their closed linear span in X. Then both X and X_0 have separating duals, and the following hold*:

(a) *the dual X_0' is poor iff $X_0'=\operatorname{lin}(e_n')$, and*
(b) *the dual X' is poor and $(X/X_0)'=\{0\}$ iff $X'=\operatorname{lin}(e_n')$,*

where (e_n') is the sequence of coordinate functionals on X_0 or X, respectively.

Proof. Let Y stand for either X or X_0. Since the projections $P_A : x \to xe_A$ $(A\subset\mathbb{N})$ are (equi-) continuous on Y, the inclusion $Y\subset\omega$ is continuous. Hence, $\operatorname{lin}(e_n')\subset Y'$ and thus Y' is separating.

The 'if' direction is obvious in either of (a) and (b).

To verify the converse implications, assume that $\dim Y'=\aleph_0$, and take any $x'\in Y'$. We claim that the set $M=\{n : x'(e_n)\neq 0\}$ is finite. Suppose it is not so, and let $(M_i)_{i\in I}$ be a family of cardinality $2^{\aleph_0}$ consisting of almost disjoint infinite subsets of M. Then the functionals $x_i'=x'\circ P_{M_i}$ on Y are obviously continuous and, moreover, they are linearly independent. In fact, let $i_1,\dots,i_k$ be distinct indices in M such that $\alpha_1x_{i_1}'+\dots+\alpha_kx_{i_k}'=0$ for some scalars $\alpha_1,\dots,\alpha_k$. Given any $j\in[k]$, there is $n_j\in M_{i_j}$ that belongs to no other of the sets $M_{i_1},\dots,M_{i_k}$. Then $0=(\alpha_1x_{i_1}'+\dots+\alpha_kx_{i_k}')(e_{n_j})=\alpha_jx_{i_j}'(e_{n_j})=\alpha_jx'(e_{n_j})$ so that $\alpha_j=0$. It follows that $\dim X_0'\geq 2^{\aleph_0}$, contradicting the assumption, and proving the claim.

Let $y'=\sum_{n\in M}x'(e_n)e_n'$. Then $(x'-y')(e_n)=0$ for all n, hence $x'-y'=0$ on X_0 (because (e_n) is a Schauder basis of X_0). If the dual of X/X_0 is trivial, then we also conclude that $x'-y'=0$ on X. This completes the proof. □

1.E. Topological vector groups. A (locally balanced) *topological vector group* (TVG) (cf. [9] and references therein) is a vector space X with a topology τ such that the addition $(x,y)\to x+y$ is continuous, while the scalar multiplication $(t,x)\to tx$ is required to be continuous only at point $(0,0)$. Then, for each scalar t, the map $x\to tx$ in X is continuous, but the maps $t\to tx$ $(x\in X)$ need not be continuous (at zero). In other terms, the topology of a TVG X is required to be compatible with the group structure of $(X,+)$ and have a base for the neighborhoods of zero consisting of balanced sets (but they need not be absorbing). If $X=(X,\tau)$ is a TVG, then

$$v(X)=v(X,\tau):=\Big\{x\in X:\ \lim_{t\to 0}tx=0\Big\}$$

is easily seen to be the largest vector subspace of X which, in the induced topology, is a TVS. Moreover, $v(X)$ is a closed subspace of X.

FACT 1.3. *Let a* TVG $X=(X,\tau)$ *have a base for the neighborhoods of zero consisting of sets that are closed in another topological vector group topology* ρ *on* X. *Then the* ρ*-closure of any bounded subset* B *of* $v(X)=v(X,\tau)$ *is again a bounded subset of* $v(X)$.

Proof. Let U be a ρ-closed τ-neighborhood of zero in X. Then $aB\subset U$ for some $a>0$. Since the map $x\to ax$ is an autohomeomorphism of (X,ρ), $a\overline{B}^{\rho}=\overline{aB}^{\rho}\subset U$. This also shows that $\overline{B}^{\rho}\subset v(X)$. □

The case most important for us is when X is an *FG-normed space*, that is, has a topology determined by an *FG-norm* $\|\cdot\|: X\to[0,\infty]$ (note that the value ∞ is allowed) that satisfies all but the third condition in the definition of an F-norm. Then, obviously,

$$v(X)=v(X,\|\cdot\|)=\Big\{x\in X:\ \lim_{t\to 0}\|tx\|=0\Big\},$$

and $\|\cdot\|$ on $v(X)$ is an F-norm. Hence, if the FG-normed space $(X,\|\cdot\|)$ is complete, $v(X)$ is an F-space.

Many FG-norms arise as suprema of sequences of F-seminorms.

FACT 1.4. *If* $(\|\cdot\|_n)$ *is a total sequence of* F*-seminorms on a vector space* X, *then* $\|\cdot\|:=\sup_n\|\cdot\|_n$ *is an* FG*-norm on* X, *and*

$$v_0(X):=\Big\{x\in X:\ \lim_{n\to\infty}\|x\|_n=0\Big\}$$

is a closed subspace of $v(X)=v(X,\|\cdot\|)$.

Proof. Simply note that the FG-seminorm $\limsup_n\|\cdot\|_n$ is $\le\|\cdot\|$, hence continuous on $(X,\|\cdot\|)$, and that $v_0(X)$ is its kernel. □

1.F. l_∞- and c_0-sums of F-spaces. Given a sequence (F_k) of F-spaces $F_k=(F_k,\|\cdot\|_k)$, the formula

$$\|x\|=\sup_k\|x_k\|_k\quad\text{for } x=(x_k)$$

defines a complete FG-norm $\|\cdot\|$ in the product F of the F_k's, and $v(F)=\{x\in F:\|tx\|\to\ 0 \text{ as } t\to 0\}$ is the largest subspace of F on which $\|\cdot\|$ is an F-norm. (Note of caution: condition $\|tx\|\to 0$ as $t\to 0$ is used here, not the weaker condition that $\|x\|<\infty$!) Equipped with this F-norm, $E_\infty=v(F)$ is an F-space, called the l_∞*-sum* of the sequence (F_k). The closed subspace E_0 of F consisting of all $x=(x_k)$ with $\lim_k\|x_k\|_k=0$ is automatically a closed subspace of E_∞ (Fact 1.4), hence an F-space, and is called the c_0*-sum* of the sequence (F_k). These l_∞- and c_0-sums are often denoted $(\sum_k F_k)_\infty$ and $(\sum_k F_k)_0$, respectively.

1.G. $L_0(\mu)$ **spaces.** Let (S,Σ,μ) be a positive measure space. $L_0(\mu)\equiv L_0(S)$ is the space of all (Σ-) measurable scalar functions on S (with the usual identification of functions that are equal μ-almost everywhere). We equip $L_0(\mu)$ with the topology τ_μ of convergence in measure μ $\underline{\text{on } S}$. It converts $L_0(\mu)$ into a complete metrizable TVG. A base of balanced neighborhoods of zero for τ_μ is given by the sets

$$\{f\in L_0(\mu):\mu(s(f,\varepsilon))\le\varepsilon\},\quad \varepsilon>0.$$

Alternatively, τ_μ is determined by the monotone FG-norm $\|\cdot\|_\mu$ defined by

$$\|f\|_\mu=\inf\{\varepsilon>0:\mu(s(f,\varepsilon))\le\varepsilon\}.$$

Note that $|f_n|\uparrow|f|$ implies $\|f_n\|_\mu\uparrow\|f\|_\mu$.

$$\tilde{L}_0(\mu):=v(L_0(\mu),\|\cdot\|_\mu)=\Big\{f\in L_0(\mu):\lim_{t\to 0}\|tf\|_\mu=0\Big\}$$

with the F-norm $\|\cdot\|_\mu$ is an F-space, by Section 1.E. Moreover, as easily seen, if $f\in L_0(\mu)$, then

$$f\in\tilde{L}_0(\mu)\quad\Longleftrightarrow\quad\lim_{r\to\infty}\mu(s(f,r))=0,$$

which in turn is equivalent to the existence of an $r>0$ with $\mu(s(f,r))<\infty$; that is, f is bounded outside a set of finite μ measure. Note that the spaces $L_p(\mu)$ for $0<p\le\infty$ are continuously embedded in $\tilde{L}_0(\mu)$.

If the measures space (S,Σ,μ) is finite, $\tilde{L}_0(\mu)=L_0(\mu)$. In this case, some 'integral' F-norms defining τ_μ and thus equivalent to $\|\cdot\|_\mu$ (see Fact 1.6 below) are often used, e.g.,

$$\|f\|_\mu^0=\int_S\min(1,|f|)\,d\mu.$$

Note that

$$\max(\|f\|_\mu,\|f\|_\mu^0)\le\mu(s(f))\le\mu(S).$$

We stress that, for us, $\|\cdot\|_\mu^0$ is $\underline{\text{the standard}}$ F-norm in all of the L_0-spaces over finite measures that will enter the scene in the course of the paper.

FACT 1.5. *If μ is a finite measure, then for all $f\in L_0(\mu)$ and scalars $r,t>0$,*

(a) $\mu(s(f,r))\le\|r^{-1}f\|_\mu^0$, *and*
(b) $\|tf\|_\mu^0\le\mu(s(f,r))+\min(1,tr)\mu(S)$.

Proof. This is easy:

$$\mu(s(f,r))=\int_{s(f,r)}\min(1,|r^{-1}f|)\,d\mu\le\|r^{-1}f\|_\mu^0$$

and

$$\begin{aligned}\|tf\|_\mu^0&=\int_{s(f,r)}\min(1,|tf|)\,d\mu+\int_{S\setminus s(f,r)}\min(1,|tf|)\,d\mu\\&\le\mu(s(f,r))+\min(1,tr)\mu(S).\end{aligned}$$

□

FACT 1.6. *If $\mu(S) \le m < \infty$, then for every $f \in L_0(\mu)$,*

$$\frac{1}{m}(\|mf\|_\mu)^2 \le \|f\|_\mu^0 \le \big(1+\mu(S)\big)\|f\|_\mu,$$

where one may replace m with 1 when $m \le 1$.

Proof. If $\|f\|_\mu < \varepsilon$, then from (b) above with $r = \varepsilon$ and $t = 1$ one gets $\|f\|_\mu^0 \le (1+\mu(S))\varepsilon$. From this the second inequality follows.

To verify the other inequality, assume first that $m \le 1$, and take any $\varepsilon > 0$ such that $\|f\|_\mu^0 \le \varepsilon^2 \le m$. Then $s(f,\varepsilon) \subset \{s : \min(1,|f(s)|) \ge \varepsilon\}$ and, therefore, $\mu(s(f,\varepsilon)) \le \varepsilon^{-1}\|f\|_\mu^0 \le \varepsilon$. It follows that $(\|f\|_\mu)^2 \le \|f\|_\mu^0$.

In the general case, let $\nu = m^{-1}\mu$. Then, by the case already proved, $(\|f\|_\nu)^2 \le \|f\|_\nu^0$. But, as easily verified, $\|f\|_\nu^0 = m^{-1}\|f\|_\mu^0$ and $\|f\|_\nu = m^{-1} \times \|mf\|_\mu$. Substituting this into the previous inequality, one gets the first inequality of the fact. □

If S is a finite nonempty set, then it is usually considered with its uniform probability measure μ_S. Thus, $\mu_S(A) = |A|/|S|$ for all $A \subset S$ and

$$\|f\|^0 = \int_S \min(1,|f|)\,d\mu_S = \frac{1}{|S|}\sum_{s\in S}\min(1,|f(s)|)$$

is the standard F-norm in $L_0(S)$. In the case $S \subset \mathbb{N}$, we shall usually treat μ_S as a measure on $\mathbb{N}$ by setting

$$\mu_S(A) = \mu_S(A\cap S) = \frac{|A\cap S|}{|S|} \quad \text{for } A \subset \mathbb{N}.$$

Finally, let us note the following.

FACT 1.7. *If μ is a finite positive measure on $\mathbb{N}$, and $S := s(\mu)$ is its support, then the F-space $L_0(\mu) = L_0(S,\mu)$, with its standard F-norm $\|x\|_\mu^0 = \int_{\mathbb{N}} \min(1,|x|)\,d\mu$, is either finite dimensional or isomorphic to ω, depending on whether S is finite or not.*

NOTE. For each $x \in \omega$, the equivalence class $[x] \in L_0(\mu)$ is represented by $x|S \in L_0(S,\mu)$.

2. Quasi-L_0-like spaces

Let $E = (E,\tau)$ be a TVS. We shall say that E (or its topology τ) is

- *L_0-like* if for every neighborhood U of zero in E there is an $m \in \mathbb{N}$ such that $E \subset U^{(m)} := U + \cdots + U$ (m summands);
- *quasi-L_0-like* if for every neighborhood U of zero in E there is a finite-codimensional subspace L in E and an $m \in \mathbb{N}$ such that $L \subset U^{(m)}$. (Since $\overline{L} \subset L + U$, and hence $\overline{L} \subset U^{(m+1)}$, L may be assumed closed.)

Note that for E to be L_0-like is the same as to be an additively bounded subset of itself, see [30, Sec. 0.3.10]; sometimes, as in [13, p. 56] or as in Section 12 of this paper, additively bounded sets are also called metrically bounded.

If μ is a finite atomless positive measure, $L_0(\mu)$ is L_0-like. Many of the F-spaces discussed later in this paper are quasi-L_0-like, but not L_0-like.

Fact 2.1. *If a* TVS *E is (quasi-) L_0-like, so is E equipped with any weaker vector topology, and so is each quotient space of E.*

The following is a fairly known fact, at least for the $L_0(\mu)$ spaces mentioned above. It can be shown by simplifying the proof of our next result.

Proposition 2.2. *Let p be a subadditive functional on an L_0-like* TVS *E. If p is bounded in a neighborhood of zero, then it is bounded on E.*

In consequence, if p is a continuous α-seminorm on E, then $p=0$.

Proposition 2.3. *Let E be a quasi-L_0-like* TVS, *and p a subadditive functional on E. If p is bounded in a neighborhood U of zero, and a subspace $L \subset X$ of finite-codimension is such that $L \subset U^{(m)}$ for some m, then p is bounded on L. In particular, if p is an α-seminorm, then $p=0$ on L.*

In consequence, if p is a continuous α-seminorm on E, then $\ker p$ *is of finite codimension and, therefore, p is equivalent to a seminorm.*

Proof. By the assumption, there is a constant $c>0$ such that $|p(x)| \le c$ whenever $x \in U$. If $x \in L$, then $x = x_1 + \cdots + x_m$ for some $x_1, \dots, x_m \in U$ and, therefore,

$$p(x) \le \sum_{i=1}^{m} p(x_i) \le mc \quad \text{and} \quad -p(x) \le p(-x) - p(0) \le mc + c$$

so that $|p(x)| \le (m+1)c$. If p is an α-seminorm, then $p(tx) = t^\alpha p(x) \le (m+1)c$ for all $t>0$, whence $p(x)=0$. This proves the main assertion.

Now, if p is a continuous α-seminorm then, by the previous part, $\ker p$ is of finite codimension and, of course, closed. Let $M \subset E$ be a finite-dimensional subspace complementary to $\ker p$, and $P : E \to M$ the associated (continuous) projection. To get a required seminorm q, choose any norm r on M and then extend it to all of E by setting $q(x) = r(Px)$ for $x \in E$. □

Corollary 2.4. *For a* TVS *$E=(E,\tau)$, let τ^{pc} denote the finest locally pseudoconvex topology in E that is weaker than τ.*

(a) *If E is L_0-like, then $E' = \{0\}$ and τ^{pc} is the trivial topology.*
(b) *If E is quasi-L_0-like, then $\tau^{\mathrm{pc}} = \sigma(E,E')$.*

Proof. (a) follows from Proposition 2.2.

(b) τ^{pc} is determined by the family of all continuous α-seminorms p, where α may depend on p. By Proposition 2.3, each of those p's is equivalent to a seminorm with a finite-codimensional kernel. Hence, τ^{pc} is locally convex and $\sigma(E,E') \le \tau^{\mathrm{pc}}$. The converse inequality holds because if

q is a continuous seminorm on E with a finite-codimensional kernel, then $q \le \max\{|x'_1|, \dots, |x'_k|\}$ for some $x'_1, \dots, x'_k$ in E', by the Hahn–Banach theorem. □

COROLLARY 2.5. *Let $E = (E, \tau)$ by an infinite-dimensional metrizable quasi-L_0-like* TVS *with separating dual. Then E has poor dual, $\sigma(E, E') = \tau^{c} = \tau^{pc}$ is metrizable, and $(E, \sigma(E, E))$ is isomorphic to a dense subspace of ω.*

Proof. This follows from the assumptions and Corollary 2.4(b), using the facts collected in Section 1.D. For future reference, in the proof of Corollary 11.3, we give a direct argument showing that $\dim X' = \aleph_0$.

Let (U_n) be a base for the neighborhoods of zero in E, and (L_n) a sequence of finite-codimensional closed subspaces of E such that $L_n \subset U_n^{(m_n)}$ for some m_n $(n \in \mathbb{N})$. For each n, choose a finite set $F_n \subset E'$ such that L_n is the intersection of the kernels of the functionals in F_n. Now, if $x' \in E'$ then, by Proposition 2.3, $\ker x' \supset L_n$ for some n, and hence $x' \in \operatorname{lin} F_n$. Therefore, E' is the linear span of the union of the F_n's, hence $\dim E' = \aleph_0$. □

COROLLARY 2.6. *Let $E = (E, \tau)$ be a quasi-L_0-like F-space with separating dual. Then a locally pseudoconvex closed subspace Z of E is complemented iff it is either finite-dimensional or isomorphic to ω.*

Proof. Since E' is separating, all finite-dimensional subspaces of E are complemented. So let Z be an infinite-dimensional locally pseudoconvex closed subspace of E.

Before proceeding, recall from Corollary 2.5 that $\tau^{pc} = \sigma(E, E')$ and that (E, τ^{pc}) is isomorphic to a dense subspace of ω. Hence, the completion F of (E, τ^{pc}) is isomorphic to ω.

Assume that there is a continuous linear projection P from E onto Z. Then P is also continuous when E is considered with the topology τ^{pc}, hence P has an extension to a continuous linear operator $\tilde{P}$ from F onto Z. Thus, Z is isomorphic to a quotient of ω and, consequently, Z is isomorphic to ω itself (see [2, Th. 4]).

Conversely, assume that Z is isomorphic to ω. Then Z is a minimal space (see [6, Prop. 3.1(c)] and its proof for more information), hence $\tau|Z = \tau^{pc}|Z$. It follows that there is a continuous linear projection Q from F onto Z, and then $Q|E$ is automatically a continuous projection from E onto Z. □

COROLLARY 2.7. *A quasi-L_0-like F-space with separating dual is an $F\omega$-space.*

By essentially the same proof as for Corollary 2.6, one shows the following corollary.

COROLLARY 2.8. *In an $F\omega$-space, a locally convex closed subspace is complemented iff it is either finite-dimensional or isomorphic to ω.*

3. General submeasures on a set

Let S be an arbitrary set. A set function $\eta : \mathcal{P}(S) \to [0,\infty]$ is a *submeasure* on S if it is subadditive, nondecreasing, and $\eta(\emptyset) = 0$; we let

$$\mathcal{Z}(\eta) := \{A \subset S : \eta(A) = 0\}$$

stand for the ideal of all η-zero sets. We shall often assume that η is *lower semicontinuous* (lsc), that is,

$$\text{(lsc)} \qquad \eta(A_n) \uparrow \eta(A) \quad \text{whenever } A_n \uparrow A.$$

It is easy to see that if S is countable, this condition is implied by its apparently weaker form in which the sets A_n are assumed to be finite. Also, it follows from (lsc) that η is countably subadditive. (For more information on submeasures on rings of sets, see [3].)

A submeasure η on S is said to be *strongly nonatomic* if for every $\varepsilon > 0$ there is a finite partition $A_1, \dots, A_k$ of S such that $\eta(A_j) < \varepsilon$ for each j. Then, obviously, $\mathcal{F}(S) \subset \mathcal{Z}(\eta)$ and $\eta(S) < \infty$. (See [17, Sec. 6] for more information and other notions of nonatomicity for submeasures.)

From now on, we shall abbreviate *strongly nonatomic* to *nonatomic.*

The following fact is implicit in [17] as an immediate consequence of Propositions 3.1, 3.4, and 6.3 stated in that paper, but the proof given below is straightforward.

FACT 3.1. *If a submeasure η on S is nonatomic, then every infinite set $A \subset S$ contains an infinite set $B \in \mathcal{Z}(\eta)$.*

Proof. Assume $A \notin \mathcal{Z}(\eta)$ and apply the nonatomicity of η to produce a decreasing sequence (A_n) of subsets of A such that $0 < \eta(A_n) < 1/n$. Next, choose a sequence of distinct points (c_n) such that $c_n \in A_n$, and then define $B_n = A_n \cup \{c_1, \dots, c_n\}$. Since $\eta(B_n) = \eta(A_n) < 1/n$, $B = \bigcap_n B_n \in \mathcal{Z}(\eta)$ and $c_n \in B$ for all n. ☐

For every submeasure η on S, we define its *core* submeasure $\eta^\bullet$ by

$$\eta^\bullet(A) := \inf\{\eta(A \setminus F) : F \in \mathcal{F}(A)\};$$

it will play an important role in what follows. It can be viewed as the inverse image of the quotient submeasure $\hat{\eta}$ on the algebra $\mathcal{P}(S)/\mathcal{F}(S)$. Note that $\eta^\bullet \le \eta$ and $\mathcal{F}(S) \subset \mathcal{Z}(\eta^\bullet)$, and that $\eta^\bullet$ is the largest submeasure satisfying these two conditions. Hence, $\eta^\bullet = \eta$ iff $\mathcal{F}(S) \subset \mathcal{Z}(\eta)$; in particular, $(\eta^\bullet)^\bullet = \eta^\bullet$.

REMARK 3.2. The core of a submeasure, without giving it any name, has already been used implicitly or explicitly in some earlier works, see e.g., [28] or [29] (the definition of $\mathrm{Exh}(\phi)$), or [18] (where $\|A\|_\phi$ stands for our $\phi^\bullet(A)$).

FACT 3.3. *Let η be a submeasure on S. If*

$$(*) \qquad \inf_{s \in S \setminus F} \eta(s) > 0 \quad \textit{for some } F \in \mathcal{F}(S),$$

then $\mathcal{Z}(\eta^\bullet) = \mathcal{F}(S)$. *Conversely, if* $\mathcal{Z}(\eta^\bullet) = \mathcal{F}(S)$ *and* η *is countably subadditive, then* η *satisfies condition* $(*)$.

Proof. The first statement is obvious. To prove the other one, suppose $(*)$ is false. Then there is an infinite set $A = \{a_n : n \in \mathbb{N}\} \subset S$ such that $\eta(a_n) \to 0$, and we may of course assume that $\sum_n \eta(a_n) < \infty$. Then, using countable subadditivity of η, it is easily seen that $\eta(A \setminus \{a_1, \dots, a_n\}) \to 0$ as $n \to \infty$. Hence, $A \in \mathcal{Z}(\eta^\bullet)$, which is impossible. □

FACT 3.4. *Let* η *be a submeasure on* S *and let* $A \subset S$. *Then* $\eta^\bullet(A) = 0$ *iff there is a countable set* $A_0 \subset A$ *such that* $\eta(A \setminus A_0) = 0$ *and* η *is order continuous on* A_0, *that is, whenever* $A_n \subset A_0$ *and* $A_n \downarrow \emptyset$, *then* $\eta(A_n) \to 0$.

In particular, if the set A *is countable, then* $\eta^\bullet(A) = 0$ *iff* η *is order continuous on* A.

Proof. 'Only if.' Let $\eta^\bullet(A) = 0$. Then there is an increasing sequence (F_n) in $\mathcal{F}(A)$ such that $\eta(A \setminus F_n) \to 0$. Let A_0 be the union of the F_n's. Then $\eta(A \setminus A_0) \le \eta(A \setminus F_n)$ for each n, hence $\eta(A \setminus A_0) = 0$. Now, if $A_n \subset A_0$ and $A_n \downarrow \emptyset$, then for each n choose a maximal m_n such that $F_{m_n} \subset A_0 \setminus A_n$. Then $\eta(A_n) \le \eta(A_0 \setminus F_{m_n}) \to 0$ because $m_n \to \infty$.

'If.' Let (F_n) be an increasing sequence in $\mathcal{F}(A_0)$ with union A_0. Then $\eta(A \setminus F_n) \le \eta(A \setminus A_0) + \eta(A_0 \setminus F_n) = \eta(A_0 \setminus F_n) \to 0$ because $\eta(A \setminus A_0) = 0$ and $A_0 \setminus F_n \downarrow \emptyset$. It follows that $\eta^\bullet(A) = 0$. □

FACT 3.5. *Let* η *be a submeasure on* S, *and let* $A_n \downarrow \emptyset$. *Then*

$$\lim_{n\to\infty} \eta^\bullet(A_n) = \lim_{n\to\infty} \eta(A_n).$$

In consequence, for every scalar function f *on* S,

$$\lim_{r\to\infty} \eta^\bullet(s(f,r)) = \lim_{r\to\infty} \eta(s(f,r)).$$

Proof. Evidently, the limits exist in $[0,\infty]$. Now, take any $a > \lim_n \eta^\bullet(A_n)$ and fix k such that $\eta^\bullet(A_k) < a$. By the definition of $\eta^\bullet$, there is a finite set $F \subset A_k$ with $\eta(A_k \setminus F) < a$. Since $A_n \downarrow \emptyset$, there is $m > k$ for which $A_m \subset A_k \setminus F$, and then $\eta(A_m) < a$. This proves the inequality $\ge$ between the two limits. Since $\eta \ge \eta^\bullet$, the other inequality is obvious. □

We shall say that a submeasure η on S is *core-nonatomic* if its core $\eta^\bullet$ is nonatomic. Note that then $\eta(S \setminus F) < \infty$ for some $F \in \mathcal{F}(S)$.

From Fact 3.1 and the relevant definitions, one easily gets the following.

COROLLARY 3.6. *If a submeasure* η *on* S *is core-nonatomic, then in every infinite set* $A \subset S$ *one can find a decreasing sequence* (A_n) *of its infinite subsets such that* $\eta(A_n) \to 0$ *as* $n \to \infty$. *In consequence,* $\eta(s_n) \to 0$ *as* $n \to \infty$ *for every sequence* (s_n) *of distinct points in* S.

A direct characterization of core-nonatomicity is very simple.

FACT 3.7. *A submeasure η on S is core-nonatomic iff for every $\varepsilon > 0$ there is a finite partition $A_0, A_1, \dots, A_k$ of S such that $A_0 \in \mathcal{F}(S)$ and $\eta(A_j) < \varepsilon$ for each $1 \le j \le k$.*

Proof. 'If.' Defining $B_1 = A_0 \cup A_1$ and $B_j = A_j$ for $j > 1$, one has $\eta^\bullet(B_j) = \eta^\bullet(A_j) \le \eta(A_j) < \varepsilon$ for all $1 \le j \le k$.

'Only if.' Assume $\eta^\bullet$ is nonatomic. Thus, given $\varepsilon > 0$, there is a partition $B_1, \dots, B_k$ of S with $\eta^\bullet(B_j) < \varepsilon$ for each j. By the definition of $\eta^\bullet$, for each j there is $F_j \in \mathcal{F}(B_j)$ with $\eta(B_j \setminus F_j) < \varepsilon$. Then $A_0 := F_1 \cup \dots \cup F_k \in \mathcal{F}(S)$, and $\eta(A_j) < \varepsilon$, where $A_j := B_j \setminus F_j$, for $j = 1, \dots, k$. □

If η and γ are submeasures on a set S, then γ is said to be η*-continuous*, denoted $\gamma \ll \eta$, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\gamma(A) < \varepsilon$ whenever $\eta(A) < \delta$. If also $\eta \ll \gamma$, then η and γ are called *equivalent*, denoted $\eta \sim \gamma$.

FACT 3.8. *For any submeasures η and γ on a set S, if $\gamma \ll \eta$, then $\gamma^\bullet \ll \eta^\bullet$. In consequence, if $\eta \sim \gamma$, then also $\eta^\bullet \sim \gamma^\bullet$.*

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be as in the definition above. If $\eta^\bullet(A) < \delta$, then $\eta(A \setminus F) < \delta$ for some $F \in \mathcal{F}(A)$. Hence, $\gamma^\bullet(A) \le \gamma(A \setminus F) < \varepsilon$. □

If η is a submeasure on a countable set S, then the definition of $\eta^\bullet$ can be equivalently given in the form

$$\eta^\bullet(A) = \lim_{n\to\infty} \eta(A \setminus F_n),$$

where (F_n) is any fixed increasing sequence of finite sets with union S. Hence, $\eta^\bullet = 0$ iff $\eta(S \setminus F_n) \to 0$ iff η is order continuous, that is, $\eta(A_n) \to 0$ whenever $A_n \downarrow \emptyset$ (see Fact 3.4).

FACT 3.9. *Let η be a lsc submeasure on a countable set S, and let $A \subset S$.*

(a) *If $\eta^\bullet(A) > a > 0$, then A is the union of a disjoint sequence (A_k) of finite sets with $\inf_k \eta(A_k) \ge a$.*
(b) *If A is the union of a sequence (A_k) of sets with $\inf_k \eta(A_k) > a > 0$, and every $F \in \mathcal{F}(A)$ is disjoint from some A_k, then $\eta^\bullet(A) > a$.*

Proof. (a) We may assume that $S = \mathbb{N}$. Fix any b such that $a < b < \eta^\bullet(A)$. As $\eta \ge \eta^\bullet$ and η is lsc, there is $m_1 \in \mathbb{N}$ such that $\eta(A \cap [1, m_1)) > b$. Since then, by the definition of $\eta^\bullet$, $\eta(A \setminus [1, m_1)) > b$, there is $m_2 > m_1$ with $\eta(A \cap [m_1, m_2)) > b$. But again, $\eta(A \setminus [1, m_2)) > b$, hence there exists $m_3 > m_2$ such that $\eta(A \cap [m_2, m_3)) > b$. We continue in this manner to find $1 = m_0 < m_1 < \cdots$ with $\eta(A \cap [m_{k-1}, m_k)) > b$ for all $k \ge 1$, and then the sets $A_k := A \cap [m_{k-1}, m_k)$ are as required.

(b) Fix any b with $a < b < \inf_k \eta(A_k)$. Take any $F \in \mathcal{F}(A)$ and next select k so that $A_k \subset A \setminus F$; then $b < \eta(A_k) \le \eta(A \setminus F)$. From this it follows that $\eta^\bullet(A) > a$. □

REMARK. If $S = \mathbb{N}$, the extra assumption on (A_k) in (b) can be written as $\sup_k \min A_k = \infty$.

4. Submeasures on $\mathbb{N}$ defined by sequences of measures

Let $\boldsymbol{\mu} = (\mu_n)$ be a sequence of finite positive measures on $\mathbb{N}$. It gives rise to the submeasures (or densities) $d_{\boldsymbol{\mu}}$ and $\bar{d}_{\boldsymbol{\mu}}$ on $\mathbb{N}$ defined by

$$d_{\boldsymbol{\mu}}(A) = \limsup_{n\to\infty} \mu_n(A) \quad \text{and} \quad \bar{d}_{\boldsymbol{\mu}}(A) = \sup_n \mu_n(A).$$

Clearly, $d_{\boldsymbol{\mu}} \le \bar{d}_{\boldsymbol{\mu}}$ and $\bar{d}_{\boldsymbol{\mu}}$ (but not $d_{\boldsymbol{\mu}}$, in general) is lsc, that is,

$$\bar{d}_{\boldsymbol{\mu}}(A_n) \uparrow \bar{d}_{\boldsymbol{\mu}}(A) \quad \text{whenever } A_n \uparrow A;$$

consequently, it is countably subadditive.

We will sometimes assume one or more of the following conditions on $\boldsymbol{\mu}$.

(A) For each $k \in \mathbb{N}$, $\mu_n(k) \to 0$ as $n \to \infty$ or, equivalently, $\mathcal{F} \subset \mathcal{Z}(d_{\boldsymbol{\mu}})$.
($\bar{\mathrm{A}}$) $\sup_k \mu_n(k) \to 0$ as $n \to \infty$.
(B) The sequence $\boldsymbol{\mu}$ is uniformly bounded, that is, $\bar{d}_{\boldsymbol{\mu}}(\mathbb{N}) < \infty$.
(C) For each $k \in \mathbb{N}$, $\bar{d}_{\boldsymbol{\mu}}(k) = \sup_n \mu_n(k) \to 0$ as $k \to \infty$.

It is not hard to see that ($\bar{\mathrm{A}}$) is equivalent to (A) & (C).

The fact below, though very simple, is quite useful.

FACT 4.1. *$(\bar{d}_{\boldsymbol{\mu}})^\bullet \le d_{\boldsymbol{\mu}} \le \bar{d}_{\boldsymbol{\mu}}$, hence $(\bar{d}_{\boldsymbol{\mu}})^\bullet = (d_{\boldsymbol{\mu}})^\bullet$, and $(\bar{d}_{\boldsymbol{\mu}})^\bullet = d_{\boldsymbol{\mu}}$ iff $\boldsymbol{\mu}$ satisfies condition* (A).

Proof. Take any $A \subset \mathbb{N}$ and $a > d_{\boldsymbol{\mu}}(A)$. Then there is k such that $\mu_n(A) < a$ for all $n > k$. Next, by the countable additivity of the μ_n's, there is $F \in \mathcal{F}(A)$ such that $\mu_n(A \setminus F) < a$ for all $n \le k$. Hence $\mu_n(A \setminus F) < a$ for all n and thus $\bar{d}_{\boldsymbol{\mu}}(A \setminus F) \le a$. In consequence, $(\bar{d}_{\boldsymbol{\mu}})^\bullet(A) \le d_{\boldsymbol{\mu}}(A)$. The other inequality is trivial.

If (A) is assumed, $A \subset \mathbb{N}$ and $F \in \mathcal{F}(A)$, then $d_{\boldsymbol{\mu}}(F) = 0$ so that $d_{\boldsymbol{\mu}}(A) = d_{\boldsymbol{\mu}}(A \setminus F) \le \bar{d}_{\boldsymbol{\mu}}(A \setminus F)$. Hence $d_{\boldsymbol{\mu}}(A) \le (\bar{d}_{\boldsymbol{\mu}})^\bullet(A)$. On the other hand, if $d_{\boldsymbol{\mu}} \le (\bar{d}_{\boldsymbol{\mu}})^\bullet$, then $\mathcal{F} \subset \mathcal{Z}((\bar{d}_{\boldsymbol{\mu}})^\bullet) \subset \mathcal{Z}(d_{\boldsymbol{\mu}})$, and $\boldsymbol{\mu}$ must satisfy (A). $\square$

The result below is a strengthened version of Fact 3.9(a) for the special case of $\eta = \bar{d}_{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}$ satisfying (A). It can be proved by an easy induction.

FACT 4.2. *Assume that $\boldsymbol{\mu} = (\mu_n)$ satisfies condition* (A), *and let $A \subset \mathbb{N}$ with $d_{\boldsymbol{\mu}}(A) > 0$. Then, given sequences $d_{\boldsymbol{\mu}}(A) > a_k \uparrow d_{\boldsymbol{\mu}}(A)$ and $0 < \varepsilon_k \downarrow 0$, there exist sequences $1 \le n_1 < n_2 < \cdots$ and $1 = m_0 < m_1 < \cdots$ in $\mathbb{N}$ such that for every $k \ge 1$,*

$$\mu_{n_k}\big(A \cap [m_{k-1}, m_k)\big) > a_k \quad \textit{and} \quad \mu_{n_k}\big(\mathbb{N} \setminus [m_{k-1}, m_k)\big) < \varepsilon_k.$$

Consequently, A is the union of the disjoint finite sets $A_k := A \cap [m_{k-1}, m_k)$ satisfying $\bar{d}_{\boldsymbol{\mu}}(A_k) > a_k$ for each $k \ge 1$.

It is worthwhile to note the following fact. Its first assertion is implicit in [17, Props. 6.3 and 5.2].

FACT 4.3. *If $d_{\boldsymbol{\mu}}$ is nonatomic, then $\boldsymbol{\mu}$ satisfies conditions* ($\bar{\mathrm{A}}$) *and* (B); *in consequence*, $(\bar{d}_{\boldsymbol{\mu}})^{\bullet} = d_{\boldsymbol{\mu}}$, *and $\bar{d}_{\boldsymbol{\mu}}$ is core-nonatomic. Conversely, if $\boldsymbol{\mu}$ satisfies condition* (A) *and $\bar{d}_{\boldsymbol{\mu}}$ is core-nonatomic, then $d_{\boldsymbol{\mu}}$ is nonatomic.*

NOTE. In general, '$d_{\boldsymbol{\mu}}$ is nonatomic' is a stronger requirement than '$\bar{d}_{\boldsymbol{\mu}}$ is core-nonatomic', see Remark 4.4(f).

Proof of Fact 4.3. If ($\bar{\mathrm{A}}$) fails, then one can easily find strictly increasing sequences (n_k) and (m_k) in $\mathbb{N}$ such that, for some $\varepsilon > 0$, $\mu_{n_k}(m_k) \geq \varepsilon$ for every k. Let $A := \{m_k : k \in \mathbb{N}\}$. Then $d_{\boldsymbol{\mu}}(B) \geq \varepsilon$ for every infinite subset B of A which is impossible in view of Fact 3.1. As for (B), since $d_{\boldsymbol{\mu}}$ is nonatomic, $d_{\boldsymbol{\mu}}(\mathbb{N}) < \infty$, and hence $\bar{d}_{\boldsymbol{\mu}}(\mathbb{N}) < \infty$. To finish, appeal to Fact 4.1. □

REMARKS 4.4. (a) As an important special case, let us point out that every sequence $\mathbf{F} = (F_n)$ of finite nonempty subsets of $\mathbb{N}$ determines densities $d_{\mathbf{F}}$ and $\bar{d}_{\mathbf{F}}$ on $\mathbb{N}$ by the formulas

$$d_{\mathbf{F}}(A) = \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|} = \limsup_{n\to\infty} \mu_{F_n}(A)$$

and

$$\bar{d}_{\mathbf{F}}(A) = \sup_n \frac{|A \cap F_n|}{|F_n|} = \sup_n \mu_{F_n}(A).$$

Clearly, $\bar{d}_{\mathbf{F}}(\mathbb{N}) = 1$, and $\bar{d}_{\mathbf{F}}$ is strictly positive iff (F_n) covers $\mathbb{N}$. If $|F_n| \to \infty$, then the sequence (μ_{F_n}) satisfies conditions ($\bar{\mathrm{A}}$), (B) and (C), and thus in this case $d_{\mathbf{F}} = (\bar{d}_{\mathbf{F}})^{\bullet}$.

(b) In particular, if $F_n = [n]$ for each $n \in \mathbb{N}$, then $d_{\mathbf{F}}$ is the *standard* (*upper*) *density* on $\mathbb{N}$. In this case, we write simply d_n, d, $\bar{d}$, and $\mathcal{Z}$ instead of μ_{F_n}, $d_{\mathbf{F}}$, $\bar{d}_{\mathbf{F}}$, and $\mathcal{Z}(d)$, respectively, and refer to the sets $A \in \mathcal{Z}$ as sets *of density zero*. Note that if $A = \{n_1 < n_2 < \cdots\}$, then $d(A) = \limsup_{k\to\infty}(k/n_k)$.

(c) Another particular case is when $F_n = [2^{n-1}, 2^n)$ for $n \in \mathbb{N}$. Then, writing δ_n, δ and $\bar{\delta}$ instead of μ_{F_n}, $d_{\mathbf{F}}$ and $\bar{d}_{\mathbf{F}}$, one can easily verify that $\frac{1}{2}\delta \leq d \leq 2\delta$ and $\frac{1}{2}\bar{\delta} \leq \bar{d} \leq 2\bar{\delta}$. (This follows also from the relations between the F-norms $\|\cdot\|$ and $|||\cdot|||$ established in the proof of Proposition 13.1.) Thus the densities d and δ are equivalent. In particular, $\mathcal{Z} = \mathcal{Z}(\delta)$ (cf. [13, Prop. 3.2]).

(d) By using arithmetic progressions $A_j = \{j + (k-1)r : k \in \mathbb{N}\}$, where $r \in \mathbb{N}$ and $j = 1, \ldots, r$, one easily sees that the standard density d is nonatomic (and $\bar{d}$ is core-nonatomic). Hence, so is the density δ, but for this case a more general result is easily available: The density $d_{\mathbf{F}}$ determined by a *disjoint* sequence $\mathbf{F} = (F_n)$ is nonatomic iff $|F_n| \to \infty$ as $n \to \infty$. For a more advanced result that can be applied in the case of densities $d_{\boldsymbol{\mu}}$, see [17, Th. 6.8]. A very general (in fact, best possible) result that characterizes those sequences $\mathbf{F}$ which, irrespectively of any set-theoretical interrelations between their terms, determine a nonatomic density, was established in [14]: *If there is a constant* $\gamma \geq 0$ *such that* $|\{n : |F_n| = m\}| \leq 2^{\gamma m}$ *for each* $m \in \mathbb{N}$, *then* $d_{\mathbf{F}}$ *is nonatomic.*

(e) One of our major open problems is that we do not know if there exists a strictly positive lsc submeasure η on $\mathbb{N}$, preferably core-nonatomic, that is not equivalent to a submeasure of type $\bar{d}_{\boldsymbol{\mu}}$.

This is possible for a general submeasure η on $\mathbb{N}$ (the reader is referred to [17] for the necessary explanations): Let $\mathcal{L}$ be the ideal generated by lacunary sets, and ν any strictly positive finite measure on $\mathbb{N}$. Define a submeasure η by $\eta(A) = \nu(A)$ if $A \in \mathcal{L}$, and $\eta(A) = 1 + \nu(A)$ otherwise. Then $\mathcal{Z}(\eta^\bullet) = \mathcal{L}$ has (ASP) but not (NP) by [17, Ex. 4.2], so it also fails to have (AP) by [17, Th. 3.5]. Suppose there is a $\boldsymbol{\mu}$ with $\eta \sim \bar{d}_{\boldsymbol{\mu}}$. Then $\eta^\bullet \sim (\bar{d}_{\boldsymbol{\mu}})^\bullet$ so that $\mathcal{L} = \mathcal{Z}((\bar{d}_{\boldsymbol{\mu}})^\bullet)$. However, since $\bar{d}_{\boldsymbol{\mu}}$ is lsc and, consequently, countably subadditive, the ideal $\mathcal{Z}((\bar{d}_{\boldsymbol{\mu}})^\bullet)$ has (AP), by [29], proof of the implication (iii) $\Longrightarrow$ (i) on p. 59 (see also [15]). A contradiction.

(f) With every strictly positive σ-finite measure μ on $\mathbb{N}$ one may associate a sequence $\boldsymbol{\mu} = (\mu_n)$ of finite measures by letting $\mu_n(A) := \mu(A \cap [1, n])$. Then $\boldsymbol{\mu}$ fails to satisfy (A), and is uniformly bounded iff μ is finite. Furthermore, $\bar{d}_{\boldsymbol{\mu}} = d_{\boldsymbol{\mu}} = \mu$, and $(\bar{d}_{\boldsymbol{\mu}})^\bullet(A) = 0$ if $\mu(A \setminus F) < \infty$ for some $F \in \mathcal{F}$, and $(\bar{d}_{\boldsymbol{\mu}})^\bullet(A) = \infty$, otherwise. In particular, if μ is finite, then $(\bar{d}_{\boldsymbol{\mu}})^\bullet = 0$; thus $\bar{d}_{\boldsymbol{\mu}}$ is trivially core-nonatomic, while $d_{\boldsymbol{\mu}} = \mu$ is not nonatomic.

5. FG-norms in ω determined by sequences of measures

Let $\boldsymbol{\mu} = (\mu_n)$ be a sequence of finite positive measures on $\mathbb{N}$ such that

$$0 < \bar{d}_{\boldsymbol{\mu}}(k) = \sup_n \mu_n(k) < \infty \quad \text{for each } k \in \mathbb{N}$$

or, equivalently, such that $0 < \bar{d}_{\boldsymbol{\mu}}(F) < \infty$ for all $\emptyset \neq F \in \mathcal{F}$. We shall refer to such $\boldsymbol{\mu}$'s as *admissible.*

In the present section, only the strict positivity of $\bar{d}_{\boldsymbol{\mu}}$ is of importance. We use $\boldsymbol{\mu}$ to define two sequences of F-seminorms in ω, and associate a 'sup' FG-norm to each of them.

For every $x = (\xi_j)$ in ω, let (cf. Section 1.G)

$$\|x\|_{\mu_n}^0 = \int_{\mathbb{N}} \min(1, |x|)\, d\mu_n \quad \text{and} \quad \|x\|_{\boldsymbol{\mu}}^0 = \sup_n \|x\|_{\mu_n}^0$$

as well as

$$\|x\|_{\mu_n} = \inf\{\varepsilon > 0 : \mu_n(s(x, \varepsilon)) \leq \varepsilon\} \quad \text{and} \quad \|x\|_{\boldsymbol{\mu}} = \sup_n \|x\|_{\mu_n}.$$

It is not hard to check that

$$\|x\|_{\boldsymbol{\mu}} = \|x\|_{\bar{d}_{\boldsymbol{\mu}}},$$

where $\|\cdot\|_{\bar{d}_{\boldsymbol{\mu}}}$ is the FG-norm

$$\|x\|_{\bar{d}_{\boldsymbol{\mu}}} = \inf\{\varepsilon > 0 : \bar{d}_{\boldsymbol{\mu}}(s(f, \varepsilon)) \leq \varepsilon\}$$

defining in ω the topology $\tau_{\bar{d}_{\boldsymbol{\mu}}}$ of convergence in submeasure $\bar{d}_{\boldsymbol{\mu}}$. Thus, for any sequence (x_k) in ω, $\|x_k\|_{\bar{d}_{\boldsymbol{\mu}}} \to 0$ iff $\bar{d}_{\boldsymbol{\mu}}(s(x_k,\varepsilon)) \to 0$ for each $\varepsilon > 0$.

It is also easily seen that the sequence of F-seminorms $(\|\cdot\|_{\mu_n}^0)$ determines the topology of ω, and that $\|\cdot\|_{\boldsymbol{\mu}}^0$ is a monotone FG-norm on ω. Thus, $\|\cdot\|_{\boldsymbol{\mu}}^0$ is subadditive, vanishes only at zero, and $\|x\|_{\boldsymbol{\mu}}^0 \le \|y\|_{\boldsymbol{\mu}}^0$ whenever $x,y \in \omega$ and $|x| \le |y|$. Evidently, the topology $\tau_{\boldsymbol{\mu}}^0$ defined by $\|\cdot\|_{\boldsymbol{\mu}}^0$ is stronger than the usual topology of ω.

Moreover, since each $\|\cdot\|_{\mu_n}^0$ is a continuous function on ω, the FG-norm $\|\cdot\|_{\boldsymbol{\mu}}^0$ is lower semicontinuous on ω. That is, if $x_k \to x$ in ω, then $\|x\|_{\boldsymbol{\mu}}^0 \le \liminf_k \|x_k\|_{\boldsymbol{\mu}}^0$; in other terms, closed $\|\cdot\|_{\boldsymbol{\mu}}^0$-balls are closed in ω. It follows that $(\omega, \|\cdot\|_{\boldsymbol{\mu}}^0)$ is complete (by [19, Th. 3.2.4] adapted to TVG's).

The same is true for the F-seminorms $\|\cdot\|_{\mu_n}$, the FG-norm $\|\cdot\|_{\boldsymbol{\mu}}$ and its topology $\tau_{\boldsymbol{\mu}}$, and thus also $(\omega, \|\cdot\|_{\boldsymbol{\mu}})$ is a complete TVG.

Note that in the case of $\boldsymbol{\mu} = (d_n)$,

$$\|x\|_{d_n}^0 = \frac{1}{n}\sum_{j=1}^{n} \min(1, |\xi_j|).$$

We now give a few simple facts concerning the F-seminorms and FG-norms defined above. First, note that directly from the definitions it follows that

$$\|x\|_{\mu_n}^0 \le \mu_n(s(x)), \qquad \|x\|_{\boldsymbol{\mu}}^0 \le \bar{d}_{\boldsymbol{\mu}}(s(x))$$

and

$$\|x\|_{\mu_n}^0 \le \mu_n(\mathbb{N})\|x\|_\infty, \qquad \|x\|_{\boldsymbol{\mu}}^0 \le \bar{d}_{\boldsymbol{\mu}}(\mathbb{N})\|x\|_\infty.$$

Likewise for $\|\cdot\|_{\mu_n}$ and $\|\cdot\|_{\boldsymbol{\mu}}$, but with $\mu_n(\mathbb{N})$ and $\bar{d}_{\boldsymbol{\mu}}(\mathbb{N})$ replaced by 1.

Moreover, for any scalar t and set $A \subset \mathbb{N}$, one has

$$\|te_A\|_{\mu_n}^0 = \min(1,|t|)\mu_n(A) \quad \text{and} \quad \|te_A\|_{\boldsymbol{\mu}}^0 = \min(1,|t|)\bar{d}_{\boldsymbol{\mu}}(A).$$

Next, as direct consequences of Facts 1.5 and 1.6, we have the following.

FACT 5.1. *For each $x \in \omega$, $r,t > 0$, and $n \in \mathbb{N}$,*

(a) $\mu_n(s(x,r)) \le \|r^{-1}x\|_{\mu_n}^0$, *and*
(b) $\|tx\|_{\mu_n}^0 \le \mu_n(s(x,r)) + \min(1,tr)\mu_n(\mathbb{N})$

whence

(a′) $\bar{d}_{\boldsymbol{\mu}}(s(x,r)) \le \|r^{-1}x\|_{\boldsymbol{\mu}}^0$, *and*
(b′) $\|tx\|_{\boldsymbol{\mu}}^0 \le \bar{d}_{\boldsymbol{\mu}}(s(x,r)) + \min(1,tr)\bar{d}_{\boldsymbol{\mu}}(\mathbb{N})$.

FACT 5.2. *If $\mu_n(\mathbb{N}) \le m$, then for every $x \in \omega$,*

$$\frac{1}{m}(\|mx\|_{\mu_n})^2 \le \|x\|_{\mu_n}^0 \le \big(1+\mu_n(\mathbb{N})\big)\|x\|_{\mu_n},$$

where one may replace m with 1 when $m \le 1$.

Hence, if $\boldsymbol{\mu}$ is uniformly bounded and $\mu_n(\mathbb{N}) \le m < \infty$ for all n, then

$$\frac{1}{m}(\|mx\|_{\boldsymbol{\mu}})^2 \le \|x\|_{\boldsymbol{\mu}}^0 \le (1+m)\|x\|_{\boldsymbol{\mu}}.$$

The corollary below follows easily from Fact 5.1, while its second part is also immediate from Fact 5.2.

COROLLARY 5.3. $\tau_{\boldsymbol{\mu}} = \tau_{\bar{d}_{\boldsymbol{\mu}}} \le \tau_{\boldsymbol{\mu}}^0$, *that is,* $\|\cdot\|_{\boldsymbol{\mu}} = \|\cdot\|_{\bar{d}_{\boldsymbol{\mu}}}$ *is weaker than* $\|\cdot\|_{\boldsymbol{\mu}}^0$, *and the converse relation holds provided* $\boldsymbol{\mu}$ *is uniformly bounded.*

Thus, if $\boldsymbol{\mu}$ *is uniformly bounded, then* $\tau_{\boldsymbol{\mu}} = \tau_{\bar{d}_{\boldsymbol{\mu}}} = \tau_{\boldsymbol{\mu}}^0$, *that is,* $\|\cdot\|_{\boldsymbol{\mu}}^0$ *and* $\|\cdot\|_{\boldsymbol{\mu}} = \|\cdot\|_{\bar{d}_{\boldsymbol{\mu}}}$ *are equivalent.*

FACT 5.4. *Each of the F-seminorms or FG-norms $\|\cdot\|$ considered here has the property that $\|x_k\| \uparrow \|x\|$ whenever $|x_k| \uparrow |x|$.*

In particular, if $x = (\xi_j) \in \omega$ and $x^k = \sum_{j=1}^k \xi_j e_j$ $(k \in \mathbb{N})$, then

$$\|x\|_{\mu_n}^0 = \sup_k \|x^k\|_{\mu_n}^0 \quad (n \in \mathbb{N}) \quad \textit{and} \quad \|x\|_{\boldsymbol{\mu}}^0 = \sup_k \|x^k\|_{\boldsymbol{\mu}}^0.$$

Likewise for $\|\cdot\|_{\mu_n}$ and $\|\cdot\|_{\boldsymbol{\mu}} = \|\cdot\|_{\bar{d}_{\boldsymbol{\mu}}}$.

We finish with a simple consequence of Facts 3.5 and 4.1.

FACT 5.5. *For each $x \in \omega$,*

$$\lim_{r\to\infty} d_{\boldsymbol{\mu}}(s(x,r)) = \lim_{r\to\infty} \bar{d}_{\boldsymbol{\mu}}(s(x,r)).$$

6. The F-lattices $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$

Let $\boldsymbol{\mu} = (\mu_n)$ be an admissible sequence of measures on $\mathbb{N}$, and $(\omega, \|\cdot\|_{\boldsymbol{\mu}}^0)$ the complete FG-normed space from Section 5. Then, as we know from Section 1.E,

$$\lambda_0(\boldsymbol{\mu}) := v(\omega, \|\cdot\|_{\boldsymbol{\mu}}^0) = \Big\{x \in \omega : \lim_{t\to 0} \|tx\|_{\boldsymbol{\mu}}^0 = 0\Big\}$$

is the largest linear subspace on which $\|\cdot\|_{\boldsymbol{\mu}}^0$ induces a vector topology, and $\lambda_0(\boldsymbol{\mu})$ is closed in $(\omega, \|\cdot\|_{\boldsymbol{\mu}}^0)$. Thus, $\lambda_0(\boldsymbol{\mu})$, equipped with the F-norm $\|\cdot\|_{\boldsymbol{\mu}}^0$, is an F-space whose topology is stronger than the topology of coordinate-wise convergence. By Fact 1.4,

$$\lambda_{00}(\boldsymbol{\mu}) := v_0(\omega, \|\cdot\|_{\boldsymbol{\mu}}^0) = \Big\{x \in \omega : \lim_{n\to\infty} \|x\|_{\mu_n}^0 = 0\Big\}$$

is a closed subspace of $\lambda_0(\boldsymbol{\mu})$. Moreover, both $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ are solid subspaces (or ideals) in ω, and the F-norm $\|\cdot\|_{\boldsymbol{\mu}}^0$ is monotone. Thus, $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ are F-lattices. Far more important than $\lambda_{00}(\boldsymbol{\mu})$ is the Lebesgue subspace $\lambda_0^c(\boldsymbol{\mu})$ of $\lambda_0(\boldsymbol{\mu})$ (the largest one on which the F-norm $\|\cdot\|_{\boldsymbol{\mu}}^0$ is order continuous); it can be defined as

$$\lambda_0^c(\boldsymbol{\mu}) := \Big\{x \in \lambda_0(\boldsymbol{\mu}) : \lim_{m\to\infty} \|x - x^m\|_{\boldsymbol{\mu}}^0 = 0\Big\}.$$

Note that for a set $A \subset \mathbb{N}$, e_A is in $\lambda_0(\boldsymbol{\mu})$ (resp., $\lambda_{00}(\boldsymbol{\mu})$) iff $\bar{d}_{\boldsymbol{\mu}}(A) < \infty$ (resp., $d_{\boldsymbol{\mu}}(A) = 0$). Hence, by the admissibility of $\boldsymbol{\mu}$, $e_A \in \lambda_0(\boldsymbol{\mu})$ for each $A \in \mathcal{F}$. In particular, all the unit vectors e_k are in $\lambda_0(\boldsymbol{\mu})$, and the support of $\lambda_0(\boldsymbol{\mu})$ is all of $\mathbb{N}$. On the other hand, $e_k \in \lambda_{00}(\boldsymbol{\mu})$ iff $d_{\boldsymbol{\mu}}(k) = 0$, and $\lambda_{00}(\boldsymbol{\mu}) \neq \{0\}$ iff such k exist.

Proposition 6.1. *The sequence (e_k) of unit vectors is a basis for $\lambda_0^c(\boldsymbol{\mu})$, and its subsequence $(e_k)_{k\in K}$, where $K = \{k : d_{\boldsymbol{\mu}}(k) = 0\}$, is a basis for $\lambda_{00}(\boldsymbol{\mu})$. In consequence, $\lambda_{00}(\boldsymbol{\mu}) = \lambda_0^c(\boldsymbol{\mu})$ iff $\boldsymbol{\mu}$ satisfies condition* (A).

Proof. The first assertion is obvious. To verify the other one, take any x in $\lambda_{00}(\boldsymbol{\mu})$ and note that $s(x) \subset K$. Given $\varepsilon > 0$, choose $k \in \mathbb{N}$ so that $\|x\|_{\mu_n}^0 \le \varepsilon$ for all $n > k$. Next, select $m \in \mathbb{N}$ with $\mu_n((m,\infty)) \le \varepsilon$ for $n \le k$. Then $\|x - x^m\|_{\mu_n}^0 \le \|x\|_{\mu_n}^0 \le \varepsilon$ for $n > k$, and $\|x - x^m\|_{\mu_n}^0 \le \mu_n((m,\infty)) \le \varepsilon$ for $n \le k$. Hence, $\|x - x^m\|_{\boldsymbol{\mu}}^0 \le \varepsilon$, which completes the proof. □

If $\boldsymbol{\mu} = (d_n)$, the sequence defining the standard density d (Remark 4.4(b)), we simplify the notation $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ to

$$\lambda_0 \quad \text{and} \quad \lambda_{00} = \lambda_0^c.$$

Remark 6.2. We do not discuss here the analogous F-spaces arising from $(\omega, \|\cdot\|_{\boldsymbol{\mu}})$ because for $\boldsymbol{\mu}$ uniformly bounded, the most interesting case, we would get the same spaces, with equivalent F-norms $\|\cdot\|_{\boldsymbol{\mu}}$ and $\|\cdot\|_{\boldsymbol{\mu}}^0$, by Fact 5.2. For the general case, see Remark 7.19. (Note of caution: The space $v_0(\omega, \|\cdot\|_{\boldsymbol{\mu}})$ is not the same as $\lambda_{00}(\bar{d}_{\boldsymbol{\mu}})$, defined in the next section.)

Given two admissible sequences $\boldsymbol{\mu} = (\mu_n)$ and $\boldsymbol{\nu} = (\nu_n)$ of measures on $\mathbb{N}$, it is natural to call them *equivalent* if $\lambda_0(\boldsymbol{\mu}) = \lambda_0(\boldsymbol{\nu})$, and *strictly equivalent* if, in addition, $\lambda_{00}(\boldsymbol{\mu}) = \lambda_{00}(\boldsymbol{\nu})$. In either case, also the F-norms $\|\cdot\|_{\boldsymbol{\mu}}^0$ and $\|\cdot\|_{\boldsymbol{\nu}}^0$ have to be equivalent (Fact 1.1), hence the submeasures $\bar{d}_{\boldsymbol{\mu}}$ and $\bar{d}_{\boldsymbol{\nu}}$ as well, because $\bar{d}_{\boldsymbol{\mu}}(A) = \|e_A\|_{\boldsymbol{\mu}}^0$ and $\bar{d}_{\boldsymbol{\nu}}(A) = \|e_A\|_{\boldsymbol{\nu}}^0$.

Note that if $\boldsymbol{\nu}$ is obtained from $\boldsymbol{\mu}$ by repeating each μ_n infinitely many times, then $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are equivalent, but not strictly.

Proposition 6.3. *For every admissible sequence $\boldsymbol{\mu} = (\mu_n)$, there is a strictly equivalent admissible sequence $\boldsymbol{\nu} = (\nu_n)$ such that, for each n, ν_n has a finite support and $\nu_n(\mathbb{N}) = \mu_n(\mathbb{N})$. Moreover, $\boldsymbol{\nu}$ can be chosen so that the ν_n's have pairwise disjoint supports provided it is so for the μ_n's.*

Proof. Fix any sequence $\varepsilon_n \to 0$ in $(0,1)$ and for each n choose a finite subset M_n of $S_n := s(\mu_n)$ so that

$$\mu_n(S_n \setminus M_n) < \varepsilon_n \min(1, \mu_n(S_n)) =: r_n.$$

Note that $\mu_n(S_n) < (1-\varepsilon_n)^{-1}\mu_n(M_n)$. Next, arrange $\mathbb{N} \setminus \bigcup_n M_n$ in a sequence $k_1 < k_2 < \cdots$ and assume, for instance, that it is infinite. Then for each n

define a measure ν_n on $\mathbb{N}$ by

$$\nu_n(A) = c_n\big(\mu_n(A \cap M_n) + r_n\gamma_n(A)\big)$$

where γ_n is the Dirac measure at point k_n and

$$c_n := \frac{\mu_n(S_n)}{\mu_n(M_n) + r_n}.$$

It is easily seen that $(1+\varepsilon_n)^{-1} \le c_n \le (1-\varepsilon_n)^{-1}$ for each n.

Clearly, $\boldsymbol{\nu} = (\nu_n)$ is admissible and $\nu_n(\mathbb{N}) = \mu_n(\mathbb{N})$ for each n.

Now, for each $x \in \omega$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|x\|^0_{\nu_n} &= c_n\biggl(\int_{M_n} \min(1,|x|)\,d\mu_n + r_n \min(1,|\xi_{k_n}|)\biggr)\\ &\le c_n(\|x\|^0_{\mu_n} + \varepsilon_n)\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\|x\|^0_{\nu_n} &= c_n\biggl(\int_{S_n} \min(1,|x|)\,d\mu_n - \int_{S_n\setminus M_n} \min(1,|x|)\,d\mu_n + r_n \min(1,|\xi_{k_n}|)\biggr)\\ &\ge c_n(\|x\|^0_{\mu_n} - \varepsilon_n).\end{aligned}$$

Thus,

$$\frac{1}{1+\varepsilon_n}(\|x\|^0_{\mu_n} - \varepsilon_n) \le \|x\|^0_{\nu_n} \le \frac{1}{1-\varepsilon_n}(\|x\|^0_{\mu_n} + \varepsilon_n).$$

It follows that $\lambda_{00}(\boldsymbol{\mu}) = \lambda_{00}(\boldsymbol{\nu})$ and, using a simple argument, that $\lambda_0(\boldsymbol{\mu}) = \lambda_0(\boldsymbol{\nu})$. The 'moreover' assertion is obvious from the construction. $\square$

We now show that when $\boldsymbol{\mu}$ is uniformly bounded, there is an alternative description of the space $\lambda_0(\boldsymbol{\mu})$, and its subspaces $\lambda_0^c(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$. Let us point out that the assumption of uniform boundedness of $\boldsymbol{\mu}$, i.e., $\bar{d}_{\boldsymbol{\mu}}(\mathbb{N}) < \infty$, is used only for the 'if' parts of the characterizations.

Proposition 6.4. *Let $\boldsymbol{\mu}$ be uniformly bounded. Then for any $x \in \omega$,*

(a) $x \in \lambda_{00}(\boldsymbol{\mu})$ *iff* $s(x,r) \in \mathcal{Z}(d_{\boldsymbol{\mu}})$ *for all* $r > 0$;
(b) $x \in \lambda_0^c(\boldsymbol{\mu})$ *iff* $s(x,r) \in \mathcal{Z}((\bar{d}_{\boldsymbol{\mu}})^\bullet)$ *for all* $r > 0$;
(c) $x \in \lambda_0(\boldsymbol{\mu})$ *iff* $\lim_{r\to\infty} d_{\boldsymbol{\mu}}(s(x,r)) = 0$.

Moreover, a subset B of $\lambda_0(\boldsymbol{\mu})$ is bounded iff

$$\lim_{r\to\infty} \sup_{x\in B} \bar{d}_{\boldsymbol{\mu}}(s(x,r)) = 0.$$

Proof. (a) follows from parts (a)–(b) of Fact 5.1, and (c) from its parts (a′)–(b′).

The latter are also used to verify part (b) of the proposition. Take any $x \in \lambda_0^c(\boldsymbol{\mu})$, $r > 0$, and $m \in \mathbb{N}$, and note that $s(x,r)\setminus[m] = s(x - x^m, r)$. Hence, $\bar{d}_{\boldsymbol{\mu}}(s(x,r)\setminus[m]) \le \|r^{-1}(x-x^m)\|^0_{\boldsymbol{\mu}}$. Letting $m\to\infty$ we get $(\bar{d}_{\boldsymbol{\mu}})^\bullet(s(x,r)) = 0$.

To prove the converse implication in (b), take any $0<\varepsilon<1$ and next choose m so that $\bar{d}_{\boldsymbol{\mu}}(s(x,\varepsilon)\setminus[m])<\varepsilon$. Then

$$\|x-x^m\|^0_{\boldsymbol{\mu}}\le \bar{d}_{\boldsymbol{\mu}}\big(s(x,r)\setminus[m]\big)+\min(1,\varepsilon)\bar{d}_{\boldsymbol{\mu}}(\mathbb{N})<\big(1+\bar{d}_{\boldsymbol{\mu}}(\mathbb{N})\big)\varepsilon.$$

Thus, $\|x-x^m\|^0_{\boldsymbol{\mu}}\to 0$ as $m\to\infty$, and $x\in\lambda_0^c(\boldsymbol{\mu})$.

The last assertion follow from Fact 5.1(a′)–(b′), and Fact 5.5. □

Observe that the condition in (c) (to the right of 'iff') holds for all $x\in l_\infty$, while those in (a) and (b) hold for all $x\in c_0$ provided $\boldsymbol{\mu}$ satisfies (A). With the latter proviso, we now show that the uniform boundedness assumption of $\boldsymbol{\mu}$ is also necessary for the above characterizations to be true.

Proposition 6.5. *If $c_0\subset\lambda_0(\boldsymbol{\mu})$, then $\boldsymbol{\mu}$ is uniformly bounded, while if $c_0\subset\lambda_{00}(\boldsymbol{\mu})$, then $\boldsymbol{\mu}$ satisfies also condition* (A).

Proof. Assume $c_0\subset\lambda_0(\boldsymbol{\mu})$ and suppose $\boldsymbol{\mu}$ is not uniformly bounded, or $d_{\boldsymbol{\mu}}(\mathbb{N})=\infty$. Apply Fact 4.2 to find a strictly increasing sequence (n_k) in $\mathbb{N}$, and a sequence (A_k) of disjoint intervals in $\mathbb{N}$ with $1<\alpha_k:=\mu_{n_k}(A_k)\to\infty$. Let $x=\sum_k\alpha_k^{-1/2}e_{A_k}$ (coordinate-wise). Then $x\in c_0$, and yet $x\notin\lambda_0(\boldsymbol{\mu})$ because $\|x\|^0_{\mu_{n_k}}\ge\alpha_k^{-1/2}\alpha_k=\alpha_k^{1/2}\to\infty$, hence $\|x\|^0_{\boldsymbol{\mu}}=\infty$.

If $c_0\subset\lambda_{00}(\boldsymbol{\mu})$, then $(e_n)\subset\lambda_{00}(\boldsymbol{\mu})$; that is, (A) has to be satisfied. □

Remarks 6.6. (a) Part (a) of Proposition 6.4 can be interpreted as saying that $\lambda_{00}(\boldsymbol{\mu})$ consists precisely of those sequences $x=(\xi_j)$ that converge to zero along the filter-base $\mathcal{Z}(d_{\boldsymbol{\mu}})^c=\{\mathbb{N}\setminus A: A\in\mathcal{Z}(d_{\boldsymbol{\mu}})\}$. Likewise for (b).

(b) For $\boldsymbol{\mu}$ uniformly bounded, there are some striking analogies between the construction of the space $\lambda_0(\boldsymbol{\mu})$ together with its description in Proposition 6.4(c) and the construction of $\tilde{L}_0(\mu)$ together with its characterization in Section 1.G. In our framework, $(\omega,\|\cdot\|^0_{\boldsymbol{\mu}})$ plays the role of $(L_0(\mu),\|\cdot\|^0_\mu)$ and $\lambda_0(\boldsymbol{\mu})$ that of $\tilde{L}_0(\mu)$; a counterpart for $\lambda_0^c(\boldsymbol{\mu})$ is somewhat hidden—it is the largest Lebesgue subspace of $\tilde{L}_0(\mu)$ (cf. Remark 7.9). Also, in view of Corollary 5.3, the topology in $(\omega,\|\cdot\|^0_{\boldsymbol{\mu}})$ or $\lambda_0(\boldsymbol{\mu})$ corresponds to that of convergence in μ measure.

7. A generalization: The F-lattices $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$

Proposition 6.4 and Corollary 5.3 (or Fact 5.2) are somewhat surprising. They say that, for $\boldsymbol{\mu}$ uniformly bounded, the spaces $\lambda_0(\boldsymbol{\mu})$, $\lambda_0^c(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ along with their topologies depend only on the densities $d_{\boldsymbol{\mu}}$ and $\bar{d}_{\boldsymbol{\mu}}$, respectively, while these densities (or equivalent densities) may, a priori, come from different sequences $\boldsymbol{\mu}=(\mu_n)$. Furthermore, if also condition (A) is assumed, then $\lambda_0^c(\boldsymbol{\mu})=\lambda_{00}(\boldsymbol{\mu})$ and everything depends ultimately on $\bar{d}_{\boldsymbol{\mu}}$, because $d_{\boldsymbol{\mu}}=(\bar{d}_{\boldsymbol{\mu}})^\bullet$ as shown in Fact 4.1.

Following up this observation, we consider a generalization of $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ for $\boldsymbol{\mu}$ uniformly bounded and satisfying (A), in which the roles of $\bar{d}_{\boldsymbol{\mu}}$ and $d_{\boldsymbol{\mu}}$ are taken up by a submeasure η and its core $\eta^\bullet$. The relationship between arbitrary spaces $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ and those to be introduced is discussed in a greater detail at the end of the section, in Remark 7.19.

In what follows

$$\eta \text{ is a } \underline{\text{strictly positive}} \text{ submeasure on } \mathbb{N}$$

satisfying

$$\text{(lsc)} \qquad \eta(A_n) \uparrow \eta(A) \quad \text{whenever } A_n \uparrow A.$$

We will sometimes refer to such submeasures as *admissible*. In particular, any submeasure of type $\bar{d}_{\boldsymbol{\mu}}$, as well as any strictly positive (but not necessarily finite) measure on $\mathbb{N}$ can serve as η (for the latter case, see Example 7.15 and Remark 4.4(f)).

Then we use the FG-norm

$$\|x\|_\eta := \inf\{\varepsilon > 0 : \eta(s(x,\varepsilon)) \le \varepsilon\}$$

to define in ω the topology τ_η of convergence in η-submeasure. (Recall that $s(x,\varepsilon) = \{j : |\xi_j| > \varepsilon\}$; we will often write $\eta(|x| > \varepsilon)$ in place of $\eta(s(x,\varepsilon))$.) Using (lsc) one easily checks that the inf in the definition of $\|x\|_\eta$ is attained. In consequence,

$$\|x\|_\eta \le \varepsilon \quad \iff \quad \eta(s(x,\varepsilon)) \le \varepsilon.$$

We go on to collect some simple facts concerning $\|\cdot\|_\eta$.

Fact 7.1. *For each $x \in \omega$, scalar t and set $A \subset \mathbb{N}$,*

$$\|x\|_\eta \le \min(\eta(s(x)), \|x\|_\infty) \quad \text{and} \quad \|te_A\|_\eta = \min(|t|, \eta(A)).$$

In consequence, τ_η is stronger than the topology of coordinate convergence, and weaker than the topology of uniform convergence on $\mathbb{N}$.

Proof. If $\varepsilon > \eta(s(x))$, then $\eta(|x| > \varepsilon) \le \eta(s(x)) < \varepsilon$; hence $\|x\|_\eta \le \eta(s(x))$. If $\varepsilon > \|x\|_\infty$, then $\eta(|x| > \varepsilon) = \eta(\emptyset) < \varepsilon$; hence $\|x\|_\eta \le \|x\|_\infty$. This proves the first inequality. In consequence, we also have the inequality $\le$ between the quantities in the other relation. Suppose that the inequality is strict, and let $\|te_A\|_\eta < \varepsilon < \min(|t|, \eta(A))$. Then $\eta(A) = \eta(|te_A| > \varepsilon) \le \varepsilon < \eta(A)$; a contradiction. To finish, note that $\min(|\xi_j|, \eta(j)) \le \|x\|_\eta \le \|x\|_\infty$ for each $x \in \omega$ and $j \in \mathbb{N}$. □

Next, using the condition (lsc), one easily gets an analog of Fact 5.4.

Fact 7.2. *The FG-norm $\|\cdot\|_\eta$ has the property that $\|x_k\|_\eta \uparrow \|x\|_\eta$ whenever $|x_k| \uparrow |x|$. In particular, if $x \in \omega$ and $x^k = xe_{[k]}$, then $\|x^k\|_\eta \uparrow \|x\|_\eta$.*

Fact 7.3. *The FG-norm $\|\cdot\|_\eta$ is lower semicontinuous on ω and hence the FG-normed space $(\omega, \|\cdot\|_\eta)$ is complete.*

Proof. Let $x_k \to x$ in ω, and let $\delta < \|x\|_\eta$. Then, by the previous fact, there is $m \in \mathbb{N}$ with $\|xe_{[m]}\|_\eta > \delta$. Next, as $x_k \to x$ coordinate-wise, there is k_0 such that, for $k \geq k_0$, $\eta(|x_k e_{[m]}| > \delta) = \eta(|xe_{[m]}| > \delta)$ and hence $\|x_k\|_\eta \geq \|x_k e_{[m]}\|_\eta > \delta$. It follows that $\liminf_k \|x_k\|_\eta \geq \|x\|_\eta$. □

We refer the reader to [16] for a study of connectedness type properties of the topological vector-lattice group $\omega(\eta) = (\omega, \|\cdot\|_\eta)$. We will not need them in what follows; nonetheless, it may be of some interest to compare Fact 7.4 below with [16, Th. 5].

We now introduce the F-lattice

$$\lambda_0(\eta) := v(\omega, \|\cdot\|_\eta) = \Big\{x \in \omega : \lim_{t\to 0} \|tx\|_\eta = 0\Big\}.$$

An analog of Proposition 6.4(c) is easy to verify:

Fact 7.4. *If $x \in \omega$, then*

$$x \in \lambda_0(\eta) \quad \textit{iff} \quad \lim_{r\to\infty} \eta(s(x,r)) = 0,$$

where, by Fact 3.5, $\eta(s(x,r))$ may be replaced with $\eta^\bullet(s(x,r))$.

Next, motivated by Proposition 6.4(b), we define the subspace

$$\lambda_{00}(\eta) = \{x \in \omega : s(x,r) \in \mathcal{Z}(\eta^\bullet),\ \forall r > 0\}$$

of $\lambda_0(\eta)$. Note that $\lambda_{00}(\eta) = \lambda_0(\eta) = \omega$ when $\eta^\bullet = 0$.

Fact 7.5. *$\lambda_{00}(\eta)$ is a closed subspace of $\lambda_0(\eta)$, and thus it is an F-space.*

Proof. Suppose that $(x_k) \subset \lambda_{00}(\eta)$, $x \in \lambda_0(\eta)$, and $\|x - x_k\|_\eta \to 0$. Fix $r > 0$ and any $0 < \varepsilon < r$, and next choose k_0 such that $\|x - x_k\|_\eta < \varepsilon$ for $k \geq k_0$. Then

$$\begin{aligned}\eta^\bullet(s(x,r)) &\leq \eta^\bullet\big(s(x_k, r-\varepsilon)\big) + \eta^\bullet(|x - x_k| > \varepsilon)\\ &\leq \eta(|x - x_k| > \varepsilon) < \varepsilon.\end{aligned}$$

It follows that $\eta^\bullet(s(x,r)) = 0$. Thus, $x \in \lambda_{00}(\eta)$. □

Remark 7.6. Let also γ be an admissible submeasure on $\mathbb{N}$. If $\eta \sim \gamma$, then $\eta^\bullet \sim \gamma^\bullet$, by Fact 3.8. It follows that then $\lambda_0(\eta) = \lambda_0(\gamma)$ and $\lambda_{00}(\eta) = \lambda_{00}(\gamma)$, and that the FG-norms $\|\cdot\|_\eta$ and $\|\cdot\|_\gamma$ are equivalent. Conversely, if $\lambda_0(\eta) = \lambda_0(\gamma)$, then $\eta \sim \gamma$. In fact, the F-norms $\|\cdot\|_\eta$ and $\|\cdot\|_\gamma$ must be equivalent, by the closed graph theorem. From this, using Fact 7.1, one easily deduces that $\eta \sim \gamma$. Also note that if μ is any finite positive measure on $\mathbb{N}$, then $\mu \ll \eta$ and, in consequence, $\eta \sim \eta + \mu$.

Fact 7.7. *$l_\infty \subset \lambda_0(\eta)$, $c_0 \subset \lambda_{00}(\eta)$, and the inclusions are continuous.*

Proof. The first inclusion and the continuity of both follow from the inequality $\|\cdot\|_\eta \leq \|\cdot\|_\infty$. The other inclusion follows from the definition of $\lambda_{00}(\eta)$ and the inclusion $\mathcal{F} \subset \mathcal{Z}(\eta^\bullet)$. *Alternatively*, use Fact 1.1. □

FACT 7.8. *The sequence (e_n) of unit vectors is a Schauder basis of $\lambda_{00}(\eta)$.*

Proof. If $x \in \lambda_{00}(\eta)$, and $\varepsilon > 0$ is given, then $\eta^\bullet(s(x,\varepsilon)) = 0$, hence one can choose $m \in \mathbb{N}$ with $\eta(s(x,\varepsilon) \setminus [m]) < \varepsilon$. Then $\|x - x^n\|_\eta < \varepsilon$ for all $n \geq m$, thus proving that (e_n) is a Schauder basis for $\lambda_{00}(\eta)$. □

REMARK 7.9. In view of Fact 7.8, $\lambda_{00}(\eta)$ could as well be defined as the closed linear span of (e_n) in $\lambda_0(\eta)$, or as the largest subspace of $\lambda_0(\eta)$ which has the Lebesgue property (or has order continuous F-norm).

In view of Facts 7.7 and 7.8, c_0 is a dense subspace of $\lambda_{00}(\eta)$. An analogous relation holds between l_∞ and $\lambda_0(\eta)$ as well.

FACT 7.10. *l_∞ is a dense subspace of $\lambda_0(\eta)$.*

Proof. Take any $x \in \lambda_0(\eta)$ and $\varepsilon > 0$. By Fact 7.4, there exists $m \in \mathbb{N}$ with $\eta(s(x,m)) < \varepsilon$. Denote $B = s(x,m)$, $C = \mathbb{N} \setminus B$, and let $y = xe_B$, $z = xe_C$. Then $z \in l_\infty$ and $\|x - z\|_\eta = \|y\|_\eta \leq \eta(B) < \varepsilon$. □

A familiar relation between the basis (e_n) of c_0 and the space l_∞ has an analog in the present setting.

FACT 7.11. *If $x = (\xi_j) \in \omega$, then the series $\sum_j \xi_j e_j$ in $\lambda_{00}(\eta)$ has bounded partial sums iff $x \in \lambda_0(\eta)$.*

Proof. Let $x = (\xi_j)$ and $x^n = \sum_{j=1}^n \xi_j e_j$ $(n \in \mathbb{N})$. Then, by Fact 7.2, $\|\delta x\|_\eta = \sup_n \|\delta x^n\|_\eta$ for every $\delta > 0$, and from this the assertion follows. □

The following result says, in particular, that λ_{00} (as well as λ_0) does not have the Zero-density Convergence Property (see the Introduction), in a very strong sense.

COROLLARY 7.12. *For each $x = (\xi_j) \in \omega$, the series $\sum_j \xi_j e_j$ in $\lambda_{00}(\eta)$ has all its subseries of $\eta^\bullet$-zero density convergent. However, if $x \notin \lambda_0(\eta)$, then the sequence of partial sums of the series is unbounded.*

Proof. Let $A \in \mathcal{Z}(\eta^\bullet)$. Given $\varepsilon > 0$, there is $F \in \mathcal{F}(A)$ with $\eta(A \setminus F) < \varepsilon$. If $B \in \mathcal{F}(A \setminus F)$, then $\|xe_B\|_\eta \leq \eta(B) < \varepsilon$. Hence, the series $\sum_{j \in A} \xi_j e_j$ converges unconditionally in $\lambda_{00}(\eta)$. To conclude, apply Fact 7.11. □

As in [7], a sequence (x_n) in a TVS is said to be *irregular* if $t_n x_n \to 0$ for every sequence (t_n) of scalars.

PROPOSITION 7.13. *The following are equivalent.*

(a) $\eta(n) \to 0$ *as* $n \to \infty$.
(b) *The basis (e_n) of $\lambda_{00}(\eta)$ is irregular.*
(c) *$\lambda_{00}(\eta)$ is an $F\omega$-space.*
(d) *$\lambda_{00}(\eta)$ has poor dual.*

Proof. The equivalence of (a) and (b) follows from the equality $\|t_n e_n\|_\eta = \min(|t_n|, \eta(n))$, while the mutual equivalence of (b), (c), and (d) is a direct consequence of [7, Prop. 2.2]. □

REMARK 7.14. Since η is countably subadditive, it is clear that if $\eta(n) \to 0$, then every infinite set $B \subset \mathbb{N}$ has an infinite subset A on which η is order continuous, and hence $\eta^\bullet(A) = 0$ (cf. Fact 3.4). Using this observation, one can prove that (a) implies (d) by the same argument as in the first part of the proof of Theorem 10.1 below.

EXAMPLE 7.15. Let η be a positive measure on $\mathbb{N}$ such that $\eta(\mathbb{N}) = \infty$, $0 < \eta(n) < \infty$ for all n, and $\eta(n) \to 0$ as $n \to \infty$. If $A \subset \mathbb{N}$, then it is clear that $\eta^\bullet(A) = 0$ if $\eta(A) < \infty$, and $\eta^\bullet(A) = \infty$ otherwise. Thus, $\mathcal{Z}(\eta^\bullet) = \{A \subset \mathbb{N} : \eta(A) < \infty\}$. It now follows from the definition of $\lambda_{00}(\eta)$ and Fact 7.4 (recalling that η is a measure, cf. the characterization of $\tilde{L}_0(\mu)$ in Section 1.G) that

$$\lambda_{00}(\eta) = \{x \in \omega : \eta(s(x,r)) < \infty \text{ for all } r > 0\},$$
$$\lambda_0(\eta) = \{x \in \omega : \eta(s(x,r)) < \infty \text{ for some } r > 0\}.$$

Moreover, by Proposition 7.13 (see also Remark 7.14), the dual of $\lambda_{00}(\eta)$ is poor (and thus $\lambda_{00}(\eta)$ is an $F\omega$-space). We are going to show that it is not so for the dual of $\lambda_0(\eta)$.

Take any set $N \subset \mathbb{N}$ with $\eta(N) = \infty$. Then, evidently, the family $\mathcal{B}_N = \{B \subset N : \eta(N \setminus B) < \infty\}$ is a filter-base in $\mathbb{N}$. Let $\mathcal{U}_N$ be an ultrafilter in $\mathbb{N}$ containing it. Note that if $x \in \lambda_0(\eta)$, then x is bounded on some set $B \in \mathcal{B}_N$. Hence, we may define a linear functional u_N on $\lambda_0(\eta)$ by

$$u_N(x) = \lim_{\mathcal{U}_N} x.$$

If $x \in \lambda_0(\eta)$ and $\|x\|_\eta < \varepsilon$, then $\{j \in N : |\xi_j| \le \varepsilon\} \in \mathcal{B}_N$ so that $|u_N(x)| \le \varepsilon$. Hence, $u_N \in \lambda_0(\eta)'$. Clearly, $u_N(e_N) = 1$, $u_N(x) = 0$ for all $x \in \lambda_0(\eta)$ with $s(x) \subset \mathbb{N} \setminus N$, and $u_N = 0$ on $\lambda_{00}(\eta)$. In consequence, $u_N(e_K) = 0$ if $K \subset \mathbb{N}$ is almost disjoint from N, i.e., $|N \cap K| < \infty$.

Now, consider a partition of $\mathbb{N}$ into finite sets F_n with $\eta(F_n) > 1$ $(n \in \mathbb{N})$, and a family $(M_i)_{i \in I}$ of cardinality $2^{\aleph_0}$ consisting of almost disjoint infinite subsets of $\mathbb{N}$. For each $i \in I$, let $N_i = \bigcup_{n \in M_i} F_n$, and denote by u_i a functional constructed as above for $N = N_i$. Then $u_i(e_{N_j}) = 1$ if $i = j$, and 0 otherwise. It follows easily that the family of functionals $(u_i)_{i \in I}$ in $\lambda_0(\eta)'$ is linearly independent so that $\dim \lambda_0(\eta)' \ge 2^{\aleph_0}$.

Finally, if $\mu_n(A) := \eta(A \cap [n])$ and $\boldsymbol{\mu} = (\mu_n)$, then $\eta = \bar{d}_{\boldsymbol{\mu}} = d_{\boldsymbol{\mu}} \ne \eta^\bullet$ (cf. Remark 4.4(f)) and $\lambda_{00}(\boldsymbol{\mu}) = \{0\}$. □

Given a solid sequence F-space X and a set $A \subset \mathbb{N}$, we denote

$$X(A) = \{x \in X : s(x) \subset A\},$$

and consider the subspace $X(A)$ with the F-norm and topology induced from X. Clearly, $X(A)$ is a closed ideal in X, and it is the range of the (continuous) natural projection $x \to xe_A$ in X.

A feeling one gets from part (c) of the next result is that the spaces $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$ are most interesting and L_0-like when $\eta^\bullet \neq 0$ and $\eta(n) \to 0$. It will become even stronger when we come later to the spaces over core-nonatomic submeasures η with $\eta^\bullet \neq 0$ (then $\eta(n) \to 0$ automatically).

THEOREM 7.16. *Let A be an infinite subset of $\mathbb{N}$.*

(a) *If $\eta^\bullet(A) = 0$, then $\lambda_{00}(\eta)(A) = \omega(A)$ and the topology τ_η in $\lambda_{00}(\eta)(A)$ coincides with that induced from ω.*
(b) *If $\eta^\bullet(A) > 0$, then both the inclusions $\lambda_{00}(\eta)(A) \subset \lambda_0(\eta)(A) \subset \omega(A)$ are proper, the topology τ_η in $\lambda_{00}(\eta)(A)$ is strictly stronger than that induced from ω, and the subspace $\lambda_0(\eta)(A)$ is nonseparable.*
(c) *If $\eta^\bullet(A) > 0$ and $\eta(n) \to 0$ as $A \ni n \to \infty$, then the subspace $\lambda_{00}(\eta)(A)$ is not locally pseudoconvex, and its Schauder basis $(e_n)_{n\in A}$ is irregular.*
(d) *If $\eta^\bullet(A) > 0$ and $\inf_{n\in A} \eta(n) > 0$, then $\lambda_{00}(\eta)(A) = c_0(A)$ and $\lambda_0(\eta)(A) = l_\infty(A)$, and the topology τ_η on $\lambda_0(\eta)(A)$ is the one induced from l_∞.*

Proof. We may assume without loss of generality that $A = \mathbb{N}$.

(a) It was already noted earlier that if $\eta^\bullet(\mathbb{N}) = 0$, then $\lambda_{00}(\eta) = \omega$ and η is order continuous (see Fact 3.4 or the paragraph before Fact 3.9). In view of Fact 7.1, it remains to show that τ_η is weaker than the topology of ω. To see this, take any $\varepsilon > 0$, and next choose $m \in \mathbb{N}$ so that $\eta(\mathbb{N} \setminus [m]) < \varepsilon$. Now, if $x \in \omega$ and $|xe_{[m]}| \le \varepsilon$, then

$$\eta(|x| > \varepsilon) = \eta\big(|xe_{(m,\infty)}| > \varepsilon\big) \le \eta(\mathbb{N} \setminus [m]) < \varepsilon.$$

Hence, $\|x\|_\eta \le \varepsilon$, and we are done.

(b) Since $e_\mathbb{N} \in \lambda_0(\eta) \setminus \lambda_{00}(\eta)$, and $x = (j)_{j\in\mathbb{N}} \notin \lambda_0(\eta)$, the inclusions are proper. Next, by Fact 3.9(a), there is a sequence (F_k) of disjoint finite sets with $\varepsilon := \inf_k \eta(F_k) > 0$. Clearly, the sequence $y_k := e_{F_k}$ $(k \in \mathbb{N})$ is in $\lambda_{00}(\eta)$ and converges to zero coordinate-wise, but not in τ_η because, by Fact 7.1, $\|y_k\|_\eta \ge \min(1, \varepsilon)$ for all k. Finally, for each $N \subset \mathbb{N}$ let $F(N) := \bigcup_{k\in N} F_k$; note that $e_{F(N)} \in \lambda_0(\eta)$ (Fact 7.7). If $M, N \subset \mathbb{N}$ and $M \neq N$, then $\|e_{F(M)} - e_{F(N)}\|_\eta \ge \|y_k\|_\eta \ge \min(1, \varepsilon)$ for some k. It follows that $\lambda_0(\eta)$ is nonseparable.

(c) By Proposition 7.13, $x_n := 2^n e_n \to 0$ in $\lambda_0(\eta)$. Let (y_k) be the sequence used above. Then $(y_k) \subset \lambda_{00}(\eta)$ and $\|y_k\|_\eta \not\to 0$. Note that $y_k = \sum_{j\in F_k} 2^{-j} x_j$. Now, if p is any continuous α-seminorm on $\lambda_{00}(\eta)(A)$, then

$$p(y_k) \le \sum_{j\in F_k} 2^{-\alpha j} p(x_j) \le (1 - 2^{-\alpha})^{-1} \max_{j\in F_k} p(x_j),$$

hence $p(y_k) \to 0$ as $k \to \infty$. From this the assertion follows.

(d) Since $\delta := \inf_{n\in\mathbb{N}} \eta(n) > 0$, $\mathcal{Z}(\eta^\bullet) = \mathcal{F}$ and hence $\lambda_{00}(\eta) = c_0$. Moreover, it follows from Fact 7.1 that $\min(\|x\|_\infty, \delta) \le \|x\|_\eta \le \|x\|_\infty$ for all $x \in \omega$. Therefore, $\lambda_0(\eta) = l_\infty$, and $\|\cdot\|_\eta$ and $\|\cdot\|_\infty$ are equivalent. □

REMARKS 7.17. (a) For the space $\lambda_{00} = \lambda_{00}(\bar{d})$, part (c) is also a consequence of Corollary 7.12 and [13, Cor. 4.7]. Moreover, if η is core-nonatomic (which implies that $\eta(n) \to 0$, see Fact 3.7), then it will be shown in Corollary 11.2 that every continuous α-seminorm on $\lambda_{00}(\eta)$ or $\lambda_0(\eta)$ is equivalent to a seminorm and has a finite-codimensional kernel. This will enable us to give part (c) a much stronger form (see Corollary 11.4).

(b) The last assertion in part (b) will be considerably strengthened in Corollary 8.3.

(c) In part (c), the assumption about $(\eta(n))_{n\in A}$ cannot be replaced by $\inf_{n\in A} \eta(n) = 0$. For, to prove non-pseudoconvexity of $\lambda_{00}(\eta)$, we need a sequence $(m_n) \subset A$ with $\eta(m_n) \to 0$ and such that $\eta^\bullet(\{m_n : n \in \mathbb{N}\}) > 0$, and there is no way to achieve this using the weaker assumption.

(d) Since η is strictly positive, $\inf_n \eta(n) > 0$ iff $\mathcal{Z}(\eta^\bullet) = \mathcal{F}$, by Fact 3.3.

(e) The role played by $\eta^\bullet$ may become clearer when calculating the quotient F-norm of $\lambda_0(\eta)/\lambda_{00}(\eta)$ in Proposition 9.1(a).

We now give an application of λ_{00} to a different type of problems. For a solid sequence Banach space E, consider its subspace

$$E(\mathcal{Z}) := \{x \in E : s(x) \in \mathcal{Z}\}.$$

It was shown in [10, Thm. 2] and, with a different proof in [11, Thm. 5.1], that $E(\mathcal{Z})$ is barrelled. In fact, as was observed in [11, Remark after Thm. 5.1], the space $E(\mathcal{Z})$ is even p-barrelled for every $0 < p \le 1$. That is, every closed-graph linear operator from $E(\mathcal{Z})$ into any p-Banach space is continuous. The F-space λ_{00} (or λ_0) allows us to show the following proposition.

PROPOSITION 7.18. *If E is an infinite-dimensional solid sequence Banach (or F-) space, then $E(\mathcal{Z})$ is not ultrabarrelled. Thus, a closed-graph linear operator from $E(\mathcal{Z})$ to an F-space need not be continuous.*

Proof. We assume, as we may, that E contains all the unit vectors e_n. Fix an element $a = (\alpha_n) \in E$ with all $\alpha_n \ne 0$ and define a linear operator $T : E(\mathcal{Z}) \to \lambda_{00}$ by

$$Ty = (n\alpha_n^{-1}\eta_n) \quad \text{for } y = (\eta_n) \in E(\mathcal{Z}).$$

Note that $Ty \in \lambda_{00}$ because $\operatorname{supp} Ty \in \mathcal{Z}$; see Proposition 6.4(a). Since the coordinates on E are continuous, the map T is continuous when λ_{00} is taken with the topology induced from ω. Hence, T has closed graph. However, it is not continuous when λ_{00} is considered with its F-norm topology: The sequence $a^n = \sum_{j=1}^n \alpha_j e_j$ in $E(\mathcal{Z})$ is bounded, while the sequence $Ta^n = \sum_{j=1}^n j e_j$ is not bounded in λ_{00} (see Fact 7.11). □

Undoubtedly, it is highly desirable to have a clear view of the relations between spaces based on sequences of measures $\boldsymbol{\mu}$ and those based on submeasures η. Hopefully, it is provided by the following.

REMARK 7.19. If $\boldsymbol{\mu} = (\mu_n)$ is an admissible sequence of measures on $\mathbb{N}$, then $\lambda_0(\boldsymbol{\mu}) \subset \lambda_0(\bar{d}_{\boldsymbol{\mu}}) = v(\omega, \|\cdot\|_{\boldsymbol{\mu}})$ and $\lambda_0^c(\boldsymbol{\mu}) \subset \lambda_{00}(\bar{d}_{\boldsymbol{\mu}})$, with continuous inclusions. The inclusions were shown in Proposition 6.4(b) and (c) (where $\bar{d}_{\boldsymbol{\mu}}(\mathbb{N}) < \infty$ was not needed for that), while their continuity follows from the first part of Corollary 5.3 (or from Fact 1.1). Moreover,

(a) $\lambda_0(\boldsymbol{\mu}) = \lambda_0(\eta)$ *or/and* $\lambda_0^c(\boldsymbol{\mu}) = \lambda_{00}(\eta)$ *for an admissible submeasure* η *iff* $\boldsymbol{\mu}$ *is uniformly bounded.*

'If': This follows from Corollary 5.3 with $\eta = \bar{d}_{\boldsymbol{\mu}}$. 'Only if': In either case, $c_0 \subset \lambda_0(\boldsymbol{\mu})$; apply Proposition 6.5.

(b) $\lambda_{00}(\boldsymbol{\mu}) = \lambda_{00}(\eta)$ *for an admissible submeasure* η *iff* $\boldsymbol{\mu}$ *is uniformly bounded and satisfies condition* (A).

'If': In view of Fact 4.1, $d_{\boldsymbol{\mu}} = (\bar{d}_{\boldsymbol{\mu}})^\bullet$ so that $\lambda_0^c(\boldsymbol{\mu}) = \lambda_{00}(\boldsymbol{\mu}) = \lambda_{00}(\eta)$ with $\eta = \bar{d}_{\boldsymbol{\mu}}$, by Proposition 6.4(a) or (b). 'Only if': $c_0 \subset \lambda_{00}(\boldsymbol{\mu})$; apply Proposition 6.5.

Thus, if $\boldsymbol{\mu}$ is uniformly bounded and satisfies condition (A), then

$$\lambda_0(\boldsymbol{\mu}) = \lambda_0(\bar{d}_{\boldsymbol{\mu}}) \quad \text{and} \quad \lambda_0^c(\boldsymbol{\mu}) = \lambda_{00}(\boldsymbol{\mu}) = \lambda_{00}(\bar{d}_{\boldsymbol{\mu}})$$

with equivalent F-norms $\|\cdot\|_{\boldsymbol{\mu}}^0$ and $\|\cdot\|_{\boldsymbol{\mu}} = \|\cdot\|_{\bar{d}_{\boldsymbol{\mu}}}$. In particular,

$$\lambda_0 = \lambda_0(\bar{d}) \quad \text{and} \quad \lambda_{00} = \lambda_{00}(\bar{d}).$$

Consequently, for such $\boldsymbol{\mu}$'s, all the results that we prove about general spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$ are valid also for the spaces $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$.

Let us also note that, for this setting, the most interesting situation pointed out before Theorem 7.16 ($\eta^\bullet \neq 0$ and $\eta(k) \to 0$) arises when, additionally, $d_{\boldsymbol{\mu}}(\mathbb{N}) > 0$ and $\boldsymbol{\mu}$ satisfies condition (C). By Fact 4.3, all these requirements are met when $d_{\boldsymbol{\mu}}$ is nonatomic and $d_{\boldsymbol{\mu}}(\mathbb{N}) > 0$.

In other cases, one has to carefully distinguish between the spaces $\lambda_0(\boldsymbol{\mu})$ and $\lambda_0^c(\boldsymbol{\mu})$ or $\lambda_{00}(\boldsymbol{\mu})$, and their counterparts $\lambda_0(\bar{d}_{\boldsymbol{\mu}})$ and $\lambda_{00}(\bar{d}_{\boldsymbol{\mu}})$.

EXAMPLES 7.20. (a) Let $\boldsymbol{\mu} = (\mu_n)$, where μ_n is the measure of mass n concentrated at the point $n \in \mathbb{N}$. Then $\boldsymbol{\mu}$ satisfies (A), but is not uniformly bounded. Further, $\bar{d}_{\boldsymbol{\mu}}(A) = \sup A$ if $A \neq \emptyset$, $(\bar{d}_{\boldsymbol{\mu}})^\bullet = d_{\boldsymbol{\mu}}$ vanishes on $\mathcal{F}$ and is ∞ elsewhere. One easily sees that $\lambda_0(\boldsymbol{\mu}) = \{x \in \omega : \sup_n n|\xi_n| < \infty\}$ and $\lambda_{00}(\boldsymbol{\mu}) = \{x \in \omega : \lim_n n|\xi_n| = 0\}$, and that their F-norm is equivalent to the norm $\|x\| := \sup_n n|\xi_n|$. On the other hand, $\lambda_0(\bar{d}_{\boldsymbol{\mu}}) = l_\infty$ and $\lambda_{00}(\bar{d}_{\boldsymbol{\mu}}) = c_0$, and their F-norm is equivalent to the sup norm $\|\cdot\|_\infty$. Clearly, the inclusions $\lambda_0(\boldsymbol{\mu}) \subset \lambda_0(\bar{d}_{\boldsymbol{\mu}})$ and $\lambda_0^c(\boldsymbol{\mu}) \subset \lambda_{00}(\bar{d}_{\boldsymbol{\mu}})$ are both proper, and neither is an isomorphic embedding.

(b) Let $\boldsymbol{\mu} = (\mu_n)$ be associated with the counting measure μ on $\mathbb{N}$ in the sense of Remark 4.4(f). Then $\lambda_0(\boldsymbol{\mu}) = \lambda_0^c(\boldsymbol{\mu}) = l_1$ and $\lambda_{00}(\boldsymbol{\mu}) = \{0\}$, while $\lambda_0(\bar{d}_{\boldsymbol{\mu}}) = l_\infty$ and $\lambda_{00}(\bar{d}_{\boldsymbol{\mu}}) = c_0$.

8. Simple embeddings into $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$

We recall that the notation $X(A)$ was introduced before Theorem 7.16.

THEOREM 8.1. *Let (A_k) be a disjoint sequence of nonempty sets in $\mathbb{N}$ with union A. Denote $y_k = e_{A_k}$ $(k \in \mathbb{N})$, and define an injective linear operator $T : \omega \to \omega$ by*

$$Tx = \sum_{k=1}^{\infty} \xi_k y_k \quad (\textit{coordinate-wise sum}) \textit{ for } x = (\xi_k) \in \omega.$$

(a) *If $A \in \mathcal{Z}(\eta^\bullet)$, then T is an isomorphic embedding of ω into $\lambda_{00}(\eta)$. If, moreover, the A_k's are singletons, then T is an isomorphism of ω onto $\lambda_{00}(\eta)(A) = \lambda_0(\eta)(A)$.*

(b) *If $\delta = \inf_k \eta(A_k) > 0$ (so that $\eta^\bullet(A) > 0$, by Fact 3.9(b)), then T maps l_∞ into $\lambda_0(\eta)$ isomorphically; moreover, if $A_k \in \mathcal{Z}(\eta^\bullet)$ for all k, then*

$$\lambda_{00}(\eta) \cap T(l_\infty) = T(c_0),$$

and if $A_k \notin \mathcal{Z}(\eta^\bullet)$ for all k, then

$$\lambda_{00}(\eta) \cap T(l_\infty) = \{0\}.$$

Proof. Note that the sequence (y_k) is basic in $\lambda_0(\eta)$. Take any $x \in \omega$, and denote $Tx = y = (\eta_j)$.

(a) Since $s(y) \subset A \in \mathcal{Z}(\eta^\bullet)$, we have $y \in \lambda_{00}(\eta)$ by the definition. Next, as the series $\sum_j \eta_j e_j$ converges unconditionally to y (Fact 7.8), so does the series $\sum_k \xi_k y_k$. In consequence, T establishes equivalence of the basis (e_k) of ω and the basic sequence (y_k) in $\lambda_{00}(\eta)$. Therefore, T is an isomorphism from ω onto $\overline{\operatorname{lin}}(y_k)$ in $\lambda_{00}(\eta)$. To finish, appeal to Theorem 7.16(a).

(b) Let $x \in l_\infty$. Then $y \in \lambda_0(\eta)$ and $\|y\|_\eta \le \|y\|_\infty = \|x\|_\infty$ so that T maps l_∞ into $\lambda_0(\eta)$ continuously. If $\|x\|_\infty = 1$, choose k so that $|\xi_k| \ge \frac{1}{2}$. Then $\|y\|_\eta \ge \|\xi_k y_k\|_\eta = \min(|\xi_k|, \eta(A_k)) \ge \min(\frac{1}{2}, \delta)$. Thus, also the inverse map $(T|l_\infty)^{-1}$ is continuous.

Now assume also that $A_k \in \mathcal{Z}(\eta^\bullet)$ for all k. Then $y_k \in \lambda_{00}(\eta)$ for all k, hence $T(c_0) \subset \lambda_{00}(\eta)$. If $x \notin c_0$, then $|\xi_k| > \varepsilon$ for some $\varepsilon > 0$ and all k in an infinite set $K \subset \mathbb{N}$. Then $s(y, \varepsilon) \supset \bigcup_{k \in K} A_k$, where the latter set is of positive $\eta^\bullet$ density (see Fact 3.9(b)). Hence, by definition, $y \notin \lambda_{00}(\eta)$. Thus $T(c_0) = \lambda_{00}(\eta) \cap T(l_\infty)$.

The other subcase of (b) follows directly from the definition of $\lambda_{00}(\eta)$. $\square$

REMARK 8.2. The argument used while proving part (b) is of standard character and, under suitable assumptions, works in general topological Riesz spaces, see for instance [1, Th. 10.7] and [12, Th. 2.7].

COROLLARY 8.3. *If $A \subset \mathbb{N}$ and $\eta^\bullet(A) > 0$, then there exist isomorphic embeddings $T : l_\infty \to \lambda_0(\eta)(A)$ such that $\lambda_{00}(\eta)(A) \cap T(l_\infty) = T(c_0)$, as well as such that $\lambda_{00}(\eta)(A) \cap T(l_\infty) = \{0\}$.*

Proof. By Fact 3.9(a), one can find a partition of A into a sequence (F_k) of finite sets with $\inf_k \eta(F_k) > 0$. Then part (b) of the theorem applies when $A_k := F_k$ $(A_k \in \mathcal{Z}(\eta^\bullet))$, as well as when $A_k := \bigcup_{m\in M_k} F_m$ $(A_k \notin \mathcal{Z}(\eta^\bullet))$ for all k, where (M_k) is any disjoint sequence of infinite subsets of $\mathbb{N}$. □

COROLLARY 8.4. *If $\eta^\bullet \neq 0$, then $\lambda_{00}(\eta)$ is uncomplemented in $\lambda_0(\eta)$.*

Proof. By the previous corollary, there is an isomorphism $T : l_\infty \to \lambda_0(\eta)$ with $T(c_0) \subset \lambda_{00}(\eta)$. Suppose there exists a continuous linear projection P of $\lambda_0(\eta)$ onto $\lambda_{00}(\eta)$. Then $PT : l_\infty \to \lambda_{00}(\eta)$ is a continuous operator with $PTe_n = Te_n \not\to 0$. In consequence, by [5] or [4], $\lambda_{00}(\eta)$ contains an isomorphic copy of l_∞, which is absurd because $\lambda_{00}(\eta)$ is separable (Fact 7.8). □

Our next result is a complement to Theorem 8.1 for the case where the A_k's are singletons and the spaces considered are $\lambda_0 = \lambda_0(\bar{d})$ and $\lambda_{00} = \lambda_{00}(\bar{d})$. Its extension to general spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$ is likely to require the map $k \to n_k$ to have some special properties relevant to η.

THEOREM 8.5. *Let $A = \{n_1 < n_2 < \cdots\}$ be an infinite subset of $\mathbb{N}$, and define an injective linear operator $T : \omega \to \omega$ by setting*

$$Tx = \sum_{k=1}^{\infty} \xi_k e_{n_k} \quad (\textit{coordinate-wise sum}) \textit{ for } x = (\xi_k) \in \omega.$$

(a) *T maps λ_0 into $\lambda_0(A)$ (resp., λ_{00} into $\lambda_{00}(A)$) continuously*;

(b) *T maps λ_0 onto $\lambda_0(A)$ (resp., λ_{00} onto $\lambda_{00}(A)$) isomorphically iff the lower density of A is positive, that is, $\delta := \liminf_{n\to\infty} d_n(A) > 0$.*

Proof. We will use the F-seminorms $\|\cdot\|_n = \|\cdot\|^0_{d_n}$ $(n \in \mathbb{N})$ and the F-norm $\|\cdot\| = \|\cdot\|^0_{\boldsymbol{\mu}}$ of Section 5 for $\boldsymbol{\mu} = (d_n)$ (see Remark 4.4(b)).

(a) Assume $x = (\xi_k)$ is in λ_0 (resp., λ_{00}), and denote $Tx = y = (\eta_j)$. Given n, let $m_n = \max\{k : n_k \le n\}$. Then

$$\begin{aligned}\|y\|_n &= \frac{1}{n}\sum_{j=1}^{n} \min(1, |\eta_j|) = \frac{m_n}{n}\frac{1}{m_n}\sum_{k=1}^{m_n} \min(1, |\xi_k|) \\ &= \frac{m_n}{n}\|x\|_{m_n} = d_n(A)\cdot\|x\|_{m_n} \le \|x\|_{m_n}.\end{aligned}$$

This implies that $y \in \lambda_0$ (resp., $y \in \lambda_{00}$), and that $\|y\| \le \|x\|$, and (a) follows.

(b) The 'if' part. Assume $\delta > 0$. Take any $y = (\eta_j)$ in $\lambda_0(A)$ (resp., $\lambda_{00}(A)$), and denote $T^{-1}y = x = (\xi_k)$. Fix n_0 so that $d_m(A) > \frac{1}{2}\delta$ for $m \geq n_0$. Then, for $m \geq n_0$,

$$\frac{1}{2}\delta\|x\|_m \leq d_{n_m}(A) \cdot \|x\|_m = \frac{m}{n_m}\frac{1}{m}\sum_{k=1}^{m}\min(1, |\xi_k|)$$
$$= \frac{1}{n_m}\sum_{j=1}^{n_m}\min(1, |\eta_j|) = \|y\|_{n_m}.$$

It follows that x is in λ_0 (resp., λ_{00}), and that $\|y\| \geq \frac{1}{2}\delta \sup_{m \geq n_0}\|x\|_m$. Since $\|x\|_m \leq n_0\|x\|_{n_0}$ for $m < n_0$, we have in fact $\|y\| \geq \frac{1}{2}\delta n_0^{-1}\|x\|$. Hence, T^{-1} maps $\lambda_0(A)$ (resp., $\lambda_{00}(A)$) into λ_0 (resp., λ_{00}) continuously.

The 'only if' part. Assume $\delta = 0$. We claim that then the map $(T|\lambda_{00})^{-1}$ is not continuous at zero. Fix $\varepsilon > 0$. Since $\delta = 0$, there is p such that $d_{n_p}(A) = p/n_p < \varepsilon$. If $0 \leq q \leq p$, then

$$d_{n_{p+q}}(A) = \frac{p+q}{n_{p+q}} \leq \frac{2p}{n_p} < 2\varepsilon.$$

It follows that $d_s(A) < 2\varepsilon$ for $n_p \leq s \leq n_{2p}$. Let y be the characteristic function of the set $\{n_k : p < k \leq 2p\}$. Then $y \in \lambda_{00}(A)$ and

$$\|y\| \leq \max\{d_s(A) : n_p < s \leq n_{2p}\} < 2\varepsilon.$$

Clearly, $x = T^{-1}(y)$ is simply the characteristic function of the interval $(p, 2p]$. Hence, $x \in \lambda_{00}$, and $\|x\| = \frac{1}{2}$. Thus, our claim has been verified. □

Corollary 8.6. *For every $r \in \mathbb{N}$, each of the spaces λ_0 and λ_{00} is isomorphic to the direct sum of r copies of itself. Moreover, λ_{00} admits an infinite Schauder decomposition into subspaces each of which is isomorphic to λ_{00}.*

Proof. The sets $A_i := \{i + (k-1)r : k \in \mathbb{N}\}$ ($i \in [r]$) form a partition of $\mathbb{N}$, and each A_i satisfies the condition from Theorem 8.5(b) with $\delta = 1/(r+1)$. Hence $\lambda_0(A_i) \approx \lambda_0$ and $\lambda_0 = \lambda_0(A_1) \oplus \cdots \oplus \lambda_0(A_r)$ (topologically). Likewise for the case of λ_{00}.

To prove the 'moreover' part, it is enough to construct a partition of $\mathbb{N}$ into an infinite sequence of sets $A_k \subset \mathbb{N}$ with a positive lower density. This can be done, e.g., as follows. Given an infinite set $A = \{n_1 < n_2 < \cdots\}$ in $\mathbb{N}$, let $A' := \{n_{2j-1} : j \in \mathbb{N}\}$. Then define $A_1 = \mathbb{N}'$, $A_2 = (\mathbb{N} \setminus A_1)'$, $A_3 = (\mathbb{N} \setminus (A_1 \cup A_2))'$, and so on. □

9. The quotient space $\lambda_0(\eta)/\lambda_{00}(\eta)$

As will be seen from Theorem 9.2 below (combined with Corollary 2.4(a)), in many cases the quotient $\lambda_0(\eta)/\lambda_{00}(\eta)$ turns out to be highly non-locally pseudoconvex, in particular, its dual is trivial.

It is not hard to get a formula for the quotient F-norm in $\lambda_0(\eta)/\lambda_{00}(\eta)$; in the proposition below we also treat the case of $\lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$.

PROPOSITION 9.1.

(a) *The quotient F-norm on $\lambda_0(\eta)/\lambda_{00}(\eta)$ is given by the formula*

$$\|\hat{x}\| = \|x\|_{\eta^\bullet}, \quad \textit{where } \hat{x} = x + \lambda_{00}(\eta).$$

Consequently, $\|\hat{x}\| \le \eta^\bullet(s(x))$ for every $x \in \lambda_0(\eta)$.

(b) *If an admissible sequence $\boldsymbol{\mu} = (\mu_n)$ of measures on $\mathbb{N}$ satisfies condition* (A), *then the quotient F-norm on $\lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$ is given by the formula*

$$\|\hat{x}\| = \limsup_{n\to\infty} \|x\|^0_{\mu_n}, \quad \textit{where } \hat{x} = x + \lambda_{00}(\boldsymbol{\mu}).$$

Consequently, $\|\hat{x}\| \le d_{\boldsymbol{\mu}}(s(x))$ for every $x \in \lambda_0(\boldsymbol{\mu})$.

Proof. Observe that $\|\hat{x}\|$ is the limit of the decreasing sequence $(\|x - x^n\|_\eta)$, resp., $(\|x - x^n\|^0_{\boldsymbol{\mu}})$, where $x^n = xe_{[n]}$ for $n \in \mathbb{N}$.

(a) First, note that $s(x - x^n, \varepsilon) = s(x, \varepsilon) \setminus [n]$ for all $\varepsilon > 0$ and $n \in \mathbb{N}$.

Now, given $\varepsilon > 0$, there is n such that $\|\hat{x}\| < \varepsilon \implies \|x - x^n\|_\eta < \varepsilon \implies \eta(s(x,\varepsilon) \setminus [n]) < \varepsilon \implies \eta^\bullet(s(x,\varepsilon)) < \varepsilon \implies \|x\|_{\eta^\bullet} \le \varepsilon$. Also the converse implications hold for some n provided, we start with $\|x\|_{\eta^\bullet} < \varepsilon$ and in the last two inequalities of the chain replace $<$ by $\le$. From this the desired equality follows.

(b) Choose a strictly increasing sequence (m_n) so that

$$\varepsilon_n := \max\{\mu_k(\mathbb{N} \setminus [m_n]) : k \in [n]\} \to 0 \quad \text{as } n \to \infty.$$

Then

$$\begin{aligned}
\|\hat{x}\| &= \lim_{n\to\infty} \|x - x^{m_n}\|^0_{\boldsymbol{\mu}} \\
&= \lim_{n\to\infty} \max\Big(\max_{1\le k\le n} \big\|xe_{[m_n+1,\infty)}\big\|^0_{\mu_k}, \sup_{k>n} \big\|xe_{[m_n+1,\infty)}\big\|^0_{\mu_k}\Big) \\
&\le \lim_{n\to\infty} \max\Big(\varepsilon_n, \sup_{k>n} \|x\|^0_{\mu_k}\Big) = \limsup_{n\to\infty} \|x\|^0_{\mu_n}.
\end{aligned}$$

Thus, $\|\hat{x}\| \le \limsup_n \|x\|^0_{\mu_n}$.

To prove the converse inequality, take any constants a, b with $\|\hat{x}\| < a < b$. Then $\|x - x^m\|^0_{\boldsymbol{\mu}} < a$ for some m. In view of condition (A), there is n such that $\mu_k([m]) < b - a$ for all $k > n$. It follows that for all $k > n$,

$$\|x\|^0_{\mu_k} = \big\|xe_{[m]}\big\|^0_{\mu_k} + \|x - x^m\|^0_{\mu_k} < (b - a) + \|x - x^m\|^0_{\boldsymbol{\mu}} < b.$$

Hence, $\limsup_n \|x\|^0_{\mu_n} \le b$, and the desired converse inequality follows. $\square$

THEOREM 9.2. *If η is core-nonatomic, then $\lambda_0(\eta)/\lambda_{00}(\eta)$ is L_0-like.*

Proof. Define an F-seminorm p on $\lambda_0(\eta)$ by $p(x) = \|\hat{x}\|$. By Proposition 9.1, $p(x) \le \eta^\bullet(s(x))$ for every $x \in \lambda_0(\eta)$. In view of Fact 2.1, it is enough to show that the F-seminormed space $(\lambda_0(\eta), p)$ is L_0-like.

Fix $\varepsilon > 0$ and let $U = \{x \in \lambda_0(\eta) : p(x) < \varepsilon\}$. Since $\eta^\bullet$ is nonatomic, there is a partition $A_1, \dots, A_m$ of $\mathbb{N}$ with $\eta^\bullet(A_j) < \varepsilon$ for $j \in [m]$. Now, take any x in $\lambda_0(\eta)$ and, for $j \in [m]$, let $x_j = xe_{A_j}$. Then $p(x_j) \le \eta^\bullet(s(x_j)) \le \eta^\bullet(A_j) < \varepsilon$ for each j, and $x = x_1 + \cdots + x_m$. Thus, $\lambda_0(\eta) \subset U^{(m)}$. □

Nevertheless, the quotient space $\lambda_0(\eta)/\lambda_{00}(\eta)$, if nonzero (which means that $\eta^\bullet \neq 0$), contains a lot of isomorphic copies of Banach spaces, in particular, all separable Banach spaces. This is shown in the following theorem.

THEOREM 9.3. *If $\eta^\bullet \neq 0$, then $\lambda_0(\eta)/\lambda_{00}(\eta)$ contains isomorphic copies of l_∞ and l_∞/c_0.*

Proof. We prove each of the cases independently, despite the known fact that $l_\infty \subset l_\infty/c_0$ isomorphically.

Let p be as in the proof of Theorem 9.2, and let

$$Q : \lambda_0(\eta) \to \lambda_0(\eta)/\lambda_{00}(\eta)$$

be the quotient map. By Fact 3.9(a), there is a disjoint sequence (F_k) in $\mathcal{F}$ such that $\eta(F_k) > \varepsilon$ for every k and some $\varepsilon > 0$.

(a) Let (M_k) be any disjoint sequence of infinite subsets of $\mathbb{N}$, and let $A_k := \bigcup_{m\in M_k} F_m$ for each $k \in \mathbb{N}$. Then the operator T defined as in Theorem 8.1 maps l_∞ into $\lambda_0(\eta)$ isomorphically, and $\lambda_{00}(\eta) \cap T(l_\infty) = \{0\}$. Take any $x = (\xi_j) \in l_\infty$ with $\|x\|_\infty \le 1$, and denote $Tx = y = (\eta_j)$. Then it is obvious that $p(y) \le \|x\|_\infty$. On the other hand, if $\|x\|_\infty = 1$ and k is such that $|\xi_k| \ge \frac{1}{2}$, then for every finite set $B \subset \mathbb{N}$ there is $m \in M_k$ with $B \cap F_m = \emptyset$ and, therefore,

$$\|y - ye_B\|_\eta \ge \|\xi_k e_{F_m}\|_\eta = \min(|\xi_k|, \eta(F_m)) \ge \min\left(\frac{1}{2}, \varepsilon\right).$$

Hence, $p(y) \ge \min(\frac{1}{2}, \varepsilon)$. Thus $QT : l_\infty \to \lambda_0(\eta)/\lambda_{00}(\eta)$ is an isomorphic embedding.

(b) Let T be as in (a) with the sets $A_k = F_k$. Then T maps l_∞ into $\lambda_0(\eta)$ isomorphically, and $T(c_0) = \lambda_{00}(\eta) \cap T(l_\infty)$. Take any $x = (\xi_j) \in l_\infty$ with $\|x\|_\infty \le 1$, and denote $Tx = y = (\eta_j)$. Then for every m, denoting $y_m = \sum_{j\le m} \xi_j e_{F_j}$ and $x^m = \sum_{j\le m} \xi_j e_j$,

$$p(y) = p(y - y_m) \le \|y - y_m\|_\eta \le \|y - y_m\|_\infty = \|x - x^m\|_\infty.$$

As $m \to \infty$, this yields

$$p(y) \le \limsup_{j\to\infty} |\xi_j| =: q(x),$$

where q thus defined is the inverse image of the quotient norm in l_∞/c_0.

On the other hand, if $q(x) > \frac{2}{3}$, then $|\xi_m| > \frac{1}{2}$ for all m in an infinite set M of indices. Note that $|\eta_j| = |\xi_m| > \frac{1}{2}$ whenever $j \in F_m$, $m \in M$. Hence, calculations like those in (a) will show that $p(y) \geq \min(\frac{1}{2}, \varepsilon)$.

It follows that the operator that assigns to each element $x + c_0$ of l_∞/c_0 the corresponding element $Tx + \lambda_{00}(\eta)$ of $\lambda_0(\eta)/\lambda_{00}(\eta)$ ($x \in l_\infty$) is an isomorphic embedding of l_∞/c_0 into $\lambda_0(\eta)/\lambda_{00}(\eta)$. □

10. The duals of $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$

They are, of course, separating, because $\lambda_0(\eta) \subset \omega$ continuously, but often poor, as was already seen in Proposition 7.13. For a general result related to the theorem below, see Fact 1.2.

THEOREM 10.1. *Let η be core-nonatomic. Then the dual space $\lambda_{00}(\eta)'$ is poor; in fact, a linear functional on $\lambda_{00}(\eta)$ is continuous iff it is a finite linear combination of the coordinates. Hence, the weak and Mackey topologies of $\lambda_{00}(\eta)$ coincide with the topology induced from ω.*

The same is true for the F-space $\lambda_0(\eta)$, and thus both $\lambda_{00}(\eta)$ and $\lambda_0(\eta)$ are $F\omega$-spaces.

Proof. Let $u \in \lambda_{00}(\eta)'$ and denote $t_j = u(e_j)$. Then, clearly, $u(x) = \sum_{j=1}^{\infty} t_j\xi_j$ for every $x = (\xi_j) \in \lambda_{00}(\eta)$. If the set $\{j : t_j \neq 0\}$ were infinite, then it would contain an infinite set $A \in \mathcal{Z}(\eta^\bullet)$ (see Fact 3.1). Then the sequence $x = (\xi_j)$, where $\xi_j = j/t_j$ for $j \in A$, and $\xi_j = 0$ otherwise, would belong to $\lambda_{00}(\eta)$, and we would have $u(x) = \infty$!

Alternatively, note that $\eta(n) \to 0$ as $n \to \infty$, by Corollary 3.6, and appeal to Proposition 7.13.

Let now $u \in \lambda_0(\eta)'$. By the first part of the proof applied to $u|\lambda_{00}(\eta)$, the set $S = \{j : u(e_j) \neq 0\}$ is finite. Define a functional $v \in \lambda_0(\eta)'$ by

$$v(x) = u(x) - \sum_{j \in S} u(e_j)\xi_j.$$

Clearly, $v|\lambda_{00}(\eta) = 0$. From Theorem 9.2 and Corollary 2.4(a) it now follows that $v = 0$, which completes the proof. □

REMARKS 10.2. (a) We will prove later, in Theorem 11.1, that $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$ in Theorem 10.1 are in fact quasi-L_0-like, from which the theorem itself will easily be deduced (see Corollary 11.3).

(b) For $\eta^\bullet \neq 0$ nonatomic, the $F\omega$-spaces $\lambda_0(\eta)$, as seen from Corollary 8.3, provide a negative answer to the question at the end of [7] whether an $F\omega$-space must have a basis or at least be separable.

(c) The dual of $\lambda_0(\eta)$ is not always poor, see Example 7.15.

As a straightforward consequence of Theorem 10.1 and Corollary 2.8, we have the following result. (It will be strengthened in Corollary 11.5 below.)

COROLLARY 10.3. *Let η be core-nonatomic. Then, in either of the spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$, a locally convex closed subspace is complemented iff it is either finite-dimensional or isomorphic to ω. In particular, no copy of l_∞ in $\lambda_0(\eta)$, and no copy of c_0 in $\lambda_{00}(\eta)$ is complemented.*

Using Theorem 10.1, we now show that the Orlicz–Pettis theorem fails in $\lambda_0(\eta)$ when η is core-nonatomic and $\eta^\bullet \neq 0$.

COROLLARY 10.4. *Let η be core-nonatomic and $\eta^\bullet \neq 0$. Then the series $\sum_{n=1}^\infty e_n$ is subseries convergent in $\lambda_0(\eta)$ with its weak topology, but is not convergent in $\lambda_0(\eta)$ with its original topology.*

Proof. We have the subseries weak convergence because it is the same as the subseries coordinate convergence. Next, if (F_n) is a sequence provided by Fact 3.9(a) for $A = \mathbb{N}$ and any $0 < a < \min(1, \eta^\bullet(\mathbb{N}))$, then $\| \sum_{j \in F_n} e_j \|_\eta = \|e_{F_n}\|_\eta = \min(1, \eta(F_n)) > a$ for all n. □

REMARK 10.5. For some time, it was open (see [21, p. 222]) whether the Orlicz–Pettis theorem is valid for an F-space E with separating dual, i.e., whether every series in E that is subseries $\sigma(E, E')$-convergent must be convergent. Examples answering the problem in the negative have been provided in [25] and [26]. The spaces $\lambda_0(\eta)$ in the corollary above, the simplest of which is $\lambda_0 = \lambda_0(\bar{d})$, are a new class of examples which seem to be more elementary and simpler than the previous ones.

11. Quasi-L_0-like spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$

In all of this section, we assume that η is core-nonatomic.

THEOREM 11.1. *Each of the spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$ is quasi-L_0-like. In fact, for every zero-neighborhood U in $\lambda_0(\eta)$ there are $m, k \in \mathbb{N}$ such that $\lambda_0(\eta)([k, \infty)) \subset U^{(m)}$. Likewise for the space $\lambda_{00}(\eta)$.*

Proof. Consider, for instance, $\lambda_0(\eta)$.

Fix any $\varepsilon > 0$ and let $U = \{x \in \lambda_0(\eta) : \|x\|_\eta < \varepsilon\}$. By the core-nonatomicity of η, see Fact 3.7, there is a finite partition $A_0, A_1, \dots, A_m$ of $\mathbb{N}$ such that $A_0 \in \mathcal{F}$ and $\eta(A_j) < \varepsilon$ for $j \in [m]$. Clearly, we may assume that $A_0 = [1, k)$ for some $k \in \mathbb{N}$.

Let $L = \lambda_0(\eta)([k, \infty))$. If $x \in L$, then $x = x_1 + \dots + x_m$, where $x_j = x e_{A_j}$ and $\|x_j\|_\eta \le \eta(A_j) < \varepsilon$. Therefore, $L \subset U^{(m)}$. (Actually, $L = (L \cap U)^{(m)}$.) □

From Theorem 11.1 one deduces the following (cf. Proposition 2.3).

COROLLARY 11.2. *Let p be a subadditive functional on $\lambda_0(\eta)$ (or $\lambda_{00}(\eta)$). If p is bounded in a neighborhood of zero, then there exists k such that p restricted to the subspace $\lambda_0(\eta)([k, \infty))$ (resp., $\lambda_{00}(\eta)([k, \infty))$) is bounded.*

In consequence, if p is a continuous α-seminorm on $\lambda_0(\eta)$ (or $\lambda_{00}(\eta)$), then $p=0$ on $\lambda_0(\eta)([k,\infty))$ (resp., $\lambda_{00}(\eta)([k,\infty))$) for some k so that $\ker p$ is of finite codimension and, therefore, p is equivalent to a seminorm.

We now can provide an alternative proof of Theorem 10.1.

COROLLARY 11.3. *Every continuous linear functional on $\lambda_0(\eta)$ (or $\lambda_{00}(\eta)$) is a finite linear combination of the coordinates.*

Proof. For the space $\lambda_0(\eta)$. We proceed as in the proof of Corollary 2.5. Let $x' \in \lambda_0(\eta)'$. Then, by Corollary 11.2, $\ker x' \supset L_k := \lambda_0(\eta)([k,\infty))$ for some k. Since L_k is the intersection of the kernels of the coordinate functionals e_i' for $1 \le i < k$, it follows that x' is a linear combinations of those functionals. □

We also can give Theorem 7.16(c) a much stronger form (but note that also the assumption on $\eta^\bullet$ is stronger).

COROLLARY 11.4. *In each of the spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$, the finest locally pseudoconvex topology that is weaker than the original topology τ_η coincides with the weak topology of the space, and in case $\eta^\bullet \neq 0$ it is strictly weaker than τ_η.*

Proof. All of this follows from Theorem 11.1 and Corollary 2.4(b), except for the last assertion which can be verified using a sequence (F_n) as in the proof of Corollary 10.4. □

In consequence, as a particular case of Corollary 2.6, we have a considerable strengthening of Corollary 10.3:

COROLLARY 11.5. *In each of the spaces $\lambda_0(\eta)$ and $\lambda_{00}(\eta)$, a locally pseudoconvex closed subspace is complemented iff it is either finite-dimensional or isomorphic to ω.*

12. Bounded subsets of $\lambda_0(\eta)$

We give here a few simple results on bounded and metrically bounded subsets of $\lambda_0(\eta)$. The first is a direct consequence of Facts 1.3 and 7.3.

PROPOSITION 12.1. *For every bounded subset B of $\lambda_0(\eta)$, its closure $\overline{B}^\omega$ in ω is a bounded subset of $\lambda_0(\eta)$. In particular, the F-lattice $\lambda_0(\eta)$ has the Levi property.*

One might think that the result above holds because $\lambda_0(\eta)$ has a base for the neighborhoods of zero consisting of sets which are closed in ω. In general, it is not so, as shown by the following.

EXAMPLE 12.2. Take an admissible sequence $\boldsymbol{\mu}=(\mu_n)$ of probability measures on $\mathbb{N}$ that have pairwise disjoint supports $S_n = s(\mu_n)$ and satisfy condition ($\bar{\mathrm{A}}$) (see p. 639). (For example, one may take $\boldsymbol{\mu} = (\mu_{S_n})$, where (S_n)

is any disjoint sequence of finite nonempty sets covering $\mathbb{N}$ and $|S_n| \to \infty$, see Remark 4.4(a).) Then the space $\lambda_0(\boldsymbol{\mu})$ does not have a base at zero consisting of sets that are closed in ω. That is, for every neighborhood U of zero in $\lambda_0(\boldsymbol{\mu})$, its closure in ω is not a subset of $\lambda_0(\boldsymbol{\mu})$. Below, we write $\|\cdot\|_n$ and $\|\cdot\|$ for $\|\cdot\|^0_{\mu_n}$ and $\|\cdot\|^0_{\boldsymbol{\mu}}$, respectively.

It is enough to verify this when $U = \{x \in \lambda_0(\boldsymbol{\mu}) : \|x\| < \varepsilon\}$, where $0 < \varepsilon < 1$. Applying Fact 4.2, we can find a strictly increasing sequence (n_k) and a sequence (F_k) of finite sets such that $F_k \subset S_{n_k}$, $\mu_{n_k}(F_k) > \varepsilon$, and $\mu_{n_k}(\mathbb{N} \setminus F_k) < \frac{1}{4}\varepsilon$ for all k. In view of condition $(\bar{\mathrm{A}})$, this can be done so that $\mu_{n_k}(j) < \frac{1}{4}\varepsilon$ for all k and j. Using this, it is now easy to select sets $A_k \subset F_k$ so that $\frac{1}{4}\varepsilon < \mu_{n_k}(A_k) \le \frac{1}{2}\varepsilon$ for each k. Then $x := \sum_k k e_{A_k} \notin \lambda_0(\boldsymbol{\mu})$. In fact, for every $m \in \mathbb{N}$, $\|m^{-1}x\| \ge \|m^{-1}x\|_{n_m} \ge \|e_{A_m}\|_{n_m} = \mu_{n_m}(A_m) > \frac{1}{4}\varepsilon$, hence $\|m^{-1}x\| \not\to 0$ as $m \to \infty$. However, for every r, if $x_r = \sum_{k\le r} k e_{A_k}$, then $\|x_r\|_m = \mu_m(\bigcup_{k\le r} A_k) = 0$ if $m > n_r$ or $m \notin \{n_1, n_2, \dots\}$, and is $< \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon$ if $m = n_j \le n_r$. Thus, $\|x_r\| < \varepsilon$. Therefore, $(x_r) \subset U$, $x_r \to x$ in ω, and nevertheless $x \notin \lambda_0(\boldsymbol{\mu})$.

Recall that a set B in a topological vector space is said to be *metrically* (or additively) *bounded* if each continuous F-seminorm is bounded on B.

Proposition 12.3. *Let η be core-nonatomic. Then a subset of $\lambda_0(\eta)$ is metrically bounded iff it is coordinate-wise bounded* (*or weakly bounded*).

Proof. The 'only if' part is obvious.

'If': Assume a set $B \subset \lambda_0(\eta)$ is coordinate-wise bounded, and let p be a continuous F-seminorm on $\lambda_0(\eta)$. By Corollary 11.2, there exist $k \in \mathbb{N}$ and $C > 0$ such that $p(x) \le C$ for all $x \in \lambda_0(\eta)$ with $Q_k(x) = 0$, where Q_k is the natural projection onto the subspace $\lambda_0(\eta)([k])$. By the assumption on B, p is bounded on $Q_k(B)$. Hence, $\sup_{x\in B} p(x) \le \sup_{x\in B} p(Q_k(x)) + C < \infty$. □

Remark 12.4. For a general admissible submeasure η, the 'if' part if the above result is false. To see this, let η be as in Example 7.15, and let B be the set of all elements $x_n := \sum_{k=1}^n k e_k$ $(n \in \mathbb{N})$. Then $B \subset \lambda_0(\eta)$ is bounded in ω. However, for each $m \in \mathbb{N}$ there is $n > m$ such that $\eta((m,n]) > m$ and then $\eta(s(x_n, m)) > m$ so that $\|x_n\|_\eta > m$. Thus, the F-norm $\|\cdot\|_\eta$ is not bounded on B.

Corollary 12.5. *If η is core-nonatomic, then the closure in ω of a metrically bounded set $B \subset \lambda_0(\eta)$ needs not be a subset of $\lambda_0(\eta)$.*

Proof. Given *any* $x = (\xi_j) \in \omega$, the sequence (x^n) in $\lambda_{00}(\eta)$ is metrically bounded by Proposition 12.3, and $x^n \to x$ in ω. □

13. $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ as l_∞- and c_0-sums

In this section, we mostly deal with the special case of an admissible sequence $\boldsymbol{\mu}=(\mu_n)$ consisting of measures on $\mathbb{N}$ with pairwise disjoint supports

$$S_n = s(\mu_n);$$

note that the union of the S_n's is all of $\mathbb{N}$. Then for all $n\in\mathbb{N}$ and $x\in\omega(S_n)$ one has

$$\|x\| = \|x\|_n = \int_{S_n} \min(1,|x|)\,d\mu_n,$$

where we write $\|\cdot\|_n$ and $\|\cdot\|$ in place of $\|\cdot\|^0_{\mu_n}$ and $\|\cdot\|^0_{\boldsymbol{\mu}}$, respectively. It is therefore evident that $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ can be isometrically viewed as the l_∞- and c_0-sum of the spaces $L_0(\mu_n)=L_0(S_n)$:

$$\lambda_0(\boldsymbol{\mu}) = \Big(\sum_n L_0(S_n)\Big)_\infty \quad\text{and}\quad \lambda_{00}(\boldsymbol{\mu}) = \Big(\sum_n L_0(S_n)\Big)_0 .$$

(See Section 1.F for l_∞- and c_0-sums.) Note that in the case considered, $\boldsymbol{\mu}$ satisfies condition (A) and that $\bar{d}_{\boldsymbol{\mu}}=\mu_n$ and $d_{\boldsymbol{\mu}}=0$ on $\mathcal{P}(S_n)$ for each n.

13.A. The two variants of λ_0 and λ_{00} are isomorphic. For a general admissible sequence $\boldsymbol{\mu}=(\mu_n)$, it is as yet unclear when $\lambda_0(\boldsymbol{\mu})$ and/or $\lambda_{00}(\boldsymbol{\mu})$ can be represented isomorphically as $\lambda_0(\boldsymbol{\nu})$ and/or $\lambda_{00}(\boldsymbol{\nu})$, respectively, for an admissible sequence $\boldsymbol{\nu}=(\nu_n)$ of measures having disjoint supports. It is so, however, for the spaces λ_0 and λ_{00}.

PROPOSITION 13.1. *$\lambda_0=\lambda_0(\boldsymbol{\delta})$ and $\lambda_{00}=\lambda_{00}(\boldsymbol{\delta})$, where $\boldsymbol{\delta}=(\delta_k)$. Thus, λ_0 and λ_{00} are isomorphic to the l_∞- and c_0-sum of the sequence $(L_0(D_k))$ of finite-dimensional L_0-spaces, respectively, where each of the intervals $D_k=[2^k,2^{k+1})\subset\mathbb{N}$ is considered with its uniform probability measure δ_k $(k\ge 0)$.*

Proof. We keep the notation $\|\cdot\|_k=\|\cdot\|^0_{d_k}$ and $\|\cdot\|=\|\cdot\|^0_{\boldsymbol{\mu}}$ for the original FG-(semi)norms that were used in Sections 5 and 6 in the case of $\boldsymbol{\mu}=(d_n)$ to define λ_0 and λ_{00}, and have to distinguish them from those related to the l_∞- and c_0-sums in question. Thus, for each $x=(\xi_j)\in\omega$, set

$$|||x|||_k = \int_{D_k}\min(1,|x|)\,d\delta_k = \frac{1}{|D_k|}\sum_{j\in D_k}\min(1,|\xi_j|)\quad\text{for } k\ge 0,$$

and $|||x|||=\sup_k |||x|||_k$. Clearly, $|||\cdot|||$ is a monotone FG-norm on ω, and one of our tasks will be to show it is equivalent to the FG-norm $\|\cdot\|$; from this the part concerning λ_0 will follow. To prove the part concerning λ_{00}, we will need to establish suitable relations between the F-seminorms $\|\cdot\|_n$ and $|||\cdot|||_k$ in order to see that $\|x\|_n\to 0$ iff $|||x|||_k\to 0$.

Let $x\in\omega$. As easily seen,

$$(*)\qquad |||x|||_k \le (2-2^{-k})\|x\|_{2^{k+1}-1} \le 2\|x\|_{2^{k+1}-1}$$

whence $|||x||| \leq 2\|x\|$. On the other hand, given $n \in \mathbb{N}$, let m be the least integer with $n < 2^{m+1}$. Then for $0 \leq p \leq m$,

$$\begin{aligned}\|x\|_n &\leq \frac{1}{2^m} \sum_{j=1}^{2^{m+1}-1} \min(1, |\xi_j|) = \frac{1}{2^m} \left(\sum_{j=1}^{2^p-1} + \sum_{j=2^p}^{2^{m+1}-1} \right) \min(1, |\xi_j|) \\ &\leq \frac{2^p - 1}{2^m} + \frac{1}{2^m} \sum_{k=p}^{m} 2^k \, |||x|||_k \leq \frac{2^p - 1}{2^m} + 2 \max_{p \leq k \leq m} |||x|||_k .\end{aligned}$$

Thus,

$$\|x\|_n \leq \frac{2^p - 1}{2^m} + 2 \max_{p \leq k \leq m} |||x|||_k . \tag{$**$}$$

Taking $p = 0$ gives $\|x\|_n \leq 2 \max_{k \leq m} |||x|||_k$. It follows that $\|x\| \leq 2\, |||x|||$.

Therefore,

$$\frac{1}{2}\, |||x||| \leq \|x\| \leq 2\, |||x|||,$$

so that the FG-norms $\|\cdot\|$ and $|||\cdot|||$ are indeed equivalent. In consequence,

$$\lambda_0 = \Big\{ x \in \omega : \lim_{t \to 0} |||tx||| = 0 \Big\},$$

and $|||\cdot|||$ is an equivalent F-norm for λ_0. Moreover, in view of $(*)$ and $(**)$,

$$\lambda_{00} = \Big\{ x \in \omega : \lim_{k \to \infty} |||x|||_k = 0 \Big\}.$$

From this, the assertion follows. □

13.B. Isomorphic copies of l_∞ and c_0. The forthcoming Theorem 13.3 about 'well-located' copies of l_∞ and c_0 is of general character, and the type of argument used in its proof is rather standard, with a slight possibility of a new idea in the part where the operators R and T enter into play. It is applied, in particular, to the spaces $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ determined by sequences $\boldsymbol{\mu}$ of measures with disjoint supports. In fact, with the help of a trick, the disjointness of supports can eventually be dropped. We first prove the following lemma.

Lemma 13.2. *If $F_1, \ldots, F_n$ are F-spaces none of which contains a locally bounded subspace of infinite dimension, then neither does their topological direct sum $F = F_1 \oplus \cdots \oplus F_m$.*

Proof. For $1 \leq m \leq n$, let P_m denote the natural projection of F onto F_m. Suppose Y is a closed infinite-dimensional locally bounded subspace of F. Then, for each m, $P_m|Y$ is a strictly singular operator from Y into $F_m \subset F$; that is, its restriction to any infinite-dimensional subspace of Y is not an isomorphism. It follows that also their sum, which is the identity operator on Y, is strictly singular (cf. [8, Cor. 4.5]), and this is absurd. □

THEOREM 13.3. *Let E_∞ and E_0 be the l_∞- and c_0-sums, respectively, of a sequence of F-spaces $F_k = (F_k, \|\cdot\|_k)$ $(k \in \mathbb{N})$ none of which contains a locally bounded subspace of infinite dimension. Then for every closed infinite-dimensional locally bounded subspace Y of E_0 there is an isomorphic embedding $T : l_\infty \to E_\infty$ such that $T(c_0) = E_0 \cap T(l_\infty) \subset Y$. In particular, Y contains an isomorphic copy of c_0.*

Proof. We denote the (sup) F-norm of the spaces E_∞ and E_0 by $|||\cdot|||$.

For every $m \in \mathbb{N}$, let P_m be the natural projection of E_∞ onto its subspace F_m. Also, let $Q_m = P_1 + \cdots + P_m$ and $R_m = I - Q_m$ (I is the identity operator on E_∞), and set $Q_0 = 0$. Thus, for each $x \in E_0$, $Q_m x \to x$ and $R_m x \to 0$ as $m \to \infty$. Since Y is locally bounded, it has an equivalent p-norm $\|\cdot\|$ for some $0 < p \le 1$. Let S_Y denote the unit sphere in $(Y, \|\cdot\|)$, and $3\varepsilon := \inf\{|||y||| : y \in S_Y\} > 0$.

Choose any $y_1 \in S_Y$. Since $Q_m y_1 \to y$, there is m_1 with $|||Q_{m_1} y_1||| > 2\varepsilon$ and $|||R_{m_1} y_1||| < \varepsilon/2^4$. Next, as $Q_{m_1}|Y$ is not an isomorphic embedding (by the lemma), there is $y_2 \in S_Y$ with $|||Q_{m_1} y_2||| < \varepsilon/2^5$. But $Q_m y_2 \to y_2$, hence there is $m_2 > m_1$ with $|||Q_{m_2} y_2||| > 2\varepsilon$ and $|||R_{m_2} y_2||| < \varepsilon/2^5$. Continuing in this manner, we find a strictly increasing sequence (m_n) in $\mathbb{N}$ and a sequence (y_n) in S_Y such that $|||Q_{m_{n-1}} y_n||| < \varepsilon/2^{n+3}$, $|||Q_{m_n} y_n||| > 2\varepsilon$, and $|||R_{m_n} y_n||| < \varepsilon/2^{n+3}$ for each $n \ge 1$. Hence if

$$u_n := (Q_{m_n} - Q_{m_{n-1}}) y_n, \quad \text{then} \quad |||u_n||| > \varepsilon \quad \text{and} \quad \sum_{n=1}^{\infty} |||y_n - u_n||| < \frac{1}{4}\varepsilon.$$

Clearly, (u_n) is a bounded basic sequence in E_0. By [6, Th. 2.8], there is n_0 such that the sequence $(y_n)_{n \ge n_0}$ is basic and equivalent to $(u_n)_{n \ge n_0}$. Of course, we may assume $n_0 = 1$.

Now, let $u := \sum_n u_n$ (coordinate-wise sum). Then

$$|||tu||| = \sup_m \|tu\|_m = \sup_n \sup_{m_{n-1} < m \le m_n} \|P_m(tu)\|_m = \sup_n |||tu_n|||,$$

and since the sequence (u_n) is bounded, we see that $|||tu||| \to 0$ as $t \to 0$. Hence, $u \in E_\infty$.

Since the sequence (u_n) is formed by pairwise disjoint 'pieces' of u with $|||u_n||| > \varepsilon$, it is easy to see that the operator

$$S : (\alpha_n) \to \sum_n \alpha_n u_n \quad \text{(coordinate-wise sum)}$$

is an isomorphic embedding of l_∞ into E_∞. In fact, if $a = (\alpha_n) \in l_\infty$ and $\|a\|_\infty \le \delta$, then $|||Sa||| \le |||\delta u|||$, and the continuity of S follows. On the other hand, if $\|a\|_\infty = 1$ and n is chosen so that $|\alpha_n| \ge \frac{1}{2}$, then $|||Sa||| \ge |||\alpha_n u_n||| \ge |||\frac{1}{2} u_n||| > \frac{1}{2}\varepsilon$. In consequence, S is as required.

We next observe that the operator $R : l_\infty \to E_0$ defined by the equality

$$Ra = \sum_{n=1}^{\infty} \alpha_n (y_n - u_n) \quad (\text{convergence in } E_0)$$

is continuous and that $|||Ra||| \le \frac{1}{4}\varepsilon$ whenever $\|a\|_\infty \le 1$.

It follows that also the operator $T := R + S : l_\infty \to E_\infty$ is continuous, and that it can be given by the formula

$$Ta = \sum_{n=1}^{\infty} \alpha_n y_n,$$

where the series on the right-hand side is convergent coordinate-wise.

Now, if $a \in l_\infty$ and $\|a\|_\infty = 1$, then

$$|||Ta||| \ge |||Sa||| - |||Ra||| > \frac{1}{2}\varepsilon - \frac{1}{4}\varepsilon = \frac{1}{4}\varepsilon;$$

hence T is an isomorphic embedding.

Obviously, $T(c_0) \subset Y$. Suppose $Ta \in E_0$ for some $a \in l_\infty$. Then also $Sa = Ta - Ra \in E_0$, hence $|||a_n u_n||| \to 0$ which is impossible if $a \notin c_0$. $\square$

Corollary 13.4. *If E_0 has a locally bounded subspace of infinite dimension, then E_0 is uncomplemented in E_∞.*

Proof. As for Corollary 8.4. $\square$

The simplest (and possibly known) case of the theorem is the following.

Corollary 13.5. *Every closed infinite-dimensional subspace Y of c_0 contains a copy of c_0 that has an 'extension' to a copy of l_∞ in l_∞. More precisely, there is an isomorphic embedding $T : l_\infty \to l_\infty$ such that $T(c_0) = c_0 \cap T(l_\infty) \subset Y$.*

From Theorem 13.3, taking into account Fact 1.7, one obtains the following corollary.

Corollary 13.6. *Let $\boldsymbol{\mu} = (\mu_n)$ be an admissible sequence of measures on $\mathbb{N}$ having pairwise disjoint supports. Then for every closed infinite-dimensional locally bounded subspace Y of $\lambda_{00}(\boldsymbol{\mu})$ there is an isomorphic embedding $T : l_\infty \to \lambda_0(\boldsymbol{\mu})$ with $T(c_0) = \lambda_0(\boldsymbol{\mu}) \cap T(l_\infty) \subset Y$.*

Surprisingly, the disjointness of supports is superfluous. To get rid of it, we need a proposition along with the isometry constructed in its proof.

Proposition 13.7. *For every admissible sequence $\boldsymbol{\mu} = (\mu_n)$ of measures on $\mathbb{N}$ there is an admissible sequence $\boldsymbol{\nu} = (\nu_n)$ of measures on $\mathbb{N}$ having disjoint supports, and a linear isometric embedding $V : \lambda_0(\boldsymbol{\mu}) \to \lambda_0(\boldsymbol{\nu})$ such that $V(\lambda_{00}(\boldsymbol{\mu})) = V(\lambda_0(\boldsymbol{\mu})) \cap \lambda_{00}(\boldsymbol{\nu})$.*

Proof. For the sequence (S_n) of the supports of μ_n's, let (S'_n) be a partition of $\mathbb{N}$ such that $|S'_n| = |S_n|$ for all n. Next, for each n, choose a bijection $\varphi_n : S'_n \to S_n$ and define a measure ν_n on $\mathbb{N}$ by $\nu_n(A) = \mu_n(\varphi_n(A \cap S'_n))$. Trivially, the sequence $\boldsymbol{\nu} = (\nu_n)$ is admissible and the measures ν_n have pairwise disjoint supports S'_n. Now, define a linear map $V : \omega \to \omega$ by $Vx = (x\circ\varphi_n)_{n\in\mathbb{N}}$; that is, $Vx \in \omega$ is such that $(Vx)|S'_n = (x|S_n)\circ\varphi_n$ for each n. Clearly, V is injective. Moreover, for each n,

$$\|x\|^0_{\mu_n} = \int_{S_n} \min(1, |x|)\, d\mu_n = \int_{S'_n} \min(1, |Vx|)\, d\nu_n = \|Vx\|^0_{\nu_n}$$

and, consequently, $\|x\|^0_{\boldsymbol{\mu}} = \|Vx\|^0_{\boldsymbol{\nu}}$. It follows that V is as required. $\square$

THEOREM 13.8. *Let $\boldsymbol{\mu} = (\mu_n)$ be an arbitrary admissible sequence of measures on $\mathbb{N}$. Then for every closed infinite-dimensional locally bounded subspace Z of $\lambda_{00}(\boldsymbol{\mu})$ there is an isomorphic embedding $R : l_\infty \to \lambda_0(\boldsymbol{\mu})$ such that $R(c_0) = \lambda_{00}(\boldsymbol{\mu}) \cap R(l_\infty) \subset Z$.*

Proof. We use the notation introduced in the previous proposition and its proof. Clearly, $Y := V(Z) \subset V(\lambda_{00}(\boldsymbol{\mu}))$ is a locally bounded closed subspace of $\lambda_{00}(\nu)$.

By the proof of Theorem 13.3, there is a sequence (z_n) in Z such that if $y_n := V(z_n)$, then the operator $T : l_\infty \to \lambda_0(\boldsymbol{\nu})$ defined by $Ta = \sum_n \alpha_n y_n$ (coordinate-wise sum) is an isomorphic embedding and, moreover, $T(c_0) = \lambda_{00}(\boldsymbol{\nu}) \cap T(l_\infty) \subset Y$. Now, if $a = (\alpha_n) \in l_\infty$ then, for each k, $(Ta)|S'_k = \sum_n \alpha_n(y_n|S'_k) = \sum_n \alpha_n((z_n|S_k) \circ \varphi_k)$ so that the series $\sum_n \alpha_n z_n$ converges coordinate-wise on each of the sets S_k, hence on $\mathbb{N}$. Let $Ra := \sum_n \alpha_n z_n$ (coordinate-wise) for $a \in l_\infty$. By the above, $(Ta)|S'_k = ((Ra)|S_k) \circ \varphi_k$ for each k and, consequently, $T = V \circ R$. Since V is injective, R is an isomorphic embedding of l_∞ into $\lambda_0(\boldsymbol{\mu})$. It is now easy to finish the proof. $\square$

13.C. Setting for what follows. In the remaining part of this section, we restrict attention to admissible sequences $\boldsymbol{\mu} = (\mu_n)$ of *probability* measures on $\mathbb{N}$ having pairwise *disjoint* supports (S_n) and such that (see p. 639)

($\bar{\mathrm{A}}$) $\lim_{n\to\infty} \sup_j \mu_n(j) = 0$.

Then, obviously, $\boldsymbol{\mu}$ satisfies conditions (A), (B), and (C), and $|S_n| \to \infty$. In consequence, $d_{\boldsymbol{\mu}} = (\bar{d}_{\boldsymbol{\mu}})^\bullet$ (Fact 4.1), and it is not hard to see that $\bar{d}_{\boldsymbol{\mu}}$ is corenonatomic. Thus, all of our previous results about spaces $\lambda_0(\boldsymbol{\mu}) = \lambda_0(\bar{d}_{\boldsymbol{\mu}})$ and $\lambda_{00}(\boldsymbol{\mu}) = \lambda_{00}(\bar{d}_{\boldsymbol{\mu}})$ hold in the present situation.

Let $\boldsymbol{\mu}$ be as specified above. Then $\boldsymbol{\mu}$ can be represented over the interval $[0, 1]$ by a sequence $\mathfrak{I} = (\mathcal{I}_n)$ where, for each n, $\mathcal{I}_n = (I_{n,j} : j \in S_n)$ is a family of disjoint intervals with union $[0, 1]$ chosen so that $\lambda(I_{n,j}) = \mu_n(j)$ for all $j \in S_n$. (Here, λ denotes Lebesgue measure.) We shall say that a function g on $[0, 1]$ is $\mathcal{I}_n$-*simple* if it is constant on each $I_{n,j}$ $(j \in S_n)$, and denote by $\Omega(\mathfrak{I})$

the linear space of all sequences (f_n) such that, for each n, f_n is an $\mathcal{I}_n$-simple function.

Recall that the F-space $L_0 = L_0[0,1]$ is considered with its standard F-norm $\|f\| = \int_0^1 \min(1,|f|)\,d\lambda$. For the following to be true, condition $(\bar{\mathrm{A}})$ is essential (but not the disjointness of the $s(\mu_n)$'s).

FACT 13.9. *If* $\mathfrak{I} = (\mathcal{I}_n)$ *is a representation of* $\boldsymbol{\mu}$, *then for every* $f \in L_0$ *there is* $(f_n) \in \Omega(\mathfrak{I})$ *such that* $\|f - f_n\| \to 0$.

Proof. Since $\boldsymbol{\mu}$ satisfies $(\bar{\mathrm{A}})$, $r_n := \sup_j \mu_n(j) \to 0$ as $n \to \infty$.

Let $f \in L_0$, and fix a sequence of reals $0 < \varepsilon_k \to 0$. Then there exists a sequence (g_k) of continuous functions on $[0,1]$ such that $\|f - g_k\| < \varepsilon_k$ $(k \in \mathbb{N})$. For each k, let $\delta_k > 0$ be such that $|g_k(t) - g_k(t')| < \varepsilon_k$ whenever $t, t' \in [0,1]$ and $|t - t'| < \delta_k$. Choose a strictly increasing sequence (n_k) in $\mathbb{N}$ so that $r_n < \delta_k$ for all $n \ge n_k$ $(k \in \mathbb{N})$. Now, define a sequence (f_n) of functions on $[0,1]$ as follows. If $1 \le n < n_1$, let $f_n = 0$. If $n_k \le n < n_{k+1}$ for some k, pick any points $t_{n,j} \in I_{n,j}$ $(j \in S_n)$, and let $f_n(t) = g_k(t_{n,j})$ for $t \in I_{n,j}$ $(j \in S_n)$. Then, for each n, the function f_n is $\mathcal{I}_n$-simple, and $\|f - f_n\| \le \|f - g_k\| + \|g_k - f_n\| < 2\varepsilon_k$ for all $n \in [n_k, n_{k+1})$, $k \in \mathbb{N}$. $\square$

Fix a representation $\mathfrak{I} = (\mathcal{I}_n)$ of $\boldsymbol{\mu}$, and write $\Omega = \Omega(\mathfrak{I})$. Then, for each $x \in \omega$ and $n \in \mathbb{N}$, let $f_{x,n}$ denote the $\mathcal{I}_n$-simple function with $f_{x,n}|I_{n,j} = \xi_j$ for all $j \in S_n$. Clearly, the map $x \to (f_{x,n})$ is a linear bijection of ω onto Ω, and it is obvious that

$$\|x\|^0_{\mu_n} = \|f_{x,n}\| \quad (n \in \mathbb{N}) \quad \text{and} \quad \|x\|^0_{\boldsymbol{\mu}} = \sup_n \|f_{x,n}\|.$$

Hence, $x \to f_{x,n}$ is an isometry of $L_0(\mu_n) = L_0(S_n, \mu_n)$ into L_0 $(n \in \mathbb{N})$.

The equalities above suggest considering Ω with the F-seminorms $(\|\cdot\|_n)$ and the FG-norm $\|\cdot\|$ defined by

$$\|(f_n)\|_n = \|f_n\| \quad (n \in \mathbb{N}) \quad \text{and} \quad \|(f_n)\| = \sup_n \|f_n\|.$$

Denote by Ω_b, Ω_c, and Ω_0 the closed subspaces of Ω consisting of sequences (f_n) that are, respectively, bounded, convergent, or convergent to zero in L_0. Obviously, $\Omega_b = v(\Omega, \|\cdot\|)$, $\Omega_0 = v_0(\Omega, \|\cdot\|)$, and $\Omega_0 \subset \Omega_c \subset \Omega_b$. Let us also define a closed subspace $\lambda_{0c}(\boldsymbol{\mu})$ of $\lambda_0(\boldsymbol{\mu})$ as consisting of all x with $(f_{x,n}) \in \Omega_c$. (Of course, $\lambda_{0c}(\boldsymbol{\mu})$ depends on the chosen representation $\mathfrak{I}$ of $\boldsymbol{\mu}$.) Trivially, $\lambda_{00}(\boldsymbol{\mu}) \subset \lambda_{0c}(\boldsymbol{\mu}) \subset \lambda_0(\boldsymbol{\mu})$.

FACT 13.10. *The map* $x \to (f_{x,n})$ *is a linear isometry of* $\lambda_0(\boldsymbol{\mu})$, $\lambda_{0c}(\boldsymbol{\mu})$ *and* $\lambda_{00}(\boldsymbol{\mu})$ *onto* Ω_b, Ω_c *and* Ω_0, *respectively.*

13.D. Non-normable locally bounded subspaces of $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$. The sequence $\boldsymbol{\mu} = (\mu_n)$ and its representation $\mathfrak{I} = (\mathcal{I}_n)$ over the interval $[0,1]$ are as explained in the previous subsection.

In view of Theorem 13.8, one might be tempted to think that $\lambda_{00}(\boldsymbol{\mu})$ and $\lambda_0(\boldsymbol{\mu})$ have no 'genuine,' that is, non-normable, locally bounded subspaces. It turned out not to be so, as will be seen in Theorem 13.12 below. Before proceeding, we explain some notation concerning spaces l_p.

For each $0<p<1$ and $n\in\mathbb{N}$, we consider the p-Banach space l_p and its n-dimensional version l_p^n with their usual p-norm $\|\cdot\|_p$, and we view l_p^n as subspace of l_p via the natural isometric embedding. We denote by B_p and S_p (resp., B_p^n and S_p^n) the closed unit ball and the unit sphere in l_p (resp., l_p^n). The same notation will be used in the case of the Banach spaces l_p and l_p^n for $p\geq 1$. (There is a slight conflict in denoting the spheres S_p and the supports S_n of μ_n's in a similar way, but hopefully it will not cause any confusion.)

In the theorem below, we shall make a crucial use of the following.

Proposition 13.11. *For each* $0<p\leq 2$, *there is an isomorphic embedding* $J: l_p\to L_0$.

This is commonly considered a standard fact, though in the literature usually only embeddings of l_p into L_q for $0<q<p\leq 2$ are discussed, see e.g., [31] (for embeddings of L_p into L_q see e.g., [22] and [23, p. 197], and for general results of this type, including case $q=0$, see [20, Prop. 7.1, Th. 7.2]). Since it is the case of $q=0$ that is of importance here, we indicate how it can be simply deduced from the more familiar case of $q>0$.

Now, if $q>0$, then one can obtain even an isometric embedding of l_p into L_q as follows (see [31, III.16]): Take a sequence (f_n) of independent p-stable functions (the characteristic function of each is $\exp(-|x|^p)$ when $p<2$) on the unit interval $[0,1]$ with Lebesgue measure λ as the underlying probability space (cf. the final paragraph in [22]). Then for each finitely nonzero sequence $a=(\alpha_n)\in l_p$ with $a\in S_p$ one shows that $f:=\sum_n \alpha_n f_n$ is in L_q and is p-stable so that $\|f\|_q=C_q$ (a constant). It follows that one may define an operator $J: l_p\to L_q$ by $Ja=c_q\sum_n \alpha_n f_n$ (with a suitable constant $c_q>0$) and that it is a desired embedding.

Now, J is also continuous as an operator from l_p into L_0, and since all the functions Ja for $a\in S_p$ have the same distribution, there exist $\varepsilon>0$ and $\delta>0$ such that for each $a\in S_p$ one has $\lambda(|Ja|>\varepsilon)>\delta$ or, in terms of the L_0's F-norm, $\|Ja\|>\min(1,\varepsilon)\delta$. In consequence, J is an isomorphic embedding of l_p into L_0. (Note, by the way, that J is *the same* linear operator for all $0\leq q<p$, so also the ranges of these formally different operators are the same and have the same topology.)

In what follows, we will be concerned with the l_∞-sum F_p, and the c_0-sum E_p, of the spaces l_p^n for $n\in\mathbb{N}$ and $0<p\leq 2$. Thus,

$$F_p=\left(\sum_{n=1}^{\infty} l_p^n\right)_\infty \quad \text{and} \quad E_p=\left(\sum_{n=1}^{\infty} l_p^n\right)_0,$$

and both F_p and E_p are p-Banach spaces if $0<p<1$ (resp., Banach spaces if $1\le p\le 2$), under the p-norm (resp., norm) $\|\cdot\|$ defined by the formula

$$\|(a_n)\| = \sup_n \|a_n\|_p \quad \text{for all } (a_n) \text{ in } F_p \text{ or } E_p.$$

Note that if $a=(\alpha_i)\in l_p$ and $a_n=(\alpha_1,\dots,\alpha_n)$ for $n\in\mathbb{N}$, then $\|a\|_p = \sup_n \|a_n\|_p = \|(a_n)\|$. Hence, the map $U: a\to (a_n)$ is a linear isometric embedding of l_p into F_p. (By Theorem 13.3, E_p cannot contain a copy of l_p.)

Let us also note that, for $0<p<1$, the spaces E_p and F_p are not locally convex, hence not normable. To see this it is enough to check that the convex hull C of the closed unit ball B of E_p is not bounded. It is indeed so because, for every n, if x_n is the arithmetic mean of the unit vectors in l_p^n, then $x_n\in C$ and $\|x_n\| = \|x_n\|_p = n^{1-p}\to\infty$ as $n\to\infty$.

THEOREM 13.12. *For each $0<p\le 2$, there is an isomorphic embedding $T: F_p\to\lambda_0(\boldsymbol{\mu})$ such that $T(E_p)\subset\lambda_{00}(\boldsymbol{\mu})$, and hence there is also an isomorphic embedding $V: l_p\to\lambda_0(\boldsymbol{\mu})$.*

Proof. For each n, write $L_0(S_n)$ instead of $L_0(S_n,\mu_n)$.

Choose an isomorphic embedding $J: l_p\to L_0$ (see Proposition 13.11). Thus, J is a linear operator from l_p into L_0 with $J(B_p)$ bounded so that

$$\beta(r) := \sup_{a\in B_p} \|rJa\| = \sup_{a\in S_p} \|rJa\| \to 0 \quad \text{as } r\to 0,$$

and such that for some $\delta\in(0,1)$ one has

$$\|Ja\| > \delta \quad \forall a\in S_p.$$

For a moment, fix $n\in\mathbb{N}$. Then, by Fact 13.9, there is $k=k_n\in\mathbb{N}$ such that for $i\in[n]$ one can find an $\mathcal{I}_k$-simple function $f_i=f_{n,i}$ with

$$\|Je_i - f_i\| < \delta/(2n)^2.$$

Define an operator $T_n: l_p^n\to L_0$ by

$$T_n a = \sum_{i=1}^n \alpha_i f_i, \quad \text{where } a=(\alpha_1,\dots,\alpha_n)\in l_p^n.$$

Then for each $a\in S_p^n$ one has

$$\|Ja - T_n a\| \le \sum_{i=1}^n \|Je_i - f_i\| < \delta/(2n) < 1/n$$

and

$$\|T_n a\| \ge \|Ja\| - \|Ja - T_n a\| > \frac{1}{2}\delta.$$

Thus, T_n is an isomorphic embedding. Moreover, for $0<r<1$ and all $a\in B_p^n$,

$$\|rT_n a\| \le \|rJa\| + \|r(Ja - T_n a)\| \le \beta(r) + 1/n.$$

By identifying each function $f_i = f_{n,i}$ with the element $u_{n,i}$ in $\lambda_0(\boldsymbol{\mu})(S_k) = L_0(S_k)$ such that $f_{n,i}|I_{k,j} = u_{n,i}(j)$ for $j \in S_k$, and recalling that $k = k_n$, what we have got above leads to the following: There is an isomorphic embedding $T_n : l_p^n \to L_0(S_{k_n})$,

$$T_n a = \sum_{i=1}^{n} \alpha_i u_{n,i}, \quad \text{where } a = (\alpha_1, \dots, \alpha_n) \in l_p^n,$$

such that

$$\|T_n a\| > \frac{1}{2}\delta \quad \text{for all } \alpha \in S_p^n$$

and

$$\|r T_n a\| \le \beta(r) + 1/n \quad \text{for } 0 < r < 1 \text{ and all } a \in B_p^n.$$

Of course, we may assume that the sequence (k_n) is strictly increasing.

Now, let us define an operator $T : F_p \to \omega$ as follows: if $a = (a_n) \in F_p$, then

$$Ta = \sum_{n=1}^{\infty} T_n a_n \quad \text{(coordinate-wise)}.$$

Thus, Ta is the element of ω whose coordinates with indices in S_{k_n} coincide with those of $T_n a_n$ $(n \in \mathbb{N})$, and all the other coordinates are zero.

We first show that T maps F_p into $\lambda_0(\boldsymbol{\mu})$ and is continuous, or equivalently, that T maps the unit ball B of F_p to a bounded subset of $\lambda_0(\boldsymbol{\mu})$. To this aim, take any $\varepsilon > 0$, and next choose $\rho \in (0,1)$ so that $\beta(\rho) < \frac{1}{2}\varepsilon$, and $m \in \mathbb{N}$ so that $1/m < \frac{1}{2}\varepsilon$. Let $a = (a_n) \in B$. Then $a_n \in B_p^n$ for each n, and

$$\|\rho T_n a_n\| \le \beta(\rho) + 1/n < \varepsilon \quad \text{for all } n \ge m.$$

Now, for the finite number of bounded sets $T_n(B_p^n)$ $(n < m)$, we can choose $0 < r < \rho$ so that $\|r T_n a\| < \varepsilon$ whenever $n < m$ and $a \in B_p^n$. Then for each $a \in B$ one has $\|r T_n a_n\| < \varepsilon$ for all n and, consequently, $\|r Ta\| \le \varepsilon$. Thus, $T(B)$ is indeed a bounded subset of $\lambda_0(\boldsymbol{\mu})$.

We next verify that T is an isomorphism. Take any $a = (a_n) \in F_p$ with $\|a\| = 1$; then $\|a_m\|_p \ge \frac{1}{2}$ for some m, whence

$$\|Ta\| \ge \|T_m a_m\| > \frac{1}{4}\delta,$$

and we are done. (We would get $> \frac{1}{2}\delta$ if $(a_n) \in E_p$.)

Finally, we show that $T(E_p) \subset \lambda_{00}(\boldsymbol{\mu})$. Let $a \in E_p$. For each n, denote $r_n = \|a_n\|_p$ and choose $s_n \in S_p^n$ so that $a_n = r_n^{1/p} s_n$. (Replace $1/p$ with 1 if $1 \le p \le 2$.) Since $r_n \to 0$, for all sufficiently large n one has

$$\|T_n a_n\| = \|r_n^{1/p} T_n s_n\| \le \beta(r_n^{1/p}) + 1/n,$$

and hence $\|T_n a_n\| \to 0$. Thus, $Ta \in \lambda_{00}(\boldsymbol{\mu})$.

To prove the second assertion it is enough to set $V := TU$, where U is the natural isometric embedding of l_p into F_p defined earlier. $\square$

13.E. Copies of L_0 in $\lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$. Again, the sequence $\boldsymbol{\mu} = (\mu_n)$ and its representation $\mathfrak{I} = (\mathcal{I}_n)$ over $[0,1]$ are as explained in Section 13.C.

THEOREM 13.13. *There exists a linear isometric embedding of $L_0 = L_0[0,1]$ into the quotient space $\lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$.*

Proof. Let $Q : \lambda_0(\boldsymbol{\mu}) \to \lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$ be the quotient map. By Proposition 9.1(b), if $x \in \lambda_0(\boldsymbol{\mu})$, then $\|Qx\| = \limsup_n \|x\|_n$, where $\|x\|_n$ is the F-norm of $x|S_n \in L_0(S_n)$.

We first give a proof avoiding the formalities from Section 13.C.

By Fact 13.9, given $f \in L_0$, we may find a sequence (f_n) in L_0 so that each f_n is $\mathcal{I}_n$-simple and $\|f - f_n\| \to 0$. Denote by x_f the element of ω such that $x_f(j) =$ the (constant) value of f_n on the interval $I_{n,j}$ ($j \in S_n$, $n \in \mathbb{N}$). Since

$$\|tx_f\|_n = \|tf_n\| \le \|tf\| + \|t(f - f_n)\|,$$

it is clear that $\|tx_f\| = \sup_n \|tx_f\|_n \to 0$ as $t \to 0$. Thus, $x_f \in \lambda_0(\boldsymbol{\mu})$. Let (f_n') be another sequence satisfying the same requirements as (f_n). Then $\|f_n - f_n'\|_n \to 0$. In consequence, if x_f' denotes the element of $\lambda_0(\boldsymbol{\mu})$ determined by (f_n'), then $x_f - x_f' \in \lambda_{00}(\boldsymbol{\mu})$. Define a map $T : L_0 \to \lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$ by setting $Tf = Qx_f$ for each $f \in L_0$. It is easy to check that T is linear. Since $\|f\| - \|f - f_n\| \le \|x_f\|_n \le \|f\| + \|f - f_n\|$, it follows that

$$\|Tf\| = \|Qx_f\| = \limsup_{n\to\infty} \|x_f\|_n = \|f\|$$

for all $f \in L_0$.

Alternatively, consider the closed subspace $\lambda_{0c}(\boldsymbol{\mu})$ of $\lambda_0(\boldsymbol{\mu})$ introduced before Fact 13.10. By Fact 13.9, the linear map $R : \lambda_{0c}(\boldsymbol{\mu}) \to L_0$ defined by $Rx = \lim_n f_{x,n}$ is onto and $\ker R = \lambda_{00}(\boldsymbol{\mu})$. Since $\|Rx\| = \lim_n \|f_{x,n}\| = \lim_n \|x|S_n\| = \|Qx\| \le \|x\|$, it is continuous, and $\hat{R} : \lambda_{0c}(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu}) \to L_0$, where $\hat{R} \circ (Q|\lambda_{0c}(\boldsymbol{\mu})) = R$, is a linear isometry onto. $\square$

REMARKS 13.14. (a) Don't expect too much. The quotient $\lambda_0(\boldsymbol{\mu})/\lambda_{00}(\boldsymbol{\mu})$ cannot be isomorphic to L_0 because it is nonseparable (or because L_0 contains no copies of c_0, see [13, Th. 2.4], and the paragraph preceding it for references, while the quotient has them in abundance, as seen from Theorem 9.3).

(b) As both $\lambda_{00}(\boldsymbol{\mu})$ and L_0 are separable, so is $\lambda_{0c}(\boldsymbol{\mu})$. Moreover, the dual $\lambda_{0c}(\boldsymbol{\mu})'$ is poor, as follows from Proposition 7.13 (or Theorem 10.1) and the fact that the dual of L_0 is trivial.

(c) However, one should not think that if a closed subspace F of $\lambda_0(\boldsymbol{\mu})$ contains $\lambda_{00}(\boldsymbol{\mu})$, then F' is poor. To see this, let a subspace L of $\lambda_0(\boldsymbol{\mu})$ be isomorphic to l_p for some $0 < p \le 2$ (Theorem 13.12). Then, in view of Corollary 13.6 (or Theorem 13.8), L and $\lambda_{00}(\boldsymbol{\mu})$ are totally incomparable, that is, they have no isomorphic subspaces of infinite dimension. Therefore, $\dim(L \cap \lambda_{00}(\boldsymbol{\mu})) < \infty$, and we as well may assume that $L \cap \lambda_{00}(\boldsymbol{\mu}) = \{0\}$.

Then, by [6, Th. 4.1], the subspace $F := \lambda_{00}(\boldsymbol{\mu}) + L$ is closed, and $F = \lambda_{00}(\boldsymbol{\mu}) \oplus L$ (topologically). Since L' is not poor, neither is F'.

13.F. A locally convex Schwartz subspace of $\lambda_{00}(\mathbf{S})$. A very special case of the sequences $\boldsymbol{\mu} = (\mu_n)$ considered in the preceding two subsections arises when we fix a sequence $\mathbf{S} = (S_n)$ of disjoint finite nonempty sets with union $\mathbb{N}$ and $|S_n| \to \infty$, each S_n being taken with its uniform probability measure $\mu_n = \mu_{S_n}$. Then the spaces $\lambda_0(\boldsymbol{\mu})$ and $\lambda_{00}(\boldsymbol{\mu})$ we are interested in can be represented as

$$\lambda_0(\mathbf{S}) := \Big(\sum_n L_0(S_n)\Big)_\infty \quad \text{and} \quad \lambda_{00}(\mathbf{S}) := \Big(\sum_n L_0(S_n)\Big)_0.$$

Of course, as far as isomorphic (or even isometric) properties are concerned, we may assume with no loss of generality, that (S_n) is a sequence of consecutive intervals in $\mathbb{N}$. Let us assume this, and denote $m_n = \min S_n$, $\sigma_n = |S_n|$,

$$r_n = n^{\sigma_n} \quad \text{and} \quad z_n = \sum_{i=1}^{\sigma_n} n^i e_{m_n+i-1}.$$

Obviously, (z_n) is an unbounded basic sequence in $\lambda_{00}(\mathbf{S})$. Before proceeding, recall that a TVS X is *Schwartz* if for each zero-neighborhood U there is a zero-neighborhood V such that for any $\varepsilon > 0$ one can find a finite set $A \subset X$ with $V \subset A + \varepsilon U$ (see e.g., [27, Sec. 6.3]). We are going to show the following theorem.

THEOREM 13.15. *The subspace $Z = \overline{\mathrm{lin}}(z_n)$ of $\lambda_{00}(\mathbf{S})$ is a locally convex Schwartz space that is non-isomorphic to ω. In particular, it has no normable subspaces of infinite dimension.*

The following will be of key importance in the proof of the theorem.

PROPOSITION 13.16. *For any sequence of scalars (α_n), the following are equivalent.*

(a) *The series $\sum_n \alpha_n z_n$ converges in $\lambda_{00}(\mathbf{S})$.*
(b) *$\alpha_n z_n \to 0$ in $\lambda_{00}(\mathbf{S})$ (i.e., $\|\alpha_n z_n\| = \|\alpha_n z_n\|_n \to 0$).*
(c) *The sequence $(\alpha_n z_n)$ is bounded in $\lambda_{00}(\mathbf{S})$.*
(d) $\liminf_{n\to\infty}(-\ln|\alpha_n|/\ln r_n) \ge 1$.

Proof. Statements (a) and (b) are equivalent because $\lambda_{00}(\mathbf{S}) = (\sum_n L_0(S_n))_0$ and $z_n \in L_0(S_n)$. That (b) implies (c) is trivial.

(c) $\Longrightarrow$ (d): Suppose (d) is false. Then for some $c < 1$ and an infinite set $K \subset \mathbb{N}$, $|\alpha_n| r_n^c > 1$ for all $n \in K$. Fix $\varepsilon > 0$ such that $c + \varepsilon < 1$, and for each $n \in \mathbb{N}$ set

$$t_n = r_n^{-\varepsilon} \quad \text{and} \quad J_n = \{i \in \mathbb{N} : (c+\varepsilon)\sigma_n \le i \le \sigma_n\}.$$

Note that for $n \in K$ and $i \in J_n$, $t_n|\alpha_n|n^i \geq |\alpha_n|r_n^c \geq 1$. Now $t_n \to 0$, but for all $n \in K$

$$\begin{aligned} \|t_n\alpha_n z_n\|_n &\geq \sigma_n^{-1} \sum_{i \in J_n} \min(1, t_n|\alpha_n|n^i) \\ &\geq \sigma_n^{-1}\big(\sigma_n - (c+\varepsilon)\sigma_n\big) = 1 - (c+\varepsilon) > 0, \end{aligned}$$

contradicting (c).

(d) $\Longrightarrow$ (b): Fix $\varepsilon > 0$ and next choose n_0 so that $|\alpha_n|r_n^{1-\varepsilon} \leq 1$ for all $n \geq n_0$. For such n, let i_n denote the largest integer $\leq (1-\varepsilon)\sigma_n$. If $1 \leq i < i_n$, then $|\alpha_n|n^i \leq n^{-1}$. Therefore, as easily seen,

$$\|\alpha_n z_n\|_n \leq n^{-1} + 2\sigma_n^{-1} + \varepsilon \quad \text{for all } n \geq n_0.$$

It follows that $\|\alpha_n z_n\|_n \to 0$ as $n \to \infty$. □

Proof of Theorem 13.15. As easily seen, condition (d) in Proposition 13.16 is equivalent to each of the following:

(d′) For every $c < 1$ there is n_0 such that $|\alpha_n|r_n^c \leq 1$ for all $n \geq n_0$.
(d″) For every $c < 1$, $\sup_n |\alpha_n|r_n^c < \infty$.

Denote by W the space of all scalar sequences $a = (\alpha_n)$ satisfying these conditions. Equipped with the sequence of *norms* $[\cdot]_k$ defined by

$$[a]_k = \sup_n |\alpha_n|r_n^{c_k}, \quad \text{where } c_k = k/(k+1) \ (k \in \mathbb{N}),$$

W becomes a Fréchet space. By Proposition 13.16, the basis (e_n) of W and the basic sequence (z_n) in $\lambda_{00}(\mathbf{S})$ are equivalent. In consequence, the spaces W and $Z \subset \lambda_{00}(\mathbf{S})$ are isomorphic. Thus, Z is locally convex and $Z \not\approx \omega$ because all the bases of ω are equivalent (see [2, Th. 5]). Finally, denoting $a_{k,n} = r_n^{c_k}$, it is clear that $a_{k,n}/a_{k+1,n} \to 0$ as $n \to \infty$. Hence, Z is Schwartz, by [27, Prop. 6.3.3], and *a fortiori* a Montel space (see [27, Prop. 6.3.2]). Therefore, it has no normable subspaces of infinite dimension. (Note: For the proof of [27, Prop. 6.3.3] to be correct one needs $a_{m,n} > 0$; otherwise the set K_L may fail to be compact.) □

REMARK 13.17. That Z has no nontrivial normable subspaces can also be shown directly as follows. First, note that every bounded block sequence (u_n) of the basic sequence (z_n) converges to zero. In fact, by the monotonicity of the F-norm $\|\cdot\|$, also the sequence of all the terms $\alpha_n z_n$ that occur in the blocks u_n has to be bounded. Then, by Proposition 13.16, the series $\sum_n \alpha_n z_n$ converges and, consequently, $u_n \to 0$.

Now suppose a subspace $Y \subset Z$ is normable and $\dim Y = \infty$. Then, as in the proof of Theorem 13.3, one can produce a bounded block sequence (u_n) of (z_n) with $\inf_n \|u_n\| > 0$, thus contradicting the first part.

14. Bounded multiplier property in $\lambda_0(\boldsymbol{\mu})$

We recall that an F-space $X = (X, \|\cdot\|)$ is said to have the *Bounded Multiplier Property* (BMP), if whenever a series $\sum_n x_n$ in X is unconditionally (or subseries) convergent, also each of the series $\sum_n t_n x_n$, where $(t_n) \in l_\infty$, is convergent. Not every F-space has this property, as shown by an example due to Rolewicz and Ryll–Nardzewski, see [27, Th. 3.8.3]. Also recall that a series $\sum_n x_n$ in an F-space is unconditionally convergent iff it satisfies the following Cauchy type condition: for every $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $\|\sum_{n\in F} \varepsilon_n x_n\| \le \varepsilon$ for every $F \in \mathcal{F}(\mathbb{N})$ disjoint from $[m]$ and all $\varepsilon_i = \pm 1$.

It is known that all the spaces $L_0(\mu)$, with the topology of convergence in measure on sets of finite μ measure, as well as all spaces of Orlicz type, have the (BMP) (see [24] for this and references to the literature). Below, we will need the following special case of [24, Lemma 2].

PROPOSITION 14.1. *If (S, Σ, μ) is a finite measure space, $f_1, \ldots, f_n \in L_0(\mu)$ and $|t_i| \le 1$ for $i \in [n]$, then*

$$\Big\|\sum_{i=1}^n t_i f_i\Big\|_\mu^0 \le c \max_{(\varepsilon_i)} \Big\| 8 \sum_{i=1}^n \varepsilon_i f_i \Big\|_\mu^0,$$

where $c = 8$ in the real case, and $c = 16$ in the complex case.

In what follows, $\boldsymbol{\mu} = (\mu_n)$ is an admissible sequence of measures on $\mathbb{N}$, and F is a closed ideal in $\lambda_0^c(\boldsymbol{\mu})$. Thus, $F = \overline{\mathrm{lin}}\{e_k : k \in K\}$ for some $K \subset \mathbb{N}$.

PROPOSITION 14.2. *If $z_1, \ldots, z_n \in \lambda_0(\boldsymbol{\mu})$ and $|t_i| \le 1$ for $i \in [n]$, then*

$$\Big\|\sum_{i=1}^n t_i z_i\Big\|_{\boldsymbol{\mu}}^0 \le c \max_{(\varepsilon_i)} \Big\| 8 \sum_{i=1}^n \varepsilon_i z_i \Big\|_{\boldsymbol{\mu}}^0,$$

where $c = 8$ in the real case, and $c = 16$ in the complex case.

Likewise for sequences in the quotient space $\lambda_0(\boldsymbol{\mu})/F$.

Proof. The case of $\lambda_0(\boldsymbol{\mu})$ is immediate from the preceding proposition, so let us proceed to the 'quotient case.' First of all observe that if $x \in \lambda_0(\boldsymbol{\mu})$, then for the element $\hat{x} = x + F$ of the quotient space one has

$$(*) \qquad \|\hat{x}\| = \lim_{m\to\infty} \|x - x e_{K\cap[m]}\|_{\boldsymbol{\mu}}^0 = \lim_{m\to\infty} \|x e_{\mathbb{N}\setminus K\cap[m]}\|_{\boldsymbol{\mu}}^0.$$

We first show that if $z_1, \ldots, z_n \in \lambda_0(\boldsymbol{\mu})/F$, then for every $\varepsilon > 0$ there is a sequence $y_1, \ldots, y_n$ in $\lambda_0(\boldsymbol{\mu})$ such that $z_i = \hat{y}_i$ for $i \in [n]$ and

$$\Big\|\sum_{i=1}^n \varepsilon_i y_i\Big\|_{\boldsymbol{\mu}}^0 \le \Big\|\sum_{i=1}^n \varepsilon_i z_i\Big\| + \varepsilon \quad \text{for all } (\varepsilon_i) \in \{-1, 1\}^n.$$

Choose $x_1,\dots,x_n \in \lambda_0(\boldsymbol{\mu})$ such that $z_i = \hat{x}_i$ for each i. Since there are only finitely many sequences $(\varepsilon_i) \in \{-1,1\}^n$, using $(*)$ we can find m so large that

$$\Big\| \sum_{i=1}^n \varepsilon_i x_i e_{\mathbb{N}\setminus K\cap[m]} \Big\|_{\boldsymbol{\mu}}^0 = \Big\| \Big(\sum_{i=1}^n \varepsilon_i x_i\Big) e_{\mathbb{N}\setminus K\cap[m]} \Big\|_{\boldsymbol{\mu}}^0 \le \Big\| \sum_{i=1}^n \varepsilon_i z_i \Big\| + \varepsilon$$

for each (ε_i). Then $y_i := x_i e_{\mathbb{N}\setminus K\cap[m]}$ $(i \in [n])$ are as required.

Now, again let $z_1,\dots,z_n \in \lambda_0(\boldsymbol{\mu})/F$. Take any $\varepsilon > 0$ and apply what was just shown to the sequence $8z_1,\dots,8z_n$, denoting the resulting elements in $\lambda_0(\boldsymbol{\mu})$ by $8y_1,\dots,8y_n$. Then

$$\Big\| \sum_{i=1}^n t_i z_i \Big\| \le \Big\| \sum_{i=1}^n t_i y_i \Big\|_{\boldsymbol{\mu}}^0 \le c \max_{(\varepsilon_i)} \Big\| 8 \sum_{i=1}^n \varepsilon_i y_i \Big\|_{\boldsymbol{\mu}}^0 \le c \max_{(\varepsilon_i)} \Big\| 8 \sum_{i=1}^n \varepsilon_i z_i \Big\| + c\varepsilon$$

and the desired inequality follows by allowing $\varepsilon \to 0$. □

THEOREM 14.3. *The spaces $\lambda_0(\boldsymbol{\mu})$ and $\lambda_0(\boldsymbol{\mu})/F$ have the* (BMP).

Proof. Let $\sum_n z_n$ be an unconditionally convergent series in either of the spaces, and let $\|(t_n)\|_\infty \le 1$. Then from Proposition 14.2 it follows that the series $\sum_n t_n z_n$ is Cauchy, hence convergent. □

REMARKS 14.4. (a) The results that have been proved above remain valid for the space $\lambda_0(\bar{d}_{\boldsymbol{\mu}})$ with the F-norm $\|\cdot\|_{\boldsymbol{\mu}}$ (see Sections 5 and 6, and Remark 7.19). In fact, for each $n \in \mathbb{N}$ the definition of the F-seminorm $\|\cdot\|_{\mu_n}$ can be written in the form $\|x\|_{\mu_n} = \inf\{\varepsilon > 0 : \rho_n(x/\varepsilon) \le \varepsilon\}$, where $\rho_n(x) := \mu_n(|x| > 1)$ is a monotone disjointly additive modular on $L_0(\mu_n)$ (see [24, p. 652]). Now, by [24, Lemma 2], an exact analog of the estimate in Proposition 14.1 holds for ρ_n replacing $\|\cdot\|_\mu^0$, and this in turn leads to the estimate $\|\sum_i t_i z_i\|_{\mu_n} \le c \max_{(\varepsilon_i)} \|\sum_i \varepsilon_i z_i\|_{\mu_n}$ for a constant $c > 0$ (see [24, Th. 2] and its proof for details). We may then continue as before.

(b) We do not know whether the general spaces $\lambda_0(\eta)$ have the (BMP); this may be related to the question in Remark 4.4(e). In fact, we do not know whether every solid sequence F-space has it, but this seems to be very unlikely.

(c) A quotient of an F-space with (BMP) need not have it. If E is an F-space, then there exists an F-space X of Orlicz type, and thus having the (BMP), with E being isomorphic to a quotient of X (see [30, Prop. 0.3.11], and [27, Sec. 2.5] for this and other related results). Taking for E the F-space of Rolewicz and Ryll–Nardzewski mentioned above, one verifies the claim.

15. Conclusion

We finish the paper by summarizing its main results in the case of our 'model' spaces λ_0 and λ_{00}. (For some of the results, take Proposition 13.1 into account.) The following hold:

(a) λ_{00} and λ_0 are quasi-L_0-like, in particular, non-locally pseudoconvex, and their duals are poor.
(b) The Orlicz–Pettis theorem fails in λ_0.
(c) λ_{00} contains copies of c_0 that are extendable to copies of l_∞ in λ_0, and λ_{00} is not complemented in λ_0.
(d) Every infinite-dimensional locally bounded closed subspace of λ_{00} contains a copy of c_0 that is extendable to a copy of l_∞ in λ_0.
(e) The quotient λ_0/λ_{00} is L_0-like and contains copies of l_∞/c_0 and L_0.
(f) λ_0 contains a copy of the space l_p for each $0 < p \leq 2$.
(g) λ_{00} has a closed Schwartz subspace of infinite dimension that is non-isomorphic to ω.
(h) λ_0 and λ_0/λ_{00} have the Bounded Multiplier Property.

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Lech Drewnowski, Faculty of Mathematics and Computer Science, A. Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland

E-mail address: drewlech@amu.edu.pl

Iwo Labuda, Department of Mathematics, University of Mississippi, Oxford-University, MS 38677, USA

E-mail address: mmlabuda@olemiss.edu