Illinois Journal of Mathematics
Volume 52, Number 4, Winter 2008, Pages 1385–1397
S 0019-2082

HELIX, SHADOW BOUNDARY AND MINIMAL SUBMANIFOLDS

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Abstract. We give conditions for the shadow boundary of a Riemannian submanifold M to be regular. We prove that a helix hypersurface is ruled. By studying some relations between these natural submanifolds, we show that a minimal helix shadow boundary hypersurface of M is totally geodesic in M.

1. Introduction

Let N be a Riemannian manifold and let $M \subset N$ be a Riemannian submanifold. Let us assume that $Y : M \longrightarrow TN$ is a vector field along M. We say that M is a helix submanifold with respect to Y when the angle between each tangent space of M and Y is constant, equivalently, the tangent component of Y with respect to M has constant length. The shadow boundary of M with respect to Y consists of those points in M where Y is tangent to M. Because such definitions are so general, it is natural to restrict the vector field. In this work, we will assume that Y is parallel with respect to the submanifold.

In the joint work with Di Scala [5], we investigated helix submanifolds of Euclidean spaces. In particular, we obtained a local classification of helix hypersurfaces. In [6], Dillen and Munteanu classified helix surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with respect to the parallel global vector field in the direction of $\mathbb{R}$ and where $\mathbb{H}^2$ is the hyperbolic plane. The authors called them constant angle surfaces. The authors of [7] give the corresponding classification for helix surfaces in $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ is the standard unitary sphere.

In the context of Affine Differential Geometry, Blaschke classified convex analytic surfaces with planar shadow boundaries; see [12], p. 61. In [4], Choe gives the definition of shadow boundary of Riemannian submanifolds, calling

Received February 14, 2008; received in final form February 18, 2009.
The author was supported by Conacyt.
2000 *Mathematics Subject Classification.* Primary 53C40, 53C42.

it horizon. Using the generalized Morse index theorem, he related this concept with the index of stability of a complete minimal surface in $\mathbb{R}^3$. More recently, Ghomi solved the shadow problem formulated by Wente. He used the very close concept of shadow. In my previous work [13], I studied shadow boundaries of Euclidean submanifolds.

In this manuscript, I offer a new perspective that goes beyond Riemannian ambients with a global parallel vector field. I will present four results in the context of those Riemannian submanifolds $M \subset N$ that admits a parallel vector field Y along them. Let me explain them as follows. The helix and shadow boundary will be with respect to Y. Our first result is Theorem 2.1, which proves that if M is a helix hypersurface with respect to Y, then M is ruled: for every point in it passes a geodesic of the ambient N contained in M. This generalizes Lemma 2.5 in [5] where the ambient is an Euclidean space. See [6] and [7] for the case when the ambient is $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, respectively. The next result, Theorem 3.1, gives a generic condition over the second fundamental form of M, for the shadow boundary to be regular, i.e., a submanifold. This extends Theorem 1.1 in [13], where the ambient is again an Euclidean space.

The third result, Theorem 3.2, shows that we can obtain helix submanifolds as a shadow boundary: If the shadow boundary is totally geodesic in M, then it is a helix submanifold. This property was the motivation to study helix submanifolds in this work. The second part proves that if a submanifold $L \subset M$ is a helix orthogonal to Y, then it is contained in the shadow boundary if and only if it is totally geodesic in M. Theorem 1.2 in [13] is a consequence of this result. Finally, Theorem 4.1, says: If $L \subset M$ is a shadow boundary, minimal in M, and it is a helix submanifold, then L is totally geodesic in M. In this last result, it is assumed that there is some technical condition over the mean curvature vector field of $L \subset N$.

2. Helix submanifolds

In this manuscript, we will work with C^∞ manifolds and C^∞ immersed Riemannian submanifolds. The manifold N will denote a connected Riemannian manifold with metric g. We denote the induced covariant derivative by ∇.

The next definition is a natural extension of the concept of parallel vector field on a Riemannian manifold. The case when the submanifold has dimension one is well known. For higher dimensional submanifolds, I do not know the reference where to find it.

Definition 2.1. Let M be a Riemannian immersed submanifold of N and let $Y : M \longrightarrow TN$ be a vector field along M. We will say that Y is a *parallel vector field along* M, if $\nabla_W Y = 0$ for every $W \in TM$. We will denote the set of all these vector fields by $\mathfrak{X}_0(N, M)$. We could call Y, also, an *extrinsic* *parallel vector field.* If M is connected, Y is constant. So, we will assume that $\|Y\| = 1$.

Let us observe that we are taking the derivative of the extrinsic vector field Y along tangent directions of the submanifold M. Equivalently, Y is parallel along some submanifold if and only if it is invariant under the parallel transport in N along curves contained in the submanifold (see Besse's book [3], p. 282 for details in the case that Y is global). In the case that the manifold N admits a global parallel vector field Y, then the restriction of Y to any immersed submanifold of N is parallel along such submanifold. The conditions for the existence of a global parallel vector field Y on N are well known, N should be locally a Riemannian product with a factor locally isometric to $\mathbb{R}$ (see the work of Welsh in [15] and [16]). To read more comments about this definition, see Remark 2.1.

The next definition is also a natural extension of the classic concept of general helix in $\mathbb{R}^3$ which appears in a basic course of differential geometry: a curve in $\mathbb{R}^3$ which makes constant angle with respect to a fixed direction. These kind of curves have been studied also when the ambient is a Riemannian or a Lorentzian three manifold (see [1] and [8]).

In the following definition, a helix submanifold might have higher dimension or codimension.

Definition 2.2. Let M be a Riemannian submanifold of N and let $Y \in \mathfrak{X}_0(N, M)$ be a parallel vector field along M. We say that M is a *helix submanifold*, of N, *with respect* to Y if the following function $h : M \longrightarrow \mathbb{R}$ is constant.

$$h(x) = \max\{g(w, Y(x)) \mid w \in T_xM, g(w, w) = 1\}. \tag{1}$$

Let us observe that

$$\begin{aligned} h(x) &= g\left(\frac{\tan(Y(x))}{(g(\tan(Y(x)), \tan(Y(x))))^{1/2}}, Y(x)\right) \\ &= (g(\tan(Y(x)), \tan(Y(x))))^{1/2}, \end{aligned}$$

where $\tan(Y)$ is the orthogonal projection of Y on TM. So, M is a helix if and only if $\tan(Y)$ has constant length, i.e., the angle $0 \le \tan^{-1}(h(x)) \le \pi/2$ between TM and Y is constant. So an alternative name for a helix submanifold could be constant angle submanifold.

Any Riemannian manifold M can be isometrically immersed as a helix submanifold of the Euclidean space with angle $\pi/2$. So the interesting case is when the angle is not $\pi/2$.

Let us see some examples below:

(1) Two elementary examples: a circular cylinder and any cone of revolution in $\mathbb{R}^3$ are helix submanifolds with respect to a constant vector field parallel to their axis. In [5], we described a method to construct, locally, any

immersed helix hypersurface in Euclidean space $\mathbb{R}^n$: they are ruled. For higher codimension, we have also a local characterization in particular, they can be nonorientable. See [6] and [7] for the local characterization and construction in the case of helix surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, respectively. In these cases, the direction is the global parallel vector field induced the factor $\mathbb{R}$.

(2) Let $M \subset N$ be a connected and totally geodesic submanifold. If $Y \in \mathfrak{X}_0(N, M)$, then M is a helix submanifold of N with respect to Y. To prove this, let us observe that TM and Y are invariant under parallel transport on N, along curves contained on M. So the angle between Y and TM is constant.

(3) We are going to see that the 2-dimensional torus is the only compact, connected, and orientable surface that can be immersed as a helix submanifold with angle different from $\pi/2$. Let M be a connected, orientable, and compact surface immersed in N. If M is a helix of N, then M is diffeomorphic to a torus or the angle is $\pi/2$.

Proof: Let us assume that M is a helix with respect to $Y \in \mathfrak{X}_0(N, M)$. If Y is orthogonal to M, we are done. Otherwise, by definition, $\tan(Y)$ has nonzero constant length. Since M is compact and orientable, we conclude by Poincare–Hopf's theorem (see [10]) that M has zero Euler characteristic. This proves that M is a torus.

For general Riemannian hypersurfaces which are helix, we have the following result. We will call a Riemannian submanifold *ruled* if through each point of it, there is a geodesic of the ambient contained in the submanifold. The next result proves that any helix hypersurface is ruled.

THEOREM 2.1. *Let M be a connected hypersurface in a Riemannian manifold N. Let us assume that M is a helix submanifold of N with respect to $Y \in \mathfrak{X}_0(N, M)$, and the following will hold.*

(a) *If Y is orthogonal to M at some point, then M is totally geodesic submanifold of N.*

(b) *If Y is tangent to M at some point, then M is locally a Riemannian product $\mathbb{R} \times M_2$, and the integral curves of Y are geodesics in the ambient.*

(c) *If Y is transversal (nonorthogonal) to M at some point, then M is ruled.*

Proof. Let ∇ and ∇^M, be the Levi–Civita connections of N and M, respectively, and let $II(\cdot,\cdot)$ be the second fundamental form of $M \subset N$. Since M is a helix, the angle between Y and M is constant.

(a) We have that Y is parallel along M and orthogonal to M. So M is a totally geodesic submanifold of N.

(b) Let us observe that Y is a parallel vector field on M, then by Welsh's work in [15], M is locally isometric to a Riemannian product. The integral curves of Y are geodesics in M, i.e., $\nabla^M_Y Y$. By Remark 2.1, $II(Y, Y) = 0$. So the integral curves of Y are also geodesics in the ambient: $\nabla_Y Y = \nabla^M_Y Y + II(Y,Y)$.

(c) In this case, Y is transversal to M in any point. Let $Y_0 = \tan(Y)$, $Y_1 = \mathrm{nor}(Y)$ be the orthogonal projection of Y into TM and $TM^\perp$, respectively. Let $\alpha \subset M$ be an integral curve of Y_0, i.e., $\dot{\alpha}(t) = Y_0(\alpha(t))$. First, we want to prove that the integral curves of $Y_0 = \tan(Y)$ are geodesics in M, i.e., $\nabla^M_{Y_0} Y_0 = 0$. Since Y is parallel along M, $0 = \nabla_X Y = \nabla_X Y_0 + \nabla_X Y_1$. Gauss and Weingarten formulas say that $\nabla_X Y_0 = \nabla^M_X Y_0 + II(X, Y_0)$ and $\nabla_X Y_1 = -A_{Y_1}(X)$, where A_{Y_1} is the shape operator, and $X \in TN$. Taking the tangent and normal components of $\nabla_X Y$, we have that $\nabla^M_X Y_0 - A_{Y_1}(X) = 0$ and $II(X, Y_0) = 0$. Finally, let us see that $A_{Y_1}(Y_0) = 0$. In particular, Weingarten implies that $g(\nabla_{Y_0} Y_1, X) = -g(A_{Y_1}(Y_0), X) = -g(Y_1, II(Y_0, X)) = 0$. This proves that $\nabla_{Y_0} Y_1$ has not tangent component. So $A_{Y_1}(Y_0) = 0$, and therefore, $\nabla^M_{Y_0} Y_0 = A_{Y_1}(Y_0) = 0$, i.e., α is geodesic in M. Finally, let us see that these integral lines of Y_0 are geodesics in N. Equivalently, we have to verify that Y is orthogonal to $\nabla_{\dot{\alpha}}\dot{\alpha}$.

$$0 = \frac{d}{dt} g(\dot{\alpha}, Y_0(\alpha)) = \frac{d}{dt} g(\dot{\alpha}, Y(\alpha)) = g(\nabla_{\dot{\alpha}}\dot{\alpha}, Y) + g(\dot{\alpha}, \nabla_{\dot{\alpha}} Y) = g(\nabla_{\dot{\alpha}}\dot{\alpha}, Y),$$

where $\nabla_{\dot{\alpha}} Y = 0$ because Y is parallel along M. Since α is geodesic in M, $\nabla_{\dot{\alpha}}\dot{\alpha}$ is orthogonal to M. Thus, the latter equality implies that $0 = g(\nabla_{\dot{\alpha}}\dot{\alpha}, Y) = g(\nabla_{\dot{\alpha}}\dot{\alpha}, Y_1)$. To finish, let us observe that $TN = TM \oplus \langle Y_1 \rangle$, because Y transversal to the hypersurface M. So, $\nabla_{\dot{\alpha}}\dot{\alpha} = 0$. Therefore, the integral lines of Y_0 (the tangent component of Y) are geodesics in N. $\square$

In [5], we proved this result when the ambient is an Euclidean space with its standard metric. The similar result is contained in [6] and [7] for the case of helix surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, respectively. So, this first theorem says that these helix surfaces are foliated by geodesics in its ambient, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.

More comments and examples about extrinsic parallel vector fields and helix submanifolds.

Remark 2.1. Let us analyze what happens when $Y \in \mathfrak{X}_0(N, M)$ is tangent or orthogonal to M. Let ∇^M be the Riemannian induced connection on M. Let $II_x : T_xM \times T_xM \longrightarrow T_xM^\perp$ be the second fundamental form of $M \subset N$ at $x \in M$. Finally, let $\nabla^\perp$ be the induced normal connection on $TM^\perp$.

First, let us see the case when M has codimension one. Let us assume that $Y : M \longrightarrow TN$ is a parallel vector field along M. If Y is tangent to M (i.e., $Y \in \mathfrak{X}(M)$), then Y is a parallel vector field on M. If Y is orthogonal to M, then it is parallel with respect to the normal connection $\nabla^\perp$ on $TM^\perp$. Let us observe that the converse assertions are false. But they are true if we add some extra conditions.

Let $X : M \longrightarrow TM$ be a parallel vector field on M (i.e., $\nabla^M X = 0$) and let us assume that for every $x \in M$, $X(x)$ is in the relative nullity of II_x (i.e. $II_x(X(x), \cdot) = 0$). Then X is parallel along M, i.e., $\nabla_W Z = 0$ for every $W \in T_x M$.

Now, let us see what happens when $Z : M \longrightarrow TM^\perp$ satisfies $\nabla^\perp Z = 0$ (i.e., Z is normal parallel), and for every $x \in M$, $g(Z(x), II_x(\cdot,\cdot)) = 0$. Then Z is parallel along M. These properties can be proved by using the Gauss and Weingarten's formulas. The extra conditions are sufficient and necessary for T and Z be parallel vector fields along M.

These observations tell us that to be a parallel vector field along a submanifold is a strong condition. This is supported also by the next property, whose proof is standard.

Let $M \subset N$ be a Riemannian submanifold of codimension r. Let $X_j : M \longrightarrow TN$, $j = 1, \dots, r$, be parallel vector fields along M such that, for every $x \in M, \{X_1(x), \dots, X_r(x)\}$ is a basis of $T_x M^\perp$. Then M is a totally geodesic submanifold of N.

Proposition 2.1. *If M is a compact helix of $N = \mathbb{R} \times M_2$ with respect to $X = \partial_t$, then X is orthogonal to M.*

Proof. Since M is compact, the projection π_1 of M into $\mathbb{R}$ is compact so the set $\pi_1(M) \subset \mathbb{R}$ has a maximum denoted by t_0. Let $x \in M$ be such that $\pi_1(x) = t_0$. It is standard to see that $t_0 \times M_2 = \pi_1^{-1}(t_0)$ is tangent to M in x. We deduce from this that $T_x M \subset T_x(t_0 \times M_2)$. Let us observe that X is orthogonal to $t_0 \times M_2$, then, X is orthogonal to M at x. Since M is a helix, X is orthogonal to M. $\square$

In general, a compact helix submanifold M, with respect to a global parallel vector field X on N, is not necessarily orthogonal to X: Let $N = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ be the standard 3-dimensional flat torus. Let us take $M := \mathbb{S}^1 \times \mathbb{S}^1 \times \{t\}$, and let Y be any global parallel vector field of N nonorthogonal to M.

Example 2.1. Let us consider a connected hypersurface M in $N = \mathbb{R}^{n+1}$. Let $Y \in \mathfrak{X}(N)$ be a constant vector field. If M is a helix submanifold of N, then:

(a) M is contained in a hyperplane (orthogonal to Y) of N when Y is orthogonal to M,

(b) M is not compact (otherwise, Y would be orthogonal to M),

(c) M is orientable (nor(Y) induces an orientation),

(d) M has zero Gauss–Kronecker curvature (the Gauss map of M is singular).

If $M \subset \mathbb{R}^{n+1}$ is not a hypersurface, but is compact, we can conclude that M is contained in a hyperplane orthogonal to Y.

To finish this section, let me ask the following natural question.

First, we need to consider the following property: Let us take $N = \mathbb{R}^3$ and let $v \in N$ be a nonzero vector. If $M^2 \subset N$ is a complete minimal surface which is a helix with respect to $Y = v$, then M is a plane.

Proof: By Example 2.1, M has zero Gauss–Kronecker curvature, but in dimension 2 it is the Gaussian curvature. Since M is minimal, it easy to see that M is a plane.

The latter argument is not valid if $N = \mathbb{R}^{n+1}$, with $n \geq 3$: In $\mathbb{R}^4$, there are minimal hypersurfaces with zero Gauss–Kronecker curvature (see [11]), like $M^3 = M' \times \mathbb{R}$, where M' is minimal in $\mathbb{R}^3$.

QUESTION 2.1. If $M^n \subset N^{n+1}$ is minimal and a helix submanifold, is it a totally geodesic submanifold of N?

3. Shadow boundary and helix

The next definition was used by Choe. In [4], he gives the definition of shadow boundary of Riemannian submanifolds calling it horizon. He used this concept to study the stability index of complete minimal surfaces in $\mathbb{R}^3$.

Blaschke used the name of shadow boundaries for the case of convex analytic surfaces in $\mathbb{R}^3$.

DEFINITION 3.1. Let M be a Riemannian immersed submanifold of N, and let $Y : M \longrightarrow TN$ be a parallel vector field along M (i.e. $Y \in \mathfrak{X}_0(N, M)$). The *shadow boundary* of M *with respect to* Y is the following subset of M.

$$S\partial(M, Y) = \{x \in M \mid Y(x) \in T_xM\}. \tag{2}$$

The shadow boundary is a natural subset of M, it is the locus where the extrinsic vector field Y is tangent to M. In general, this subset $S\partial(M, Y) \subset M$ is closed, so, if M is compact it is also compact. This subset, $S\partial(M, Y)$, is not always a submanifold of M. It may be empty (when Y is nowhere tangent to M), or equal to M (when Y is anywhere tangent to M). See Example 3.1 below for other property of shadow boundaries.

Finally, when $N = \mathbb{R}^n$, any constant vector field Y on N is parallel along any submanifold. In this context, if M is a compact submanifold, the shadow boundary of M is nonempty, with respect to any such vector field Y.

The second fundamental form of $M \subset N$ at $x \in M$ is a symmetric bilinear tensor, which we denote by $II_x : T_xM \times T_xM \longrightarrow T_xM^{\perp}$. So II_x is a bilinear application for every $x \in M$.

Let Y be a parallel vector field along M. Let $x \in M$ be a point such that $Y(x) \in T_xM$. Then we can consider the following linear application:

$$II(Y(x), \cdot) : T_xM \longrightarrow T_xM^{\perp}.$$

If this transformation is surjective, we will say that $II(Y(x), \cdot)$ is surjective. In particular, if $\mathrm{cod}M = 1$, the latter condition is equivalent to $II(Y(x), \cdot) \neq 0$.

Theorem 3.1. *Let M be a submanifold of dimension n and codimension k in N, with $n \geq k$. Let Y be a parallel vector field along M. If $II(Y(y),\cdot)$ is surjective for every $y \in S\partial(M,Y)$, then $S\partial(M,Y)$ is a submanifold of dimension $n-k$ in M.*

Proof. Let ∇ be the covariant derivative of N. Let us take $p \in S\partial(M,Y)$, and let $U \subset M$ be a open neighborhood of p. Our goal is to verify that $S\partial(M,Y) \cap U$ is a submanifold of M.

Let $\xi_j : U \longrightarrow TU^{\perp}$, $j = 1,\dots,k$, be a basis of orthonormal vector fields (U is such that there exist these vector fields). Let us consider the next function $F : U \longrightarrow \mathbb{R}^k$, given by

$$F(x) = (g(Y(x),\xi_1(x)),\dots,g(Y(x),\xi_k(x))).$$

It is clear that $F^{-1}(0) = S\partial(M,Y) \cap U$. We are going to prove that $0 \in \mathbb{R}^k$ is a regular value of F. We need verify that for every $x \in S\partial(M,Y) \cap U$, $F_{*x} : T_xM \longrightarrow \mathbb{R}^k$ is surjective. Let $(y_1,\dots,y_n)$ be local coordinates in U. Let us calculate the next derivatives in these coordinates, $\frac{\partial F}{\partial y_l} = (\frac{\partial}{\partial y_l} g(Y(x),\xi_1(x)), \dots, \frac{\partial}{\partial y_l} g(Y(x),\xi_k(x)))$, for every $1 \leq l \leq n$. Since Y is parallel,

$$\frac{\partial}{\partial y_l} g(Y(x),\xi_j(x)) = g(\nabla_{\partial y_l} Y,\xi_j) + g(Y,\nabla_{\partial y_l}\xi_j) = g(Y,\nabla_{\partial y_l}\xi_j).$$

Let us apply Weingarten's formula, which says that $\nabla_{\partial y_l}\xi_j = -A_{\xi_j}(\partial y_l) + \nabla^{\perp}_{\partial y_l}\xi_j$. In conclusion,

$$\frac{\partial}{\partial y_l} g(Y(x),\xi_j(x)) = g(Y,-A_{\xi_j}(\partial y_l)) = -g(II(Y,\partial y_l),\xi_j(x)),$$

for every $x \in S\partial(M,Y)$, $1 \leq j \leq k$, and $1 \leq l \leq n$.

Now, we are ready to see that the next matrix

$$(F_{*x})_{jl} = -(g(II(Y,\partial y_l),\xi_j(x)))$$

has rank k. Let us assume that the row vectors are linearly dependent, i.e., we have the following condition $\sum_{j=1}^{k} a_j g(II(Y,\partial y_l),\xi_j(x)) = 0$, for every $1 \leq l \leq n$, and where $a_j \in \mathbb{R}$ are constants. We can rewrite this expression as

$$g\left(II(Y,\partial y_l), \sum_{j=1}^{k} a_j \xi_j(x) \right) = 0,$$

for every $1 \leq l \leq n$. Since $II(Y,\cdot)$ is surjective, $\sum_{j=1}^{k} a_j\xi_j(x) = 0$, therefore $a_j = 0$. Which proves that 0 regular value of F. Then we can conclude that $F^{-1}(0) \cap U$ is a submanifold U of dimension $n-k$. $\square$

A special case of Theorem 3.1 is when $\dim N = 2\dim M$. The conclusion in this situation is that $S\partial(M,Y)$ is a discrete subset of M. So, if M were compact, $S\partial(M,Y)$ would be a finite set of points in M.

In [13], we proved that when $M^n \subset \mathbb{R}^{n+1}$ has nowhere zero Gauss–Kronecker curvature, then for every $v \in \mathbb{R}^{n+1}$, $S\partial(M,v)$ is a submanifold of M of codimension one.

In the particular case of surfaces in $\mathbb{R}^3$, the smoothness of the shadow boundary is investigated in [9].

We need to recall the next basic concept of submanifolds. Let $L \subset N$ be a Riemannian submanifold. Let us take $x \in L$, then L is called a *totally geodesic submanifold of* N*, at the point* x, if every geodesic γ of L through x satisfies $\nabla_{\dot\gamma}\dot\gamma_{|x} = 0$.

In her work on Affine Differential Geometry [14], Schwenk used conditions similar to those in first part of the next theorem. The second part was the original motivation to consider helix submanifolds in this work.

THEOREM 3.2. *Let $M^n \subset N^{n+k}$ $(n \geq 2)$ be a submanifold of codimension k $(k \geq 0)$. Let L be a hypersurface of M, which is nowhere totally geodesic of N, and let us take $Y \in \mathfrak{X}_0(N,M)$. If Y is orthogonal to L, then $L \subset S\partial(M,Y)$ if and only if L is a totally geodesic submanifold of M. If $L \subset S\partial(M,Y)$ is a totally geodesic submanifold of M and Y is not orthogonal to L, then L is a helix submanifold of N with respect to Y.*

Proof. $(\Longrightarrow)$ Let us take $x \in L$, since $\dim(T_xL^\perp \cap T_xM) = 1$ and by hypothesis, $Y(x) \in T_xL^\perp \cap T_xM$, we obtain that $\langle Y(x)\rangle = T_xL^\perp \cap T_xM$. Therefore, we have the following equality for every $x \in L$,

$$T_xM = T_xL \oplus (T_xL^\perp \cap T_xM) = T_xL \oplus \langle Y(x)\rangle. \tag{3}$$

Let $\gamma \subset L$ be a geodesic and let $x \in \gamma$ be any point. Hence, $\nabla_{\dot\gamma}\dot\gamma$ is orthogonal to L, i.e., $\nabla_{\dot\gamma}\dot\gamma \in T_xL^\perp$. Let us prove that γ is a geodesic of M. By equality (3), we just have to verify that $\nabla_{\dot\gamma}\dot\gamma$ is orthogonal to $Y(x)$: We know that $g(Y(\gamma(t)),\dot\gamma) = 0$, this implies that

$$g(Y(\gamma(t)),\nabla_{\dot\gamma}\dot\gamma) + g(\nabla_{\dot\gamma}Y(\gamma(t)),\dot\gamma) = \frac{d}{dt}g(Y(\gamma(t)),\dot\gamma) = 0.$$

Then $\nabla_{\dot\gamma}\dot\gamma$ is orthogonal to M, so γ is a geodesic of M.

$(\Longleftarrow)$ In this implication, we will assume that $k = 1$. We have to see that $Y(x) \in T_xM$, for every $x \in L$. Since L is not a totally geodesic submanifold of N at x, there exists a geodesic γ of L through x with $\nabla_{\dot\gamma}\dot\gamma_{|x} \neq 0$. By hypothesis, γ is also a geodesic of M. So, $\nabla_{\dot\gamma}\dot\gamma \in (T_\gamma M)^\perp$. Let us prove that $Y(x)$ is orthogonal to $\nabla_{\dot\gamma}\dot\gamma$. For this, let us observe that $g(\dot\gamma, Y(x)) = 0$. Therefore,

$$g(Y(\gamma(t)),\nabla_{\dot\gamma}\dot\gamma) + g(\nabla_{\dot\gamma}Y(\gamma(t)),\dot\gamma) = \frac{d}{dt}g(Y(\gamma(t)),\dot\gamma) = 0.$$

But $\nabla_{\dot\gamma}Y(\gamma(t)) = 0$, because Y is parallel along L. Since M is of codimension one, $g(Y(\gamma(t)),\nabla_{\dot\gamma}\dot\gamma) = 0$ implies that $Y(x) \in T_xM$.

Finally, let us prove the second part of the theorem. If $Y(x) \in T_xL$, for every $x \in L$, then L is a helix. Otherwise, there exist $p \in L$ such that $Y(p) \notin T_pL$. So,

$$T_pL \oplus \langle Y(p)\rangle \subset T_pM. \tag{4}$$

We are going to verify that the angle between T_xL and $Y(x)$ is constant, for every x in L. Let γ be any geodesic of L from p to x, hence, it is also geodesic of M. Now, let us consider the parallel transport τ in M, along γ, from p to x. Therefore, $\tau : T_pM \longrightarrow T_xM$ is an isometry. So, τ transforms the latter equation (4), in $T_xL \oplus \langle Y(x)\rangle \subset T_xM$. Since the parallel transport is an isometry, the angle between T_xL and $Y(x)$ is equal to the angle between T_pL and $Y(p)$. □

In Theorem 3.2, the condition that L is not totally geodesic in N at any point is important to prove that $L \subset S\partial(M,Y)$. We can see this with the next example: $N = \mathbb{R}^n$, M a hyperplane, L a linear subspace of codimension one in M. Finally, let $Y = v$ be any constant vector field orthogonal to M. In this example, the relation $L \subset S\partial(M,Y)$ is false.

Example 3.1. Let us consider the next property of shadow boundaries, which could be useful to study the shadow of higher codimensional submanifolds. Let M be the Riemannian product $M_1 \times M_2$, of two submanifolds $M_1 \subset N_1$ and $M_2 \subset N_2$. Let us take $Y = (Y_1, Y_2)$ where $Y_j \in \mathfrak{X}_0(N_j, M_j)$. Then

$$S\partial(M,Y) = S\partial(M_1,Y_1) \times S\partial(M_2,Y_2).$$

Let us apply this to the submanifold $M = S^1 \times S^1 \subset \mathbb{R}^2 \times S^2$. We can see that the only possibilities for $S\partial(M,Y)$ are $S^1 \times S^1$, $\{p,-p\} \times S^1$ or $\emptyset$, where $p \in S^1$.

4. Minimal shadow boundaries

We need the next lemma, which is due to Chen, see [2].

Let us recall that the mean curvature vector of a Riemannian submanifold L of N, is the trace of the second fundamental form of $L \subset N$. When this vector field is constant zero, we say that L is minimal in N.

Lemma 4.1 (Chen's lemma). *Let L^n be a submanifold of M^s, where M is a submanifold of N^m. Then L is minimal in M if and only if the mean curvature vector field of $L \subset N$ is orthogonal to M.*

Lemma 4.2. *Let $M^n \subset N$ be a Riemannian immersed submanifold and let $L^{n-1} \subset M$ be a submanifold such that $L \subset S\partial(M,Y)$, where Y is parallel along M and transverse to L. Let H be the mean curvature vector field of $L \subset N$. Then L is minimal in M if and only if $g(H,Y) = 0$.*

Proof. By hypothesis $Y(x) \in T_xM$ for every $x \in L$. By Lemma 4.1, if L is minimal in M then H is orthogonal to M. So $H(x)$ is orthogonal to $Y(x)$, i.e., $g(H,Y)=0$.

Now, let us assume that $g(H,Y)=0$. By definition, H is orthogonal to L. To apply Lemma 4.1, we need prove that H is orthogonal to M. Since Y is transversal to L, $T_xM = T_xL \oplus \langle Y(x)\rangle$ for every $x \in L$. Now it is clear that H is orthogonal to M. Then L is minimal in M. □

We will say that a Riemannian submanifold $L \subset N$ has *exhaustive mean curvature vector* at the point $p \in L$, if $T_pL \subset V_p$, where V_p is the vector subspace of T_pN generated by the following set: vectors in T_pN that are obtained by the parallel transport in N of the mean curvature vectors $H(x)$ (for every $x \in L$), along curves in L from x to p. For example, a closed hypersurface L of an Euclidean ambient satisfies this condition, at any point, because the parallel transport in such ambient is just a translation, and the mean curvature vector of a compact L is nonconstant zero.

Theorem 4.1. *Let $M \subset N$ be a Riemannian immersed submanifold and let us take $Y \in \mathfrak{X}_0(N,M)$. Let $L \subset S\partial(M,Y)$ be a transversal helix hypersurface of M with respect to Y, and let us assume that L has exhaustive mean curvature vector in N. If L is minimal in M, then L is a totally geodesic in M.*

Proof. Let H be the mean curvature vector field of $L \subset N$. Let $p \in L$ be a point such that the mean curvature vector H, of $L \subset N$, is exhaustive at p. Let V_p be the vector subspace of T_pN generated by the next set: vectors obtained by parallel transport, in N, of the mean curvature vectors $H(x)$ of $L \subset N$, with $x \in L$. The parallel transport is along curves contained in L from x to the point p.

The main goal will be to see that Y is orthogonal to L. It is important to use the equation $T_pL \subset V_p$, which follows by definition of an exhaustive mean curvature vector. Then the initial step is to prove that Y is orthogonal to V_p. We can apply Lemma 4.2 (L is minimal in M) to deduce that $\langle H(q), Y(q)\rangle = 0$, for every $q \in L$, i.e., $Y(q)$ is orthogonal to $H(q)$.

The vector field Y is invariant under parallel transport in N along curves contained in M, and in particular, along curves in L. Therefore, $Y(p)$ is orthogonal to V_p. Since $T_pL \subset V_p$, $Y(p)$ is orthogonal to T_pL. But L is a helix with respect to Y, so the angle between TM and Y is constant, i.e., $Y(q)$ is orthogonal to T_qL, for every $q \in L$.

Finally, we can apply first part of Theorem 3.2, which says that if $L \subset S\partial(M,Y)$ and Y is orthogonal to L, then L is a totally geodesic submanifold of M. □

Example 4.1. We are going to construct a hypersurface M of $N = \mathbb{R}^{n+2}$, which contains a minimal submanifold in some shadow boundary.

Let $L^n \subset \mathbb{R}^{n+2}$ be a submanifold and let $Y = v \in \mathbb{R}^{n+2} - \{0\}$ be a vector such that:

- v is transverse to $L : v \notin T_x L$ for every $x \in L$,
- $\langle H, v\rangle = 0$, where H is the mean curvature vector field of $L \subset N$.
- $L_{\varepsilon,v} = \{y = x + tv \in \mathbb{R}^{n+2} \mid x \in L, |t| < \varepsilon\}$ is a submanifold, where $\varepsilon = \varepsilon(x)$ denotes a positive smooth function of L.

Then L is a minimal submanifold of $M := L_{\varepsilon,v}$. If L is compact, ε can be a constant function. Proof: this is consequence of Lemma 4.2. We should verify the hypothesis of such theorem. By hypothesis, the mean curvature vector of $L \subset N$ is orthogonal to Y. Finally, since $M = L_{\varepsilon,v}$ is a "ruled" neighborhood of L in direction v, $S\partial(M, v) = L_{\varepsilon,v} = M$, and then $L \subset S\partial(M, v)$.

We want to finish with the following question. Let M be a compact hypersurface of $N = \mathbb{R}^{n+1}$ transversal to a constant vector field Y on N. Let us assume that L is a hypersurface in M, such that $L \subset S\partial(M, Y)$ and L is contained in a hyperplane H of N (so L is a hypersurface of H). But every closed hypersurface L of H has exhaustive mean curvature vector in H. So there exists $p \in L$ and a subspace $V_p \subset T_p H$ such that $T_p L \subset V_p$ (where V_p is as in the definition of exhaustive mean curvature vector). Since H is totally geodesic in N, the parallel transport in H of vectors w in TH coincides with the parallel transport of w in N. By the same reason, the mean curvature vector of $L \subset H$ coincides with the mean curvature vector of $L \subset N$. Then L has exhaustive mean curvature vector in N. By the latter theorem, if L is minimal in M, L is totally geodesic in M. Our final question will be important, in which L is not contained in a hyperplane or with a nonexhaustive mean curvature vector.

Question 4.1. Does a closed hypersurface in $\mathbb{R}^{n+1}$ with some minimal and nontotally geodesic shadow boundary L exist?

Acknowledgments. The author wishes to express his gratitude to Luis Hernández Lamoneda for his support and helpful suggestions. The author is grateful to the referee for helpful comments which were very useful to improve the exposition in this manuscript.

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