Illinois Journal of Mathematics
Volume 52, Number 4, Winter 2008, Pages 1249–1258
S 0019-2082

ON QUASICONFORMAL INVARIANCE OF CONVERGENCE AND DIVERGENCE TYPES FOR FUCHSIAN GROUPS

KATSUHIKO MATSUZAKI

Abstract. We characterize convergence and divergence types for Fuchsian groups in terms of the critical exponent of convergence and modified functions of the Poincaré series for certain subgroups associated with ends of the quotient Riemann surfaces. As an application of this result, we prove that convergence and divergence type are not invariant under a quasiconformal automorphism of the unit disk.

1. Background and statement of results

In this paper, we show that divergence and convergence types for Fuchsian groups are not necessarily quasiconformally invariant. Notations and the definition of the types of Fuchsian groups are given in this section after the statement of the theorem. For other results on quasiconformal invariance of certain classes of Fuchsian groups, see for example [8].

Theorem 1. *There exist Fuchsian groups* Γ *of convergence type and* $\Gamma_\#$ *of divergence type such that the conjugation* $f^{-1}\Gamma f$ *coincides with* $\Gamma_\#$ *for a quasiconformal automorphism* f *of the unit disk* $\mathbb{B}^2$.

A Fuchsian group Γ is a discrete group of orientation-preserving isometric automorphisms of the hyperbolic plane $\mathbb{H}^2$, which acts properly discontinuously on $\mathbb{H}^2$. We assume that Γ is torsion-free for the sake of simplicity, and consequently, that it acts freely (but this assumption is not essential). The unit disk $\mathbb{B}^2 \subset \mathbb{C}$ with a conformal metric $\rho(z)|dz| = 2|dz|/(1-|z|^2)$ is a model of the hyperbolic plane and Γ acts on $\mathbb{B}^2$ as a group of Möbius transformations. The boundary S^1 of the model $\mathbb{B}^2$ is located at infinity of the hyperbolic plane and the action of Γ extends to S^1.

Received November 15, 2007; received in final form March 11, 2008.
2000 *Mathematics Subject Classification.* Primary 30F35, 37F30. Secondary 37F35.

The *Poincaré series* of dimension $s \geq 0$ for a Fuchsian group Γ with respect to the origin $0 \in \mathbb{B}^2$ is defined by

$$P_s(\Gamma) = \sum_{\gamma \in \Gamma} \exp(-s\rho(0, \gamma(0))),$$

where ρ also denotes the hyperbolic distance on $\mathbb{B}^2$. The *critical exponent of convergence* for Γ is defined by

$$\delta(\Gamma) = \inf\{s \geq 0 \mid P_s(\Gamma) < \infty\},$$

which is in the interval $[0,1]$. This is an index measuring the distribution of the orbit of Γ, which is closely related to geometric structure of the associated hyperbolic Riemann surface $R = \mathbb{H}^2/\Gamma$. We say that Γ is of *divergence type* if $P_{\delta(\Gamma)}(\Gamma) = \infty$ and of *convergence type* if $P_{\delta(\Gamma)}(\Gamma) < \infty$. Finitely generated Fuchsian groups (=geometrically finite Fuchsian groups) are all of divergence type.

It is known that, as in the following proposition, convergence and divergence type is a quasiconformal invariant if $\delta(\Gamma) = 1$. Note that the Hopf–Tsuji theorem asserts that the condition $P_1(\Gamma) = \infty$ is equivalent to a property that $R = \mathbb{H}^2/\Gamma$ has no Green function, as well as to a property that the conical limit set has full 1-dimensional measure on S^1. See [12, Theorem 6.3.3].

PROPOSITION 2. *Let* Γ *be a Fuchsian group with* $\delta(\Gamma) = 1$ *and* $\Gamma_\# = f^{-1}\Gamma f$ *is another Fuchsian group that is quasiconformally conjugate to* Γ. *If* Γ *is of convergence type, then so is* $\Gamma_\#$, *and if* Γ *is of divergence type then so is* $\Gamma_\#$.

Actually, the invariance of divergence at exponent 1 is seen from the invariance of the property that R has no Green function, which is a famous result due to Pfluger (see [14, p. 221]). The invariance of critical exponent 1 has been proved by Fernández and Rodríguez [7] from the invariance of the bottom of spectra of the Laplacian on R. Proposition 2 makes a contrast to Theorem 1, and it, in particular, shows that the example of Fuchsian groups Γ and $\Gamma_\#$ in Theorem 1 should satisfy $\delta(\Gamma) < 1$ and $\delta(\Gamma_\#) < 1$.

To prove Theorem 1, we characterize convergence and divergence types of Fuchsian groups by considering the modified Poincaré series defined below. This idea comes from recent work of Anderson, Falk and Tukia [2], [6]. The modification of the Poincaré series arises in construction of the Patterson measure.

For a Fuchsian group Γ, a probability measure μ on S^1 is said to be a Γ-*invariant conformal measure* of dimension $s \geq 0$ if

$$\mu(\gamma(E)) = \int_E |\gamma'(x)|^s \, d\mu(x)$$

for any Borel measurable set E on S^1 and for any $\gamma \in \Gamma$. For $s > \delta(\Gamma)$, consider the sum of the weighted Dirac measures

$$\sum_{\gamma \in \Gamma} \exp(-s\rho(0,\gamma(0)))D_{\gamma(0)}.$$

Dividing this by the total mass, we have a probability measure on the compact space $\overline{\mathbb{B}^2}$. Then, taking $s \downarrow \delta(\Gamma)$, we have a weak limit of a subsequence of the measures. If Γ is of divergence type, this limit is a Γ-invariant conformal measure of dimension $\delta(\Gamma)$ and has support in the limit set $\Lambda(\Gamma)$. In general, a Γ-invariant conformal measure of dimension $\delta(\Gamma)$ that has support in $\Lambda(\Gamma)$ is called the *Patterson measure* for Γ. The above construction is precisely the way one obtains the Patterson measure for a Fuchsian group of divergence type.

On the other hand, if Γ is of convergence type, we have to modify the weight $\exp(-s\rho(0,\gamma(0))$ by using a continuous, nondecreasing function $h:(0,\infty) \to (0,\infty)$ that satisfies the following:

(a) The *modified Poincaré series*

$$P_s^h(\Gamma) = \sum_{\gamma \in \Gamma} h(\rho(0,\gamma(0)))\exp(-s\rho(0,\gamma(0)))$$

converges for $s > \delta(\Gamma)$ and diverges for $s \leq \delta(\Gamma)$;

(b) For every $\varepsilon > 0$, there exists a constant $r_0 > 0$ such that $h(t+r) \leq e^{\varepsilon t}h(r)$ for all $r \geq r_0$ and $t > 0$;

(c) There exists a constant $C > 0$ such that $h(r+t) \leq Ch(r)h(t)$ for all $r > 0$ and $t > 0$.

A function h satisfying these properties is called a *Patterson function* for Γ. Note that, if Γ is of divergence type, then the constant function $h(t) \equiv 1$ can be a Patterson function for Γ. It is proved by Patterson [13] that a Patterson function exists for any Fuchsian group Γ and hence, by the same construction using the modified Poincaré series instead, there is the Patterson measure for Γ even if Γ is of convergence type. See also [16], [12, Section 3.1], and [5]. The above definition of a Patterson function is due to [6].

Concerning characterization of Fuchsian groups of divergence type, the following result is originally due to Patterson [13] and Sullivan [16], [17]. Since the 1-dimensional measure on S^1 is the Patterson measure for a Fuchsian group Γ with $P_1(\Gamma) = \infty$, this is a generalization of the Hopf–Tsuji theorem. See also [9, Section 4].

PROPOSITION 3. *The following conditions are equivalent to each other for any nonelementary Fuchsian group* Γ:

(1) Γ *is of divergence type*;

(2) *The Patterson measure has full measure on the conical limit set of* Γ.

Moreover, if $\delta(\Gamma) \geq 1/2$, *then condition* (3) *is also equivalent to* (1) *and* (2):

(3) *For the hyperbolic Laplacian* Δ *on* $R=\mathbb{H}^2/\Gamma$ *and the constant* $\lambda=\delta(\Gamma)(1-\delta(\Gamma))$, *the Green function with respect to the operator* $\Delta-\lambda$ *does not exist on* R.

However, these properties are described by global geometric structure on $R=\mathbb{H}^2/\Gamma$ and it is difficult to see whether they are quasiconformally invariant or not. On the other hand, our criterion (Theorem 4 below) will be given by using subgroups Γ_E associated with ends E of R. Here an *end* is a connected component E of the complement of a compact subsurface W of R that is not relatively compact. (Precisely speaking, this should be called a neighborhood of a topological end of R.) We will say that E is outside W in this situation. The end subgroup Γ_E with respect to E is the image of the fundamental group of E under the homomorphism $\pi_1(E)\to\pi_1(R)\cong\Gamma$ (up to conjugacy) induced by the inclusion map. The limit set for Γ_E is investigated in [2].

Theorem 4. *The following conditions are equivalent for a nonelementary Fuchsian group* Γ:

(1) Γ *is of divergence type*;

(4) *There exists a Patterson function* h *for* Γ *such that the modified Poincaré series* $P^h_{\delta(\Gamma)}(\Gamma')$ *converges for every subgroup* Γ' *of* Γ *with* $\Lambda(\Gamma')\subsetneqq\Lambda(\Gamma)$;

(5) *There exists a Patterson function* h *for* Γ *and a compact subsurface* W *of* $R=\mathbb{H}^2/\Gamma$ *such that* $P^h_{\delta(\Gamma)}(\Gamma_E)$ *converges for the end subgroup* Γ_E *with respect to every end* E *outside* W.

As an application of Theorem 4, we have Theorem 1, which will be demonstrated in Section 3. In the next section, we show Theorem 4, which is a new formulation of existing results. Actually, in the proof of Theorem 1, we use these results rather than Theorem 4. Nevertheless, this theorem seems to have its own interest.

2. Criteria for convergence and divergence types

Theorem 4 follows from two results (Lemmas 5 and 7) below. To obtain the necessary condition to be of divergence type, that is, the implication (1) $\Rightarrow$ (4), we need the following lemma, which has been formulated in [9, Lemma 30].[1] We give a complete proof for it here. Note that, though the statement itself is given for Fuchsian groups, this is true for Kleinian groups of any dimension by the same proof.

Lemma 5. *Let* Γ' *be a subgroup of a Fuchsian group* Γ *such that the limit set* $\Lambda(\Gamma')$ *is a proper subset of* $\Lambda(\Gamma)$. *If* Γ' *is of divergence type, then* $\delta(\Gamma')<\delta(\Gamma)$.

Proof. Let μ be a Patterson measure for Γ. Since $\Lambda(\Gamma)\supsetneqq\Lambda(\Gamma')$, we can choose an interval $I\subset\Omega(\Gamma')$ such that $\mu(I)>0$ and $I\cap\gamma(I)=\emptyset$ for any

[1] The original note is [19]. After completing the present paper, the author has found [18] which also proves this result.

nontrivial element $\gamma \in \Gamma'$. Then $\sum_{\gamma\in\Gamma'} \mu(\gamma(I)) = \mu(\bigcup_{\gamma\in\Gamma'} \gamma(I)) \le 1$. Since

$$\sum_{\gamma\in\Gamma'} \mu(\gamma(I)) = \sum_{\gamma\in\Gamma'} \int_I |\gamma'(x)|^{\delta(\Gamma)}\, d\mu(x) = \int_I \sum_{\gamma\in\Gamma'} |\gamma'(x)|^{\delta(\Gamma)}\, d\mu(x),$$

this implies that $\sum_{\gamma\in\Gamma'} |\gamma'(x)|^{\delta(\Gamma)} < \infty$ for almost every $x \in I$. Here, by

$$|\gamma'(x)| = \frac{1+|\gamma(0)|}{|x-\gamma^{-1}(0)|^2}\bigl(1-|\gamma(0)|\bigr) \ge \frac{1}{4}\bigl(1-|\gamma(0)|\bigr)$$

and $1-|\gamma(0)| \ge e^{-\rho(0,\gamma(0))}$ for every γ, we see that the Poincaré series $P_{\delta(\Gamma)}(\Gamma')$ of dimension $\delta(\Gamma)$ converges. On the other hand, $P_{\delta(\Gamma')}(\Gamma')$ of dimension $\delta(\Gamma')$ diverges, for Γ' is of divergence type. Hence, we obtain $\delta(\Gamma') < \delta(\Gamma)$. □

An example of a Fuchsian group of convergence type arises from a nontrivial self-covering of a Riemann surface. In the next paragraph, we illustrate this situation. We apply Lemma 5 in such a particular case. A stronger result than Corollary 6 below is obtained in [11].

A pair of pants is a hyperbolic surface homeomorphic to a three-punctured sphere with three geodesic boundary components. Choose a pair of pants P whose boundary components b_0, b_1, and b_2 have the same length. First, glue two copies of P along the 2 boundary components b_1 and b_2 of P, which results in a hyperbolic surface R_1 with 5 boundary components. Next, glue four copies of P along the 4 boundary components coming from b_1 and b_2 of P, which results in a hyperbolic surface R_2 with 9 boundary components. Continuing this process infinitely many times, we have a hyperbolic surface R_∞ with the boundary component b_0. Let Γ be a Fuchsian group such that $R = \mathbb{H}^2/\Gamma$ is the Nielsen extension of R_∞ beyond b_0. On the other hand, for an end E of R that is one of the two components of $R_\infty - P$, the end subgroup Γ_E of Γ is properly contained in Γ but it is conformally conjugate to Γ.

Corollary 6. *If there exists an end E of $R = \mathbb{H}^2/\Gamma$ such that the end subgroup Γ_E is properly contained in Γ but it is conformally conjugate to Γ, then Γ is of convergence type. More generally, if there exists a subgroup Γ' such that $\Lambda(\Gamma') \subsetneq \Lambda(\Gamma)$ and Γ' is conformally conjugate to Γ, then Γ is of convergence type.*

Proof. Suppose to the contrary that Γ is of divergence type. Because Γ' is conformally conjugate to Γ, then $\delta(\Gamma') = \delta(\Gamma)$ and Γ' is of divergence type as well. However, this contradicts Lemma 5 asserting $\delta(\Gamma') < \delta(\Gamma)$. □

The following sufficient condition for Γ to be of divergence type is given by Falk and Tukia [6, Corollary] as an application of the arguments for end subgroups developed in [2].

Lemma 7. *Let Γ be a Fuchsian group of convergence type. Let $\{E_1, \dots, E_n\}$ be the family of all ends outside a compact subsurface W of $R = \mathbb{H}^2/\Gamma$. Then,* *for every Patterson function h for Γ, there exists an end E_i $(1 \leq i \leq n)$ such that the modified Poincaré series $P^h_{\delta(\Gamma)}(\Gamma_{E_i})$ for the end subgroup Γ_{E_i} diverges.*

In general, if a subgroup Γ' of Γ satisfies $\delta(\Gamma') < \delta(\Gamma)$, then $P^h_{\delta(\Gamma)}(\Gamma')$ converges for every Patterson function h for Γ. Hence, Lemma 7 in particular claims the following.

COROLLARY 8. *If there exists a compact subsurface W of $R = \mathbb{H}^2/\Gamma$ such that $\delta(\Gamma_E) < \delta(\Gamma)$ for the end subgroup Γ_E, with respect to every end E outside W, then Γ is of divergence type.*

REMARK. The converse of Corollary 8 is not true. Indeed, let R be a Riemann surface of infinite genus that is a cyclic cover of a compact Riemann surface of genus 2 and let Γ be the Fuchsian group corresponding to R. Since we know that R admits no Green function, $\delta(\Gamma) = 1$ and Γ is of divergence type. However, it can be also proved that $\delta(\Gamma_E) = 1$ for any end subgroup Γ_E.

At this stage, we can verify the statement of Theorem 4 as follows.

Proof of Theorem 4. (1) $\Rightarrow$ (4): Suppose that Γ is of divergence type. We take $h(t) \equiv 1$ as a Patterson function for Γ. Let $\Gamma' \subset \Gamma$ be any subgroup with $\Lambda(\Gamma') \subsetneq \Lambda(\Gamma)$. When Γ' is of divergence type, we have $\delta(\Gamma') < \delta(\Gamma)$ by Lemma 5. Hence, the Poincaré series $P_{\delta(\Gamma)}(\Gamma')$ converges. When Γ' is of convergence type, $P_{\delta(\Gamma)}(\Gamma')$ also converges since so does $P_{\delta(\Gamma')}(\Gamma')$ by definition. In any case, $P^h_{\delta(\Gamma)}(\Gamma')$ converges for $h \equiv 1$.

(4) $\Rightarrow$ (5): For this implication, it suffices to see that there exists a compact subsurface W in R such that the end subgroup Γ_E for each end E outside W satisfies $\Lambda(\Gamma_E) \subsetneq \Lambda(\Gamma)$. However, since R has a noncyclic fundamental group, we can always find such W. (5) $\Rightarrow$ (1): This follows directly from Lemma 7. □

REMARK. The implication (4) $\Rightarrow$ (5) is valid only for Fuchsian groups. The other implications are valid for Kleinian groups even in higher dimension.

Finally in this section, we note another sufficient condition for Γ to be of divergence type in terms of the exhaustion of $R = \mathbb{H}^2/\Gamma$. A *canonical exhaustion* of a Riemann surface R is an increasing sequence $\{R_n\}_{n\in\mathbb{N}}$ of compact subsurfaces of R such that each connected component of the complement of R_n is an end having exactly one relative boundary component. It is well known that any topologically infinite Riemann surface R admits a canonical exhaustion (see [1, p. 144]). For the canonical exhaustion of R, the increasing sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ of finitely generated Fuchsian subgroups corresponding to $\pi_1(R_n)$ defines an exhaustion of Γ, that is, $\bigcup_{n\in\mathbb{N}} \Gamma_n = \Gamma$. Then this gives a strictly increasing sequence of the critical exponents

$$\delta(\Gamma_1) < \delta(\Gamma_2) < \cdots < \delta(\Gamma_n) < \cdots < \delta(\Gamma) = \lim_{n\to\infty} \delta(\Gamma_n).$$

The strictness of the inequalities is a consequence of Lemma 5. The continuity $\delta(\Gamma)=\lim_{n\to\infty}\delta(\Gamma_n)$ is a consequence of the lower semicontinuity of the critical exponent under geometric convergence due to Sullivan [16, Corollary 6]. See also [9, Lemma 21]. Hence, we have another corollary to Lemma 7.

COROLLARY 9. *Let* $\{R_n\}_{n\in\mathbb{N}}$ *be a canonical exhaustion of a Riemann surface* $R=\mathbb{H}^2/\Gamma$ *and* $\{\Gamma_n\}_{n\in\mathbb{N}}$ *the corresponding exhaustion of* Γ. *If there exists* $n\in\mathbb{N}$ *such that, for every end* E *outside* R_n, *the end subgroup* Γ_E *satisfies* $\delta(\Gamma_E)\leq\delta(\Gamma_n)$, *then* Γ *is of divergence type. Equivalently, if* Γ *is of convergence type, then, for every* $n\in\mathbb{N}$, *there exists an end* E *outside* R_n *such that* $\delta(\Gamma_E)>\delta(\Gamma_n)$.

This tells us that, if the dominant structure with respect to the critical exponent is concentrated on a compact part of R, then Γ is of divergence type.

3. Quasiconformal noninvariance of the type

We prove quasiconformal noninvariance of convergence and divergence types by constructing a pair of Riemann surfaces.

Proof of Theorem 1. First, we make our Fuchsian group Γ as follows. Let $\hat{R}$ be a complete hyperbolic torus with one hole such that the meridian geodesic c and the boundary-parallel geodesic b have the same length ℓ sufficiently small. Take a cyclic cover R_0 of $\hat{R}$ that has no genus and that has infinitely many lifts of b and c. Let Γ_0 denote the Fuchsian group corresponding to R_0. Take one of the lifts of c in R_0 denoted by c_0, which divides R_0 into two connected components, and consider a subgroup Γ of Γ_0 that corresponds to one of the components. Then set $R=\mathbb{H}^2/\Gamma$ and denote the lifts of b and c in R by $\{b_i\}_{i\geq1}$ and $\{c_i\}_{i\geq0}$ in order. See Figure 1.

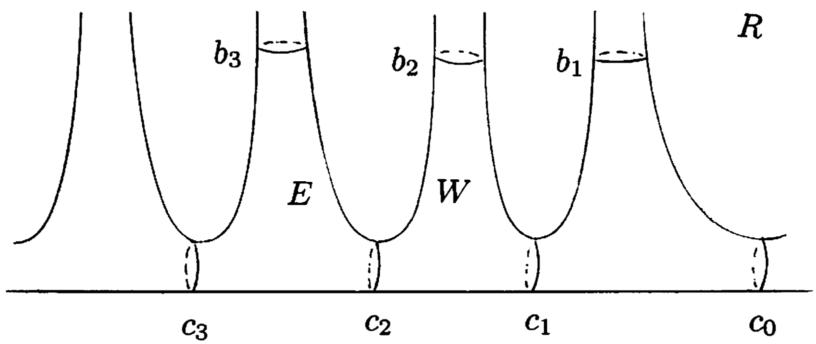

FIGURE 1. The Riemann surface R.

In the above construction, if we take a dividing curve c_2 instead of c_0, we have another subgroup Γ' of Γ_0. This Γ' is conformally conjugate to Γ though it can be regarded as a subgroup of Γ. Actually, Γ' is the end subgroup Γ_E for the end E that is cut off by c_2. Since $\Lambda(\Gamma') \subsetneqq \Lambda(\Gamma)$ in this case, Corollary 6 asserts that Γ is of convergence type.

We will show that the critical exponent of Γ satisfies $\delta(\Gamma) < 1$. By the Elstrodt–Patterson–Sullivan theorem [17] (see also [9, Theorem 17]), the bottom $\lambda(\Gamma)$ of the spectra of the Laplacian on the hyperbolic surface $R = \mathbb{H}^2/\Gamma$ is represented as $\lambda(\Gamma) = \delta(\Gamma)(1-\delta(\Gamma))$ when $\delta(\Gamma) \geq 1/2$ and $\lambda(\Gamma) = 1/4$ otherwise. Also, it is estimated by the isoperimetric constant $h(\Gamma) = \sup A(D)/L(\partial D)$ as $\lambda(\Gamma) \geq 1/\{4h(\Gamma)^2\}$, where the supremum is taken over all relatively compact subdomain D in R with smooth boundary; A and L indicate the hyperbolic area and length, respectively. See [4, IV.3]. If we restrict the above subdomains D to those D^* bounded by simple closed geodesics, we have another isoperimetric constant $h^*(\Gamma) = \sup A(D^*)/L(\partial D^*)$, but they are related as $h^*(\Gamma) \leq h(\Gamma) \leq h^*(\Gamma) + 1$ (see [7] and [10]). Since our Riemann surface R is planar, if D^* is bounded by n geodesic boundary components, then $A(D^*) = 2\pi(n-2)$ by the Gauss–Bonnet formula. Also, since ℓ is chosen to be sufficiently small, we see that $L(\partial D^*) \geq n\ell$. Thus, we have

$$h^*(\Gamma) = \sup_n \frac{2\pi(n-2)}{n\ell} = \frac{2\pi}{\ell}.$$

Applying this estimate to the above inequalities, we have

$$\lambda(\Gamma) \geq \frac{\ell^2}{4(2\pi+\ell)^2} > 0.$$

Finding a constant $d \in (\frac{1}{2}, 1)$ satisfying

$$d(1-d) = \frac{\ell^2}{4(2\pi+\ell)^2} < \frac{1}{4},$$

we conclude that $\delta(\Gamma) \leq d < 1$.

Next, we make a quasiconformal conjugate $\Gamma_\#$ of Γ so that $\Gamma_\#$ is of divergence type. The hyperbolic surface $R_\# = \mathbb{H}^2/\Gamma_\#$ is obtained by deforming R as follows. Let W be a compact subsurface in R bounded by simple closed geodesics c_0, b_1, b_2, and c_2. The end E is outside W. We deform R to $R_\#$ so that the lengths of c_0, b_1 and another simple closed geodesic c_1 become shorter but the lengths of the other b_i and c_i $(i \geq 2)$ remain unchanged. The corresponding subsurfaces with geodesic boundaries in $R_\#$ are denoted by $W_\#$ and $E_\#$, respectively. Note that the end $E_\#$ is conformally equivalent to E. The other ends outside $W_\#$ are annuli, which are negligible. Since the end subgroup Γ_E is conformally conjugate to Γ, so is $\Gamma_{E_\#}$, and hence $\delta(\Gamma_{E_\#}) = \delta(\Gamma) \leq d$. On the other hand, $\delta(\Gamma_\#)$ can be arbitrarily close to 1, and hence greater than d if we make c_0, b_1 and c_1 sufficiently short. This is because, if the lengths of the three boundary geodesics of a pair of pants tends to 0, its geometry approaches that of the three-punctured sphere whose Fuchsian group has the critical exponent 1. Then we have $\delta(\Gamma_{E_\#}) < \delta(\Gamma_\#)$ and, by Corollary 8, we see that $\Gamma_\#$ is of divergence type. □

Remark. In the above example, by changing the length ℓ of all the geodesics b_i and c_i in R, we consider a quasiconformally equivalent family of Riemann surfaces $\{R_\ell\}_{\ell>0}$ and the corresponding Fuchsian groups $\{\Gamma_\ell\}_{\ell>0}$. Note that each Γ_ℓ is of convergence type by the same reason as above. As we will see in the next paragraph, the critical exponents $\delta(\Gamma_\ell)$ move continuously with respect to ℓ. Also, we have $\lim_{\ell\to 0}\delta(\Gamma_\ell) = 1$ by the same reason as above. Hence, there exists some ℓ' such that $\delta(\Gamma_{\ell'}) = \delta(\Gamma_\#)$. This means that $\Gamma_{\ell'}$ and $\Gamma_\#$ are quasiconformally conjugate and have the same critical exponent, but their types are different.

The critical exponent $\delta(\Gamma_\ell)$ is equal to the Hausdorff dimension of the conical limit set of Γ_ℓ. This fact is true for nonelementary Kleinian groups of any dimension by Bishop and Jones [3] (see also [15] and [9, Theorem 1]), but it can be proved relatively easily for Fuchsian groups because any Fuchsian group can be represented as the geometric limit of an increasing sequence of geometrically finite Fuchsian groups (see [16, Corollary 27] and [12, Theorem 9.3.9]). Hence, we can reduce the problem to the continuity of the Hausdorff dimension of the conical limit set of Γ_ℓ. Since the conical limit set is preserved by quasiconformal deformation of a Fuchsian group (see [8, Lemma 7.2]) and since the variation of the Hausdorff dimension is well estimated by the maximal dilatation of a quasiconformal automorphism, we obtain that continuity.

References

[1] L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Univ. Press, 1960. MR 0114911

[2] J. Anderson, K. Falk and P. Tukia, *Conformal measures associated to ends of hyperbolic n-manifolds*, Quart. J. Math. **58** (2007), 1–15. MR 2305045

[3] C. Bishop and P. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math. **179** (1997), 1–39. MR 1484767

[4] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press, Orlando, FL, 1984. MR 0768584

[5] K. Falk and B. Stratmann, *Constructing restricted Patterson measures for geometrically infinite Kleinian groups*, Acta Math. Sinica (Engl. Ser.) **22** (2006), 431–446. MR 2214365

[6] K. Falk and P. Tukia, *A note on Patterson measures*, Kodai Math. J. **29** (2006), 227–236. MR 2247432

[7] J. Fernández and J. Rodríguez, *The exponent of convergence of Riemann surfaces. Bass Riemann surfaces*, Ann. Acad. Sci. Fenn. **15** (1990), 165–183. MR 1050789

[8] K. Matsuzaki, *Ergodic properties of discrete groups; inheritance to normal subgroups and invariance under quasiconformal deformations*, J. Math. Kyoto Univ. **33** (1993), 205–226. MR 1203896

[9] K. Matsuzaki, *Dynamics of Kleinian groups—the Hausdorff dimension of limit sets*, Selected papers on classical analysis, AMS Translation Series (2), vol. 204, Amer. Math. Soc., Providence, RI, 2001, pp. 23–44. MR 1846445

[10] K. Matsuzaki, *Isoperimetric constants for conservative Fuchsian groups*, Kodai Math. J. **28** (2005), 292–300. MR 2153917

[11] K. Matsuzaki and Y. Yabuki, *The Patterson-Sullivan measure and proper conjugation for Kleinian groups of divergence type*, Ergodic Theory Dynam. Systems **29** (2009), 657–665.

[12] P. Nicholls, *The ergodic theory of discrete groups*, LMS Lecture Note Series, vol. 143, Cambridge Univ. Press, 1989. MR 1041575

[13] S. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136** (1976), 241–273. MR 0450547

[14] L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Springer, New York, 1970. MR 0264064

[15] B. Stratmann, *The exponent of convergence of Kleinian groups; on a theorem of Bishop and Jones*, Fractal geometry and stochastics III, Progr. Probab., vol. 57, Birkhäuser, Basel, 2004, pp. 93–107. MR 2087134

[16] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 171–202. MR 0556586

[17] D. Sullivan, *Related aspects of positivity in Riemannian geometry*, J. Differential Geom. **25** (1987), 327–351. MR 0882827

[18] F. Dal'bo, J. Otal and M. Peigné, *Séries de Poincaré des groupes géométriquement finis*, Israel J. Math. **118** (2000), 109–124. MR 1776078

[19] K. Matsuzaki, *A remark on the critical exponent of Kleinian groups*, Analysis and geometry of hyperbolic spaces (Kyoto, 1997). Sūrikaisekikenkyūsho Kōkyūroku **1065** (1998), 106–107. MR 1698110

Katsuhiko Matsuzaki, Department of Mathematics, Okayama University, Tsushima-naka, Okayama 700-8530, Japan

E-mail address: matsuzak@math.okayama-u.ac.jp