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BMO-TEICHMÜLLER SPACES

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ABSTRACT. We show that the complex dilatation of the Douady-Earle extension of a strongly quasisymmetric homeomorphism produces a Carleson measure. As an application, we study the BMO-Teichmüller theory compatible with a Fuchsian group.

1. Introduction

Let G be a Fuchsian group, i.e., a group of Möbius transformations acting properly discontinuously on the unit disk $\mathbb{D}$. For such a group we define $M(G)$ as

$$M(G) = \left\{ \mu \in L^\infty(\mathbb{D}) : \ \|\mu\|_\infty < 1 \, \text{and} \, \forall g \in G, \ \mu = \frac{\overline{g}'}{g'} \mu \circ g \right\}.$$

If $\mu \in M(G)$, then there exists a unique quasiconformal self-mapping f^μ of $\mathbb{D}$ fixing $1, -1, i$ and satisfying

$$\frac{\partial f^\mu}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$$

in $\mathbb{D}$. Similarly there exists a unique quasiconformal homeomorphism of the plane f_μ which is holomorphic outside $\mathbb{D}$ with the normalization

$$f_\mu(z) = z + \frac{b_1}{z} + \cdots$$

at ∞ and such that in $\mathbb{D}$ we have again

$$\frac{\partial f_\mu}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}.$$

These homeomorphisms conjugate G respectively to a new Fuchsian group and to a quasi-Fuchsian group, i.e., a Möbius group acting properly discontinuously on the quasidisk $f_\mu(\mathbb{D})$.

The mapping f^μ has a geometric interpretation: If we denote by S the Riemann surface $\mathbb{D}/G$, then f^μ is the lift (to the universal covering) of a

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quasiconformal mapping from the Riemann surface S onto $S' = \mathbb{D}/G'$, where $G' = f^\mu \circ G \circ (f^\mu)^{-1}$. Conversely, if F is a quasiconformal homeomorphism from S to a Riemann surface S', it has a lift to a quasiconformal homeomorphism f of $\mathbb{D}$ and, replacing if necessary F by $\theta \circ F$, where $\theta : S' \to S''$ is a conformal isomorphism, we may assume that $f = f^\mu$ for some $\mu \in M(G)$.

If $\mu \in M(G)$, then f^μ has a well-defined boundary value which is a quasisymmetric homeomorphism of the unit circle. We define an equivalence relation on $M(G)$ by $\mu \sim \nu$ if $f^\mu|\partial\mathbb{D} = f^\nu|\partial\mathbb{D}$. Again this equivalence relation has a geometric interpretation: If F, G represent the quasiconformal mappings on S whose lifts are precisely f^μ, f^ν, then $\mu \sim \nu$ is equivalent to saying that $G \circ F^{-1}$ is homotopic to a conformal isomorphism between $F(S)$ and $G(S)$, the homotopy being constant on the (possibly empty) boundary of $F(S)$.

The Teichmüller space T_S is the quotient space $M(G)/\sim$. We refer to [11] for details about this construction.

The preceding remarks imply that the mapping $[\mu] \mapsto f^\mu$ is well defined and injective from T_S into $\mathrm{QS}(G)$, the set of quasisymmetric homeomorphisms h of the unit circle such that $h \circ G \circ h^{-1}$ is a Möbius group (more precisely, the trace on the unit circle of a Möbius group). A deep theorem of Tukia [13] asserts that this mapping is also onto, so that one may identify T_S with $\mathrm{QS}(G)$.

There is a similar description of the Teichmüller space in terms of f_μ. We call a quadratic differential for the group G a holomorphic mapping φ in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that

$$\forall g \in G,\ \varphi = \varphi \circ g (g')^2.$$

If $\mu \in M(G)$, it is easy to see that the Schwarzian derivative

$$S_{f_\mu}(z) = (\log f'_\mu)'' - \frac{1}{2}(\log f_\mu)'^2$$

is a quadratic differential for G. In [11] it is shown that the mapping $[\mu] \mapsto S_{f_\mu}$ is well defined and injective on T_S. The image of this mapping is included in $T(G)$, the space of Schwarzian derivatives of injective holomorphic functions in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ having a quasiconformal extension to $\mathbb{C}$ which are quadratic differentials for G. A theorem due to Lehto and Tukia [11] asserts that this mapping is a bijection onto $T(G)$. This is the so-called Bers embedding; it allows us to identify the Teichmüller space T_S with $T(G)$, a space of quadratic differentials.

Both theorems (the identification of T_S with $\mathrm{QS}(G)$ and with $T(G)$) have been given a simplified proof using the Douady-Earle extension theorem.

The aim of this paper is to follow the same idea, i.e., to use the Douady-Earle extension theorem to prove analogs of the above statements in the setting of the BMO-Teichmüller theory, introduced by Astala and the second author [3]. Before stating these results, we recall the basics of this non-standard Teichmüller theory.

2. BMO-Teichmüller theory

A positive measure m in the unit disk is called a Carleson measure if

$$\sup_{I\subset\partial\mathbb{D}\,\text{interval}} m(C(I))/|I| < +\infty,$$

where $C(I)=\{rz:\, z\in I,\, (1-|I|/(2\pi)\leq r\leq 1\}$. We will also need Carleson measures on $\mathbb{C}-\overline{\mathbb{D}}$; the reader will easily guess their proper definition. We then define $\mathrm{CM}(\mathbb{D})$ as the set of measurable functions μ in the unit disk such that

$$\frac{|\mu|^2(z)}{1-|z|}dxdy$$

is a Carleson measure.

An homeomorphism of the unit circle is called strongly quasisymmetric if it is absolutely continuous at every scale, i.e., if

$$\forall\epsilon>0,\ \exists\delta>0;\ \forall I \text{ interval},\ \forall E\subset I \text{ Borel},\ |E|\leq\delta|I|\Rightarrow|h(E)|\leq\epsilon|h(I)|.$$

We denote by SQS the set of strongly quasisymmetric homeomorphisms of the circle. SQS is a group; more precisely, it is the group of homeomorphisms h such that $V_h: b\mapsto b\circ h$ is an isomorphism of the space $\mathrm{BMO}(\partial\mathbb{D})$; see [5], [10]. We recall the definition of this space:

$$\mathrm{BMO}(\partial\mathbb{D})=\{b\in L^2(\partial\mathbb{D});\ \sup_I V_I(b)<+\infty\},$$

where $V_I(b)$ is the variance of b on the interval I.

Naturally a strongly quasisymmetric homeomorphism is quasisymmetric, but the converse is far from being true since a quasisymmetry may be totally singular.

Let us denote by $M(1)$, $T(1)$ the spaces $M(G)$, $T(G)$ for $G=\{I\}$. The following theorem holds:

Theorem 1. *The following are equivalent:*

(1) $\mu\in M(1)\cap\mathrm{CM}(\mathbb{D})$.
(2) $f^\mu\in\mathrm{SQS}(\partial\mathbb{D})$.
(3) $S_{f_\mu}\in T(1)$ *and* $|S_{f_\mu}|^2(|z|-1)^3dxdy$ *is a Carleson measure in* $\mathbb{C}\setminus\overline{\mathbb{D}}$.

The equivalence (1) $\Leftrightarrow$ (2) is essentially due to Fefferman, Kenig and Pipher [8], while (1) $\Leftrightarrow$ (3) is due to Astala and Zinsmeister [3]. The implication (2) $\Rightarrow$ (1) must be interpreted as follows: If $h\in$ SQS then it has a quasiconformal extension to the unit disk whose complex dilatation satisfies (1). It should be noticed that a slight modification of the Beurling-Ahlfors extension does the job [8]. Similarly, the implication (3) $\Rightarrow$ (1) must be understood as follows: If f is holomorphic and injective in $\mathbb{C}\setminus\overline{\mathbb{D}}$ with a qc extension to $\mathbb{C}$ and such that $|S_f|(z)^2(|z|-1)^3dxdy$ is a Carleson measure in $\mathbb{C}\setminus\overline{\mathbb{D}}$, then it has a qc extension whose complex dilatation belongs to $M(1)\cap\mathrm{CM}(\mathbb{D})$.

Let us now consider a Fuchsian group G. Define $\mathcal{M}(G) = M(G) \cap CM(\mathbb{D})$, $SQS(G) = QS(G) \cap SQS(\partial\mathbb{D})$, $\mathcal{T}(G) = \{\varphi \in T(G); |\varphi|^2(z)(|z|-1)^3 dxdy$ is a Carleson measure on $\mathbb{C} \setminus \overline{\mathbb{D}}\}$. The same equivalence relation as in the classical case may be defined on $\mathcal{M}(G)$ and we denote by $\mathcal{T}_S$ the quotient space ($S = \mathbb{D}/G$). As a byproduct of the main result of this paper we will prove the following theorem.

Theorem 2. *The mapping $[\mu] \mapsto f^\mu$ is a bijection from $\mathcal{T}_S$ onto $SQS(G)$ while $[\mu] \mapsto S_{f_\mu}$ is bijective from $\mathcal{T}_S$ onto $\mathcal{T}(G)$.*

In the next section we introduce the Douady-Earle extension and use it to give a proof of this theorem.

We end this section with two comments:

2.1. Motivation for the BMO-Teichmüller theory. The whole Teichmüller theory as just described can be viewed geometrically as follows. In this section we take $G = \{I\}$, so that $S = \mathbb{D}$, and we put $T = T_{\mathbb{D}}$. If $[\mu] \in T$ then f_μ is a Riemann mapping (defined on $\mathbb{C} \setminus \overline{\mathbb{D}}$) on a domain which is a quasiconformal image of the disk and f^μ is then the conformal welding of the boundary of this domain, i.e., $f^\mu = \psi^{-1} o f_\mu$, where ψ is a Riemann mapping from the disk. Very loosely speaking, the theory of the universal Teichmüller space is the theory dealing with quasiconformal geometry. The situation for the BMO-Teichmüller theory is not so clear, but its starting point is the following theorem:

Theorem 3. *The following are equivalent for $\mu \in M(1)$:*

(1) $\exists \nu \in [\mu] \in \mathrm{CM}(\mathbb{D})$ *with a small norm.*
(2) $\log(f^\mu)' \in \mathrm{BMO}(\partial\mathbb{D})$ *with a small norm.*
(3) $(|\zeta|-1)^3 |S_{f_\mu}|^2 d\zeta d\overline{\zeta}$ *is a Carleson measure with small norm.*

These three conditions are equivalent to the fact that if $f_\mu(\partial\mathbb{D})$ passes through ∞ (which we may of course assume), it is the image of a line under a bilipschitz homeomorhism of the plane with constant close to 1. So at least in a neighborhood of the origin BMO-Teichmüller theory deals with bilipschitz geometry.

But this fact ceases to hold in general. In fact, Bishop and Jones [4] have characterized domains arising in Theorem 3 and the corresponding Jordan curves need not be rectifiable. The following question is still open. Let $\mu \in \mathcal{M}(1)$ be such that $f_\mu(\partial\mathbb{D})$ is the bilipschitz image of a circle or a line. Is the same true for $f_{t\mu}(\partial\mathbb{D})$, $0 < t < 1$?

2.2. Groups of convergence type. In contrast to the classical Teichmüller spaces, $\mathcal{T}_S$ can be trivial. More precisely, the latter space is reduced to 0 if and only if Brownian motion is recurrent on S, which is equivalent to the fact that the Fuchsian group G is of divergence type:

$$\sum_{\gamma \in G} (1 - |\gamma(0)|) = +\infty.$$

The reason for this is the two-dimensional version of the Mostow rigidity theorem due to Agard and Pommerenke [1], [12]: If G is of divergence type and if $h \in QS(G)$, then h must be singular. On the other hand, it has been shown in [2] that $\mathcal{T}_S$ is never trivial if G is of convergence type.

3. The Douady-Earle extension theorem

Theorem 4 ([7]). *There exists a map E mapping $QS(\partial\mathbb{D})$ into the set of quasiconformal self-maps of the unit disk such that:*

(1) $\forall h \in QS(\partial\mathbb{D}),\ E(h)|\partial\mathbb{D} = h.$
(2) $\forall h \in SQ(\partial\mathbb{D}),\ \forall \tau, \sigma \in \mathrm{Aut}(\mathbb{D}),\ E(\sigma \circ h \circ \tau) = \sigma \circ E(h) \circ \tau.$

The main step in the construction of $E(h)$ is the following fact that we mention here for later use: If $h \in SQ(\partial\mathbb{D})$ we define the function $F = F_h : \mathbb{D} \times \mathbb{D} \mapsto \mathbb{C}$ by

$$F(z, w) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{h(\zeta) - w}{1 - \overline{w} h(\zeta)} \frac{1 - |z|^2}{|z - \zeta|^2} |d\zeta|.$$

Then for any $z \in \mathbb{D}$ there exists a unique $w \in \mathbb{D}$ such that $F(z, w) = 0$. We define $E(h)(z) = w$. Notice that if $\int h = 0$ then $E(h)(0) = 0$.

Our main result is the following theorem:

Theorem 5. *If $h \in SQS(\partial\mathbb{D})$ then, if μ denotes the complex dilatation of the Douady-Earle extension $E(h)$, it holds that $\mu \in CM(\mathbb{D})$.*

The proof of this theorem will be given in the next section. We end the present section by showing that it implies Theorem 2.

Let us first consider $h \in SQS(G)$. Let μ be the complex dilatation of $E(h)$. It suffices to prove that $\mu \in M(G)$. But if $g \in G$, then, since $E(h \circ g) = E(h) \circ g$ and since $h \in Q(G)$, there exists g_1 Möbius such that $h \circ g = g_1 \circ h$. By a new application of the Douady-Earle theorem, $E(g_1 \circ h) = g_1 \circ E(h)$. It follows that $E(h)$ and $E(h) \circ g$ have the same complex dilatation, but this is equivalent to saying that $\mu \in M(G)$.

For the other part of the theorem we start with a univalent function f on $\mathbb{C} \setminus \overline{\mathbb{D}}$ such that $S_f \in T(G)$ and such that $(|\zeta| - 1)^3 |S_f(\zeta)|^2 d\zeta d\overline{\zeta}$ is a Carleson measure. Let F be a Riemann mapping from $\mathbb{D}$ onto $\mathbb{C} \setminus \overline{f(\mathbb{D})}$ and $h = F^{-1} \circ f$ the conformal welding. By Theorem 1, we have $h \in SQS(\partial\mathbb{D})$. Now the proof in [11, p. 199] gives that $h \in SQS(G)$. Let $F_1 = F \circ E(h)$. Then F_1 is a quasiconformal extension of f whose dilatation is in $\mathcal{M}(G)$. The proof is now complete.

4. Proof of Theorem 5

We adapt methods from [6]. Let h be a homeomorphism of the unit circle and H its harmonic extension to the unit disk. We assume that $\int_{-\pi}^{\pi} h(e^{it})dt = 0$, which implies that $f(0) = 0$, where $f = E(h)$ is the Douady-Earle extension of h. We also assume that h is quasisymmetric and consider a quasiconformal extension g of h. Finally we denote by ν the complex dilatation of g^{-1}.

Proposition 6. *For some universal constant $C > 0$,*

$$\iint_{\mathbb{D}} |\overline{\partial} H|^2 dxdy \leq C \iint_{\mathbb{D}} \frac{|\nu|^2}{1-|\nu|^2} dxdy.$$

Proof. We write $H = H_1 + \overline{H}_2$, where H_1, H_2 are holomorphic on $\mathbb{D}$ and vanish at 0. Then $\partial H = H_1'$, $\overline{\partial} H = \overline{H}_2'$, $|\nabla H|^2 = |\partial H|^2 + |\overline{\partial} H|^2$, $J_H = |\partial H|^2 - |\overline{\partial} H|^2$. The starting point is the inequality

$$\iint_{\mathbb{D}} |\nabla H|^2 dxdy \leq \iint_{\mathbb{D}} |\nabla g|^2 dxdy,$$

which is due to the fact that H is harmonic and that H, g have the same boundary values. On the other hand, by Stokes' formula (or by Choquet's theorem asserting that H is a self-diffeomorphism of $\mathbb{D}$), we also have

$$\iint_{\mathbb{D}} J_H dxdy = \iint_{\mathbb{D}} J_g dxdy = \pi.$$

Combining the two inequalities we get

$$\iint_{\mathbb{D}} |\overline{\partial} H|^2 dxdy \leq \iint_{\mathbb{D}} |\overline{\partial} g|^2 dxdy.$$

But

$$\iint_{\mathbb{D}} |\overline{\partial} g|^2 dxdy = \iint_{\mathbb{D}} \frac{|\overline{\partial} g|^2}{|\partial g|^2 - |\overline{\partial} g|^2} J_g dxdy = \iint_{\mathbb{D}} \frac{|\mu_g|^2}{1-|\mu_g|^2} J_g dxdy.$$

Performing then the change of variable $\zeta = g(z)$ we obtain that this integral is also equal to

$$\iint_{\mathbb{D}} \frac{|\mu_g \circ g^{-1}|^2}{1-|\mu_g \circ g^{-1}|^2} dxdy = \iint_{\mathbb{D}} \frac{|\nu|^2}{1-|\nu|^2} dxdy,$$

since $|\mu_g \circ g^{-1}| = |\mu_{g^{-1}}| = |\nu|$. □

Proposition 7. *There exists a constant $C(K)$ (where K is the constant of quasisymmetry of h) such that*

$$\frac{|\mu_f(0)|^2}{1-|\mu_f(0)|^2} \leq C \iint_{\mathbb{D}} |\overline{\partial} H|^2 dxdy.$$

Proof. We first recall from [7] that $f(z) = w$ is the unique solution of $F(z,w) = 0$, where

$$F(z,w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{it}) - w}{1 - \overline{w}h(e^{it})} \frac{1-|z|^2}{|z - e^{it}|^2} dt.$$

In [7] it is also shown that

$$\begin{aligned}
F_z(0,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it} h(e^{it}) dt = \hat{h}(1),\\
F_{\bar{z}}(0,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it} h(e^{it}) dt = \hat{h}(-1),\\
F_w(0,0) &= -1,\\
F_{\bar{w}}(0,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{it})^2 dt = \frac{1}{\pi}\int_{-\pi}^{\pi} H(e^{it})\overline{H}_2(e^{it}) dt.
\end{aligned}$$

We next compute $|\mu_f(0)|^2/(1 - |\mu_f(0)|^2)$ using the implicit function theorem and the formula $F(z, f(z)) = 0$. We get (writing F_z for $F_z(0,0)$, etc.) the system

$$F_{\bar{z}} + F_{\bar{w}}\bar{f}_{\bar{z}} + F_w f_{\bar{z}} = 0,\ \bar{F}_{\bar{z}} + \bar{F}_w f_{\bar{z}} + \bar{F}_{\bar{w}}\bar{f}_{\bar{z}} = 0,$$

whose solution is

$$f_{\bar{z}} = \frac{\bar{F}_{\bar{z}} F_{\bar{w}} - F_{\bar{z}}\bar{F}_{\bar{w}}}{|F_w|^2 - |F_{\bar{w}}|^2},\ f_z = \frac{\bar{F}_z F_{\bar{w}} - F_z \bar{F}_{\bar{w}}}{|F_w|^2 - |F_{\bar{w}}|^2},$$

and finally

$$\frac{|\mu_f(0)|^2}{1 - |\mu_f(0)|^2} = \frac{|\bar{F}_{\bar{z}} F_{\bar{w}} - F_{\bar{z}}\bar{F}_{\bar{w}}|^2}{(|F_z|^2 - |F_{\bar{z}}|^2)\,(|F_w|^2 - |F_{\bar{w}}|^2)}.$$

First of all, in [7] it was shown that

$$\begin{aligned}
|F_z|^2 - |F_{\bar{z}}|^2 &= |\hat{h}(1)|^2 - |\hat{h}(-1)|^2 > 0,\\
|F_w|^2 - |F_{\bar{w}}|^2 &= 1 - |h(e^{it})^2 dt|/(2\pi) > 0.
\end{aligned}$$

By compactness we deduce the existence of a constant $C(K)$ such that if h is K-qs, then

$$\left(|F_z|^2 - |F_{\bar{z}}|^2\right)\left(|F_w|^2 - |F_{\bar{w}}|^2\right) \geq C(K).$$

From this we deduce

$$\frac{|\mu_f(0)|^2}{1 - |\mu_f(0)|^2} \leq C(K) \left| \overline{\hat{h}(1)} \frac{1}{\pi}\int_0^{2\pi} H(e^{it})\bar{H}_2(e^{it}) dt + \hat{h}(-1)\right|^2.$$

But we have $|\hat{h}(1)| \leq 1$, $|\hat{h}(-1)| \leq \iint |\overline{\partial} H|^2$, and

$$\left|\frac{1}{\pi}\int_0^{2\pi} H(e^{it})\overline{H}_2(e^{it}) dt\right|^2 \leq \frac{1}{\pi}\int_0^{2\pi} \left|H_2(e^{it})\right|^2 dt \leq C \iint_{\mathbb{D}} \left|\overline{\partial} H\right|^2 dxdy,$$

and the proposition is proven. □

PROPOSITION 8. *There exists a constant $C(K)$ such that $\forall z \in \mathbb{D}$,*

$$\frac{\left|\mu_{f^{-1}}(z)\right|^2}{1-\left|\mu_{f^{-1}}(z)\right|^2} \le C(K) \iint_{\mathbb{D}} \frac{\left|\mu_{g^{-1}}(w)\right|^2}{1-\left|\mu_{g^{-1}}(w)\right|^2} \frac{(1-|z|)^2}{|1-\bar{w}z|^4} dudv.$$

Proof. The case $z=0$ follows from Propositions 6 and 7 and from the fact that $|\mu_{f^{-1}}(0)| = |\mu_f(0)|$. For the general case we use

$$M_1(\zeta) = \frac{\zeta + z}{1+\bar{\zeta}z},\ M_2(\zeta) = \frac{\zeta - f(z)}{1-\overline{f}(z)\zeta},$$

so that $M_1(0) = z,\ M_2 \circ f \circ M_1(0) = 0$. Let $F(\zeta) = M_2 \circ f \circ M_1(\zeta)$, $G(\zeta) = M_2 \circ g \circ M_1(\zeta)$. We have

$$|\mu_F(\zeta)| = |\mu_f(M_1(\zeta))|\,,\ |\mu_{G^{-1}}(\zeta)| = \left|\mu_{g^{-1}}(M_2^{-1}(\zeta))\right|.$$

Applying then Proposition 7 we obtain

$$\frac{|\mu_f(z)|^2}{1-|\mu_f(z)|^2} \le C \iint_{\mathbb{D}} \frac{|\mu_{g^{-1}}(M_2^{-1}(w))|^2}{1-|\mu_{g^{-1}}(M_2^{-1}(w))|^2} dudv$$

and, recalling that $\nu = \mu_{g^{-1}}$, we get the bound

$$\begin{aligned} &\le C \iint_{\mathbb{D}} \frac{|\nu(\zeta)|^2}{1-|\nu(\zeta)|^2} |M_2'(\zeta)|^2 d\zeta d\overline{\zeta} \\ &= C \iint_{\mathbb{D}} \frac{|\nu(\zeta)|^2}{1-|\nu(\zeta)|^2} \frac{(1-|f(z)|^2)^2}{|1-\overline{f}(z)\zeta|^4} d\zeta d\overline{\zeta}. \end{aligned}$$

The proposition follows by replacing z by $f^{-1}(z)$. □

THEOREM 9. *Let $h \in SQS(\partial\mathbb{D})$ and $f = E(h)$ its Douady-Earle extension. Then*

$$\frac{|\mu_f(z)|^2}{1-|z|} dxdy$$

is a Carleson measure in the unit disk.

Proof. First of all there exists $M \in \mathrm{Aut}(\mathbb{D})$ such that $M \circ f(0) = 0$. As $M \circ f = E(M \circ h)$ and $\mu_{M\circ f} = \mu_f$ we may assume that $f(0) = 0$.

Next we consider an extension g of h such that

$$\frac{|\mu_g|^2}{1-|z|} dxdy \in CM(\mathbb{D})$$

(for instance the modified Beurling-Ahlfors extension; see [8]).

Lemma 10. *If G is bilipschitz for the hyperbolic metric and if*

$$\frac{|\mu_g|^2}{1-|z|}dxdy \in CM(\mathbb{D}),$$

then the same is true for g^{-1}.

Proof. To simplify the notations we prove the analogous statement for the upper half plane $\mathbb{R}^2_+ = \{y > 0\}$. Let $I \subset \mathbb{R}$ be an interval and $C_I = I \times [0, |I|]$ be the associated Carleson box. Then by an obvious change of variables we get

$$\mathcal{I} = \iint_{C_I} \frac{|\mu_{g^{-1}}(z)|^2}{\Im(z)}dxdy = \iint_{g^{-1}(C_I)} \frac{|\mu_g(\zeta)|^2}{\Im(\zeta)}\frac{\Im(\zeta)}{\Im(g(\zeta))}J_g(\zeta)d\zeta d\overline{\zeta}.$$

But there exists a constant $\alpha = \alpha(K)$ such that

$$C_{\alpha J} \subset g^{-1}(C_I) \subset C_J,\ J = h^{-1}(I),$$

where αJ is the interval with the same center as J, but with length $\alpha|J|$.

On the other hand, by quasiconformality and the fact that g is bilipschitz for the hyperbolic metric,

$$\frac{\Im(\zeta)}{\Im(g(\zeta))}J_g(\zeta) \sim \frac{|h(I_\zeta)|}{|I_\zeta|},$$

where $I(\zeta)$ is the interval $[a, b]$ such that the triangle (a, b, ζ) is equilateral. Let then $\omega = h'$, $\varphi = \omega 1_{2J}$. By standard Carleson-type estimates [9],

$$\mathcal{I} \leq \int_J \varphi^*(x)dx,$$

where φ^* stands for the Hardy-Littlewood maximal function of φ.

By Muckenhoupt theory, there exists $C, p > 1$ such that for any interval J,

$$\frac{1}{|J|}\int_J \omega(x)^p dx \leq C\left(\frac{1}{|J|}\int_J \omega(x)dx\right)^p.$$

We may then write

$$\mathcal{I} \leq |J|^{1/p'}\left(\int_J \varphi^{*p}\right)^{1/p} \leq C|J|^{1/p'}\left(\int_J \omega^p\right)^{1/p} \leq C\int_J \omega = C|I|,$$

from which Lemma 10 follows. □

Lemma 11. *If $A(z)dzd\overline{z}$ is a Carleson measure in $\mathbb{D}$, the same is true for $B(z)dzd\overline{z}$, where*

$$B(z) = \iint_{\mathbb{D}} A(\omega)\frac{(1-|\omega|)(1-|z|)}{|1-\overline{\omega}z|^4}dudv.$$

Proof. Here again we prove the statement for $\mathbb{R}^2_+$. In this case we write $B = T(A)$, where

$$T(A)(x+iy) = \iint_{\mathbb{R}^2_+} A(w)\frac{vy}{|w - x + iy|^4}dudv.$$

By translation invariance it suffices to test the property on intervals $I = [-b, b]$. Furthermore, if $A(z)dxdy \in CM(\mathbb{R}^2_+)$ the same is true for $\lambda A(\lambda z)$ with the same norm. Since $T(\lambda A(\dot{\lambda})) = \lambda^{-1}B(\lambda^{-1}z)$, we only have to show the property for $b = 1/2$. Let $C = [-1/2, 1/2] \times [0, 1]$. Then

$$\iint_C B(x+iy)dxdy = \iint_{\mathbb{R}^2_+} vA(w)\left(\iint_C \frac{y}{|w - x + iy|^4}dxdy\right)dudv = I.$$

We put $\overline{C} = [-1, 1] \times [0, 2]$ and write $I = \mathcal{A} + \mathcal{B} = \iint_{\overline{C}} + \iint_{\mathbb{R}^2_+ \setminus \overline{C}}$. Then

$$\begin{aligned}\mathcal{B} &\leq C \iint_{\mathbb{R}^2_+ \setminus C} \frac{vA(w)}{w^4}dudv \\ &\leq C \sum_{n \geq 1} \iint_{|w| \sim 2^n} \frac{2^n A(w)}{2^{4n}}dudv \\ &\leq C \sum_{n \geq 1} 2^{-2n} \leq C.\end{aligned}$$

To estimate $\mathcal{A}$ it suffices to observe (by a simple computation) that

$$\iint_C \frac{y}{((u-x)^2 + (v+y)^2)^2}dxdy \leq \frac{C}{v}.$$

The proof of the theorem is then completed by applying all preceding propositions and lemmas. $\square$

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