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UNIQUENESS OF STARSHAPED COMPACT HYPERSURFACES WITH PRESCRIBED m-TH MEAN CURVATURE IN HYPERBOLIC SPACE

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Abstract. Let ψ be a given function defined on a Riemannian space. Under what conditions does there exist a compact starshaped hypersurface M for which ψ, when evaluated on M, coincides with the m-th elementary symmetric function of principal curvatures of M for a given m? The corresponding existence and uniqueness problems in Euclidean space have been investigated by several authors in the mid 1980s. Recently, conditions for existence were established in elliptic space and, most recently, for hyperbolic space. However, the uniqueness problem has remained open. In this paper we investigate the problem of uniqueness in hyperbolic space and show that uniqueness (up to a geometrically trivial transformation) holds under the same conditions under which existence was established.

1. Introduction

In Euclidean space $\mathbb{R}^{n+1}$ fix a point O and let $\mathbb{S}^n$ be the unit sphere centered at O. Let u denote a point on $\mathbb{S}^n$ and let (u, ρ) be the spherical coordinates in $\mathbb{R}^{n+1}$ with the origin at O. The standard metric on $\mathbb{S}^n$ induced from $\mathbb{R}^{n+1}$ is denoted by e. Let $I = [0, a)$, where $a = \text{const},\ 0 < a \leq \infty$, and $f(\rho)$ a C^∞ function on I, positive on $(0, a)$ and such that $f(0) = 0$. Introduce in $\mathbb{S}^n \times I$ the metric

$$h = d\rho^2 + f(\rho)e \tag{1}$$

and consider the resulting Riemannian space. When $a = \infty$ and $f(\rho) = \rho^2$ this space is the Euclidean space $\mathbb{R}^{n+1} \equiv \mathbb{R}^{n+1}(0)$, when $a = \infty$ and $f(\rho) = \sinh^2 \rho$ it is the hyperbolic space $\mathbb{R}^{n+1}(-1)$ with sectional curvature -1, and when $a = \pi/2,\ f(\rho) = \sin^2 \rho$, it is the elliptic space $\mathbb{R}^{n+1}(1)$ with sectional curvature $+1$. We use the notation $\mathbb{R}^{n+1}(K),\ K = 0, \pm 1$, for either of these spaces.

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Let M be a hypersurface in $\mathbb{R}^{n+1}(K)$ and m, $1 \leq m \leq n$, an integer. The m-th mean curvature, $H_m(\lambda) \equiv H_m(\lambda_1, \ldots, \lambda_n)$, of M is the normalized elementary symmetric function of order m of the principal curvatures $\lambda_1, \ldots, \lambda_n$ of M, that is,

$$H_m(\lambda) = \binom{n}{m}^{-1} \sum_{i_1 < \cdots < i_n} \lambda_{i_1} \cdots \lambda_{i_m}.$$

The subject of this paper is the following problem. Let $\psi(u, \rho),\ u \in \mathbb{S}^n,\ \rho \in I$, be a given positive function and $m,\ 1 \leq m \leq n$, a given integer. Under what conditions on ψ does there exist a smooth hypersurface M in $\mathbb{R}^{n+1}(K)$ given as $(u, z(u)),\ u \in \mathbb{S}^n,\ z > 0$, for which

$$H_m(\lambda(z(u))) = \psi(u, z(u)) \text{ for all } u \in \mathbb{S}^n? \tag{2}$$

If such a hypersurface exists, what are the conditions for uniqueness?

In analytic form this problem consists in establishing existence and uniqueness of solutions for a second order nonlinear partial differential equation on $\mathbb{S}^n$ expressing H_m in terms of z. When $m = 1$, this equation is quasilinear, and for $m > 1$ it is fully nonlinear. In particular, when $m = n$ it is of Monge-Ampère type. In Euclidean space $\mathbb{R}^{n+1}(0)$ this problem was investigated and conditions for existence and uniqueness were given by I. Bakelman and B. Kantor [2], [3] and A. Treibergs and S.W. Wei [12] when $m = 1$ (the mean curvature case), by V. Oliker [10] when $m = n$ (the Gauss curvature case), and by L. Caffarelli, L. Nirenberg and J. Spruck [6] when $1 < m < n$.

In [11] V. Oliker investigated the problem for hypersurfaces in $\mathbb{R}^{n+1}(-1)$ and $\mathbb{R}^{n+1}(1)$ when $m = n$ and gave conditions for existence and uniqueness. In [4] L. Barbosa, J. Lira and V. Oliker obtained C^0, C^1 and C^2 estimates for solutions of (2) for the elliptic space form $\mathbb{R}^{n+1}(1)$ for any $m,\ 1 \leq m \leq n$, and then, in [9], Y. Y. Li and V. Oliker, used these estimates and degree theory for fully nonlinear elliptic operators [8] to prove the existence of solutions. In the same paper [4], the authors also obtained C^0 and C^1 estimates for any $m,\ 1 \leq m \leq n$, in the hyperbolic space $\mathbb{R}^{n+1}(-1)$. Recently, Q. Jin and Y. Y. Li [7] obtained C^2 estimates for $\mathbb{R}^{n+1}(-1)$ and proved the existence for this case as well. The main results in [9] and [7] can be formulated together as follows.

Denote by Γ_m the connected component of $\{\lambda \in \mathbb{R}^n \mid H_m(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_n > 0\}$.

Definition 1.1. A positive function $z \in C^2(\mathbb{S}^n)$ is m-admissible for the operator H_m if the corresponding hypersurface $M = (u, z(u)),\ u \in \mathbb{S}^n$, is such that at every point of M the principal curvatures $(\lambda_1(z(u)), \ldots, \lambda_n(z(u)))$, where the λ_i are calculated with respect to the inner normal, belong to Γ_m.

Theorem 1.2. *Let $1 \leq m \leq n,\ K = \pm 1$, and $\psi(u, \rho)$ is a positive smooth function on the annulus $\bar{\Omega} \subset \mathbb{R}^{n+1}(K),\ \bar{\Omega}:\ u \in \mathbb{S}^n,\ \rho \in [R_1, R_2]$, where*

$0 < R_1 < R_2 < a$, and $a = \infty$ for $\mathbb{R}^{n+1}(-1)$ and $a = \pi/2$ for $\mathbb{R}^{n+1}(1)$. Suppose ψ satisfies the following conditions:

If $K = -1$,

$$\psi(u, R_1) \geq \coth^m R_1 \text{ for } u \in \mathbb{S}^n, \tag{3}$$

$$\psi(u, R_2) \leq \coth^m R_2 \text{ for } u \in \mathbb{S}^n, \tag{4}$$

and

$$\frac{\partial}{\partial \rho}\left[\psi(u, \rho) \sinh^m \rho\right] \leq 0 \text{ for all } u \in \mathbb{S}^n \text{ and } \rho \in [R_1, R_2]. \tag{5}$$

If $K = 1$,

$$\psi(u, R_1) \geq \cot^m R_1 \text{ for } u \in \mathbb{S}^n, \tag{6}$$

$$\psi(u, R_2) \leq \cot^m R_2 \text{ for } u \in \mathbb{S}^n, \tag{7}$$

and

$$\frac{\partial}{\partial \rho}\left[\psi(u, \rho) \cot^{-m} \rho\right] \leq 0 \text{ for all } u \in \mathbb{S}^n \text{ and } \rho \in [R_1, R_2]. \tag{8}$$

Then there exists a closed, smooth, embedded hypersurface M in $\mathbb{R}^{n+1}(K)$, $M \subset \bar{\Omega}$, which is a radial graph over $\mathbb{S}^n$ of an m-admissible function z and

$$H_m(\lambda(z(u))) = \psi(u, z(u)) \text{ for all } u \in \mathbb{S}^n. \tag{9}$$

Similar to the case of $\mathbb{R}^{n+1}(1)$ the proof in [7] uses degree theory. The degree theory arguments in [9] and [7] do not provide an answer to the uniqueness problem. Thus for elliptic and hyperbolic space forms this question remained open except in the case $m = n$; see [11]. The purpose of this paper is to prove uniqueness for hyperbolic space forms for all m, $1 \leq m \leq n$, under the same conditions as in Theorem 1.2. Namely, we have the following result:

Theorem 1.3. *Let $K = -1$. Then under condition (5) in Theorem 1.2 any two hypersurfaces defined by m-admissible solutions z^1 and z^2 of (9) in $\bar{\Omega}$ are related by the transformation*

$$c \tanh\left(\frac{z_1(u)}{2}\right) = \tanh\left(\frac{z_2(u)}{2}\right), \ u \in \mathbb{S}^n, \tag{10}$$

where c is a positive constant. If the inequality (5) is strict, then $c = 1$, that is, the hypersurface M in Theorem 1.2 is unique.

For $m = n$ the condition (5) is slightly less restrictive than condition c) in Theorem 1.1 of [11]. For elliptic space forms the uniqueness problem is still open, except in the case $m = n$, where condition (8) also implies uniqueness.

2. The equation of the problem

In this section we present some local formulas and lemmas valid in $\mathbb{R}^{n+1}(K)$, where $K = \pm 1$. Though our main result (Theorem 1.2) applies only to the case $K = -1$, it seems worthwhile to record here the results which are also valid for the case $K = +1$ because they may be useful in future studies of similar problems. Furthermore, the presentation in this section is carried out in a unified way simultaneously for both cases.

2.1. The main equation. First we fix our notation. Unless explicitly stated otherwise, the range for the Latin indices is $1, \ldots, n$. The summation convention over repeated lower and upper indices is assumed to be in effect. Denote by $(u^1, \ldots, u^n) = u$ smooth local coordinates on $\mathbb{S}^n$ and let $\partial_i = \partial/\partial u^i$, $i = 1, 2, \ldots, n$, be the corresponding local frame of tangent vectors such that $e(\partial_i, \partial_j) = e_{ij}$. The first covariant derivative of a function $v \in C^2(\mathbb{S}^n)$ is given by $v_i \equiv \nabla'_i v = \partial_i v$. Put $(e^{ij}) = (e_{ij})^{-1}$ and let

$$\nabla' v = v^i \partial_i, \text{ where } v^i = e^{ij} v_j.$$

For the covariant derivative of $\nabla' v$ we have

$$\nabla'_{\partial_s} \nabla' v = v_{sj} e^{ji} \partial_i + v_j \nabla'_{\partial_s}(e^{ij}\partial_i) = (v_{sj} - \Gamma'^i_{sj} v_i) e^{jk} \partial_k,$$

where

$$v_{sj} = \frac{\partial^2 v}{\partial u^s \partial u^j}$$

and Γ'^i_{sj} are the Christoffel symbols of the second kind of the metric e. The second covariant derivatives of v are defined by

$$\nabla'_{sj} v = v_{sj} - \Gamma'^i_{sj} v_i. \tag{11}$$

Next we recall some of the basic formulas derived in [4]. Let M be a hypersurface in $\mathbb{R}^{n+1}(K)$ given by $r(u) = (u, z(u))$, $u \in \mathbb{S}^n$, where $z \in C^2(\mathbb{S}^n)$ and z is positive on $\mathbb{S}^n$. The metric $g = g_{ij} du^i du^j$ induced on M from $\mathbb{R}^{n+1}(K)$ has coefficients

$$g_{ij} = f e_{ij} + z_i z_j \quad \text{and} \quad \det(g_{ij}) = f^{n-1}(f + |\nabla' z|^2) \det(e_{ij}). \tag{12}$$

The elements of the inverse matrix $(g^{ij}) = (g_{ij})^{-1}$ are

$$g^{ij} = \frac{1}{f}\left[e^{ij} - \frac{z^i z^j}{f + |\nabla' z|^2}\right]. \tag{13}$$

With the choice of the normal on M in inward direction the second fundamental form b of M has coefficients

$$b_{ij} = \frac{f}{\sqrt{f^2 + f|\nabla' z|^2}}\left[-\nabla'_{ij} z + \frac{\partial \ln f}{\partial \rho} z_i z_j + \frac{1}{2}\frac{\partial f}{\partial \rho} e_{ij}\right]. \tag{14}$$

Note that the second fundamental form of a sphere $z = \text{const} > 0$ is positive definite, since for $\mathbb{R}^{n+1}(K)$ $\partial f/\partial \rho > 0$.

The principal curvatures of M at a point $(u, z(u))$ are the eigenvalues of the second fundamental form b relative to the metric g and are the real roots, $\lambda_1(z(u)), \ldots, \lambda_n(z(u))$, of the equation

$$\det(b_{ij}(z(u)) - \lambda g_{ij}(z(u)) = 0$$

or, equivalently, of

$$\det(a^i_j(z(u)) - \lambda\delta^i_j) = 0,$$

where

$$a^i_j = g^{ik}b_{kj}, \tag{15}$$

is a self-adjoint transformation of the tangent space to M at $(u, z(u))$. The elementary symmetric function of order m, $1 \leq m \leq n$, of the principal curvatures is defined by

$$S_m(\lambda) = \sum_{i_1<\cdots<i_n} \lambda_{i_1}\cdots\lambda_{i_m} \text{ and } S_m(\lambda) = \binom{n}{m} H_m(\lambda) = F_m(a^i_j), \tag{16}$$

where F_m is the sum of principal minors of (a^i_j) of order m. Evidently,

$$F_m(a^i_j(z(u))) \equiv F(u, z, \nabla'_1, \ldots, \nabla'_n z, \nabla'_{11}z, \ldots, \nabla'_{nn}z), \tag{17}$$

and the equation (9) assumes the form

$$S_m(\lambda(z(u))) \equiv F_m(a^i_j(z(u))) = \bar{\psi}(u, z(u)), \tag{18}$$

where $\bar{\psi} \equiv \binom{n}{m}\psi$.

2.2. The conformal model of $\mathbb{R}^{n+1}(K)$ and a change of the function z. For the function $f(\rho)$, $\rho \in I$, in (1) corresponding to $\mathbb{R}^{n+1}(-1)$ or $\mathbb{R}^{n+1}(+1)$ we put

$$s(\rho) = \sqrt{f(\rho)},\ c(\rho) = \frac{ds(\rho)}{d\rho},\ t(\rho) = \frac{s(\rho)}{c(\rho)}.$$

It will be convenient to transform the function z in (17) to a function v defined by[1] $v(u) = t(z(u)/2)$. Put

$$q = \frac{2}{1 + Kv^2}.$$

Then

$$z_i = qv_i,\ \nabla'_{ij}z = q\nabla'_{ij}v - Kq^2vv_iv_j. \tag{19}$$

Put

$$\hat{g}_{ij}(v) = v^2e_{ij} + v_iv_j,\ \hat{g}^{ij}(v) = \frac{1}{v^2}\left(e^{ij} - \frac{v^iv^j}{W^2(v)}\right),\ W(v) = \sqrt{v^2 + |\nabla'v|^2}.$$

[1]This is equivalent to re-writing (1) in the conformal model of the corresponding space form in the unit ball in Euclidean space $\mathbb{R}^{n+1}$ centered at the origin.

Substitution into (13) gives

$$g^{ij}(v) = \frac{1}{q^2}\hat{g}^{ij}(v)$$

and substitution into (14) gives

$$b_{ij}(v) = q\hat{b}_{ij}(v) - Kq^2v^2\frac{\hat{g}_{ij}(v)}{W(v)},$$

where

$$\hat{b}_{ij}(v) = \frac{-v\nabla'_{ij}v + 2v_iv_j + v^2e_{ij}}{W(v)}. \tag{20}$$

Note that $\hat{g}$ and $\hat{b}$ are, respectively, the first and second fundamental forms in the Euclidean sense of the hypersurface which is a graph of v over $\mathbb{S}^n$ in the unit ball [10]. Finally, we obtain

$$a^i_j(v) = g^{ik}(v)b_{kj}(v) = \frac{\hat{a}^i_j(v)}{q} - K\frac{v^2\delta^i_j}{W(v)}, \text{ where } \hat{a}^i_j(v) = \hat{g}^{ik}(v)\hat{b}_{kj}(v). \tag{21}$$

For an m-admissible function $z \in C^2(\mathbb{S}^n)$ and $v = t(z/2)$ consider the family of functions sv, where $s > 0$ and $sv < 1$. Then

$$a^i_j(sv) = \frac{1 + Ks^2v^2}{2s}\hat{a}^i_j(v) - K\frac{sv^2}{W(v)}\delta^i_j. \tag{22}$$

Define, as before, the eigenvalues $\lambda_i(sv(u)), i = 1, \ldots, n$, of $(b_{ij}(sv(u)))$ with respect to $(g_{ij}(sv(u)))$ (which is positive definite) and consider the corresponding m-th elementary symmetric function $S_m(\lambda(sv(u)))$. Clearly, since z is m-admissible, the function v is m-admissible, that is, $\lambda(v(u)) \in \Gamma_m$.

Lemma 2.1. *Let z, v and s be as above. Put*

$$A(sv) = \frac{1 + Ks^2v^2}{s(1 + Kv^2)}, \; B(sv) = K\frac{(1 - s^2)v^2}{s(1 + Kv^2)W(v)}.$$

Then

$$\begin{aligned} S_m(\lambda(sv)) = {} & A^m(sv)S_m(\lambda(v)) \\ & + \sum_{j<m} c(n, m, j)A^j(sv)B^{m-j}(sv)S_j(\lambda(v)), \end{aligned} \tag{23}$$

where $c(n, m, j)$ are positive coefficients. Furthermore, if $K = -1$ and $s \geq 1$ or if $K = +1$ and $s \leq 1$, then

$$S_m(\lambda(sv(u))) \geq A^m(sv(u))S_m(\lambda(v(u))). \tag{24}$$

In particular, sv is m-admissible for H_m in $\mathbb{R}^{n+1}(K)$ for the corresponding choice of s.

Proof. It follows from (21) and (22) that

$$a^i_j(sv) = A(sv)a^i_j(v) + B(sv)\delta^i_j.$$

Since v is m-admissible, $S_j(v) > 0$ for each $j \le m$ (see [5]) and $A(sv) > 0$ because $sv < 1$. On the other hand, $B(sv) \ge 0$ with each choice of s as in the statement of the lemma. Then $S_m(\lambda(sv)) > 0$ in both cases. Because sv is a positive multiple of $v \in \Gamma_m$, we conclude that $sv \in \Gamma_m$ in both cases. □

We complete this section with the following lemma:

LEMMA 2.2. *Let z be m-admissible for the operator H_m and $v = t(z/2)$. Then the operator $F_m(a^i_j(v))$ is negatively elliptic on v on $\mathbb{S}^n$.*

Proof. In order to show that $F_m(a^i_j(v))$ is negatively elliptic we need to show that at any point of $\mathbb{S}^n$ we have

$$\frac{\partial F_m(a^i_j(v))}{\partial \nabla'_{ij} v}\xi^i\xi^j < 0 \text{ for all } \xi \in \mathbb{R}^n,\ \xi \ne 0. \tag{25}$$

It follows from (21) and (20) that

$$\frac{\partial F_m}{\partial \nabla'_{ij} v} = -\frac{v}{q(v)W(v)}F^j_i, \text{ where } F^j_i = \frac{\partial F_m}{\partial a^i_j(v)}. \tag{26}$$

Thus, we need to consider the matrix (F^j_i).

Fix an arbitrary point $u_0 \in \mathbb{S}^n$ and diagonalize at that point the metric $(g_{ij}(v))$ and the second fundamental form $(b_{ij}(v))$ using the orthonormal set of principal directions as a basis. Then at u_0 we have $g_{ij}(v) = \delta_{ij}$,

$$b_{ij}(v) = \begin{cases} \lambda_i(v) & \text{when } i = j, \\ 0 & \text{when } i \ne j, \end{cases}$$

and

$$a^i_j(v) = \begin{cases} \lambda_i(v) & \text{when } i = j, \\ 0 & \text{when } i \ne j. \end{cases}$$

Therefore at u_0 we have

$$F^j_i = 0 \text{ when } i \ne j \text{ and } F^i_i = \frac{\partial S_m(\lambda_1(v), \ldots, \lambda_n(v))}{\partial \lambda_i(v)}, \tag{27}$$

where no summation over i is performed. Since z, and therefore v, are m-admissible for H_m, it follows that $S_m(\lambda_1(v), \ldots, \lambda_n(v)) > 0$. Then, by a well known property of elementary symmetric functions, $\partial S_m/\partial\lambda_i(v) > 0$ for each $i = 1, 2, \ldots, n$; see [5]. Now, (25) follows from (27) and (26). □

3. Proof of Theorem 1.3

In this section we work in the hyperbolic space $\mathbb{R}^{n+1}(-1)$.

Let z_1 and z_2 be two different m-admissible solutions of (9) and M_1, M_2 the corresponding hypersurfaces on the annulus $\bar{\Omega}$. It follows from Lemma 4.1 (see the Appendix) that for any m-admissible solution z of (9) such that $R_1 \leq z(u) \leq R_2$ we have either $z(u) \equiv R_1$ or $z(u) \equiv R_2$ or

$$R_1 < z(u) < R_2 \text{ for all } u \in \mathbb{S}^n. \tag{28}$$

Assume first that

$$R_1 \leq z_k(u) < R_2 \quad \text{for } k = 1, 2 \text{ and for all } u \in \mathbb{S}^n. \tag{29}$$

The case when $z_1 \equiv R_1$ or $z_2 \equiv R_2$ is special and will be treated separately at the end of the proof.

Let $v_k(u) = t(z_k(u)/2)$, where now $t(z_k(u)/2) = \tanh(z_k(u)/2)$. Suppose $v_1 < v_2$ somewhere on $\mathbb{S}^n$; otherwise re-label v_1 and v_2. Multiply v_1 by $s \geq 1$ such that

$$sv_1(u) < 1,\ sv_1(u) \geq v_2(u) \text{ for all } u \in \mathbb{S}^n \text{ and } sv_1(\bar{u}) = v_2(\bar{u}) \text{ at some } \bar{u} \in \mathbb{S}^n.$$

By (29) there exists a neighborhood $U \subset \mathbb{S}^n$ of the point $\bar{u}$ such that $z^s = 2t^{-1}(sv_1)$ satisfies the inequality

$$R_1 < z^s(u) < R_2,\ u \in U.$$

Since $S_m(\lambda(z_1)) = \bar{\psi}(u, z_1(u))$, it follows from Lemma 2.1 that in U

$$\begin{aligned} S_m(\lambda(sv_1)) - \bar{\psi}(u, 2t^{-1}(sv_1)) \geq A^m(sv_1)\bar{\psi}(u, 2t^{-1}(v_1)) \\ - \bar{\psi}(u, 2t^{-1}(sv_1)). \end{aligned} \tag{30}$$

Put $sv_1 = \tilde{v}$. Then, using the explicit expression for $A(sv_1)$ and taking into account that $K = -1$, we get

$$\begin{aligned} S_m(\lambda(\tilde{v})) - \bar{\psi}(u, 2t^{-1}(\tilde{v})) \geq \left[\frac{1-\tilde{v}^2}{s(1-\frac{\tilde{v}^2}{s^2})}\right]^m \bar{\psi}\left(u, 2t^{-1}\left(\frac{\tilde{v}}{s}\right)\right) \\ - \bar{\psi}(u, 2t^{-1}(\tilde{v})). \end{aligned} \tag{31}$$

Put

$$Q(s) = \left[\frac{1-\tilde{v}^2}{s(1-\frac{\tilde{v}^2}{s^2})}\right]^m \bar{\psi}\left(u, 2t^{-1}\left(\frac{\tilde{v}}{s}\right)\right) - \bar{\psi}(u, 2t^{-1}(\tilde{v})).$$

Note that $Q(1) \equiv 0$. We have

$$\begin{aligned}\frac{\partial Q}{\partial s} &= -m\left[\frac{1-\tilde{v}^2}{s(1-\frac{\tilde{v}^2}{s^2})}\right]^{m-1}\frac{(1-\tilde{v}^2)(1+\frac{\tilde{v}^2}{s^2})}{s^2(1-\frac{\tilde{v}^2}{s^2})^2}\bar{\psi}\left(u, 2t^{-1}\left(\frac{\tilde{v}}{s}\right)\right)\\
&\quad-\left[\frac{1-\tilde{v}^2}{s(1-\frac{\tilde{v}^2}{s^2})}\right]^{m}\frac{2\tilde{v}}{s^2(1-\frac{\tilde{v}^2}{s^2})}\bar{\psi}_z\left(u, 2t^{-1}\left(\frac{\tilde{v}}{s}\right)\right)\\
&= -\frac{1}{s^{m+1}}\left[\frac{1-\tilde{v}^2}{1-\frac{\tilde{v}^2}{s^2}}\right]^m\left[m\frac{1+\frac{\tilde{v}^2}{s^2}}{1-\frac{\tilde{v}^2}{s^2}}\bar{\psi}\left(u, 2t^{-1}\left(\frac{\tilde{v}}{s}\right)\right)\right.\\
&\quad\left.+\frac{2\tilde{v}}{s(1-\frac{\tilde{v}^2}{s^2})}\bar{\psi}_z\left(u, 2t^{-1}\left(\frac{\tilde{v}}{s}\right)\right)\right] \geq 0,\end{aligned}$$

where $\bar{\psi}_z = \partial\bar{\psi}/\partial z$. The last inequality on the right follows from (5).

By (31), (30) and the assumption that z_2 is an m-admissible solution of (9) we have

$$F_m(a^i_j(\tilde{v})) - \bar{\psi}(u, 2t^{-1}(\tilde{v})) \geq 0 = F_m(a^i_j(v_2)) - \bar{\psi}(u, 2t^{-1}(v_2)).$$

Since F_m is negatively elliptic, $\tilde{v} \geq v_2$ in U and $\tilde{v}(\bar{u}) = v_2(\bar{u})$, we conclude from the geometric form of Aleksandrov's maximum principle [1] that $\tilde{v} \equiv v_2$ in U. By continuity, the set

$$\{u \in \mathbb{S}^n \mid \tilde{v}(u) = v_2(u)\}$$

is open and closed on $\mathbb{S}^n$. Hence, $\tilde{v}(u) = v_2(u) = sv_1(u)$ everywhere on $\mathbb{S}^n$ and the proof of uniqueness is complete in this case.

Suppose now that $z_2 \equiv R_2$ and $z_1 < R_2$ for all $u \in \mathbb{S}^n$. In this case we extend $\bar{\psi}(u,\rho)$ smoothly for $\rho > R_2$ satisfying conditions

$$\psi(u,\rho) \leq \coth^m R_2 \text{ for } u \in \mathbb{S}^n, \tag{32}$$

and

$$\frac{\partial}{\partial\rho}\left[\psi(u,\rho)\sinh^m\rho\right] \leq 0 \text{ for all } u \in \mathbb{S}^n \text{ and } \rho \geq R_2, \tag{33}$$

and we can apply the same arguments. This completes the proof of the theorem.

4. Appendix

Lemma 4.1. *Assume that the conditions of the Theorem 1.2 are satisfied except for conditions (5) and (8). Then an m-admissible solution z of (9) such that $R_1 \leq z(u) \leq R_2$ is either $\equiv R_1$ or $\equiv R_2$, or satisfies*

$$R_1 < z(u) < R_2 \textit{ for all } u \in \mathbb{S}^n. \tag{34}$$

This lemma was stated in [4] without a detailed proof. At the suggestion of the referee we provide a proof here. The proof consists in showing that the conditions of Aleksandrov's maximum principle [1] are satisfied.

Proof. Suppose, on the contrary, that there exists some $u_0 \in \mathbb{S}^n$ such that $z(u_0) = R_2$ and $z(u) \not\equiv R_2$. (The case when $z(u_0) = R_1, z(u) \not\equiv R_1$, is treated similarly.) Then z attains a maximum at u_0. Consider the family of functions

$$z(s) = (1-s)z + sR_2, \ \ s \in [0,1].$$

Obviously, $z(s)$ also attains a maximum equal to R_2 at u_0 for all $s \in [0,1]$. We will need an expression for the m-th elementary symmetric function of the hypersurface $M(s)$ defined by $z(s)$ at u_0. We have

$$\nabla' z(s) = 0 \text{ and } \nabla'_{ij}(z(s)) = (1-s)z_{ij} \text{ at } u_0, \ \ s \in [0,1]. \tag{35}$$

Put

$$\mu = \frac{1}{2f(R_2)} \frac{\partial f(z(s))}{\partial z}\bigg|_{z(s)=R_2}$$

and observe that $\mu > 0$, since $0 < R_2 < a$ and $\partial f/\partial \rho > 0$ in $\bar{\Omega}$. Using (13), (14) and (11) and noting that

$$a^i_j(R_2) = \mu \delta^i_j,$$

we obtain at the point u_0

$$a^i_j(z(s)) = \frac{e^{ik}}{f(R_2)}\left[-(1-s)z_{kj} + \mu e_{kj}\right] = (1-s)a^i_j(z) + sa^i_j(R_2).$$

Then

$$S_m(\lambda(z(s)))|_{u_0} = \sum_{p=0}^{m} (1-s)^p (\mu s)^{m-p} S_p(\lambda(z))|_{u_0}, \tag{36}$$

where $S_0 = \binom{n}{m}$. Since $S_p(\lambda(z)) > 0$ for all $p \le m$ and $\mu > 0$, it follows that $S_m(\lambda(z(s)))|_{u_0} > 0$ for all $s \in [0,1]$. By continuity, $S_m(\lambda(z(s))) > 0$ in some neighborhood U_0 of u_0 in $\mathbb{S}^n$. Then, by [5], $\partial S_m/\partial \lambda_i(z(s)) > 0$, and by shrinking U_0, if necessary, we have $\partial S_m/\partial \lambda_i(z(s)) \ge C > 0$ for all $u \in U_0$ with some fixed constant C. It follows now from Lemma 2.2 and the second expression in (19) that $-F_m(a^i_j(z(s))$ is positively elliptic in U_0 for all $s \in [0,1]$.

The above arguments establish that the function $-F_m(a^i_j(z(s)) + \bar{\psi}(u, z(s))$ satisfies the conditions (1)–(4) in §1 of [1]. (Note that our orientation of $M(s)$ is opposite to that in [1].) We need to check one more inequality. Namely, since z satisfies (9) on $\mathbb{S}^n$, we have

$$-F_m(a^i_j(z) + \bar{\psi}(u, z) = 0,$$

while, taking into account (4) for the hyperbolic space or (7) for the elliptic space, we also have

$$-F_m(a^i_j(R_2) + \bar{\psi}(u, R_2)) \leq 0.$$

Since, also, $z(u) \leq R_2$ on $\mathbb{S}^n$ and $z(u_0) = R_2$, it follows from the maximum principle in [1] that $z(u) = R_2$ everywhere on U_0. This implies that the set $\{u \in \mathbb{S}^n \mid z(u) = R_2\}$ is open on $\mathbb{S}^n$. Since it is also closed, we conclude that $z(u) = R_2$ everywhere on $\mathbb{S}^n$. $\square$

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