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ON HARMONIC NORMAL AND $Q_p^{\#}$ FUNCTIONS

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Abstract. We investigate relationships between classes of harmonic functions corresponding to the meromorphic $Q_p^{\#}$ classes. We give many analogs to the situations in the corresponding analytic and meromorphic classes, and we give some examples in which the behavior is different in the harmonic classes.

1. Introduction

Let C denote the complex plane; W the Riemann sphere; D denote the unit disk $\{z \in C\colon |z| < 1\}$; and Σ the collection of all one-to-one conformal mappings of D onto itself. If f is a meromorphic function in D, we say that f is a normal function if the family $F = \{f(\gamma(z))\colon \gamma \in \Sigma\}$ is a normal family. We denote the family of all normal meromorphic functions by N. There is a related subfamily, the so-called "little normal functions", defined by

$$N_0 = \{f\colon f \text{ meromorphic in } D \text{ and } \lim_{|z|\to 1}(1-|z|^2)f^{\#}(z) = 0\},$$

where $f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative of f.

If u is a function which is harmonic and real-valued in D, we say that u is a *normal harmonic function* if the family $F = \{u(\gamma(z))\colon \gamma \in \Sigma\}$ is a normal family. It is consequence of this definition that if u is a normal harmonic function, and if f is the analytic function $f(z) = u(z) + iv(z)$, where $v(z)$ is a harmonic conjugate of $u(z)$, then f is a normal (analytic) function (see [La1]). However, the converse is not true, since the elliptic modular function is a normal (analytic) function for which the real part is not a normal harmonic function (see [La1, p. 158]). We denote by N_h the family of all real harmonic normal functions.

Let $w \in D$ and let $g(z,w) = \log\left|\frac{1-\overline{w}z}{z-w}\right|$ be the Green's function in D with logarithmic singularity at w. Let $u^{\#}(z) = \frac{|\operatorname{grad} u(z)|}{1+|u(z)|^2}$, and let $dm(z)$ denote

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the Euclidean element of area in C. In [La1] it was shown that a real-valued function u that is harmonic in D is a normal function if and only if

$$\sup_{z\in D}(1-|z|^2)u^{\#}(z)<\infty.$$

In addition to N_h, we will be considering the following classes of functions:

$$\mathrm{UBC}_h=\Big\{u\colon u \text{ real harmonic in } D \text{ and } \sup_{a\in D}\iint_D (u^{\#}(z))^2 g(z,a)\,dm(z)<\infty\Big\},$$

$$\mathrm{UBC}_{h,0}=\Big\{u\colon u \text{ real harmonic in } D \text{ and } \lim_{|a|\to 1}\iint_D (u^{\#}(z))^2 g(z,a)\,dm(z)=0\Big\},$$

$$N_{h,0}=\Big\{u\colon u \text{ real harmonic in } D \text{ and } \lim_{|z|\to 1}(1-|z|^2)u^{\#}(z)=0\Big\},$$

$$D_h^{\#}=\Big\{u\colon u \text{ real harmonic in } D \text{ and } \iint_D (u^{\#}(z))^2\,dm(z)<\infty\Big\},$$

and, for $0<p<\infty$,

$$Q_{h,p}^{\#}=\Big\{u\colon u \text{ real harmonic in } D \text{ and } \sup_{a\in D}\iint_D (u^{\#}(z))^2 (g(z,a))^p\,dm(z)<\infty\Big\},$$

$$Q_{h,p,0}^{\#}=\Big\{u\colon u \text{ real harmonic in } D \text{ and } \lim_{|a|\to 1}\iint_D (u^{\#}(z))^2 (g(z,a))^p\,dm(z)=0\Big\}.$$

Meromorphic (and analytic) normal functions have been studied extensively by many authors (see, for example, [AnClPo], [LeVi], and [Po]). Complex-valued functions (that are not necessarily meromorphic) and real-valued functions (that are not necessarily harmonic) can also be considered as normal functions; this has been studied in [AuLa1], [AuLa2], [La1], [La2], and [La3]. Some new characterizations for the classes N and N_0 have appeared in [AuZh]. One of the goals of this paper is to investigate how these characterizations apply to harmonic normal functions. We will give results in this direction in Sections 2 and 3.

For $0 < p < \infty$, the classes

$$Q_p = \{f\colon f \text{ analytic in } D \text{ and } \sup_{a\in D} \iint_D |f'(z)|^2 (g(z,a))^p dm(z) < \infty\}$$

and the classes

$$Q_p^{\#} = \{f\colon f \text{ meromorphic in } D \text{ and } \sup_{a\in D} \iint_D (f^{\#}(z))^2 (g(z,a))^p dm(z) < \infty\}$$

were introduced in [AuLa2] and [AuXiZh]. It is possible to extend this definition to the case $p = 0$ with the interpretation that

$$Q_0 = \Big\{f\colon f \text{ analytic in } D \text{ and } \iint_D |f'(z)|^2 dm(z) < \infty\Big\}$$

and

$$Q_0^{\#} = \Big\{f\colon f \text{ meromorphic in } D \text{ and } \iint_D (f^{\#}(z))^2 dm(z) < \infty\Big\}.$$

Then Q_0 is simply the usual Dirichlet space D_A and $Q_0^{\#}$ is the spherical Dirichlet space $D_A^{\#}$. We will use these interpretations in Section 5.

It has been shown that the classes Q_p and $Q_p^{\#}$ have the nesting property, i.e., that for $0 < p < q < \infty$ both $Q_p \subset Q_q$ and $Q_p^{\#} \subset Q_q^{\#}$ hold (see [AuXiZh]). In Section 4, we will give the corresponding property for the classes $Q_{h,p}^{\#}$. In [AuLa1], it was proved that $D_h^{\#} \subset N_h$ (also see [Ko]). Chen and Gauthier improved this result by showing that $D_h^{\#} \subset N_{h,0}$ (see [ChGa, Theorem 4]). By considering the classes $Q_{h,p,0}^{\#}$, we can sharpen this result, as we will show in Section 4.

In Section 5, we will establish some relationships between a harmonic function in the class $Q_{h,p}^{\#}$ and its corresponding analytic function. These results generalize results in [La1] and [La3].

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2. Characterizations of harmonic normal functions

The result of this section is an analog of a result for meromorphic normal functions in [AuZh]. If $a \in D$, let $\phi_a(z) = (a - z)/(1 - \overline{a}z)$, and, if $0 < r < 1$, let $D(a,r) = \{z\colon |\phi_a(z)| < r\}$. Finally, let $|D(a,r)|$ denote the Euclidean area of the disk $D(a,r)$.

Theorem 1. *Let u be a real harmonic function in D, let $0 < r < 1$, $2 < p < \infty$, and $1 < q < \infty$. The following statements are equivalent:*

(A) $\quad u \in N_h,$

(B) $$\sup_{a\in D} \frac{1}{|D(a,r)|^{1-p/2}} \iint_{D(a,r)} (u^{\#}(z))^p dm(z) < \infty,$$

(C) $$\sup_{a\in D} \iint_{D(a,r)} (u^{\#}(z))^p (1-|z|^2)^{p-2} dm(z) < \infty,$$

(D) $$\sup_{a\in D} \iint_{D} (u^{\#}(z))^p (1-|z|^2)^{p-2} (1-|\phi_a(z)|^2)^q dm(z) < \infty,$$

(E) $$\sup_{a\in D} \iint_{D} (u^{\#}(z))^p (1-|z|^2)^{p-2} (g(z,a))^q dm(z) < \infty,$$

(F) $$\sup_{a\in D} \iint_{D} (u^{\#}(z))^p \left(\log \frac{1}{|z|}\right)^p |\phi_a'(z)|^2 dm(z) < \infty.$$

Proof. The proofs of the implications (A) $\Longrightarrow$ (E) $\Longrightarrow$ (D) $\Longrightarrow$ (C) $\Longleftrightarrow$ (B) and (A) $\Longrightarrow$ (F) $\Longrightarrow$ (D) follow the proof of Theorem 1 of [AuZh], with obvious modifications, so we omit these proofs here. Thus, to establish the theorem, we need only prove (C) $\Longrightarrow$ (A).

Suppose that u satisfies (C) but not (A). By [AuLa1], there are two sequences of points $\{z_n\}$ and $\{z_n'\}$ in D such that $u(z_n) \to 0$, $u(z_n') \to 1$, and the pseudohyperbolic distance $\rho(z_n, z_n') = |\phi_{z_n}(z_n')| \to 0$. Let $f = u + iv$ be an analytic function whose real part is u, and let $f_n(z) = f(\gamma_n(z)) - f(z_n)$, where $\gamma_n(z) = (z + z_n)/(1 + \overline{z}_n z)$. Then $f_n(0) = 0$ and $\operatorname{Re}(f_n(\gamma_n^{-1}(z_n'))) \to 1$, which means that $\{f_n\}$ is not a normal family in any neighborhood of $z = 0$. By a result of Zalcman [Za], there exist a sequence of points $\{\zeta_n\}$ in D, and a sequence of positive real numbers $\{\rho_n\}$ with $\zeta_n \to 0$ and $\rho_n \to 0$, and there exists a subsequence of the sequence $\{h_n(t) = f_n(\zeta_n + \rho_n t)\}$, which converges uniformly on each compact subset of C to a non-constant entire function $h(t)$. Without loss of generality, we may assume that the full sequence $\{h_n(t)\}$ converges uniformly to $h(t)$ on each compact subset of C. Now

$$\operatorname{Re}(h_n(t)) = \operatorname{Re}(f_n(\zeta_n + \rho_n t)) \to \operatorname{Re} h(t)$$

uniformly on each compact subset of C. Thus we have

$$\iint_{|t|<s} ((\operatorname{Re} h_n)^{\#}(t))^p dm(t) \to \iint_{|t|<s} ((\operatorname{Re} h)^{\#}(t))^p dm(t) > 0.$$

Since $\zeta_n \to 0$, $\rho \to 0$, and $2 < p < \infty$, we have, for each $r > 0$ and each $s > 0$,

$$\{z = \gamma_n(\zeta_n + \rho_n t) \colon |t| < s\} \subset D(\gamma_n(\zeta_n), r)$$

for n sufficiently large, and

(2.1) $$\iint_{|t|<s} ((\operatorname{Re} h_n)^{\#}(t))^p \left(\frac{1 - |\zeta_n + \rho_n t|^2}{\rho_n}\right)^{p-2} dm(t) \to \infty.$$

But

$$\begin{aligned}\iint_{|t|<s} ((\operatorname{Re} h_n)^{\#}(t))^p \left(\frac{1-|\zeta_n+\rho_n t|^2}{\rho_n}\right)^{p-2} dm(t) \\ = \iint_{|t|<s} \left(\frac{|f'(\gamma_n(\zeta_n+\rho_n t))\gamma_n'(\zeta_n+\rho_n t)|\rho_n}{1+|\operatorname{Re}(f(\gamma_n(\zeta_n+\rho_n t))-f(z_n))|^2}\right)^p \\ \times \left(\frac{1-|\gamma_n(\zeta_n+\rho_n t)|^2}{|\gamma_n'(\zeta_n+\rho_n t)|\rho_n}\right)^{p-2} dm(t) \\ \le \iint_{D(\gamma_n(\zeta_n),r)} \left(\frac{|\operatorname{grad} u(z)|}{1+|u(z)-u(z_n)|^2}\right)^p (1-|z|^2)^{p-2} dm(z).\end{aligned}$$

However, by condition (C), this last term is finite, contradicting (2.1). Hence we have shown that (C) implies (A), and the proof is complete. □

Remark 1. For $p = 2$, the implication (A) $\Longrightarrow$ (E) was proved in [AuLa1]. All of the implications cited in the first sentence of the proof are valid for $0 < p < \infty$. However, for the implication (C) $\Longrightarrow$ (A) the condition $p > 2$ is necessary (see [AuZh]).

3. Characterizations of the class $N_{h,0}$

In this section we prove a result analogous to Theorem 1 with the class $N_{h,0}$ of N_h. The corresponding result for meromorphic functions in the class N was proved in [AuZh].

Theorem 2. *Let u be a real-valued harmonic function in D, let $0 < r < 1$, $2 \le p < \infty$, and $1 < q < \infty$. The following statements are equivalent:*

(a) $$u \in N_{h,0},$$

(b) $$\lim_{|a|\to 1} \frac{1}{|D(a,r)|^{1-p/2}} \iint_{D(a,r)} (u^{\#}(z))^p dm(z) = 0,$$

(c) $$\lim_{|a|\to 1} \iint_{D(a,r)} (u^{\#}(z))^p (1-|z|^2)^{p-2} dm(z) = 0,$$

(d) $$\lim_{|a|\to 1} \iint_{D} (u^{\#}(z))^p (1-|z|^2)^{p-2} (1-|\phi_a(z)|^2)^q dm(z) = 0,$$

(e) $$\lim_{|a|\to 1} \iint_{D} (u^{\#}(z))^p (1-|z|^2)^{p-2} (g(z,a))^q dm(z) = 0,$$

(f) $$\lim_{|a|\to 1} \iint_{D} (u^{\#}(z))^p \left(\log\frac{1}{|z|}\right)^p |\phi_a'(z)|^2 dm(z) = 0.$$

Proof. The theorem will be proved if we show the implications (a) $\Longrightarrow$ (e) $\Longrightarrow$ (d) $\Longrightarrow$ (c) $\Longleftrightarrow$ (b) and (c) $\Longrightarrow$ (a) $\Longrightarrow$ (f) $\Longrightarrow$ (d). However, the proofs of (a) $\Longrightarrow$ (e) $\Longrightarrow$ (d) $\Longrightarrow$ (c) $\Longleftrightarrow$ (b) and (a) $\Longrightarrow$ (f) $\Longrightarrow$ (d) are, aside from obvious modifications, contained in the proofs of Theorems 2 and 3 of [AuZh], so we will omit these proofs. Thus, we will have proved the theorem if we prove the implication (c) $\Longrightarrow$ (a).

We first suppose that u satisfies condition (c) with $p = 2$. We claim that this implies that $u \in N_h$. If u is not in the class N_h, we can let $\{z_n\}$, $\{z'_n\}$, $\gamma_n(z)$, $\{\zeta_n\}$, $\{\rho_n\}$, $h_n(t)$, and $h(t)$ be as in the proof of the implication (C) $\Longrightarrow$ (A) of Theorem 1. Thus we have

$$\operatorname{Re} h_n(t) = u_n(\gamma(\zeta_n + \rho_n t)) - u(z_n) \to \operatorname{Re} h(t)$$

uniformly on compact subsets of C, and

$$\iint_{|t|<s} ((\operatorname{Re} h_n)^{\#}(t))^2 dm(t) \to \iint_{|t|<s} ((\operatorname{Re} h)^{\#}(t))^2 dm(t) > 0,$$

since $h(t)$ is a nonconstant entire function. Now, for $0 < r < 1$, $s > 0$, and n sufficiently large, we have $\{z = \gamma_n(\zeta_n + \rho_n t) \colon |t| < s\} \subset D(\gamma_n(\zeta_n), r)$, so that

$$\iint_{D(\gamma_n(\zeta_n),r)} \left(\frac{|\operatorname{grad} u(z)|}{1 + |u(z) - u(z_n)|^2}\right)^2 dm(z) \le K \iint_{D(\gamma_n(\zeta_n),r)} (u^{\#}(z))^2 dm(z)$$

and

$$\begin{aligned}\iint_{D(\gamma_n(\zeta_n),r)} &\left(\frac{|\operatorname{grad} u(z)|}{1 + |u(z) - u(z_n)|^2}\right)^2 dm(z) \\ &\ge \iint_{|t|<s} ((\operatorname{Re} h_n)^{\#}(t))^2 dm(t) \to \iint_{|t|<s} ((\operatorname{Re} h)^{\#}(t))^2 dm(t) > 0\end{aligned}$$

for some constant $K > 0$. Since $|z_n| \to 1$ and $|\zeta_n| \to 0$, we have $|\gamma_n(\zeta_n)| \to 1$, and hence, by condition (c) with $p = 2$,

$$\lim_{n\to\infty} \iint_{D(\gamma_n(\zeta_n),r)} (u^{\#}(z))^2 dm(z) = 0.$$

But this contradicts the previous inequality, so we have shown that $u \in N_h$.

Now suppose that u is not in $N_{h,0}$. Then there exists a sequence of points $\{a_n\}$ in D with $|a_n| \to 1$ and a constant C such that

$$\lim_{n\to\infty} (1 - |a_n|^2) u^{\#}(a_n) = C > 0.$$

Set $\gamma_n(w) = (w + a_n)/(1 + \overline{a}_n w)$ and let $u_n(w) = u(\gamma_n(w))$. Since $u \in N_h$, we may suppose that $\{u_n(w)\}$ converges uniformly on each compact subset of D to a function $u_0(w)$ which is either harmonic on D or identically infinite. We claim that $|u_0(w)|$ cannot be identically infinite.

Let f be any function that is analytic in a disk D^* with radius r_0 and center at the origin and whose image is in the right half-plane. Without loss of generality we may assume that $f(0)$ is real. Then the mapping $g(z) = \frac{f(z)-f(0)}{f(z)+f(0)}$ sends D^* into the unit disk and satisfies $g(0) = 0$, so by the Schwarz Lemma we have $|g'(0)| \le 1/r_0$. But $|g'(0)| = |f'(0)|/(2|f(0)|)$. Thus, setting $U(z) =$

$\operatorname{Re} f(z)$, we have $\frac{|\operatorname{grad} U(0)|}{2|U(0)|} \le 1/r_0$, and hence $U^\#(0) \le \frac{2|U(0)|}{r_0(1+|U(0)|^2)}$. Thus, if $\{U_n(z)\}$ is a sequence of harmonic functions which converges uniformly to infinity on D^*, we have $U_n^\#(0) \le \frac{2}{r_0|U_n(0)|} \to 0$. It follows that if $\{U_n(z)\}$ is a sequence of harmonic functions which converges uniformly to either $+\infty$ or $-\infty$ on a disk with center at the origin, then $U_n^\#(0) \to 0$.

Returning to the sequence $\{u_n(w)\}$, we conclude that if $|u_0(w)|$ is identically infinite, then $u_n^\#(0) \to 0$. Since we assumed that $u_n^\#(0) \to C > 0$, we conclude that $u_0(w)$ is harmonic in D and $u_0^\#(0) = C > 0$. Since $|\operatorname{grad} u_n(0)| \to |\operatorname{grad} u_0(0)| \ge C > 0$, we conclude that $u_0(w)$ is a nonconstant function, and hence

$$\begin{aligned}\iint_{D(a_n,r)} (u_n^\#(z))^2 dm(z) &= \iint_{D(0.r)} (u_n^\#(w))^2 dm(w)\\ &\to \iint_{D(0,r)} (u_0^\#(w))^2 dm(w) > 0.\end{aligned}$$

This contradicts (c) in the case $p = 2$ and thus proves the implication (c) $\Longrightarrow$ (a) for $p = 2$.

Now let $p > 2$. By the Hölder inequality,

$$\begin{aligned}&\iint_{D(a,r)} (u^\#(z))^2 dm(z)\\ &\qquad \le \left(\iint_{D(a,r)} (u^\#(z))^p dm(z)\right)^{2/p} \left(\iint_{D(a,r)} dm(z)\right)^{1-(2/p)}\\ &\qquad = \left(\frac{1}{|D(a,r)|^{1-(p/2)}} \iint_{D(a,r)} (u^\#(z))^p dm(z)\right)^{2/p}.\end{aligned}$$

Thus, from condition (b) with $2 < p < \infty$, which is equivalent to (c) (see [AuZh]), we get

$$\lim_{|a|\to 1} \iint_{D(a,r)} (u^\#(z))^2 dm(z) = 0.$$

But this is condition (c) with $p = 2$, and we have proved above that this implies (a). This completes the proof. □

Remark 2. For $0 < p < 2$ we do not know whether the conditions given in Theorem 2 are equivalent. The implications stated in the second sentence of the proof of Theorem 2 remain valid for all $p > 0$, but the implication (c) $\Longrightarrow$ (a) requires other methods.

4. Inclusions

In [AuLa2, Theorem 3] it was proved that $Q_{h,p}^\# = N_h$ for $p > 1$. In the previous section, we showed that $Q_{h,p,0}^\# = N_{h,0}$ for $p > 1$. From the definitions it is easy to see that $Q^{\#}_{h,1} = \mathrm{UBC}_h$ and $Q^{\#}_{h,1,0} = \mathrm{UBC}_{h,0}$. The meromorphic classes UBC and UBC_0 have been studied extensively; see, for example, [Ya1], [Ya2], and [Pa]. In this section, we will show that, for $0 < p \le 1$, the classes $Q^{\#}_{h,p}$ and $Q^{\#}_{h,p,0}$ are different and establish various inclusion relationships.

We begin with characterizations of $Q^{\#}_{h,p}$ and $Q^{\#}_{h,p,0}$.

Proposition 1. *Let $0 < p \le 1$, let u be a real-valued harmonic function in D such that $u \in N_h$, and for $a \in D$ let $\phi_a(z) = (z-a)/(1-\overline{a}z)$.*

(i) *We have $u \in Q^{\#}_{h,p}$ if and only if*

$$\sup_{a\in D} \iint_D (u^{\#}(z))^2 (1-|\phi_a(z)|^2)^p dm(z) < \infty. \tag{4.1}$$

(ii) *We have $u \in Q^{\#}_{h,p,0}$ if and only if*

$$\lim_{|a|\to 1} \iint_D (u^{\#}(z))^2 (1-|\phi_a(z)|^2)^p dm(z) = 0. \tag{4.2}$$

Proof. Since $1-|\phi_a(z)|^2 \le 2g(z,a)$, the "only if" part in (i) and (ii) follows from the definitions of $Q^{\#}_{h,p}$ and $Q^{\#}_{h,p,0}$.

To prove the "if" part, suppose that (4.1) is valid. Because $g(z,a) = \log(1/|\phi_a(z)|)$, we have $g(z,a) \ge \log 4 > 1$ for $z \in D(a, 1/4)$; that is, $|\phi_a(z)| < 1/4$, and $g(z,a) \le 4(1-|\phi_a(z)|^2)$ for $z \in D - D(a,1/4)$. Then, for a fixed $q > 1$,

$$\begin{aligned}\iint_D &(u^{\#}(z))^2 (g(z,a))^p dm(z) \\ &= \iint_{D(a,1/4)} (u^{\#}(z))^2 (g(z,a))^p dm(z) \\ &\qquad + \iint_{D-D(a,1/4)} (u^{\#}(z))^2 (g(z,a))^p dm(z) \\ &\le \iint_{D(a,1/4)} (u^{\#}(z))^2 (g(z,a))^p dm(z) \\ &\qquad + 4^p \iint_{D-D(a,1/4)} (u^{\#}(z))^2 (1-|\phi_a(z)|^2)^p dm(z) \\ &\le \iint_D (u^{\#}(z))^2 (g(z,a))^q dm(z) \\ &\qquad + 4^p \iint_{D-D(a,1/4)} (u^{\#}(z))^2 (1-|\phi_a(z)|^2)^p dm(z).\end{aligned} \tag{4.3}$$

However, since $u \in N_h$, we have, by [AuLa2, Theorem 3],

$$\sup_{a\in D} \iint_D (u^{\#}(z))^2 (g(z,a))^q dm(z) < \infty. \tag{4.4}$$

Hence it follows from (4.3) and (4.4) that $u \in Q^{\#}_{h,p}$, and the "if" part is established for (i).

Now suppose (4.2). If we fix $q > 1$ and recall that $0 < p \leq 1$, then, using the inequality $(1 - |\phi_a(z)|^2)^q \leq (1 - |\phi_a(z)|^2)^p$, we have

$$\lim_{|a|\to 1} \iint_D (u^{\#}(z))^2 (1 - |\phi_a(z)|^2)^q dm(z) = 0.$$

Hence, by Theorem 2(d), we have $u \in N_{h,0}$, and thus, by Theorem 2(e),

$$\lim_{|a|\to 1} \iint_D (u^{\#}(z))^2 (g(z,a))^q dm(z) = 0. \tag{4.5}$$

Now it follows from (4.2), (4.3), and (4.5) that

$$\lim_{|a|\to 1} \iint_D (u^{\#}(z))^2 (g(z,a))^p dm(z) = 0,$$

which means that $u \in Q_{h,p,0}$. This establishes the "if" part for (ii) and completes the proof. □

Remark 3. For $0 \leq p < \infty$, we say that a positive measure μ defined on D is a bounded p-Carleson measure, provided

$$\mu(S(I)) = O(|I|^p) \tag{4.6}$$

for all subarcs I of ∂D, where $|I|$ denotes the arc length and $S(I)$ denotes the Carleson box based on I. When $p = 1$, this gives the standard definition of a Carleson measure (see, for example, [Ba]). If the right side of (4.6) is $o(|I|^p)$, then we say that μ is a compact p-Carleson measure. In [AuStXi], it was proved that, for $0 < p < \infty$, a positive measure μ on D is a bounded p-Carleson measure if and only if

$$\sup_{a\in D} \iint_D |\phi_a'(z)|^p d\mu(z) < \infty, \tag{4.7}$$

and μ is a compact p-Carleson measure if and only if

$$\lim_{|a|\to 1} \iint_D |\phi_a'(z)|^p d\mu(z) = 0.$$

Thus, since $|\phi_a'(z)|(1 - |z|^2) = 1 - |\phi_a(z)|^2$, Proposition 1 implies that if $u \in N_h$ and $0 < p \leq 1$, then $u \in Q^{\#}_{h,p}$ if and only if $(u^{\#}(z))^2(1 - |z|^2)^p dm(z)$ is a bounded p-Carleson measure, and also that $u \in Q^{\#}_{h,p,0}$ if and only if $(u^{\#}(z))^2(1 - |z|^2)^p dm(z)$ is a compact p-Carleson measure.

Remark 4. The analog of Proposition 1 for meromorphic functions is given in [AuXiZh, Proposition 2]. Note that in the meromorphic case we do not need the additional assumption $f \in N$ to obtain the characterization of $Q^{\#}_p$. It seems reasonable to conjecture that the assumption $u \in N_h$ is not necessary in Proposition 1, but we do not know how to prove (i) without this assumption. Note that this assumption is not necessary to prove (ii).

Next, we prove an inclusion result.

LEMMA 1. *For $0 < p \le 1$ we have $Q^{\#}_{h,p} \subset N_h$.*

Proof. Let $0 < p \le 1$, and let $u \in Q^{\#}_{h,p}$, so that

$$M = \sup_{a\in D} \iint_D (u^{\#}(z))^2 (g(z,a))^p dm(z) < \infty. \tag{4.8}$$

If u is not a normal function, let $\{z_n\}$, $\{z'_n\}$, $\gamma_n(z)$, $\{\zeta_n\}$, $\{\rho_n\}$, $h_n(t)$, and $h(t)$ be as in the proof of the implication (C) $\Longrightarrow$ (A) of Theorem 1. Since $h(t)$ is a nonconstant entire function, we have, for $s > 0$,

$$\begin{aligned}\iint_{|t|<s} \left(\frac{|\operatorname{grad} \operatorname{Re} h_n(t)|}{1 + |\operatorname{Re} h_n(t)|^2}\right)^2 dm(t) \\ \to \iint_{|t|<s} \left(\frac{|\operatorname{grad} \operatorname{Re} h(t)|}{1 + |\operatorname{Re} h(t)|^2}\right)^2 dm(t) > 0.\end{aligned} \tag{4.9}$$

Since $\zeta_n \to 0$, $\rho_n \to 0$, and $u(z_n) \to 0$, we have, for some r with $0 < r < 1$,

$$\begin{aligned}\Bigg| \iint_{D(\gamma_n(\zeta_n),r)} \left(\frac{|\operatorname{grad}(u(w) - u(z_n))|}{1 + |u(w) - u(z_n)|^2}\right)^2 (g(w, \gamma_n(\zeta_n)))^p dm(w) \\ - \iint_{D(\gamma_n(\zeta_n),r)} (u^{\#}(w))^2 (g(w, \gamma_n(\zeta_n)))^p dm(w) \Bigg| \to 0.\end{aligned} \tag{4.10}$$

On the other hand, letting $w = \gamma_n(\zeta_n + \rho_n t)$, we have

$$\begin{aligned}&\iint_{D(\gamma_n(\zeta_n),r)} \left(\frac{|\operatorname{grad}(u(w) - u(z_n))|}{1 + |u(w) - u(z_n)|^2}\right)^2 (g(w, \gamma_n(\zeta_n)))^p dm(w) \\ &\ge \iint_{|t|<s} \left(\frac{|\operatorname{grad}(u(\gamma_n(\zeta_n + \rho_n t)) - u(z_n))|}{1 + |u(\gamma_n(\zeta_n + \rho_n t)) - u(z_n)|^2}\right)^2 |\rho_n \gamma'_n(\zeta_n + \rho_n t)|^2 \\ &\qquad \times (g(\gamma_n(\zeta_n + \rho_n t), \gamma_n(\zeta_n)))^p dm(t) \\ &= \iint_{|t|<s} \left(\frac{|\operatorname{grad} \operatorname{Re} h_n(t)|}{1 + |\operatorname{Re} h_n(t)|^2}\right)^2 (g(\gamma_n(\zeta_n + \rho_n t), \gamma_n(\zeta_n)))^p dm(t) \\ &\ge \inf_{|t|<s} \{(g(\gamma_n(\zeta_n + \rho_n t), \gamma_n(\zeta_n)))^p\} \iint_{|t|<s} \left(\frac{|\operatorname{grad} \operatorname{Re} h_n(t)|}{1 + |\operatorname{Re} h_n(t)|^2}\right)^2 dm(t) \\ &\to \infty,\end{aligned}$$

by (4.9) and the relation

$$\inf_{|t|<s} \{(g(\gamma_n(\zeta_n + \rho_n t), \gamma_n(\zeta_n)))^p\} \to \infty$$

(which holds since $\rho(\zeta_n+\rho_n t,\zeta_n)\to 0$ for $|t|<s$). But, by observing (4.10), we see that (4.8) is now violated, and the lemma is proved. □

Lemma 1 is actually valid for $0<p<\infty$ since, for $p>1$, we have $Q_{h,p}=N_h$ (see [AuLa2, Theorem 3]).

THEOREM 3. *For $0<p<q<\infty$, we have*

(i) $Q^{\#}_{h,p}\subset Q^{\#}_{h,q}$,

(ii) $Q^{\#}_{h,p,0}\subset Q^{\#}_{h,q,0}$.

Proof. (i) If $q>1$, then $Q^{\#}_{h,q}=N_h$, and the result follows from Lemma 1. If $0<p<q\le 1$ and $u\in Q^{\#}_{h,p}$, then, by Lemma 1, $u\in N_h$ and, by Proposition 1,

$$\sup_{a\in D}\iint_D (u^{\#}(z))^2\left(1-|\phi_a(z)|^2\right)^p dm(z)<\infty.$$

Since $0<p<q$, we have $\left(1-|\phi_a(z)|^2\right)^q\le\left(1-|\phi_a(z)|^2\right)^p$ and hence

$$\sup_{a\in D}\iint_D (u^{\#}(z))^2\left(1-|\phi_a(z)|^2\right)^q dm(z)<\infty,$$

which by Proposition 1 (since $u\in N_h$) is equivalent to $u\in Q^{\#}_{h,q}$. This completes the proof of (i).

(ii) Let $0<p<q<\infty$ and $u\in Q^{\#}_{h,p,0}$. Since $1-|\phi_a(z)|^2\le 2g(z,a)$ for all z, $a\in D$, we have

$$\begin{aligned}\lim_{|a|\to 1}&\iint_D (u^{\#}(z))^2(1-|\phi_a(z)|^2)^p dm(z)\\ &\le \lim_{|a|\to 1}2^p\iint_D (u^{\#}(z))^2(g(z,a))^p dm(z)=0.\end{aligned}$$

Again, since $(1-|\phi_a(z)|^2)^q\le(1-|\phi_a(z)|^2)^p$ for $0<p<q<\infty$, we obtain

$$\lim_{|a|\to 1}\iint_D (u^{\#}(z))^2(1-|\phi_a(z)|^2)^q dm(z)=0.$$

If $0<q<1$, this yields $u\in Q^{\#}_{h,q,0}$, by Proposition 1. If $q>1$, then applying Theorem 2(d) with $p=2$, we see that $u\in N_{h,0}=Q^{\#}_{h,q,0}$. This completes the proof. □

THEOREM 4. *The inclusions given in Theorem 3 are strict when $0<p<q$ and $p<1$.*

Proof. Let $0<p<q\le 1$ and let $f_0(z)=\sum_{n=0}^{\infty}a_n z^{2^n}$, where $a_n=2^{-n(1-p)/2}$. In [AuXiZh, Corollary 3], it was shown that f_0 is in the class $Q_{q,0}\subset Q_q$, but not in the class $Q_p^{\#}$ (and hence not in $Q^{\#}_{p,0}$). (Actually,

Corollary 3 in [AuXiZh] dealt with the case $q \le 1$, but it is clear that, if $p < 1 < q$, we can take q' such that $p < q' < 1 < q$ and we then have $f_0 \in Q^{\#}_{q'} \subset Q^{\#}_{q} = N_h$ from [AuXiZh, Theorem 2].) We can write $f_0(z) = u_0(z) + iv_0(z)$, where u_0 and v_0 are real harmonic functions. Since $|f_0'(z)| = |\operatorname{grad} u_0(z)|$ and f_0 is a bounded function, it follows that u_0 is not in either of the classes $Q^{\#}_{h,p}$ or $Q^{\#}_{h,p,0}$. Hence $u_0 \in Q^{\#}_{h,q} - Q^{\#}_{h,p}$ and so $u_0 \in Q^{\#}_{h,q,0} - Q^{\#}_{h,p,0}$. This completes the proof. □

In [AuLa1, Theorem 1] it was shown that $D^{\#}_h \subset N_h$ (see also [Ko]). Chen and Gauthier have improved this result, showing that $D^{\#}_h \subset N_{h,0}$ (see [ChGa, Theorem 4]). We can sharpen this inclusion as follows.

THEOREM 5. *We have* $D^{\#}_h \subset \bigcap_{0<p<\infty} Q^{\#}_{h,p,0}$.

Proof. If $u \in D^{\#}_h$, then

$$\iint_D (u^{\#}(z))^2 dm(z) < \infty. \tag{4.11}$$

It follows that

$$\lim_{|a|\to 1} \iint_{D(a,r)} (u^{\#}(z))^2 dm(z) = 0$$

for each fixed $r \in (0,1)$. Hence, by either Theorem 4 of [ChGa] or Theorem 2(b) (with $p = 2$) we have $u \in N_{h,0}$. Letting $0 < p < 1$ and $q > 1$, we obtain by Hölder's inequality

$$\begin{aligned}\iint_D &(u^{\#}(z))^2 (g(z,a))^p dm(z) \\ &\le \left(\iint_D (u^{\#}(z))^2 (g(z,a))^q dm(z)\right)^{p/q} \left(\iint_D (u^{\#}(z))^2 dm(z)\right)^{1-p/q}.\end{aligned} \tag{4.12}$$

Since $u \in N_{h,0}$, (4.11), (4.12), and the case $p = 2$ of Theorem 2(e) yield

$$\lim_{|a|\to 1} \iint_{D(a,r)} (u^{\#}(z))^2 (g(z,a))^p dm(z) = 0.$$

This means that $u \in Q^{\#}_{h,p,0}$, and the theorem is proved. □

We next show that the inclusion in Theorem 5 is strict.

THEOREM 6. *We have* $D^{\#}_h \ne \bigcap_{0<p<\infty} Q^{\#}_{h,p,0}$.

Proof. In [AuXiZh, Corollary 4] it was shown that the function $f_1(z) = \sum_{n=1}^{\infty} 2^{-n/2} z^{2^n}$ is a bounded function such that $f_1 \in \bigcap_{0<p<\infty} Q^{\#}_{p,0} - D_A$, where

$$D_A = \{f\colon f \text{ analytic in } D \text{ and } \iint_D |f'(z)|^2 dm(z) < \infty\}.$$

We can write $f_1(z) = u_1(z) + iv_1(z)$, where u_1 and v_1 are real harmonic functions. Because $|f_1'(z)| = |\operatorname{grad} u_1(z)|$, we have $u_1^{\#}(z) \leq |f_1'(z)|$, and hence $u_1 \in \bigcap_{0<p<\infty} Q_{h,p,0}^{\#}$. On the other hand, $|u_1(z)| \leq |f_1(z)| \leq \sum_{n=1}^{\infty} 2^{-n/2} < \infty$, which implies that u_1 is not in $D_h^{\#}$. This completes the proof. □

5. Real parts of analytic functions

In this section, we give some relationships between the classes $Q_{h,p}^{\#}$ and $Q_{h,p,0}^{\#}$ and the corresponding classes of analytic functions. The classes Q_p and $Q_p^{\#}$ have been defined in Section 1. For completeness, we recall the definition of the classes $Q_{p,0}$ and $Q_{p,0}^{\#}$ (see [AuXiZh]):

$$Q_{p,0} = \Big\{ f\colon f \text{ analytic in } D \text{ and } \lim_{|a|\to 1} \iint |f'(z)|^2 (g(z,a))^p \, dm(z) = 0 \Big\},$$

and

$$Q_{p,0}^{\#} = \Big\{ f\colon f \text{ meromorphic in } D \text{ and } \lim_{|a|\to 1} \iint_D (f^{\#}(z))^2 (g(z,a))^p \, dm(z) = 0 \Big\}.$$

We can generalize [La1, Theorem 5] as follows.

Proposition 2. *If u is a real-valued harmonic function in D, $0 < p < \infty$, $u \in Q_{h,p}^{\#}$ (or $Q_{h,p,0}^{\#}$), and if $f = u + iv$ is an analytic function, where v is the harmonic conjugate of u, then $f \in Q_p^{\#}$ (or $Q_{p,0}^{\#}$).*

Proof. The result follows easily from the facts that $|\operatorname{grad} u(z)| = |f'(z)|$ and $u^{\#}(z) \geq f^{\#}(z)$. □

We note that the converse of Proposition 2 does not hold. In fact, the following examples show that two different types of converses to this proposition are not valid.

Example 1. *There exists a function $f = u + iv$ with $f \in Q_p^{\#}$ for each $p > 0$, such that $u = \operatorname{Re}(f) \in Q_{h,p}^{\#}$ for each $p > 0$, but $v = \operatorname{Im}(f) \in Q_{h,p}^{\#}$ for no $p > 0$.*

Proof. Let f be a conformal mapping from D onto the region $W = \{w = u + iv\colon |v| < 1 + \sqrt{u}, u > 0\}$ such that f maps the real axis in D onto the real axis in W. It is clear that the region W has finite spherical area, so $f \in Q_0^{\#} \subset Q_p^{\#}$ for $p > 0$ (see [AuXiZh, Theorem 4]), and also that f is not a Bloch function, since the region W contains arbitrarily large disks. Hence, since f is univalent and f is not a Bloch function, f is not in any of the classes Q_p, $p>0$, by [AuLaXiZh, Theorem 6.1]. Since

$$|f(z)|^2 = |u(z)|^2 + |v(z)|^2 \le |u(z)|^2 + 1 + |u(z)| + 2\sqrt{|u(z)|}$$

for $|u(z)| \ge 1$, we have $|f(z)|^2 < 5|u(z)|^2$, and so

$$\frac{|\operatorname{grad} u(z)|^2}{(1+|u(z)|^2)^2} = \frac{|f'(z)|^2}{(1+|u(z)|^2)^2} \le 25\frac{|f'(z)|^2}{(1+|f(z)|^2)^2} = 25(f^{\#}(z))^2.$$

Also, for $|u(z)| \le 1$ we have $|f(z)|^2 \le 5$, which means

$$\frac{|\operatorname{grad} u(z)|^2}{(1+|u(z)|^2)^2} = \frac{|f'(z)|^2}{(1+|u(z)|^2)^2} \le |f'(z)|^2 \le 36(f^{\#}(z))^2.$$

Hence it follows that $u \in Q^{\#}_{h,p}$ for each $p>0$.

Since f maps the real axis onto the real axis, it follows that if $r>0$, $a_r = f^{-1}(r)$, and $D(a_r, 1/e) = \{z \in D\colon g(z,a_r) > 1\}$, then $f(D(a_r,1/e))$ contains a Euclidean disk $\Delta(r,\rho_r) = \{w\colon |w-r| < \rho_r\}$ with radius $\rho_r = \sqrt{r}/e$ and center at r, where $\rho_r \to \infty$ as $r \to \infty$. Thus, for $p>0$,

$$\begin{aligned}
&\iint_{D(a_r,1/e)} \frac{|\operatorname{grad} v(z)|^2}{(1+|v(z)|^2)^2}(g(z,a_r))^p\,dm(z)\\
&\qquad = \iint_{f(D(a_r,1/e))} \frac{1}{(1+|v|^2)^2}(g_W(w,r))^p\,dm(w)\\
&\qquad \ge \iint_{f(D(a_r,1/e))} \frac{1}{(1+|v|^2)^2}\,dm(w),
\end{aligned}$$

where $w = u+iv$ and $g_W(w,r)$ is the Green's function in W with logarithmic singularity at r. Thus, for $\sqrt{r} > 2e$,

$$\begin{aligned}
\iint_{f(D(a_r,1/e))} \frac{1}{(1+|v|^2)^2}\,dm(w) &\ge \iint_{\Delta(r,\rho_r)} \frac{1}{(1+|v|^2)^2}\,dm(w)\\
&\ge \int_{r-\sqrt{r}/(2e)}^{r+\sqrt{r}/(2e)} 2\left(\int_0^1 \frac{1}{(1+|v|^2)^2}\,dv\right)du\\
&= (2\sqrt{r}/e)\int_0^1 \frac{1}{(1+|v|^2)^2}\,dv \to \infty.
\end{aligned}$$

It follows that $v(z)$ is not in $Q^{\#}_{h,p}$ for any $p>0$ (and hence not in $Q^{\#}_{h,p,0}$ either for any choice of $p>0$). This proves the result. □

In Example 1, we had $f \in Q^{\#}_p$ and $u \in Q^{\#}_{h,p}$ for each $p>0$, but $v \notin Q^{\#}_{h,p}$ for any choice of $p>0$. In the following example, we show that we can have $f \in Q^{\#}_p$, while neither u nor v are in $Q^{\#}_{h,p}$.

EXAMPLE 2. *There exists an analytic function f with $f \in Q_p^{\#}$ for each $p > 0$ such that neither $u(z) = \mathrm{Re}(f(z))$ nor $v(z) = \mathrm{Im}(f(z))$ is in any of the classes $Q_{h,p}^{\#}$, $p > 0$.*

Proof. Let $\{D_n\}$ be a sequence of disjoint disks of radius r_n centered on the positive real axis such that $r_n \to \infty$. Let $D_n' = \{w = iz\colon z \in D_n\}$ denote the disk D_n, rotated so that its center falls on the imaginary axis in the upper half-plane. For $n > 1$, D_n and D_n' have disjoint closures. For each n, let S_n be a thin channel from the center of D_n to the center of D_n', and let S_n' be a thin channel from the center of D_n' to the center of D_{n+1}. Let $\Omega = \bigcup_{n=1}^{\infty}(D_n \cup S_n \cup D_n' \cup S_n')$ (so that Ω is simply connected), and let f be a conformal mapping from the unit disk D onto Ω. If the disk D_n has center w_n and radius r_n, then the Green's function on the disk D_n with singularity at w_n is $g_{D_n}(w, w_n) = \ln \left|\frac{r_n}{w - w_n}\right|$, and if $g_\Omega(w, w_n)$ is the Green's function on Ω with singularity at w_n, then $g_\Omega(w, w_n) \geq g_{D_n}(w, w_n)$. We can now repeat the proof given for Example 1 to show that u is not in $Q_{h,p}^{\#}$ for any positive p. By a similar argument using the disks D_n', we see that v is not in $Q_{h,p}^{\#}$ for any positive p. But $f(D) = \Omega$ has finite spherical area, so $f \in Q_0^{\#} \subset Q_p^{\#}$ for each $p > 0$. This completes the proof. □

It is easy to modify the construction in Example 2 so that, for each α with $0 \leq \alpha < 2\pi$, the function $g(z) = e^{i\alpha} f(z)$ is in $Q_p^{\#}$ for each $p > 0$, but neither $u(z) = \mathrm{Re}\, g(z)$ nor $v(z) = \mathrm{Im}\, g(z)$ is in any of the classes $Q_{h,p}^{\#}$. We simply need to arrange the disks $\{D_n\}$ so that each ray from the origin forming a rational angle with the positive real axis passes through the centers of infinitely many of these disks. We then create, for each n, a suitable channel S_n connecting the centers of the disks D_n and D_{n+1}. Applying the argument in the proof of Example 2 to the conformal mapping f of the unit disk D onto the region $\Omega' = \bigcup_{n=1}^{\infty}(D_n \cup S_n)$, we then obtain the result.

Proposition 2 states that if f is analytic and $u(z) = \mathrm{Re}(f(z)) \in Q_{h,p}^{\#}$ for some positive p, then $f \in Q_p^{\#}$. However, as the next example shows, the condition $u \in Q_{h,p}^{\#}$ does not imply that $f \in Q_p$.

EXAMPLE 3. *There exists an analytic function f such that both $u(z) = \mathrm{Re}(f(z))$ and $v(z) = \mathrm{Im}(f(z))$ satisfy $u \in Q_{h,p}^{\#}$ and $v \in Q_{h,p}^{\#}$ for each $p > 0$, while $f \notin Q_p$ for all p. (Thus, by Proposition 2, we have $f \in Q_p^{\#} - Q_p$ for each $p > 0$.)*

Proof. Let f be a conformal mapping from the unit disk D onto the region $\Lambda = \{w = u + iv\colon 0 < u < v < 2u\}$. Then, for $w \in \Lambda$,

$$\frac{1}{\sqrt{5}}|w| \leq |u|, |v| \leq \sqrt{5}|w|$$

and $|\operatorname{grad} u(z)|^2 = |f'(z)|^2 = |\operatorname{grad} v(z)|^2$, where $f(z) = u(z) + iv(z)$. Thus,

$$(u^{\#}(z))^2 \leq 25(f^{\#}(z))^2, \qquad (v^{\#}(z))^2 \leq 25(f^{\#}(z))^2.$$

Now $f(D) = \Lambda$ has finite spherical area, so $f \in Q_0^{\#} \subset Q_p^{\#}$ for each $p > 0$. Thus, we have both $u \in Q_{h,p}^{\#}$ and $v \in Q_{h,p}^{\#}$, for each $p > 0$. But f is a conformal mapping which is not a Bloch function, so f is not in any Q_p for $p > 0$, by [AuLaXiZh, Theorem 6.1]. Further, $f(D) = \Lambda$ does not have finite planar area, so f is not in Q_0. This completes the proof. □

PROPOSITION 3. *Let $0 < p < \infty$, and let u and v be harmonic conjugate functions such that the corresponding analytic function $f = u + iv$ is in Q_p (or $Q_{p,0}$). Then both u and v are in $Q_{h,p}^{\#}$ (or $Q_{h,p,0}^{\#}$).*

Proof. The result follows easily from the inequality $u^{\#}(z) \leq |\operatorname{grad} u(z)| = |f'(z)|$. □

The last part of this inequality shows that if we define classes $Q_{h,p}$ and $Q_{h,p,0}$ in the natural way, using $|\operatorname{grad} u(z)|$ in place of $u^{\#}(z)$, then $u \in Q_{h,p}$ (or $Q_{h,p,0}$) and $u(z) = \operatorname{Re}(f(z)$ if and only if $f \in Q_p$ (or $Q_{p,0}$). We will not pursue this direction further here.

The following result generalizes Lemma 1 in [La3].

PROPOSITION 4. *Let $0 < p < \infty$, let u be a function that is harmonic in D such that $u \in Q_{h,p}^{\#}$ (or $Q_{h,p,0}^{\#}$), and let v be a conjugate harmonic function of u. Then $f(z) = e^{u(z)+iv(z)}$ is in $Q_p^{\#}$ (or $Q_{p,0}^{\#}$).*

Proof. We note that $e^x + e^{-x} \geq 1 + x^2$ for all real x, which implies $e^{u(z)} + e^{-u(z)} \geq 1 + |u(z)|^2$. Also, note that $f'(z) = f(z)\frac{d}{dz}(u(z) + iv(z))$, so $|f'(z)| = |f(z)||\operatorname{grad} u(z)|$. Thus, for $0 < p < \infty$,

$$\begin{aligned}
&\iint_D (f^{\#}(z))^2 (g(z,a))^p dm(z) \\
&\qquad = \iint_D \left(\frac{|f'(z)|}{1 + |f(z)|^2}\right)^2 (g(z,a))^p dm(z) \\
&\qquad = \iint_D |\operatorname{grad} u(z)|^2 \left(\frac{|f(z)|}{1 + |f(z)|^2}\right)^2 (g(z,a))^p dm(z) \\
&\qquad = \iint_D |\operatorname{grad} u(z)|^2 \left(\frac{1}{|f(z)| + \frac{1}{|f(z)|}}\right)^2 (g(z,a))^p dm(z) \\
&\qquad = \iint_D |\operatorname{grad} u(z)|^2 \frac{1}{(e^{u(z)} + e^{-u(z)})^2} (g(z,a))^p dm(z)
\end{aligned}$$

$$\leq \iint_D |\operatorname{grad} u(z)|^2 \left(\frac{1}{1+|u(z)|^2}\right)^2 (g(z,a))^p dm(z)$$
$$= \iint_D (u^{\#}(z))^2 (g(z,a))^p dm(z).$$

Since $u \in Q_{h,p}^{\#}$ (or $Q_{h,p,0}^{\#}$), it follows from the inequality above that $f \in Q_p^{\#}$ (or $Q_{p,0}^{\#}$). This completes the proof. □

References

[AnClPo] J. M. Anderson, J. Clunie, and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37.

[AuLa1] R. Aulaskari and P. Lappan, *An integral condition for harmonic normal functions*, Complex Variables **23** (1993), 213–219.

[AuLa2] ———, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal*, Complex analysis and its applications, Pitman Research Notes in Mathematics Series, vol. 305, Longman Scientific & Technical, Harlow, 1994, pp. 136–146.

[AuLaXiZh] R. Aulaskari, P. Lappan, J. Xiao, and R. Zhao, *On α-Bloch spaces and multipliers of Dirichlet spaces*, J. Math. Anal. Appl. **209** (1997), 103–121.

[AuStXi] R. Aulaskari, D. A. Stegenga, and J. Xiao, *Some subclasses of BMOA and their characterization in terms of Carleson measure*, Rocky Mountain J. Math. **26** (1996), 485–506.

[AuXiZh] R. Aulaskari, J. Xiao, and R. Zhao, *Some subspaces and subsets of BMOA and UBC*, Analysis **15** (1995), 101–121.

[AuZh] R. Aulaskari and R. Zhao, *Some characterizations of normal and little normal functions*, Complex Variables **28** (1995), 135–148.

[Ba] A. Baernstein II, *Analytic functions of bounded mean oscillation*, Aspects of Contemporary Complex Analysis, Academic Press, 1980, pp. 3–36.

[ChGa] H. Chen and P. M. Gauthier, *On strongly normal functions*, Can. Math. Bull. **39** (1996), 408–419.

[Ko] P. Koskela, *Normal solutions of elliptic equations*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 2, 363–371.

[La1] P. Lappan, *Some results on harmonic normal functions*, Math. Z. **90** (1965), 155–159.

[La2] ———, *Asymptotic values of normal harmonic functions*, Math. Z. **94** (1966), 152–156.

[La3] ———, *Fatou points of harmonic normal functions and uniformly normal functions*, Math. Z. **102** (1967), 110–114.

[LeVi] O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.

[Pa] Z. Pavicevic, *The Carleson measure and meromorphic functions of uniformly bounded characteristic*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 249–254.

[Po] Ch. Pommerenke, *On normal and automorphic functions*, Michigan Math. J. **21** (1974), 193–202.

[Ya1] S. Yamashita, *Functions of uniformly bounded characteristic*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), 349–367.

[Ya2] ———, *Image area and functions of uniformly bounded characteristic*, Comment. Math. Univ. St. Paul. **34** (1985), 37–44.

[Za] L. Zalcman, *A heuristic principle in complex function theory*, Amer. Math. Monthly **82** (1975), 813–817.

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