

## ABSOLUTE-VALUABLE BANACH SPACES

JULIO BECERRA GUERRERO, ANTONIO MORENO GALINDO, AND ÁNGEL  
RODRÍGUEZ PALACIOS

ABSTRACT. Absolute-valuable Banach spaces are introduced as those Banach spaces which underlie complete absolute-valued algebras. Examples and counterexamples are given. It is proved that every Banach space can be isometrically enlarged to an absolute-valuable Banach space, which has the same density character as the given Banach space, and whose dual space is also absolute-valuable. It is also shown that every weakly countably determined Banach space different from  $\mathbb{R}$  can be renormed in such a way that neither it nor its dual are absolute-valuable. Hereditarily indecomposable Banach spaces are examples of Banach spaces which cannot be renormed as absolute-valuable Banach spaces.

### 1. Introduction

Let  $\mathbb{K}$  denote the field of real or complex numbers. By an *absolute-valued algebra* over  $\mathbb{K}$  we mean a non-zero algebra  $A$  over  $\mathbb{K}$  endowed with a norm  $\|\cdot\|$  satisfying  $\|xy\| = \|x\|\|y\|$  for all  $x, y$  in  $A$ . The reader is referred to the survey paper [21] for a comprehensive view of the theory of absolute-valued algebras. The aim of the present paper is to study those Banach spaces which underlie complete absolute-valued algebras. Such Banach spaces will be called “absolute-valuable”. Since finite-dimensional absolute-valuable Banach spaces are well-understood from the classical work of A. Albert [1] (see Proposition 2.1 below for details), we focus our attention on the infinite-dimensional case.

We begin Section 2 by providing the reader with several natural examples and counterexamples of absolute-valuable Banach spaces. Concerning examples, it is worth mentioning that, roughly speaking, many classical Banach spaces, including all infinite-dimensional Hilbert spaces, are absolute-valuable (see Theorem 2.3 and Corollary 2.5). By contrast, it has been difficult for us

---

Received January 30, 2004; received in final form November 8, 2004.

2000 *Mathematics Subject Classification*. Primary 46B04. Secondary 46B03, 46H70.

Partially supported by Junta de Andalucía grant FQM 0199 and Projects I+D MCYT BFM2001-2335 and BFM2002-01810.

to find classical Banach spaces which are not absolute-valuable. Nevertheless, the space  $c$  of all real or complex convergent sequences is an example of such a Banach space (Proposition 2.8). As main result, we prove that every Banach space can be isometrically embedded into a suitable absolute-valuable Banach space, which has the same density character as the given Banach space, and whose dual space is also absolute-valuable (Theorem 2.11).

Section 3 is devoted to the isomorphic aspects of absolute valuability. As main result we prove that every weakly countably determined real Banach space different from  $\mathbb{R}$  can be equivalently renormed in such a way that neither it nor its dual are absolute-valuable (Theorem 3.4). We note that a Banach space is weakly countably determined whenever it is either reflexive, separable, or of the form  $c_0(\Gamma)$  for any set  $\Gamma$ . We also show that both the separable reflexive Banach space of Gowers-Maurey [12] and the non-separable reflexive one of Shelah-Steprans-Wark (see [23] and [25]) are not isomorphic to any absolute-valuable Banach space (Propositions 3.7 and 3.8).

The concluding Section 4 deals with the relation between absolute valuability and the Banach-Mazur rotation problem. We provide examples of non-Hilbert absolute-valuable almost transitive separable Banach spaces (Proposition 4.4) as well as non-Hilbert absolute-valuable transitive non-separable Banach spaces (Corollary 4.5). However, the particular case of the rotation problem whether every absolute-valuable transitive separable Banach space is a Hilbert space remains open (Problem 4.1).

Most absolute-valuable Banach spaces  $X$  arising in this paper have the additional property that  $X^*$ ,  $\mathcal{L}(X)$ , and  $\mathcal{K}(X)$  are absolute-valuable (see Theorem 2.3, Proposition 2.4, Corollary 2.5, and Theorem 2.11) or, even more, that  $\mathcal{L}(X, Y)$  and  $\mathcal{K}(X, Y)$  are absolute-valuable for every absolute-valuable Banach space  $Y$  (Proposition 4.4). As usual, given Banach spaces  $X$  and  $Y$  over  $\mathbb{K}$ , we denote by  $\mathcal{L}(X, Y)$  the Banach space of all bounded linear operators from  $X$  to  $Y$ , and by  $\mathcal{K}(X, Y)$  the closed subspace of  $\mathcal{L}(X, Y)$  consisting of all compact operators from  $X$  to  $Y$ . Moreover, we write  $X^*$ ,  $\mathcal{L}(X)$ , and  $\mathcal{K}(X)$  instead of  $\mathcal{L}(X, \mathbb{K})$ ,  $\mathcal{L}(X, X)$ , and  $\mathcal{K}(X, X)$ , respectively.

## 2. Isometric aspects of absolute valuability

By a *product* on a vector space  $X$  we mean a bilinear mapping from  $X \times X$  into  $X$ . A Banach space  $(X, \|\cdot\|)$  over  $\mathbb{K}$  is said to be *absolute-valuable* if there exists a product  $(x, y) \rightarrow xy$  on  $X$  satisfying

$$\|xy\| = \|x\|\|y\|$$

for all  $x, y$  in  $X$ . Finite-dimensional absolute-valuable Banach spaces are well-understood thanks to the next result, which follows from the work of A. A. Albert [1].

PROPOSITION 2.1. *The real Hilbert spaces of dimension 1, 2, 4 or 8 are the unique finite-dimensional absolute-valuable real Banach spaces. The complex field  $\mathbb{C}$  is the unique finite-dimensional absolute-valuable complex Banach space.*

Despite the above fact, most classical Banach spaces are absolute-valuable. For instance, this is the case for the real or complex spaces  $c_0$  and  $\ell_p$  with  $1 \leq p \leq \infty$ . More generally, we have the result given by the next proposition. Let  $Y$  be a Banach space over  $\mathbb{K}$ , and let  $\Gamma$  be an infinite set. For  $0 \leq p < \infty$ , we denote by  $\ell_p(\Gamma, Y)$  the Banach space over  $\mathbb{K}$  of all mappings  $f : \Gamma \rightarrow Y$  such that

$$\|f\|^p := \sum_{\gamma \in \Gamma} \|f(\gamma)\|^p < \infty.$$

By  $\ell_\infty(\Gamma, Y)$  we mean the Banach space over  $\mathbb{K}$  of all bounded mappings from  $\Gamma$  to  $Y$ , endowed with the sup norm, and  $c_0(\Gamma, Y)$  stands for the subspace of  $\ell_\infty(\Gamma, Y)$  consisting of those functions  $f : \Gamma \rightarrow Y$  such that  $\lim_{\gamma \rightarrow \infty} f(\gamma) = 0$  (where  $\lim_{\gamma \rightarrow \infty}$  denotes the limit along the filter of all co-finite subsets of  $\Gamma$ ). When  $Y = \mathbb{K}$ , we simply write  $\ell_p(\Gamma)$ ,  $\ell_\infty(\Gamma)$ , and  $c_0(\Gamma)$ , respectively.

PROPOSITION 2.2. *Let  $\Gamma$  be an infinite set, let  $Y$  be an absolute-valuable Banach space, and let  $X$  stand for either  $c_0(\Gamma, Y)$  or  $\ell_p(\Gamma, Y)$  ( $1 \leq p \leq \infty$ ). Then  $X$  is absolute-valuable.*

*Proof.* Choose a product  $(y, z) \rightarrow yz$  on  $Y$  converting  $Y$  into an absolute-valued algebra, and an injective mapping  $\phi : \Gamma \times \Gamma \rightarrow \Gamma$ . Given two functions  $u$  and  $v$  from  $\Gamma$  to  $Y$ , we can consider the mapping  $u \diamond v : \Gamma \rightarrow Y$  defined by

$$(u \diamond v)(\gamma) := \begin{cases} u(i)v(j) & \text{if } \gamma = \phi(i, j) \text{ for some } (i, j) \in \Gamma \times \Gamma \\ 0 & \text{if } \gamma \notin \phi(\Gamma \times \Gamma) \end{cases}.$$

Then it is straightforward that  $u \diamond v$  belongs to  $X$  whenever  $u$  and  $v$  are in  $X$ , and that  $X$  becomes an absolute-valued algebra under the product  $\diamond$ .  $\square$

Other examples of absolute-valuable Banach spaces are given in the next theorem.

THEOREM 2.3. *Let  $1 \leq p \leq \infty$ , let  $\Gamma_1$  be an infinite set, and let  $X_1$  stand for  $\ell_p(\Gamma_1)$ . Then  $X_1^*$  is absolute-valuable. Moreover, if  $\Gamma_2$  is another infinite set, and if  $X_2$  stands for  $\ell_p(\Gamma_2)$ , then  $\mathcal{L}(X_1, X_2)$ , and  $\mathcal{K}(X_1, X_2)$  are absolute-valuable.*

*Proof.* Let  $n = 1, 2$ . Fix a bijective mapping  $\phi_n : \Gamma_n \times \Gamma_n \rightarrow \Gamma_n$ . Then the mapping  $\Psi_n : X_n \rightarrow \ell_p(\Gamma_n \times \Gamma_n)$ , defined by  $\Psi_n(x) := x \circ \phi_n$  for every  $x$  in  $X_n$ , is a surjective linear isometry. On the other hand, given  $h$  in  $\ell_p(\Gamma_n \times \Gamma_n)$ , we can consider the element  $\Phi_n(h)$  of  $\ell_p(\Gamma_n, X_n)$  defined by

$[\Phi_n(h)(i)](j) := h(i, j)$  for all  $i, j$  in  $\Gamma_n$ , so that the mapping  $\Phi_n : h \rightarrow \Phi_n(h)$  becomes a linear isometry from  $\ell_p(\Gamma_n \times \Gamma_n)$  onto  $\ell_p(\Gamma_n, X_n)$ . Moreover, given  $T \in \mathcal{L}(X, Y)$  (where  $X$  and  $Y$  stand for arbitrary Banach spaces), we can consider the bounded linear operator  $T^{[n]}$  from  $\ell_p(\Gamma_n, X)$  to  $\ell_p(\Gamma_n, Y)$  defined by  $[T^{[n]}(g)](i) := T(g(i))$  for every  $g \in \ell_p(\Gamma_n, X)$  and every  $i \in \Gamma_n$  (so that we have  $\|T^{[n]}\| \leq \|T\|$ ). Finally, we consider the surjective linear isometry  $r : \ell_p(\Gamma_1, X_2) \rightarrow \ell_p(\Gamma_2, X_1)$  defined by  $[(r(f))(i)](j) = [f(j)](i)$  for every  $f \in \ell_p(\Gamma_1, X_2)$  and every  $(i, j) \in \Gamma_2 \times \Gamma_1$ .

Now, given  $F, G \in \mathcal{L}(X_1, X_2)$ , we put

$$F \square G := \Psi_2^{-1} \circ \Phi_2^{-1} \circ F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1 \in \mathcal{L}(X_1, X_2),$$

so that we have  $\|F \square G\| \leq \|F\| \|G\|$ . To see the converse inequality, recall from the proof of Proposition 2.2 that  $X_n$  becomes an absolute-valued algebra under the product  $\diamond_n$  defined by  $(x \diamond_n y)(\phi_n(i, j)) := x(i)y(j)$  for all  $x, y$  in  $X_n$  and all  $i, j$  in  $\Gamma_n$ . We claim that, for  $x, y$  in  $X_1$  and  $F, G$  in  $\mathcal{L}(X_1, X_2)$ , the equality

$$(F \square G)(x \diamond_1 y) = F(x) \diamond_2 G(y)$$

holds. Indeed, for  $i, j$  in  $\Gamma_2$ , we have

$$\begin{aligned} [(F \square G)(x \diamond_1 y)]\phi_2(i, j) &= [\Psi_2^{-1} \circ \Phi_2^{-1} \circ F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y)]\phi_2(i, j) \\ &= [\Phi_2^{-1} \circ F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y)](i, j) \\ &= [[F^{[2]} \circ r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y)](j)](i) \\ &= [F[(r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(j)]](i). \end{aligned}$$

But, for  $k$  in  $\Gamma_1$ , we have

$$\begin{aligned} [(r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(j)](k) &= [(G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(k)](j) \\ &= [G[\Phi_1 \circ \Psi_1(x \diamond_1 y)(k)]](j). \end{aligned}$$

Also, for  $t$  in  $\Gamma_1$ , we have

$$\begin{aligned} [\Phi_1 \circ \Psi_1(x \diamond_1 y)(k)](t) &= [\Psi_1(x \diamond_1 y)](k, t) \\ &= (x \diamond_1 y)(\phi_1(k, t)) \\ &= x(k)y(t) = [x(k)y](t), \end{aligned}$$

and therefore

$$\begin{aligned} [(r \circ G^{[1]} \circ \Phi_1 \circ \Psi_1(x \diamond_1 y))(j)](k) &= [G[x(k)y]](j) \\ &= x(k)[G(y)](j) \\ &= [G(y)(j)x](k). \end{aligned}$$

It follows that

$$\begin{aligned} [(F \square G)(x \diamond_1 y)]\phi_2(i, j) &= [F[G(y)(j)x]](i) \\ &= G(y)(j)F(x)(i) \\ &= [F(x) \diamond_2 G(y)]\phi_2(i, j), \end{aligned}$$

which proves the claim. Now, for  $F, G$  in  $\mathcal{L}(X_1, X_2)$  and  $x, y$  in the closed unit ball of  $X_1$ , we have

$$\|F(x)\|\|G(y)\| = \|F(x) \diamond_2 G(y)\| = \|(F \square G)(x \diamond_1 y)\| \leq \|F \square G\|,$$

and hence  $\|F\|\|G\| \leq \|F \square G\|$ . It follows that  $(\mathcal{L}(X_1, X_2), \square)$  is an absolute-valued algebra, and therefore  $\mathcal{L}(X_1, X_2)$  is absolutely valuable.

Since  $X_2$  has the approximation property, to prove that  $\mathcal{K}(X_1, X_2)$  is absolutely valuable it is enough to show that, if  $F$  and  $G$  are rank-one operators from  $X_1$  to  $X_2$ , then so is the operator  $F \square G$ . Let the elements  $F$  and  $G$  of  $\mathcal{L}(X_1, X_2)$  have one-dimensional range (say  $F = f \odot x : z \rightarrow f(z)x$  and  $G = g \odot y$ , for some  $x, y \in X_2 \setminus \{0\}$  and  $f, g \in X_1^* \setminus \{0\}$ ). A straightforward but tedious calculation (like the one in the proof of the claim above) shows that the equality  $F \square G = (f \sharp g) \odot (x \diamond_2 y)$  holds, where  $f \sharp g$  is the element of  $X_1^*$  defined by  $f \sharp g := f \circ g^{[1]} \circ \Phi_1 \circ \Psi_1$ . Therefore  $F \square G$  has one-dimensional range.

To conclude the proof, let us show that  $X_1^*$  becomes an absolute-valued algebra under the product  $\sharp$  defined in the above paragraph. But, if  $f, g$  are in  $X_1^*$ , it is enough to choose a norm-one element  $x$  in  $X_2$ , to have

$$\begin{aligned} \|f\|\|g\| &= \|f \odot x\|\|g \odot x\| = \|(f \odot x) \square (g \odot x)\| \\ &= \|(f \sharp g) \odot (x \diamond_2 x)\| = \|f \sharp g\|\|x \diamond_2 x\| = \|f \sharp g\|. \quad \square \end{aligned}$$

Repeating almost verbatim the proof of Theorem 2.3, we obtain:

**PROPOSITION 2.4.** *Let  $\Gamma_1$  and  $\Gamma_2$  be infinite sets. Then  $\mathcal{L}(c_0(\Gamma_1), c_0(\Gamma_2))$  and  $\mathcal{K}(c_0(\Gamma_1), c_0(\Gamma_2))$  are absolute-valuable.*

For later reference, we emphasize the following consequence of Theorem 2.3.

**COROLLARY 2.5.** *Every infinite-dimensional Hilbert space over  $\mathbb{K}$  is absolute-valuable. Moreover, if  $H$  and  $K$  are infinite-dimensional Hilbert spaces over  $\mathbb{K}$ , then  $\mathcal{L}(H, K)$  and  $\mathcal{K}(H, K)$  are absolute-valuable.*

According to Proposition 2.1, a finite-dimensional Banach space is absolute-valuable if and only if so is its dual. Moreover, by the same proposition,  $\mathbb{K}$  is the unique finite-dimensional absolute-valuable Banach space  $X$  over  $\mathbb{K}$  such that  $\mathcal{L}(X)$  is absolute-valuable. In view of Theorem 2.3, in the infinite-dimensional case things are not so clear, so that the following question becomes natural.

QUESTION 2.6. For an infinite-dimensional Banach space  $X$  consider the following conditions:

- (1)  $X$  is absolute-valuable.
- (2)  $X^*$  is absolute-valuable.
- (3)  $\mathcal{L}(X)$  is absolute-valuable.
- (4)  $\mathcal{K}(X)$  is absolute-valuable.

Is there some dependence between these four conditions?

For the moment, we only know that Condition 2 in Question 2.6 does not imply Condition 1. This will follow from Proposition 2.8 below. This proposition will also provide us with the first “natural” examples of infinite-dimensional non-absolute-valuable Banach spaces. Given an infinite set  $\Gamma$ , we denote by  $c(\Gamma)$  the subspace of  $\ell_\infty(\Gamma)$  consisting of those functions  $f : \Gamma \rightarrow \mathbb{K}$  such that  $\lim_{\gamma \rightarrow \infty} f(\gamma)$  does exist. Given a Hausdorff compact topological space  $E$ , we denote by  $C^\mathbb{K}(E)$  the Banach space over  $\mathbb{K}$  of all  $\mathbb{K}$ -valued continuous functions on  $E$ .

LEMMA 2.7. *Let  $\Gamma$  be an infinite set, and let  $f_1, f_2$  be in  $c(\Gamma)$  such that there exist linear isometries  $T_1, T_2 : c(\Gamma) \rightarrow c(\Gamma)$  satisfying  $T_1(f_1) = T_2(f_2)$ . Then  $\lim_{\gamma \rightarrow \infty} |f_1(\gamma)| = \lim_{\gamma \rightarrow \infty} |f_2(\gamma)|$ .*

*Proof.* After considering the one-point compactification  $\Gamma \cup \{\infty\}$  of the discrete space  $\Gamma$ , we identify  $c(\Gamma)$  with  $C^\mathbb{K}(\Gamma \cup \{\infty\})$  by putting  $f(\infty) := \lim_{\gamma \rightarrow \infty} f(\gamma)$  for every  $f \in c(\Gamma)$ . According to [15], for each linear isometry  $T : c(\Gamma) \rightarrow c(\Gamma)$  there exist a closed subset  $E_T$  of  $\Gamma \cup \{\infty\}$ , a surjective continuous mapping  $\phi_T : E_T \rightarrow \Gamma \cup \{\infty\}$ , and an  $\alpha_T \in C^\mathbb{K}(\Gamma \cup \{\infty\})$  with  $|\alpha_T(t)| = 1$  for every  $t \in E_T$  satisfying

$$T(f)(t) = f(\phi_T(t))\alpha_T(t)$$

for all  $t \in E_T$  and  $f \in C^\mathbb{K}(\Gamma \cup \{\infty\})$ . This implies that  $\infty$  belongs to  $E_T$  and that  $\phi_T(\infty) = \infty$ . Now, for the elements  $f_1, f_2 \in c(\Gamma)$  in the statement of the lemma we have

$$\begin{aligned} f_1(\infty)\alpha_{T_1}(\infty) &= f_1(\phi_{T_1}(\infty))\alpha_{T_1}(\infty) = T_1(f_1)(\infty) \\ &= T_2(f_2)(\infty) = f_2(\phi_{T_2}(\infty))\alpha_{T_2}(\infty) = f_2(\infty)\alpha_{T_2}(\infty), \end{aligned}$$

and hence  $|f_1(\infty)| = |f_2(\infty)|$ .  $\square$

PROPOSITION 2.8. *Let  $\Gamma$  be an infinite set. Then  $c(\Gamma)$  is not absolute-valuable.*

*Proof.* Assume that  $c(\Gamma)$  is an absolute-valued algebra under some product  $\diamond$ . Let  $f_1$  denote the constant function equal to 1 on  $\Gamma$ , let  $f_2$  be the characteristic function on  $\Gamma$  of a previously chosen singleton, and let  $T_1$  and  $T_2$  stand for the linear isometries from  $c(\Gamma)$  to itself defined by  $T_1(f) := f_2 \diamond f$

and  $T_2(f) := f \diamond f_1$ , respectively. Since  $T_1(f_1) = T_2(f_2)$ , Lemma 2.7 applies giving

$$1 = \lim_{\gamma \rightarrow \infty} |f_1(\gamma)| = \lim_{\gamma \rightarrow \infty} |f_2(\gamma)| = 0,$$

a contradiction.  $\square$

Despite Propositions 2.1 and 2.8, every Banach space becomes absolute-valuable up to a suitable enlargement. This will be proved in Theorem 2.11 below. Following [9, 12.1], a *tensor norm*  $\alpha$  on the class BAN of all Banach spaces assigns to each pair  $(X, Y)$  of Banach spaces a norm  $\|\cdot\|_\alpha = \alpha(\cdot; X, Y)$  on the algebraic tensor product  $X \otimes Y$  such that the following two conditions are satisfied:

- (1) For every pair  $(X, Y)$  of Banach spaces, we have  $\|\cdot\|_\epsilon \leq \|\cdot\|_\alpha \leq \|\cdot\|_\pi$  on  $X \otimes Y$ , where  $\epsilon$  and  $\pi$  denote respectively the injective and the projective tensor norm.
- (2) If  $X, Y, Z$ , and  $T$  are Banach spaces, and if  $F : X \rightarrow Z$  and  $G : Y \rightarrow T$  are bounded linear operators, then the linear operator  $F \otimes G$  from  $(X \otimes Y, \|\cdot\|_\alpha)$  to  $(Z \otimes T, \|\cdot\|_\alpha)$ , determined by

$$(2.1) \quad (F \otimes G)(x \otimes y) := F(x) \otimes G(y)$$

for every  $(x, y) \in X \times Y$ , is continuous with norm  $\leq \|F\| \|G\|$ .

Given a tensor norm  $\alpha$  on BAN, and Banach spaces  $X$  and  $Y$ , we denote by  $X \widetilde{\otimes}_\alpha Y$  the completion of the algebraic tensor product  $X \otimes Y$  under the norm  $\|\cdot\|_\alpha$ . Given Banach spaces  $X$  and  $Y$  over  $\mathbb{K}$ , we denote by  $\mathcal{F}(X, Y)$  the space of all finite-rank operators from  $X$  to  $Y$ , and by  $\overline{\mathcal{F}(X, Y)}$  the closure of  $\mathcal{F}(X, Y)$  in  $\mathcal{L}(X, Y)$ . The convention  $\mathcal{F}(X) := \mathcal{F}(X, X)$  will be subsumed.

**LEMMA 2.9.** *Let  $X$  and  $Y$  be Banach spaces over  $\mathbb{K}$ . Assume that there exists a tensor norm  $\alpha$  on BAN such that  $X \widetilde{\otimes}_\alpha X$  is linearly isometric to a quotient of  $X$  and  $Y \widetilde{\otimes}_\alpha Y$  is linearly isometric to a subspace of  $Y$ . Then  $\mathcal{L}(X, Y)$  and  $\overline{\mathcal{F}(X, Y)}$  are absolute-valuable.*

*Proof.* Let  $\Phi$  be a linear isometry from  $Y \widetilde{\otimes}_\alpha Y$  to  $Y$ , and let  $\Psi$  be a continuous linear surjection from  $X$  to  $X \widetilde{\otimes}_\alpha X$  such that the induced bijection  $X/\ker(\Psi) \rightarrow X \widetilde{\otimes}_\alpha X$  is an isometry. Note that, as a consequence of Condition 1 for tensor norms, for  $u, v$  both in either  $X$  or  $Y$  we have

$$(2.2) \quad \|u \otimes v\|_\alpha = \|u\| \|v\|.$$

Then, by (2.2) and Condition 2 for tensor norms, given  $F$  and  $G$  in  $\mathcal{L}(X, Y)$ , there exists a unique element  $F \widetilde{\otimes} G$  in  $\mathcal{L}(X \widetilde{\otimes}_\alpha X, Y \widetilde{\otimes}_\alpha Y)$  which extends  $F \otimes G : X \otimes X \rightarrow Y \otimes Y$ , and we have  $\|F \widetilde{\otimes} G\| = \|F\| \|G\|$ . It follows that, putting

$$(2.3) \quad F \square G := \Phi \circ (F \widetilde{\otimes} G) \circ \Psi \in \mathcal{L}(X, Y),$$

$(\mathcal{L}(X, Y), \square)$  becomes an absolute-valued algebra.

Let  $F$  and  $G$  be finite-rank operators from  $X$  to  $Y$ . Then, by (2.1),  $F(X) \otimes G(X)$  is a finite-dimensional subspace of  $Y \widetilde{\otimes}_\alpha Y$  containing the range of  $F \otimes G$ . Therefore, by (2.3), also  $F \square G$  has finite-dimensional range. In this way  $\mathcal{F}(X, Y)$  (and hence  $\overline{\mathcal{F}(X, Y)}$ ) becomes a subalgebra of  $(\mathcal{L}(X, Y), \square)$ .  $\square$

**COROLLARY 2.10.** *For a Banach space  $Y$  over  $\mathbb{K}$  and a tensor norm  $\alpha$  on BAN, consider the following conditions:*

- (1)  $Y \widetilde{\otimes}_\alpha Y$  is linearly isometric to a subspace of  $Y$ .
- (2)  $Y \widetilde{\otimes}_\alpha Y$  is linearly isometric to a quotient of  $Y$ .

Then we have:

- (i) *If the Banach space  $Y$  satisfies Condition (1) for some tensor norm  $\alpha$ , then  $Y$  is absolute-valuable.*
- (ii) *If the Banach space  $Y$  satisfies Condition (2) for some tensor norm  $\alpha$ , then  $Y^*$  is absolute-valuable.*
- (iii) *If the Banach space  $Y$  satisfies both Conditions (1) and (2) for the same tensor norm  $\alpha$ , then  $\mathcal{L}(Y)$  and  $\overline{\mathcal{F}(Y)}$  are absolute-valuable.*

*Proof.* Since  $Y = \mathcal{L}(\mathbb{K}, Y)$ , and  $Y^* = \mathcal{L}(Y, \mathbb{K})$ , and  $\mathbb{K} \widetilde{\otimes}_\alpha \mathbb{K}$  is linearly isometric to  $\mathbb{K}$ , the result follows from Lemma 2.9.  $\square$

Given a Banach space  $X$ , we denote by  $\text{dens}(X)$  the density character of  $X$ .

**THEOREM 2.11.** *Every Banach space  $X$  over  $\mathbb{K}$  can be isometrically regarded as a subspace of a Banach space  $Y$  over  $\mathbb{K}$  with  $\text{dens}(Y) = \text{dens}(X)$  and such that  $Y$ ,  $Y^*$ ,  $\mathcal{L}(Y)$ , and  $\mathcal{K}(Y)$  are absolute-valuable.*

*Proof.* Let  $X$  be a Banach space over  $\mathbb{K}$ , and let  $F$  denote the Hausdorff compact space consisting of the closed unit ball of  $X^*$  and the weak\* topology. Then we can consider  $X$  isometrically as a subspace of  $C^{\mathbb{K}}(F)$ . For any Hausdorff compact space  $G$ , let  $*$  denote the identity mapping on  $C^{\mathbb{K}}(G)$  or the natural involution on  $C^{\mathbb{K}}(G)$  depending on whether  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . Let  $Z$  stand for the unital closed  $*$ -invariant subalgebra of  $C^{\mathbb{K}}(F)$  generated by  $X$ . Then we have  $\text{dens}(Z) = \text{dens}(X)$  and  $Z = C^{\mathbb{K}}(E)$  for some Hausdorff compact space  $E$ . Put  $Y := C^{\mathbb{K}}(E^{\mathbb{N}})$ . Then  $Z$  is linearly isometric to a subspace of  $Y$ . Indeed, fixing  $n \in \mathbb{N}$  and denoting by  $\pi_n$  the  $n$ -coordinate projection from  $E^{\mathbb{N}}$  onto  $E$ , the mapping  $f \mapsto f \circ \pi_n$  is a linear isometry from  $Z$  into  $Y$ . It follows that  $X$  is linearly isometric to a subspace of  $Y$ . Moreover the equality  $\text{dens}(Y) = \text{dens}(Z)$  holds. Indeed, if  $\mathcal{D}$  is a dense subset of  $Z$  whose cardinal equals  $\text{dens}(Z)$ , then, by the Stone-Weierstrass theorem, the unital  $*$ -invariant subalgebra of  $Y$  generated by the set  $\{f \circ \pi_n : (f, n) \in \mathcal{D} \times \mathbb{N}\}$  is dense in  $Y$ . On the other hand, by [9, Example 4.2.(3)], the complete injective tensor product  $Y \widetilde{\otimes}_\epsilon Y$  is linearly isometric to  $C^{\mathbb{K}}(E^{\mathbb{N}} \times E^{\mathbb{N}})$ . Since  $E^{\mathbb{N}} \times E^{\mathbb{N}}$  is homeomorphic to  $E^{\mathbb{N}}$ , the actual situation is that  $Y \widetilde{\otimes}_\epsilon Y$  is linearly

isometric to  $Y$ . Since  $\epsilon$  is a tensor norm [9, 4.1], and  $Y$  has the approximation property, the proof is concluded by applying Corollary 2.10.  $\square$

As a consequence of Theorem 2.11 above we see that absolute valuability cannot imply any hereditary property of isometric or isomorphic type. This situation is not new. Indeed, the same happens with transitivity [5, Corollary 2.21] (see Section 4 below for the definition), Mazur’s intersection property, and Mazur’s  $w^*$ -intersection property of the dual [16]. In any case, we already know that, both in the finite-dimensional and the infinite-dimensional cases, absolute valuability is not isometrically innocuous (by Propositions 2.1 and 2.8), and we will see below that it is even not isomorphically innocuous (a consequence of Proposition 3.7 or 3.8).

### 3. Isomorphic aspects of absolute valuability

Let  $X$  be a Banach space over  $\mathbb{K}$ . We denote by  $B_X$  and  $S_X$  the unit closed ball and the unit sphere, respectively, of  $X$ . For  $x$  in  $S_X$ , we define the set  $D(X, x)$  of states of  $X$  relative to  $x$  by

$$D(X, x) := \{f \in B_{X^*} : f(x) = 1\}.$$

For convenience, we say that  $X$  is *almost smooth* if, for every  $x \in S_X$  and all  $\phi, \psi \in D(X, x)$ , we have  $\|\phi - \psi\| < 2$ .

LEMMA 3.1. *Let  $Y$  and  $Z$  be non-zero almost smooth Banach spaces, and put  $X := Y \overset{\ell_1}{\oplus} Z$ . Then  $X$  is not absolute-valuable.*

*Proof.* The key observation is that the elements of  $S_Y \cup S_Z$  are characterized in  $X$  as those elements  $x$  of  $S_X$  such that there exist  $\phi, \psi$  in  $D(X, x)$  satisfying  $\|\phi - \psi\| = 2$ . To prove this, let us identify  $X^*$  with  $Y^* \overset{\ell_\infty}{\oplus} Z^*$ . For  $y$  in  $S_Y$ , we can choose  $f \in D(Y, y)$  and  $g \in S_{Z^*}$ , so that  $\phi := (f, g)$  and  $\psi := (f, -g)$  belong to  $D(X, y)$  and satisfy  $\|\phi - \psi\| = 2$ . Conversely, if  $x = (y, z)$  is in  $S_X$  and satisfies  $y \neq 0 \neq z$ , then we see that

$$D(X, x) = \left\{ (f, g) : (f, g) \in D\left(Y, \frac{y}{\|y\|}\right) \times D\left(Z, \frac{z}{\|z\|}\right) \right\},$$

and hence, since  $Y$  and  $Z$  are almost smooth, for all  $\phi, \psi \in D(X, x)$  we have  $\|\phi - \psi\| < 2$ .

Let  $T : X \rightarrow X$  be a linear isometry, and let  $y$  be in  $S_Y$ . By the above paragraph and the Hahn-Banach theorem, there exist  $\phi, \psi$  in  $D(X, T(y))$  with  $\|\phi - \psi\| = 2$ . Again by the above paragraph, this implies  $T(y) \in S_Y \cup S_Z$ . Now assume that  $X$  is an absolute-valued algebra under some product  $\diamond$ . Let us fix  $y$  in  $S_Y$ . For  $x$  in  $S_X$ , the mapping  $t \rightarrow x \diamond t$  from  $X$  to  $X$  is a linear isometry, and hence we have  $x \diamond y \in S_Y \cup S_Z$ . Since  $X \diamond y$  is a subspace of  $X$ , and  $Y \cap Z = 0$ , it follows that either  $X \diamond y \subseteq Y$  or  $X \diamond y \subseteq Z$ . But both possibilities

are contradictory because, if one of them happened (say  $X \diamond y \subseteq Y$ ), then the mapping  $x \rightarrow x \diamond y$  would be a linear isometry from the non almost smooth Banach space  $X$  to the almost smooth Banach space  $Y$ .  $\square$

Again, for convenience, we introduce *almost rotund* Banach spaces as those Banach spaces  $X$  such that there is no segment of length 2 contained in  $S_X$ .

LEMMA 3.2. *Let  $Y$  and  $Z$  be non-zero almost rotund Banach spaces over  $\mathbb{K}$ , and put  $X := Y \overset{\ell_\infty}{\oplus} Z$ . Then  $X$  is not absolute-valuable.*

*Proof.* First note that elements of  $S_Y \cup S_Z$  lie in  $S_X$  and are mid-points of segments in  $B_X$  of length 2. Indeed, if  $y$  is in  $S_Y$ , then, choosing  $z \in S_Z$ , we have  $y = \frac{1}{2}[(y, z) + (y, -z)]$  with  $(y, z), (y, -z) \in B_X$  and  $\|(y, z) - (y, -z)\| = 2$ . As a partial converse, we claim that, if  $x$  belongs to  $S_X$  and is a mid point of a segment in  $B_X$  of length 2, then  $\min\{\|\pi_Y(x)\|, \|\pi_Z(x)\|\} < 1$ , where  $\pi_Y$  and  $\pi_Z$  stand for the natural projections from  $X$  to  $Y$  and  $Z$ , respectively. Indeed, if  $x = (y, z)$  is in  $S_X$  (say for example  $y \in S_Y$ ) and if  $x = \frac{1}{2}[(y_1, z_1) + (y_2, z_2)]$  with  $(y_1, z_1), (y_2, z_2) \in B_X$  and  $\|(y_1, z_1) - (y_2, z_2)\| = 2$ , then, by the almost rotundness of  $Y$ , we have  $\|y_1 - y_2\| < 2$  and hence  $\|\frac{z_1 - z_2}{2}\| = 1$ , so that, by the almost rotundness of  $Z$ , we have  $\|z_1 + z_2\| < 2$  and therefore  $\|z\| < 1$ .

It follows from the above paragraph that, if  $T : X \rightarrow X$  is a linear isometry and if  $y$  is in  $S_Y$ , then each of the possibilities  $\|\pi_Y T(y)\| = 1$  and  $\|\pi_Z T(y)\| = 1$  excludes the other. Now assume that  $X$  is an absolute-valued algebra under some product  $\diamond$ . Let us fix  $y$  in  $S_Y$ . Since for  $x$  in  $S_X$  the mapping  $t \rightarrow x \diamond t$  from  $X$  to  $X$  is a linear isometry, we deduce that  $S_X = A \cup B$ , where

$$\begin{aligned} A &:= \{x \in S_X : \|\pi_Y(x \diamond y)\| = 1\} \quad \text{and} \\ B &:= \{x \in S_X : \|\pi_Z(x \diamond y)\| = 1\} \end{aligned}$$

are disjoint closed subsets of  $S_X$ . Since  $S_X$  is connected, we have that either  $A = S_X$  or  $B = S_X$ . But both possibilities are contradictory because, if one of them happened (say  $A = S_X$ ), then the mapping  $x \rightarrow \pi_Y(x \diamond y)$  would be a linear isometry from the non almost rotund Banach space  $X$  to the almost rotund Banach space  $Y$ .  $\square$

REMARK 3.3. In the case  $\mathbb{K} = \mathbb{R}$ , particular cases of Lemmas 3.1 and 3.2 can be reformulated in more classical terms, giving rise to new “natural” examples of non absolute-valuable Banach spaces. To this end, we recall that a *compact convex set* is a convex compact subset of some Hausdorff locally convex space, and that, given a compact convex set  $K$ , the real Banach space of all real-valued affine continuous functions on  $K$  (with the sup norm) is usually denoted by  $\mathcal{A}(K)$ . Let  $Y$  be a non-zero real Banach space, and let  $K_Y$  stand for the compact convex set  $(B_{Y^*}, w^*)$ . It is well-known and easy to see that  $\mathcal{A}(K_Y)$  is linearly isometric to  $\mathbb{R} \overset{\ell_1}{\oplus} Y$ . Then, by Lemma 3.1,  $\mathcal{A}(K_Y)$

fails to be absolute-valuable provided  $Y$  is almost smooth. Moreover, noticing that almost rotundness of  $Y^*$  implies almost smoothness of  $Y$ , the same holds provided  $Y^*$  is almost rotund, with the additional information that, in this case  $\mathcal{A}(K_Y)^*$  fails also to be absolute-valuable (thanks to Lemma 3.2).

To emphasize particular outstanding cases of the facts just mentioned, recall that the Banach space  $Y$  is said to be *smooth* if, for every  $y$  in  $S_Y$ ,  $D(Y, y)$  is reduced to a singleton, and *rotund* if every element in  $S_Y$  is an extreme point of  $B_Y$ . It follows from the above paragraph that, if  $Y$  is smooth, then  $\mathcal{A}(K_Y)$  is not absolute-valuable, and if actually  $Y^*$  is rotund, then  $\mathcal{A}(K_Y)^*$  fails also to be absolute-valuable.

A Banach space  $X$  is called *weakly countably determined* if there exists a countable collection  $\{K_n\}_{n \in \mathbb{N}}$  of  $w^*$ -compact subsets of  $X^{**}$  such that for every  $x$  in  $X$  and every  $u$  in  $X^{**} \setminus X$  there is  $n_0$  such that  $x \in K_{n_0}$  and  $u \notin K_{n_0}$ . If  $X$  is either reflexive, separable, or of the form  $c_0(\Gamma)$  for any set  $\Gamma$ , then  $X$  is weakly countably determined. In fact, the class of weakly countably determined Banach spaces is hereditary, and contains the non hereditary class of weakly compactly generated Banach spaces (see [10, Example VI.2.2] for details).

**THEOREM 3.4.** *Every weakly countably determined real Banach space, different from  $\mathbb{R}$ , is isomorphic to a real Banach space  $X$  such that both  $X$  and  $X^*$  are not absolute-valuable.*

*Proof.* Let  $Z$  be a weakly countably determined real Banach space different from  $\mathbb{R}$ . Take a closed maximal subspace  $P$  of  $Z$ . Since  $P$  is also a weakly countably determined real Banach space,  $P$  is isomorphic to a Banach space  $Y$  such that  $Y^*$  becomes rotund [10, Theorem VII.1.16]. Since such a space  $Y$  is smooth, it follows from Lemma 3.1 that the Banach space  $X := \mathbb{R} \overset{\ell_1}{\oplus} Y$  is not absolute-valuable. Moreover, clearly,  $X$  is isomorphic to  $Z$ . Since  $X^* = \mathbb{R} \overset{\ell_\infty}{\oplus} Y^*$  and  $Y^*$  is rotund, it follows from Lemma 3.2 that  $X^*$  is not absolute-valuable.  $\square$

A consequence of Proposition 2.1 is that every finite-dimensional Banach space over  $\mathbb{K}$ , different from  $\mathbb{K}$ , is isomorphic to a Banach space  $X$  such that both  $X$  and  $X^*$  are not absolute-valuable. We do not know if the same remains true when finite dimensionality is altogether removed. In fact we are at present unable to answer the following simpler question.

**QUESTION 3.5.** Is every infinite-dimensional Banach space over  $\mathbb{K}$  isomorphic to a non-absolute-valuable Banach space?

**REMARK 3.6.** Theorem 3.4 gives a partial affirmative answer to Question 3.5. However, one of the main tools in its proof has not been fully

exploited. Indeed, Lemma 3.1 actually shows that a Banach space  $Z$  over  $\mathbb{K}$  is isomorphic to a non-absolute-valuable Banach space whenever  $Z$  is different from  $\mathbb{K}$  and has an almost smooth equivalent renorming. Thus one might wonder whether every Banach space  $Z$  has such an equivalent renorming. As a matter of fact, the answer to this last question is negative even for  $\mathbb{K} = \mathbb{R}$ . A counterexample is given by the space  $Z := \ell_\infty(\Gamma)$  for any uncountable set  $\Gamma$ , since every equivalent renorming of  $Z$  has an isometric copy of  $\ell_\infty$  [20] (see also [10, Theorem II.7.12]).

By Proposition 2.1, most finite-dimensional Banach spaces are not isomorphic to any absolute-valuable Banach space. The remaining part of this section is devoted to prove the existence of infinite-dimensional Banach spaces which are not isomorphic to any absolute-valuable Banach space. Such spaces are precisely those constructed by Gowers-Maurey [12] and Shelah-Steprans-Wark (see [23] and [25]) with the common property of having “few” operators. We recall that a Banach space  $X$  is said to be *hereditarily indecomposable* if, for every closed subspace  $Y$  of  $X$ , the unique complemented subspaces of  $Y$  are the finite-dimensional ones and the closed finite-codimensional ones.

**PROPOSITION 3.7.** *There exists an infinite-dimensional separable reflexive Banach space over  $\mathbb{K}$  which is not isomorphic to any absolute-valuable Banach space.*

*Proof.* We will in fact prove that every infinite-dimensional hereditarily indecomposable Banach space over  $\mathbb{K}$  fails to be absolute-valuable. The result will follow from the existence of infinite-dimensional hereditarily indecomposable separable reflexive Banach spaces over  $\mathbb{K}$  [12, Section 3], and the clear fact that hereditary indecomposability is preserved under isomorphisms. Let  $X$  be an infinite-dimensional hereditarily indecomposable Banach space over  $\mathbb{K}$ . By [12, Corollary 19 and Theorem 21],  $X$  is not isomorphic to any of its proper subspaces. Assume that, for some product  $\diamond$  on  $X$ ,  $(X, \diamond)$  becomes an absolute-valued algebra. Then, for every nonzero element  $x$  in  $X$ , the operators of left and right multiplication by  $x$  on the algebra  $(X, \diamond)$  are isomorphisms onto their ranges, and hence they are bijective. Therefore  $(X, \diamond)$  is an absolute valued division algebra. By [26],  $X$  is finite dimensional, a contradiction.  $\square$

In the above proof we have just applied the theorem of F. B. Wright [26] that absolute-valued division algebras are finite-dimensional. It is worth mentioning that, today, such a theorem is an easy consequence of the celebrated Urbanik-Wright theorem [24] that  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$ , and  $\mathbb{O}$  are the unique absolute-valued real algebras with a unit (see [17, Proposition 1.2] for details).

**PROPOSITION 3.8.** *There exists a non-separable reflexive Banach space over  $\mathbb{K}$  which is not isomorphic to any absolute-valuable Banach space.*

*Proof.* The authors of [23] construct a non-separable Banach space  $Z$  over  $\mathbb{K}$  satisfying the following Property  $\mathcal{P}$ :

$\mathcal{P}$  Every element  $F$  of  $\mathcal{L}(Z)$  has the form  $S + \rho I_Z$ , where  $I_Z$  denotes the identity mapping on  $Z$ ,  $\rho = \rho(F)$  belongs to  $\mathbb{K}$ , and  $S = S(F) \in \mathcal{L}(Z)$  has separable range.

Very recently, H. M. Wark [25] refined the construction of [23] to show the existence of a non-separable reflexive Banach space  $Z$  satisfying  $\mathcal{P}$ . Now, since Property  $\mathcal{P}$  is preserved under isomorphisms, to prove the result it is enough to show that every non-separable Banach space  $Z$  satisfying  $\mathcal{P}$  fails to be absolute-valuable. Let  $Z$  be such a Banach space. We note that, for  $F$  in  $\mathcal{L}(Z)$ , the couple  $(\rho(F), S(F))$  given by  $\mathcal{P}$  is uniquely determined, and that the mappings  $\rho : F \rightarrow \rho(F)$  and  $S : F \rightarrow S(F)$  from  $\mathcal{L}(Z)$  to  $\mathbb{K}$  and  $\mathcal{L}(Z)$ , respectively, are linear. Since  $\ker(\rho)$  consists of those elements of  $\mathcal{L}(Z)$  which have separable range, we have  $\rho(F) \neq 0$  whenever the operator  $F$  on  $Z$  is an isomorphism onto its range. Assume that  $Z$  is an absolute-valued algebra under some product  $\diamond$ . For  $z$  in  $Z$ , denote by  $L_z$  the operator of left multiplication by  $z$  on the algebra  $(Z, \diamond)$ . Since  $L_z$  is an isomorphism onto its range whenever  $z$  is nonzero, it follows that the linear mapping  $z \rightarrow \rho(L_z)$  from  $Z$  to  $\mathbb{K}$  is injective. Therefore  $Z$  is one-dimensional, a contradiction.  $\square$

REMARK 3.9. (a) By a *normed algebra* over  $\mathbb{K}$  we mean a non-zero algebra  $A$  over  $\mathbb{K}$  endowed with a norm  $\|\cdot\|$  satisfying  $\|xy\| \leq \|x\| \|y\|$  for all  $x, y$  in  $A$ . Following [18], a *nearly absolute-valued algebra* over  $\mathbb{K}$  will be a normed algebra  $A$  over  $\mathbb{K}$  such that there exists  $\delta > 0$  satisfying  $\|xy\| \geq \delta \|x\| \|y\|$  for all  $x, y$  in  $A$ . Of course, one can think about those Banach spaces underlying complete nearly absolute-valued algebras. Such Banach spaces will be called *nearly absolute-valuable*. It is easy to see that near absolute valuability is preserved under isomorphisms. Consequently, isomorphic copies of absolute-valuable Banach spaces are nearly absolute-valuable. If  $X$  is a finite-dimensional nearly absolute-valuable Banach space over  $\mathbb{C}$  (respectively,  $\mathbb{R}$ ), then, by [18, Remark 2.8] (respectively, [11, Chapter 11]),  $X$  has dimension equal to 1 (respectively, 1, 2, 4, or 8), and therefore, by Proposition 2.1,  $X$  is isomorphic to an absolute-valuable Banach space. However, in the infinite-dimensional setting, we do not know if every nearly absolute-valuable Banach space is isomorphic to an absolute-valuable Banach space. We certainly know that, in both the finite- and the infinite-dimensional case, nearly absolute-valued algebras need not be algebra-isomorphic to absolute-valued algebras (see [18, Introduction] and [18, Example 1.1], respectively). In any case, the proof of Proposition 3.8 actually shows that the non-separable reflexive Banach space of Wark [25] cannot be isomorphic to any nearly absolute-valuable Banach space.

(b) Recently, S. A. Argyros, J. López-Abad, and S. Todorčević [2] have constructed a non-separable reflexive hereditarily indecomposable Banach space. This, together with the fact that every infinite-dimensional hereditarily indecomposable Banach space is not absolute-valuable (see the proof of Proposition 3.7), provides us with a new proof of Proposition 3.8.

#### 4. Transitivity of the norm and absolute valuability

Given a Banach space  $X$ , we denote by  $\mathcal{G}$  the group of all surjective linear isometries on  $X$ . We recall that a Banach space  $X$  is said to be *transitive* (respectively, *almost transitive*) if, for every (equivalently, some) element  $u$  in  $S_X$  we have  $\mathcal{G}(u) = S_X$  (respectively,  $\overline{\mathcal{G}(u)} = S_X$ , where  $\overline{\phantom{x}}$  means norm closure). The reader is referred to the book of Rolewicz [22] and the survey papers of Cabello [7] and Becerra-Rodríguez [5] for a comprehensive overview of known results and fundamental questions in relation to the notions just introduced. Hilbert spaces become the natural motivating examples of transitive Banach spaces, but there are also examples of non-Hilbert almost transitive separable Banach spaces, as well as of non-Hilbert transitive non-separable Banach spaces. However, the Banach-Mazur *rotation problem* [3], the question whether every transitive separable Banach space is a Hilbert space, remains unsolved to date. Since transitive finite-dimensional Banach spaces are indeed Hilbert spaces, the rotation problem is actually interesting only in the infinite-dimensional setting. Then, since infinite-dimensional Hilbert spaces are absolute-valuable (by Corollary 2.5), it is natural to raise the following strong form of the Banach-Mazur rotation problem.

**PROBLEM 4.1.** Let  $X$  be an absolute-valuable transitive separable Banach space. Is  $X$  a Hilbert space?

Unfortunately, we are unable to provide the reader with an affirmative answer to Problem 4.1, even under the reasonable additional assumption that  $\mathcal{L}(X)$  and  $\mathcal{K}(X)$  are absolute-valuable (see again Corollary 2.5). On the other hand, a negative answer to Problem 4.1 could not be reasonably expected, since such a negative answer would also provide a negative answer to the classical Banach-Mazur rotation problem. Thus we limit ourselves for the moment to a discussion of the different requirements in Problem 4.1. Indeed, we are going to show that an affirmative answer to Problem 4.1 cannot be expected if either the transitivity of  $X$  is relaxed to almost transitivity or if the separability of  $X$  is removed.

**LEMMA 4.2.** *Let  $X$  and  $Y$  be Banach spaces over  $\mathbb{K}$ . Assume that the complete projective tensor product  $X \widetilde{\otimes}_{\pi} X$  is linearly isometric to a quotient of  $X$ , and that  $Y$  is absolute-valuable. Then  $\mathcal{L}(X, Y)$  and  $\overline{\mathcal{F}(X, Y)}$  are absolute-valuable.*

*Proof.* By the assumed absolute valuability of  $Y$  and [9, 3.2.(1)], there exists a norm-one continuous linear mapping  $\Phi : Y \widetilde{\otimes}_\pi Y \rightarrow Y$  satisfying  $\|\Phi(y_1 \otimes y_2)\| = \|y_1\| \|y_2\|$  for all  $y_1, y_2$  in  $Y$ . On the other hand, the assumption on  $X$  provides us with a continuous linear surjection  $\Psi : X \rightarrow X \widetilde{\otimes}_\pi X$  such that the induced bijection  $X/\ker(\Psi) \rightarrow X \widetilde{\otimes}_\pi X$  is an isometry. Let  $F$  and  $G$  be in  $\mathcal{L}(X, Y)$ . By [9, 3.2.(4)], there exists a unique element  $F \widetilde{\otimes} G$  in  $\mathcal{L}(X \widetilde{\otimes}_\pi X, Y \widetilde{\otimes}_\pi Y)$  which extends  $F \otimes G : X \otimes X \rightarrow Y \otimes Y$ , and we have  $\|F \widetilde{\otimes} G\| \leq \|F\| \|G\|$ . Then, writing  $F \square G := \Phi \circ (F \widetilde{\otimes} G) \circ \Psi \in \mathcal{L}(X, Y)$ , the inequality  $\|F \square G\| \leq \|F\| \|G\|$  is clear. To see the converse inequality, note that, by the properties of  $\Psi$ , the equality  $\|F \square G\| = \|\Phi \circ (F \widetilde{\otimes} G)\|$  holds, and that, by the properties of  $\Phi$  and [9, 3.2.(3)], for  $x_1, x_2$  in  $B_X$  we have

$$\begin{aligned} \|F(x_1)\| \|G(x_2)\| &= \|\Phi(F(x_1) \otimes G(x_2))\| \\ &= \|(\Phi \circ (F \widetilde{\otimes} G))(x_1 \otimes x_2)\| \leq \|\Phi \circ (F \widetilde{\otimes} G)\|. \end{aligned}$$

Now  $(\mathcal{L}(X, Y), \square)$  becomes an absolute-valued algebra. Arguing as in the concluding paragraph of the proof of Lemma 2.9, we see that  $\mathcal{F}(X, Y)$  (and hence  $\overline{\mathcal{F}(X, Y)}$ ) becomes a subalgebra of  $(\mathcal{L}(X, Y), \square)$ .  $\square$

In relation to the above lemma, the following question arises naturally.

QUESTION 4.3. Let  $X$  be a Banach space such that  $\mathcal{L}(X, Y)$  and  $\overline{\mathcal{F}(X, Y)}$  are absolute-valuable, for every absolute-valuable Banach space  $Y$ . Is  $X \widetilde{\otimes}_\pi X$  linearly isometric to a quotient of  $X$ ?

PROPOSITION 4.4. *There exists a non-Hilbert absolute-valuable almost transitive separable Banach space  $X$  such that, for every absolute-valuable Banach space  $Y$ ,  $\mathcal{L}(X, Y)$  and  $\mathcal{K}(X, Y)$  are absolute-valuable.*

*Proof.* Take  $X$  equal to the non-Hilbert separable Banach space  $L_1([0, 1])$ . By [22, Theorem 9.6.4],  $X$  is almost transitive. On the other hand, by [6, Example 42.14] and [19, Theorem 2.3],  $X \widetilde{\otimes}_\pi X$  is linearly isometric to  $X$ . This last property implies that  $X$  is absolute-valuable (by Corollary 2.10) as well as the remaining part of the result (by Lemma 4.2 and the fact that  $X^*$  has the approximation property [9, 5.2 and 5.3]).  $\square$

To derive from Proposition 4.4 that Problem 4.1 cannot be answered in the affirmative if the separability is removed, let us recall some notions and results from the theory of (Banach) ultraproducts [14]. Let  $\mathcal{U}$  be an ultrafilter on a nonempty set  $I$ , and  $\{X_i\}_{i \in I}$  a family of Banach spaces. We can consider the Banach space  $\oplus_{i \in I}^{\ell_\infty} X_i$ , together with its closed subspace

$$N_{\mathcal{U}} := \{\{x_i\}_{i \in I} \in \oplus_{i \in I}^{\ell_\infty} X_i : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The quotient space  $(\oplus_{i \in I}^{\ell_\infty} X_i)/N_{\mathcal{U}}$  is called the *ultraproduct* of the family  $\{X_i\}_{i \in I}$  relative to the ultrafilter  $\mathcal{U}$ , and is denoted by  $(X_i)_{\mathcal{U}}$ . Let  $(x_i)$  stand

for the element of  $(X_i)_\mathcal{U}$  containing a given family  $\{x_i\} \in \oplus_{i \in I}^{\ell_\infty} X_i$ . It is easy to check that  $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$ . From this equality follows the folklore fact that  $(X_i)_\mathcal{U}$  is absolute-valuable if, for every  $i \in I$ ,  $X_i$  is absolute-valuable (see for instance [17, Section 3]). On the other hand, it is also folklore that, if  $I$  is countable, if  $\mathcal{U}$  is nontrivial, and if  $X_i$  is almost transitive for every  $i \in I$ , then  $(X_i)_\mathcal{U}$  is transitive (see either [8, Lemma 1.4], [13, Remark p. 479], or [5, Proposition 2.18]). When  $X_i = X$  for every  $i \in I$  and some prefixed Banach space  $X$ , the ultraproduct  $(X_i)_\mathcal{U}$  is called the *ultrapower* of  $X$  relative to  $\mathcal{U}$ , and is denoted by  $X_\mathcal{U}$ . In this case  $X_\mathcal{U}$  contains “naturally” an isometric copy of  $X$ .

Now, taking a nontrivial ultrafilter  $\mathcal{U}$  on the set of all natural numbers, and considering the ultrapower  $X_\mathcal{U}$ , where  $X$  stands for the Banach space given by Proposition 4.4, we obtain the next corollary.

**COROLLARY 4.5.** *There exists a non-Hilbert absolute-valuable transitive (non-separable) Banach space.*

**REMARK 4.6.** In view of Proposition 2.1, for finite-dimensional Banach spaces, absolute valuability is a condition much stronger than transitivity. In the infinite-dimensional setting things change drastically. Indeed, if  $H$  is an infinite-dimensional separable Hilbert space, then  $\mathcal{L}(H)$  is absolute-valuable (by Corollary 2.5) but, by [4, Theorem 4.5], it is not transitive (nor even convex-transitive). We recall that a Banach space  $X$  is said to be *convex-transitive* if, for every element  $u$  in  $S_X$ , the convex hull of  $\mathcal{G}(u)$  is dense in  $B_X$ . On the other hand, to see that an affirmative answer to Problem 4.1 is actually easier than an affirmative answer to the classical rotation problem, we would have to be sure that infinite-dimensional transitive Banach spaces need not be absolute-valuable, a fact that (although clearly plausible) is not clear for us for the moment.

**Acknowledgements.** The authors are especially grateful to Y. Benyamini for deep suggestions which allowed them to obtain the present versions of Theorems 2.11 and 3.4. They also thank M. Cabrera, G. Godefroy, G. López, J. Martínez, R. Payá, and G. Wood for fruitful remarks.

#### REFERENCES

- [1] A. A. Albert, *Absolute valued real algebras*, Ann. of Math. (2) **48** (1947), 495–501. MR 0020550 (8,561d)
- [2] S. A. Argyros, J. López-Abad, and S. Todorćević, *A class of Banach spaces with few non-strictly singular operators*, J. Funct. Anal. **222** (2005), 306–384. MR 2132394
- [3] S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, vol. 1, Warszawa, 1932.
- [4] J. Becerra and A. Rodriguez, *Transitivity of the norm on Banach spaces having a Jordan structure*, Manuscripta Math. **102** (2000), 111–127. MR 1771231 (2001g:46022)

- [5] ———, *Transitivity of the norm on Banach spaces*, Extracta Math. **17** (2002), 1–58. MR 1914238 (2003i:46022)
- [6] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York, 1973. MR 0423029 (54 #11013)
- [7] F. Cabello, *Regards sur le problème des rotations de Mazur*, Extracta Math. **12** (1997), 97–116. MR 1607201 (99m:46017)
- [8] ———, *Transitivity of  $M$ -spaces and Wood's conjecture*, Math. Proc. Cambridge Philos. Soc. **124** (1998), 513–520. MR 1636581 (99i:46007)
- [9] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co., Amsterdam, 1993. MR 1209438 (94e:46130)
- [10] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow, 1993. MR 1211634 (94d:46012)
- [11] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, and R. Remmert, *Numbers*, Graduate Texts in Mathematics, vol. 123, Springer-Verlag, New York, 1990. MR 1066206 (91h:00005)
- [12] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), 851–874. MR 1201238 (94k:46021)
- [13] P. Greim, J. E. Jamison, and A. Kamińska, *Almost transitivity of some function spaces*, Math. Proc. Cambridge Philos. Soc. **116** (1994), 475–488. MR 1291754 (95j:46037)
- [14] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313** (1980), 72–104. MR 552464 (82b:46013)
- [15] W. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous function*, Studia Math. **26** (1966), 133–136. MR 0193491 (33 #1711)
- [16] M. Jiménez Sevilla and J. P. Moreno, *Renorming Banach spaces with the Mazur intersection property*, J. Funct. Anal. **144** (1997), 486–504. MR 1432595 (98a:46024)
- [17] A. Kaidi, M. I. Ramírez, and A. Rodríguez, *Absolute-valued algebraic algebras are finite-dimensional*, J. Algebra **195** (1997), 295–307. MR 1468894 (99c:17004)
- [18] ———, *Nearly absolute-valued algebras*, Comm. Algebra **30** (2002), 3267–3284. MR 1914996 (2003e:46080)
- [19] H. E. Lacey and S. J. Bernau, *Characterizations and classifications of some classical Banach spaces*, Advances in Math. **12** (1974), 367–401. MR 0341044 (49 #5794)
- [20] J. R. Partington, *Equivalent norms on spaces of bounded functions*, Israel J. Math. **35** (1980), 205–209. MR 576470 (81h:46013)
- [21] Á. Rodríguez, *Absolute valued algebras and absolute-valuable Banach spaces*, Advanced courses of mathematical analysis I, Proc. First Intern. School, Cádiz, Spain, 2002 (edited by A. Aizpuru-Tomás and F. León-Saavedra), World Scientific Publ., 2004, pp. 99–155.
- [22] S. Rolewicz, *Metric linear spaces*, Mathematics and its Applications (East European Series), vol. 20, D. Reidel Publishing Co., Dordrecht, 1985. MR 808176 (88i:46004b)
- [23] S. Shelah and J. Steprāns, *A Banach space on which there are few operators*, Proc. Amer. Math. Soc. **104** (1988), 101–105. MR 958051 (90a:46047)
- [24] K. Urbanik and F. B. Wright, *Absolute-valued algebras*, Proc. Amer. Math. Soc. **11** (1960), 861–866. MR 0120264 (22 #11021)
- [25] H. M. Wark, *A non-separable reflexive Banach space on which there are few operators*, J. London Math. Soc. (2) **64** (2001), 675–689. MR 1865556 (2003a:46031)
- [26] F. B. Wright, *Absolute valued algebras*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 330–332. MR 0054583 (14,944b)

*E-mail address*, J. Becerra: [juliobg@ugr.es](mailto:juliobg@ugr.es)

*E-mail address*, A. Moreno: [agalindo@ugr.es](mailto:agalindo@ugr.es)

*E-mail address*, A. Rodríguez: [apalacio@ugr.es](mailto:apalacio@ugr.es)

UNIVERSIDAD DE GRANADA, FACULTAD DE CIENCIAS, DEPARTAMENTO DE MATEMÁTICA  
APLICADA, 18071-GRANADA, SPAIN