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THE SPECTRAL THEOREM FOR BIMODULES IN HIGHER RANK GRAPH C^*-ALGEBRAS

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Abstract. In this note we extend the spectral theorem for bimodules to the higher rank graph C*-algebra context. Under the assumption that the graph is row finite and has no sources, we show that a bimodule over a natural abelian subalgebra is determined by its spectrum iff it is generated by the Cuntz-Krieger partial isometries which it contains iff the bimodule is invariant under the gauge automorphisms. We also show that the natural abelian subalgebra is a masa iff the higher rank graph satisfies an aperiodicity condition.

1. Introduction

Many C*-algebras can be coordinitized—a property that proves very useful both in the study of the C*-algebra and also of its subalgebras. Coordinitization is achieved by presenting the C*-algebra as a groupoid C*-algebra. The unit space of the groupoid is associated with an abelian subalgebra which is often, though not always, a masa. (The abelian subalgebra depends on the choice of coordinates and need not be intrinsic.) A great many of the (non-self-adjoint) subalgebras of a groupoid C*-algebra either contain the "diagonal" abelian algebra or are a bimodule over it. When the groupoid is r-discrete and principal, one of the most fundamental tools used in the study of subalgebras is the spectral theorem for bimodules of Muhly and Solel [4]. Roughly speaking, this says that a bimodule is determined by the coordinates on which it is supported.

When the groupoid is not principal, it is no longer true that a bimodule is determined by its spectrum. For graph C*-algebras, [2] contains a characterization of those bimodules which are determined by their spectra: these are the bimodules which are invariant under the gauge automorphisms. (Another equivalent condition is that the bimodule be generated by the Cuntz-Krieger partial isometries which it contains.) Graph C*-algebras have been extensively

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studied in the last decade; see [5] for an excellent summary and a bibliography of relevant papers. More recently, considerable attention has turned to a multi-dimensional analog, the higher rank graph C*-algebras.

In the paper in which higher rank graph C*-algebras were first formalized [3], Kumjian and Pask modified the path groupoid model for graph C*-algebras to produce a model for higher rank graph C*-algebras. The purpose of this note is to extend the spectral theorem for bimodules as it appears in [2] for graph C*-algebras to the higher rank context. Section 2 will provide a brief review of the notation and construction of higher rank graph C*-algebras and their associated path groupoids. Section 3 is devoted to the spectral theorem for bimodules in the higher rank context. It also contains a characterization of when the "diagonal" is a masa.

2. Higher rank C*-algebras and the path groupoid

A k-graph (Λ, d) is a small category Λ together with a functor $d\colon \Lambda \to \mathbb{N}^k$ which satisfies the following factorization property: if $\lambda \in \Lambda$ and $d(\lambda) = m+n$ with $m, n \in \mathbb{N}^k$, then there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$. For $n \in \mathbb{N}^k$, we let $\Lambda^n = d^{-1}(n)$ and note that Λ^0 can be identified with the objects in Λ.

When $k = 1$, Λ is the category of finite paths from a directed graph; Λ^0 is the set of vertices; Λ^1 is the set of directed edges; and Λ^n is the set of paths of length n. A higher rank graph is a multi-dimensional analog of an ordinary directed graph.

The category Λ has range and source maps r and s (so λ is a morphism from $s(\lambda)$ to $r(\lambda)$). For each object v, and each $n \in \mathbb{N}^k$, let $\Lambda^n(v) = \{\lambda \in \Lambda \mid d(\lambda) = n, r(\lambda) = v\}$. We assume throughout this paper that each $\Lambda^n(v)$ is a finite, non-empty set. (This is usually expressed by saying that Λ is row finite and has no sources.)

A higher rank graph C*-algebra, $C^*(\Lambda)$, is the universal C*-algebra generated by a family of partial isometries $\{s_\lambda \mid \lambda \in \Lambda\}$ satisfying:

(1) $\{s_v \mid v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
(2) $s_{\lambda\mu} = s_\lambda s_\mu$, for all composable $\lambda, \mu \in \Lambda$ (i.e., for all λ, μ with $r(\mu) = s(\lambda)$),
(3) $s_\lambda^* s_\lambda = s_v$, where $v = s(\lambda)$,
(4) for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$, $s_v = \displaystyle\sum_{\lambda \in \Lambda^n(v)} s_\lambda s_\lambda^*$.

Any set of partial isometries in a C*-algebra which satisfies these four conditions is known as a Cuntz-Krieger family; if $\{t_\lambda \mid \lambda \in \Lambda\}$ is a Cuntz-Krieger family, then the map $s_\lambda \mapsto t_\lambda$ extends to a homomorphism of $C^*(\Lambda)$ to the C*-algebra generated by the t_λ.

The description above is take largely from [3], where the reader can find more detail and a number of examples. The same source provides more complete information about the path groupoid, $\mathcal{G}$, which we now summarize.

Let Ω_k denote the following k-graph:

- $\operatorname{Obj}\Omega_k = \mathbb{N}^k$.
- $\Omega_k = \{(m,n) \mid (m,n) \in \mathbb{N}^k \times \mathbb{N}^k \text{ and } m \leq n\}$.
- $r(m,n) = m$; $s(m,n) = n$.
- $d\colon \Omega_k \to \mathbb{N}^k$ by $d(m,n) = n - m$.

Infinite path space in Λ is then defined to be

$$\Lambda^\infty = \{x\colon \Omega_k \to \Lambda \mid x \text{ is a } k\text{-graph morphism}\}.$$

For $v \in \Lambda^0$, let $\Lambda^\infty(v) = \{x \in \Lambda^\infty \mid x(0) = v\}$. For each $p \in \mathbb{N}^k$, define a shift map, $\sigma^p\colon \Lambda^\infty \to \Lambda^\infty$, by $\sigma^p(x)(m,n) = x(m+p, n+p)$.

Using the factorization property, Kumjian and Pask show that $x \in \Lambda^\infty$ is determined by the values $x(0,m)$, $m \in \mathbb{N}^k$. They also show that if $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $x(0) = s(\lambda)$, then we can concatenate λ and x: there is a unique $y \in \Lambda^\infty$ such that $x = \sigma^{d(\lambda)}y$ and $\lambda = y(0, d(\lambda))$. Naturally, we write $y = \lambda x$. This leads immediately to the factorization of any infinite path $x \in \Lambda^\infty$ as a product of a finite path (an element of Λ) and an infinite tail: $x = x(0,p)\sigma^p x$, for any $p \in \mathbb{N}^k$.

For any $\lambda \in \Lambda$, let

$$\begin{aligned} Z(\lambda) &= \{\lambda x \in \Lambda^\infty \mid s(\lambda) = x(0)\} \\ &= \{y \in \Lambda^\infty \mid y(0, d(\lambda)) = \lambda\}. \end{aligned}$$

The collection $\{Z(\lambda) \mid \lambda \in \Lambda\}$ generates a topology on path space Λ^∞; in this topology each $Z(\lambda)$ is a compact, open set. The map $\lambda x \mapsto x$ is a homeomorphism of $Z(\lambda)$ onto $Z(s(\lambda))$ and each map σ^p is a local homeomorphism.

Λ^∞ will be identified with the set of units in the groupoid $\mathcal{G}_\Lambda$, which is defined by

$$\mathcal{G}_\Lambda = \{(x,n,y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid \sigma^p x = \sigma^q y \text{ and } n = p - q\}.$$

When $k = 1$, Λ^∞ reduces to the usual infinite path space and $\mathcal{G}_\Lambda$ is the usual groupoid based on shift equivalence on path space. Inversion in $\mathcal{G}_\Lambda$ is given by $(x,n,y)^{-1} = (y,-n,x)$. Composable elements consist of those with matching third and first coordinates, in which case multiplication is given by $(x,n,y)(y,m,z) = (x,n+m,z)$. Λ^∞ is identified with the space of units, $\mathcal{G}_\Lambda^0$, via $x \mapsto (x,0,x)$. A basis for a topology on $\mathcal{G}_\Lambda$ is given by the family

$$Z(\lambda,\mu) = \{(\lambda z, d(\lambda) - d(\mu), \mu z) \mid z \in \Lambda^\infty(v)\},$$

where $\lambda, \mu \in \Lambda$ and $s(\lambda) = s(\mu) = v$. The topology generated by this basis is locally compact and Haussdorff. $\mathcal{G}_\Lambda$ is then a second countable, r-discrete, locally compact groupoid; each basic open set $Z(\lambda,\mu)$ is compact. The identification of Λ^∞ with $\mathcal{G}^0_\Lambda$ is a homeomorphism. The groupoid C*-algebra, $C^*(\mathcal{G}_\Lambda)$, is isomorphic to the higher rank graph C*-algebra, $C^*(\Lambda)$.

The gauge action which appears in the spectral theorem for bimodules is an action of the k-torus $\mathbb{T}^k$ on $C^*(\Lambda)$. First, a bit of notation: if $t \in \mathbb{T}^k$ and $n \in \mathbb{N}^k$ then $t^n = t_1^{n_1} t_2^{n_2} \cdots t_k^{n_k}$. If $\{s_\lambda \mid \lambda \in \Lambda\}$ is a generating Cuntz-Krieger family, then so is $\{t^{d(\lambda)} s_\lambda \mid \lambda \in \Lambda\}$; the universal property then yields an automorphism γ_t of $C^*(\Lambda)$ such that $\gamma_t(s_\lambda) = t^{d(\lambda)} s_\lambda$, for all λ.

The fixed point algebra of the gauge action is an AF subalgebra of $C^*(\Lambda)$; it is generated by all $s_\lambda s_\mu^*$ with $d(\lambda) = d(\mu)$. The map Φ_0 of $C^*(\Lambda)$ onto the fixed point algebra given by $\Phi_0(f) = \int_{\mathbb{T}^k} \gamma_t(f)\,dt$ is a faithful conditional expectation. For details concerning this, see [3].

It is shown in [3] that $\mathcal{G}_\Lambda$ is amenable; consequently, $C^*(\mathcal{G}_\Lambda) = C^*_{\text{red}}(\mathcal{G}_\Lambda)$. Proposition II.4.2 in [6] allows us to identify the elements of $C^*(\mathcal{G}_\Lambda)$ with (some of the) elements of $C_0(\mathcal{G}_\Lambda)$, the continuous functions on $\mathcal{G}_\Lambda$ vanishing at infinity. (Note, however, that all continuous functions on $\mathcal{G}_\Lambda$ with compact support are elements of $C^*(\mathcal{G}_\Lambda)$.)

For each $m \in \mathbb{Z}^k$, let $\mathcal{G}_m$ be the set of those elements (x,n,y) in $\mathcal{G}_\Lambda$ with $n = m$. The conditional expectation Φ_0 is just restriction map to $\mathcal{G}_0$. Restriction to $\mathcal{G}_m$ is also a map of $C^*(\mathcal{G}_\Lambda)$ into itself; this is seen by observing that it is given by the norm decreassing map Φ_m defined by $\Phi_m(f) = \int_{\mathbb{T}^k} t^{-m}\gamma_t(f)\,dt$. If $\mathcal{B}$ is a closed linear subspace of $C^*(\Lambda)$ which is left invariant by the gauge automorphisms, then $\Phi_m(\mathcal{B}) \subseteq \mathcal{B}$, for each m.

3. The spectral theorem for bimodules

Throughout this section, Λ is a k-graph for which each $\Lambda^n(v)$ is finite and non-empty and $\mathcal{G}$ is the associated r-discrete locally compact groupoid. Elements of the groupoid C*-algebra (= higher rank graph C*-algebra) are viewed as continuous functions on $\mathcal{G}$. (Since k does not vary, we drop the subscript from the notation for the grouoid.) As above, we identify path space Λ^∞ with the space of units of $\mathcal{G}$; with this identification $C_0(\Lambda^\infty)$ becomes an abelian subalgebra of $C^*(\mathcal{G})$. Λ^∞ is not compact except when Λ has finitely many objects ("vertices"), hence the use of C_0.

For simplicity of notation, let $\mathcal{A}$ denote the groupoid C*-algebra and let $\mathcal{D}$ denote $C_0(\Lambda^\infty)$. At the end of the section we will discuss when $\mathcal{D}$ is a masa in $\mathcal{A}$.

Since $\mathcal{G}$ is r-discrete, the Haar system can be taken to be counting measure, and so is not mentioned explicitly. Since elements of $\mathcal{A}$ are interpreted as functions on $\mathcal{G}$, multiplication is given by a convolution type formula

$$fg(x,n,y) = \sum f(x,p,z)g(z,q,y),$$

where the sum is taken over all composable pairs (x,p,z) and (z,q,y) with $p+q=n$. (For functions in $\mathcal{A}$, the series will converge.) In particular, if $f \in \mathcal{A}$ and $g \in \mathcal{D}$,

$$gf(x,n,y) = g(x,0,x)f(x,n,y), \tag{1}$$

$$fg(x,n,y) = f(x,n,y)g(y,0,y). \tag{2}$$

For each $\lambda \in \Lambda$, let s_λ denote the characteristic function of the set $Z(\lambda, s(\lambda))$. Then $\{s_\lambda \mid \lambda \in \Lambda\}$ forms a Cuntz-Krieger family and generates $\mathcal{A}$ as a C*-algebra. This can be checked using the definition of $Z(\lambda, s(\lambda))$ and the formula given above for multiplication. Note also that, for $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$, $s_\lambda s_\mu^*$ is the characteristic function of the set $Z(\lambda, \mu)$.

If $\mathcal{B} \subseteq \mathcal{A}$ is a bimodule over $\mathcal{D}$, we define the *spectrum* of $\mathcal{B}$ to be:

$$\sigma(\mathcal{B}) = \{(x,n,y) \in \mathcal{G} \mid f(x,n,y) \neq 0 \text{ for some } f \in \mathcal{B}\}.$$

The spectrum $\sigma(\mathcal{B})$ is an open subset of $\mathcal{G}$. On the other hand, any open subset P of $\mathcal{G}$ determines a $\mathcal{D}$-module $A(P)$ given by

$$A(P) = \{f \in \mathcal{A} \mid f(x,n,y) = 0 \text{ for all } (x,n,y) \notin P\}.$$

Since P is open, if $(x,n,y) \in P$, then there is a basic open set $Z(\lambda,\mu)$ such that $(x,n,y) \in Z(\lambda,\mu) \subseteq P$. It follows that $s_\lambda s_\mu^* \in A(P)$; since $s_\lambda s_\mu^*$ has the value 1 at (x,n,y), we obtain $\sigma(A(P)) = P$, for any open subset $P \subseteq \mathcal{G}$.

It is clear that if $\mathcal{B}$ is a bimodule over $\mathcal{D}$ then $\mathcal{B} \subseteq A(\sigma(\mathcal{B}))$; equality does not always hold. A counterexample in the special case of Cuntz algebras (algebras determined by 1-graphs with only one vertex) can be found in [1]. Also, it is shown in [2] that there is a counterexample for any graph C*-algebra which is not AF. (For AF C*-algebras the Muhly-Solel spectral theorem for bimodules says that $\mathcal{B} = A(\sigma(\mathcal{B}))$ always.) Thus counterexamples exist for all 1-graphs which contain a loop.

A characterization of those bimodules which are determined by their spectra, $\mathcal{B} = A(\sigma(\mathcal{B}))$, is given in the graph C*-algebra context in [2]. The main result in this note is the extension to the higher rank context:

Theorem (Spectral Theorem for Bimodules). *Let Λ be a row finite k-graph with no sources. Let $\mathcal{G}$ be the associated path groupoid. Let $\mathcal{A} = C^*(\Lambda) = C^*(\mathcal{G})$ and $\mathcal{D} = C_0(\Lambda^\infty)$. If $\mathcal{B} \subseteq \mathcal{A}$ is a bimodule over $\mathcal{D}$, then the following are equivalent:*

1. $\mathcal{B} = A(\sigma(\mathcal{B}))$.
2. *$\mathcal{B}$ is generated by the Cuntz-Krieger partial isometries which it contains.*
3. *$\mathcal{B}$ is invariant under the gauge automorphisms.*

Proof. (1) $\Rightarrow$ (2). Assume P is an open subset of $\mathcal{G}$. Let $\mathcal{B}$ be the bimodule generated by the Cuntz-Krieger partial isometries in $A(P)$. Each such partial isometry has its support in P, so $\sigma(\mathcal{B}) \subseteq P$ and $\mathcal{B} \subseteq A(P)$. We need to show that any function f in $A(P)$ is actually in $\mathcal{B}$. We claim that it is sufficient to do this for functions which are supported on some $Z(\lambda, \mu) \subseteq P$. Indeed, it then follows readily that functions supported on compact subsets of P are in $\mathcal{B}$ (every compact subset of P is contained in a finite union of subsets of the form $Z(\lambda, \mu)$) and the compactly supported functions in $A(P)$ are dense in $A(P)$.

If f has support in $Z(\lambda, \mu)$, with the aid of convolution formulas (1) and (2) it is easy to find a function g supported in Λ^∞ such that $f = g s_\lambda s_\mu^*$. Since $s_\lambda s_\mu^* \in \mathcal{B}$ and $g \in \mathcal{D}$, $f \in \mathcal{B}$ also.

(2) $\Rightarrow$ (3). Since a gauge automorphism maps a Cuntz-Krieger partial isometry to a scalar multiple of itself, $\mathcal{B}$ is trivially left invariant when it is generated by its Cuntz-Krieger partial isometries.

(3) $\Rightarrow$ (1). Let $\mathcal{B}$ be a gauge invariant bimodule and let $P = \sigma(\mathcal{B})$. Since $\mathcal{B} \subseteq A(P)$ is automatic, we just need to show that $A(P) \subseteq \mathcal{B}$. For each $m \in \mathbb{Z}^k$, let $P_m = P \cap \mathcal{G}_m$, so that $P = \bigcup_m P_m$. Since Φ_m maps $A(P)$ onto $A(P_m)$ and, for each f, f is in the closed linear span of the $\Phi_m(f)$, we need merely show that $A(P_m) \subseteq \mathcal{B}$, for each m.

Fix m. Suppose that $\alpha, \beta \in \Lambda$ satisfy $s(\alpha) = s(\beta)$ and $d(\alpha) - d(\beta) = m$. Denote $\mathcal{G}_{\alpha,\beta} = \{(\alpha z, m, \beta w) \mid z, w \in \Lambda^\infty(s(\alpha))\}$ and $P_{\alpha,\beta} = P_m \cap \mathcal{G}_{\alpha,\beta}$. Now, by what we have just proven $A(P_m)$ is the closed linear span of the Cuntz-Krieger partial isometries which it contains. But if $s_\alpha s_\beta^*$ is one of these, then $s_\alpha s_\beta^* \in A(P_{\alpha,\beta})$, so $A(P_m)$ is the closed linear span of the $A(P_{\alpha,\beta})$. This reduces the task to showing that $A(P_{\alpha,\beta}) \subset \mathcal{B}$ for each suitable pair α, β.

We can finish the proof by transfering the problem to (a subset of) $\mathcal{G}_0$; the latter is a principal groupoid so the Muhly-Solel spectral theorem for bimodules is available. Let

$$\mathcal{G}_0(s(\alpha)) = \{(z, 0, w) \mid z, w \in \Lambda^\infty(s(\alpha))\}.$$

The map $\psi\colon \mathcal{G}_0(s(\alpha)) \to \mathcal{G}_{\alpha,\beta}$ given by $(z, 0, w) \mapsto (\alpha z, m, \beta w)$ is a homeomorphism. Let Q be the inverse image of $P_{\alpha,\beta}$ under this map. Note that $f \mapsto s_\alpha f s_\beta^*$ carries $A(Q)$ onto $A(P_{\alpha,\beta})$.

Let

$$\mathcal{C} = \{f \in A(\mathcal{G}_0(s(\alpha))) \mid s_\alpha f s_\beta^* \in \mathcal{B}\}.$$

We claim that $\mathcal{C}$ is a bimodule over $\mathcal{D}$. Since $\mathcal{D}$ is generated by projections of the form $s_\lambda s_\lambda^*$, it suffices to show that $\mathcal{C}$ is closed under multiplication left and right by such projections. Now if $f \in \mathcal{C}$, then, since $s_\alpha s_\lambda s_\lambda^* \neq 0$ exactly when $s_\lambda s_\lambda^* \leq s_\alpha^* s_\alpha$,

$$s_\alpha s_\lambda s_\lambda^* f s_\beta^* = s_\alpha s_\lambda s_\lambda^* s_\alpha^* s_\alpha f s_\beta^* = s_{\alpha\lambda} s_{\alpha\lambda}^* s_\alpha f s_\beta^* \in \mathcal{B}.$$

The last assertion uses $s_{\alpha\lambda} s_{\alpha\lambda}^* \in \mathcal{D}$ and $s_\alpha f s_\beta^* \in \mathcal{B}$. Thus $\mathcal{C}$ is a left bimodule over $\mathcal{D}$; the argument that it is a right bimodule is similar.

The definition of Q implies that $\sigma(\mathcal{C}) \subseteq Q$. The gauge invariance of $\mathcal{B}$ implies that $\sigma(\mathcal{C}) = Q$. Indeed, let $q \in Q$ and let $p = \psi(q)$. Since $p \in P$, there is $f \in \mathcal{B}$ such that $f(p) \neq 0$. Then $\Phi_m(f)(p) \neq 0$ and, by gauge invariance, $\Phi_m(f) \in \mathcal{B}$. If $g = s_\alpha^* \Phi_m(f) s_\beta$, then $g \in \mathcal{C}$ and $g(q) \neq 0$.

Since $\sigma(\mathcal{C}) = Q$ and the Muhly-Solel spectrum for bimodules holds in $A(\mathcal{G}_0)$, we have $\mathcal{C} = A(Q)$. This implies that $A(P_{\alpha,\beta}) \subseteq \mathcal{B}$. □

As mentioned earlier, $\mathcal{D} = C_0(\Lambda^\infty)$ need not be a masa in $\mathcal{A}$. For the graph C*-algebra case, it was shown in [2] that $\mathcal{D}$ is a masa if, and only if, every loop has an entrance. Kumjian and Pask [3] define an analogous condition, the *aperiodicity condition*, for higher rank graphs and use this to extend the Cuntz-Krieger uniqueness theorem. Their condition also extends the masa theorem. Here are the relevant definitions: an element $x \in \Lambda^\infty$ is *periodic* with non-zero period $p \in \mathbb{Z}^k$ if, for every $(m, n) \in \Omega$ with $m + p \geq 0$, $x(m + p, n + p) = x(m, n)$. If there is an element $n \in \mathbb{N}^k$ such that $\sigma^n(x)$ is periodic, x is *eventually periodic*; otherwise, x is *aperiodic*. Finally, Λ satisfies the *aperiodicity condition* if, for every $v \in \Lambda^0$, there is an aperiodic path $x \in \Lambda^\infty(v)$.

Note that x is eventually periodic with period p if, and only if, $(x, p, x) \in \mathcal{G}$.

Kumjian and Pask prove that Λ satisfies the aperiodicity condition if, and only if, the points in Λ^∞ with trivial isotropy are dense in $\mathcal{G}^0$ [3, Proposition 4.5]. We will show below that the aperiodicity condition is also equivalent to the assertion that $\mathcal{G}^0$ is the interior of the isotropy group bundle $\mathcal{G}^1$. (Note: in the Kumjian-Pask proposition, $\mathcal{G}^0$ is viewed as Λ^∞; we will view $\mathcal{G}^0$ as the open subset $\{(x, 0, x) \mid x \in \Lambda^\infty\}$ of $\mathcal{G}^1 = \{(x, p, x) \in \mathcal{G} \mid p \in \mathbb{Z}^k\}$.) Renault [6, Proposition II.4.7] has shown that, $C_0(\mathcal{G}^0)$ is a masa in $C^*_{\mathrm{red}}(\mathcal{G})$ if, and only if, $\mathcal{G}^0$ is the interior of $\mathcal{G}^1$. Since the path groupoid $\mathcal{G}$ is amenable, Renault's Proposition yields the masa theorem.

PROPOSITION. *Λ satisifies the aperiodicity condition if, and only if, $\mathcal{G}^0$ is the interior of $\mathcal{G}^1$.*

Proof. Assume that the aperiodicity condition holds. Let $(x, p, x) \in \mathcal{G}^1$ with $p \neq 0$. We shall show that we can approximate (x, p, x) by points in $\mathcal{G}$ which are not in $\mathcal{G}^1$. This shows that (x, p, x) is not in the interior of $\mathcal{G}^1$. Since $\mathcal{G}^0$ is an open subset of $\mathcal{G}^1$, it follows that $\mathcal{G}^0$ is the interior.

Let $Z(\alpha, \beta)$ be a neighborhood of (x, p, x). For m sufficiently large (meaning for each m_i sufficiently large), $m + p \geq 0$ and both $x(0, m)$ and $x(0, m + p)$ lie in $Z(\alpha)$ and in $Z(\beta)$. Since $(x, p, x) \in \mathcal{G}$, $\sigma^m(x) = \sigma^{m+p}(x)$ and $x(0, m)$ and $x(0, m + p)$ have a common source v. Choose y aperiodic in $\Lambda^\infty(v)$. Let $z = x(0, m)y$ and $w = x(0, m + p)y$. Then $z \neq w$ and $(z, p, w) \in Z(\alpha, \beta)$. So $(z, p, w) \notin \mathcal{G}^1$ and (z, p, w) approximates (x, p, x).

Now suppose that Λ does not satisfy the aperiodicity condition. By Proposition 4.5 in [3], there is $x \in \Lambda^\infty$ which cannot be approximated by aperiodic points. Since x must be eventually periodic there is a non-zero element p of $\mathbb{Z}^k$ such that $(x, p, x) \in \mathcal{G}$. If (x, p, x) could be approximated in the topology of $\mathcal{G}$ by points outside $\mathcal{G}^1$, it would follow that x is a limit of aperiodic points in Λ^∞—a contradiction. This shows that (x, p, x) is in the interior of $\mathcal{G}^1$ and so $\mathcal{G}^0$ is not the interior. □

This Proposition, Proposition II.4.7 in [6], and the amenability of $\mathcal{G}$ yield the following theorem.

Theorem. *$\mathcal{D}$ is a masa in $\mathcal{A}$ if, and only if Λ satisifies the aperiodicity condition.*

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