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UNIFORM APPROXIMATION ON RIEMANN SURFACES BY HOLOMORPHIC AND HARMONIC FUNCTIONS

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ABSTRACT. Let K be a compact subset of an open Riemann surface. We prove that if L is a peak set for $A(K)$, then $A(K)|L = A(L)$. We also prove that if E is a compact subset of K with no interior such that each component of E^c intersects K^c, then $A(K)|E$ is dense in $C(E)$. One consequence of the latter result is a characterization of the real-valued continuous functions that when adjoined to $A(K)$ generate $C(K)$.

1. Introduction

In this paper, we are concerned with some problems involving uniform algebras on sets in an arbitrary open Riemann surface. The original idea comes from the related problems in the complex plane. Let K be a compact set in the complex plane. Let $A(K)$ denote the algebra of functions in $C(K)$ that are holomorphic on the interior of K and let $R(K)$ denote the algebra of the functions in $C(K)$ that can be approximated uniformly on K by rational functions with poles off K.

Wermer's maximality theorem [9, Theorem II.5.1] states that the algebra $A(\bar{D})$ (where D denotes the open unit disk) is not a maximal subalgebra of $C(\bar{D})$. In other words, there is a function f in $C(\bar{D})$, but not in $A(\bar{D})$, such that the norm-closed subalgebra $A(\bar{D})\,[f]$ of $C(\bar{D})$ generated by $A(\bar{D})$ and f is not equal to $C(\bar{D})$. It is natural to ask for a characterization of the functions f in $C(\bar{D})$ for which $A(\bar{D})\,[f] = C(\bar{D})$.

In 1965, J. Wermer [18] showed that when f is continuous differentiable on a neighborhood of $\bar{D}$, then $A(\bar{D})\,[f] = C(\bar{D})$ if and only if the graph of f is polynomially convex in $\mathbb{C}^2$ and $R(E) = C(E)$, where E is the zero set of $\bar{\partial} f$.

In 1969, E.M. Čirca [8], using Wermer's technique, obtained the following theorem.

THEOREM A. *Let K be a compact set in the plane and suppose that every point of ∂K is a peak point for $R(K)$. Let $f \in C(K)$ be harmonic on the*

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interior of K, but nonholomorphic on each component of the interior of K. Then the norm-closed subalgebra of $C(K)$ generated by $R(K)$ and f is equal to $C(K)$.

Suppose Ω is an open set on the Riemann sphere, $\bar{\mathbb{C}}$, and let $H^\infty(\Omega)$ denote the algebra of bounded holomorphic functions on Ω. If f is any bounded measurable function on Ω we let $H^\infty(\Omega)\,[f]$ denote the subalgebra of $L^\infty(\Omega)$ generated by $H^\infty(\Omega)$ and f. Let $C(\bar{\Omega})$ denote the uniformly continuous functions on Ω (i.e., those with continuous extension to $\bar{\Omega}$, the closure of Ω). The following theorem is due to C.J. Bishop [4].

Theorem B. *Suppose that Ω is an open set in the Riemann sphere and that $f \in H^\infty(\Omega)$ is nonconstant on each component of Ω. Then $C(\bar{\Omega}) \subset H^\infty(\Omega)\left[\bar{f}\right]$.*

In 1991, A.J. Izzo [13] obtained the following result:

Theorem C. *Let K be a compact set in the plane, and let $u \in C(K)$ be real-valued and harmonic on the interior of K, but nonconstant on each component of the interior of K. Then $A(K)\,[u] = C(K)$.*

In this paper, we will obtain analogous results on open Riemann surfaces. Some of the techniques are similar to those used by A.J. Izzo in the complex plane.

2. Preliminaries

Let $\mathfrak{R}$ be an open Riemann surface and let K be a compact subset of $\mathfrak{R}$. Let $C(K)$ be the class of complex-valued continuous functions on K. Let $A(K)$ be the class of functions in $C(K)$ which are holomorphic in the interior of K, and let $M(K)$ be the class of functions in $C(K)$ which can be uniformly approximated on K by meromorphic functions on $\mathfrak{R}$ with poles off K. If $f \in C(K)$, $A(K)\,[f]$ denotes the algebra of functions in $C(K)$ generated by $A(K)$ and f.

We use the following notation: $\mathfrak{R}^*$ denotes the one point compactification of $\mathfrak{R}$; for $E \subset \mathfrak{R}$, ∂E denotes the boundary of E in $\mathfrak{R}$, E^0 denotes the interior of E, and E^C denotes the complement of E in $\mathfrak{R}$.

In 1967, R. C. Gunning and R. Narasimhan [12] showed that every noncompact Riemann surface can be realized in a very concrete way. More precisely, they proved the following theorem.

Theorem 2.1. *There exists a globally defined holomorphic function $\rho : \mathfrak{R} \to \mathbb{C}$ which is locally a homeomorphism.*

Thus ρ is a global uniformizing parameter on $\mathfrak{R}$. In this article, every local coordinate system will always be defined only in terms of ρ (without further notice). For example, a parametric disk D on $\mathfrak{R}$ will be an open connected set on $\mathfrak{R}$ on which ρ is one to one and $\rho(D)$ is a disk $\{z : |z - z_0| < r\}$ in $\mathbb{C}$; if $\rho(p) = z_0$, then p and r will be called, respectively, the center and the radius of the parametric disk D. For convenience we write this disk as $D(p, r)$.

Using ρ, S. Scheinberg [15] and P.M. Gauthier [11] constructed a Cauchy kernel $F(p, q)$, which is a meromorphic function on $\mathfrak{R} \times \mathfrak{R}$ satisfying:

(1) $F(q, p) = -F(p, q)$.
(2) The only singularities of F are simple poles with residues ± 1 on the diagonal.

Using this kernel, A. Boivin [6] extended to Riemann surfaces the definition of the localization operator T_ϕ of A.G. Vitushkin.

Definition 2.2. Whenever ϕ is a smooth function with compact support, we define the operator T_ϕ on the space of bounded Borel functions f by the formula

$$(T_\phi f)(q) = \phi(q) f(q) + \frac{1}{2\pi i} \iint f(p) F(p, q) \bar{\partial}\phi(p) \wedge \partial\rho(p). \tag{2.1}$$

Boivin proved the following result.

Proposition 2.3. *Let f and ϕ be as above. Then:*

(i) *$T_\phi f$ is continuous wherever f is continuous.*
(ii) *$T_\phi f$ is analytic wherever f is analytic.*
(iii) *$T_\phi f$ is analytic off the closed support of ϕ.*
(iv) *$f - T_\phi f$ is analytic on the interior of the level set $\phi^{-1}(1)$.*

Definition 2.4. If A is a uniform algebra on a compact Hausdorff space X, a subset Y of X is said to be a *peak set* for A if there is a function $f \in A$ such that $f(y) = 1$ for $y \in Y$, and $|f(x)| < 1$ for $x \in X \backslash Y$. In this situation, the function f is said to peak on Y. A point $x \in X$ is said to be a *peak point* for A if $\{x\}$ is a peak set for A.

If μ is a measure on the complex plane with compact support, the *Cauchy transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z) = \int \frac{d\mu(w)}{w - z}$$

for all z such that the integral converges absolutely. Analogously, we have (see [6] and [14]):

Definition 2.5. If μ is a finite complex Borel measure on $\mathfrak{R}$ with compact support, then the *Cauchy transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(q) = \int F(p, q) d\mu(p).$$

Lemma 2.6 ([6, Lemma 3.2]). *If μ is a measure of the form $\mu = h\omega$, where ω is the area measure on $\mathfrak{R}$ and $h \in L^\infty(\omega)$ has compact support, then $\hat{\mu}$ is continuous.*

Lemma 2.7. *If μ is a measure of the form $\mu = h\omega$, where ω is the area measure on $\mathfrak{R}$ and $h \in L^\infty(\omega)$ has compact support, then $\hat{\mu}$ is holomorphic off the compact support of h.*

Proof. Suppose E is the compact support of h. Given any point $p_0 \in E^c$, let G be an open subset of $\mathfrak{R}$ such that $E \subset G$, and $p_0 \in \bar{G}^c$. Pick a smooth function ϕ with compact support on $\bar{G}$ satisfying

$$\phi(p) = \rho(p), \quad \text{for} \quad p \in E.$$

Then we have

$$\hat{\mu}(q) = 2\pi i (T_\phi h)(q), \quad \text{for any} \quad q \in \bar{G}^c.$$

So, by Proposition 2.3, $\hat{\mu}$ is holomorphic on G^c. Hence $\hat{\mu}$ is holomorphic at p_0. This means that $\hat{\mu}$ is holomorphic off the compact support of h. □

Lemma 2.8. *Let G be a Borel set on an open Riemann surface $\mathfrak{R}$ and let μ be a finite complex measure on $\mathfrak{R}$ with compact support. Then $\hat{\mu} = 0$ a.e. on G if and only if μ annihilates all functions of the form*

$$f(q) = \int h(p) F(p,q) \partial\rho(p) \wedge \bar{\partial}\rho(p),$$

where h is a bounded Borel function on $\mathfrak{R}$ with compact support vanishing off G.

Proof. If $\hat{\mu} = 0$ a.e. on G, then by applying Fubini's theorem we have

$$\begin{aligned}
\int f(q) d\mu(q) &= \iint h(p) F(p,q) \partial\rho(p) \wedge \bar{\partial}\rho(p) d\mu(q) \\
&= \int_G h(p) \int F(p,q) d\mu(q) \partial\rho(p) \wedge \bar{\partial}\rho(p) \\
&= \int_G h(p) \hat{\mu}(p) \partial\rho(p) \wedge \bar{\partial}\rho(p) \\
&= 0.
\end{aligned}$$

On the other hand, if we pick $h(p) = \overline{\hat{\mu}(p)} \mid_G$, then $\int f(q) d\mu(q) = 0$ implies that

$$\begin{aligned}
\iint h(p) F(p,q) \partial\rho(p) \wedge \bar{\partial}\rho(p) d\mu(q) &= \int h(p) \int F(p,q) d\mu(q) \partial\rho(p) \wedge \bar{\partial}\rho(p) \\
&= \int_G \left|\hat{\mu}(p)\right|^2 \partial\rho(p) \wedge \bar{\partial}\rho(p) \\
&= 0.
\end{aligned}$$

But $dA = (-2i)^{-1}\partial\rho(p) \wedge \bar{\partial}\rho(p)$ is a (positive 2-form) area element, so $\hat{\mu} = 0$ a.e. on G. □

Lemma 2.9. *Suppose that K is a compact subset of an open Riemann surface, and μ is a measure on K that annihilates $A(K)$. Then $\hat{\mu} = 0$ a.e. off K^0.*

Proof. Suppose h is an arbitrary bounded Borel function on $\mathfrak{R}$ with compact support vanishing on K^0. Then, by Lemma 2.7, we conclude that the function

$$f(q) = \int h(p)F(p,q)\partial\rho(p) \wedge \partial\bar{\rho}(p)$$

is holomorphic on K^0, so $f \in A(K)$. Because μ annihilates $A(K)$, by Lemma 2.8 we have that $\hat{\mu}$ a.e. off K^0. □

Lemma 2.10. *Let K be a compact subset of an open Riemann surface. Then a measure μ on K is orthogonal to $M(K)$ if and only if $\hat{\mu} = 0$ on K^c.*

This is just [14, Lemma 5] and [6, Theorem 2.1].

3. The restriction of $A(K)$ to a peak set

The following theorems are the main results of this section.

Theorem 3.1. *If K is a compact subset of an open Riemann surface $\mathfrak{R}$ and L is a peak set for $A(K)$, then $A(K)|L = A(L)$.*

Theorem 3.2. *If K is a compact subset of an open Riemann surface $\mathfrak{R}$ and L is a peak set for $M(K)$, then $M(K)|L = M(L)$*

Before proving Theorem 3.1, we need some preliminaries. Let K be a compact set on an open Riemann surface $\mathfrak{R}$ and U an open set in $\mathfrak{R}$ contained in K. Let $A(K,U)$ denote the algebra of continuous functions on K that are holomorphic on U.

The following lemma, which is a generalization of [13, Lemma 1.2], will be proved using Lemma 2.8.

Lemma 3.3. *Suppose that K is a compact set on an open Riemann surface $\mathfrak{R}$, U is an open set in $\mathfrak{R}$ contained in K and N is a relatively closed subset of U which has area measure zero (i.e., $\int_N dA(p) = 0$). Then $C((K\backslash U) \cup N)$ is the closed linear span of $A(K,U)$ and the functions $p \to F(p,q_0), q_0 \in U\backslash N$.*

Proof. Suppose that μ is a measure on $(K\backslash U)\cup N$ that annihilates $A(K,U)$ and the functions $p \to F(p,q_0), q_0 \in U\backslash N$. Then obviously $\hat{\mu}(q_0) = 0$ for every $q_0 \in U\backslash N$. If h is a bounded Borel function on $\mathfrak{R}$ with compact support such that h is zero on U, then using Lemma 2.7 we conclude that the function f defined on K by

$$f(q) = \int h(p)F(p,q)\partial\rho(p) \wedge \partial\bar{\rho}(p)$$

is in $A(K,U)$, and hence is annihilated by μ. Consequently, using Lemma 2.8, we have that $\hat{\mu} = 0$ a.e. on U^c. Since $\int_N dA(p) = 0$, we conclude that $\hat{\mu} = 0$ a.e., so μ is the zero measure. This implies that $C((K\backslash U) \cup N)$ is the closed linear span of $A(K,U)$ and the functions $F(p,q_0), q_0 \in U\backslash N$. $\square$

Lemma 3.3 will not be used in its full generality. For convenience we state here the two special cases that will be used.

Corollary 3.4. *If Ω is a bounded (i.e., relatively compact) open subset of an open Riemann surface, then $C(\partial\Omega)$ is the closed linear span of $A(\Omega)$ and the functions $F(p,q_0)$, $q_0 \in \Omega$.*

Corollary 3.5. *If K is a compact set on an open Riemann surface $\mathfrak{R}$ and N is a relatively closed subset of K^0 having area measure zero, then $C(\partial K \cup N)$ is the closed linear span of $A(K)$ and the functions $F(p,q_0)$, $q_0 \in K^0\backslash N$.*

Lemma 3.6. *If K is a compact set on an open Riemann surface $\mathfrak{R}$ and L is a peak set for $M(K)$, then $\partial L \subset \partial K$.*

This is an immediate consequence of the maximum modulus principle and the definition of peak sets.

Lemma 3.7. *If K is a compact set on an open Riemann surface and L is a peak set for $A(K)$, then each component of L^c intersects K^c.*

Proof. Assume, to get a contradiction, that some component U of L^c lies entirely in K. Since L is a closed set, each component of L^c is open. It follows that ∂U is contained in L. Now let f in $A(K)$ be a function that peaks on L. Then $f|\bar{U}$ is in $A(\bar{U})$ and is 1 on ∂U. Consequently, by the maximum principle, f is 1 on all of $\bar{U}$. Hence, $\bar{U}$ is contained in L, contradicting our assumption that U is a component of L^c. $\square$

S. Scheinberg obtained the following generalization of Runge's theorem to Riemann surfaces.

Lemma 3.8 ([16, Corollary 3]). *If E is a compact subset of a Riemann surface $\mathfrak{R}$ and P contains a point from each bounded component of E^c, then every holomorphic function on E is the uniform limit on E of meromorphic functions on $\mathfrak{R}$ all of whose poles lie in P.*

LEMMA 3.9. *Let K be a compact subset of an open Riemann surface, L be a peak set for $A(K)$, and let $X = \overline{K\backslash L} \cap L$. Then $A(\overline{K\backslash L})|X = C(X)$.*

Proof. Let $f \in A(K)$ be a function that peaks on L. Then $f|(\overline{K\backslash L})$ is in $A(\overline{K\backslash L})$ and peaks on $\overline{K\backslash L} \cap L = X$. Thus, X is a peak set for $A(\overline{K\backslash L})$. Therefore, $A(\overline{K\backslash L})|X$ is closed in $C(X)$ [17, Lemma 12.3]. In addition, using Lemma 3.7, we conclude that each component of the complement of X intersects the complement of $\overline{K\backslash L}$. Note that X is contained in ∂L. Hence X is contained in ∂K (by Lemma 3.6), and consequently X is contained in $\partial(\overline{K\backslash L})$. So by Corollary 3.5 we see that the linear span of $A(\overline{K\backslash L})|X$ and the functions $F(p, q_0), q_0 \in (\overline{K\backslash L})^0$, is dense in $C(X)$. Because each component of the complement of X intersects the complement of $\overline{K\backslash L}$, Lemma 3.8 shows that each of the functions $F(p, q_0), q_0 \in (\overline{K\backslash L})^0$, can be approximated uniformly on X by meromorphic functions with poles off $\overline{K\backslash L}$. Thus, $A(\overline{K\backslash L})|X$ is dense in $C(X)$. □

Proof of Theorem 3.1. Obviously $A(K)|L \subset A(L)$. To prove the reverse inclusion, let $f \in A(L)$ be arbitrary. By Lemma 3.9, there exists a function g in $A(\overline{K\backslash L})$ that agrees with f on $\overline{K\backslash L} \cap L$. Hence, the function h on K given by

$$h(p) = \begin{cases} f(p) & \text{if} \quad p \in L, \\ g(p) & \text{if} \quad p \in \overline{K\backslash L}, \end{cases}$$

is well defined. Now h is continuous on K and holomorphic on K^0 (since $\partial L \subset \partial K$ by Lemma 3.6). Thus, h is in $A(K)$ and clearly $h|L = f$. Hence, $A(L) \subset A(K)|L$. □

COROLLARY 3.10. *If E is a peak set for $A(K)$ contained in ∂K, then $A(K)|E = C(E)$.*

This is an immediate consequence of Theorem 3.1 and Theorem 2.6 in [6].

If A is a uniform algebra on a compact Hausdorff space X, a subset Y of X is said to be an interpolation set for A if $A|Y = C(Y)$. The set Y is said to be a peak-interpolation set for A if for each nonzero $f \in C(Y)$ there is an $F \in A$ such that $F|Y = f$ and $|F(x)| < \|f\|_Y$ for all $x \in X\backslash Y$. It can be shown that a set is a peak-interpolation set if and only if it is simultaneously a peak set and an interpolation set [18, Lemma 20.1]. Thus, Corollary 3.10 can be reformulated as follows.

COROLLARY 3.10′. *If E is a peak set for $A(K)$ contained in ∂K, then E is a peak-interpolation set for $A(K)$.*

Proof of Theorem 3.2. Let G be a bounded component of L^c. Then G meets K^c, for if $G \subset K$, and f peaks on L, $f \in M(K)$, then f is holomorphic in G and equals 1 on the boundary of G, and hence $f = 1$ in G, a contradiction. It follows from Lemma 3.8 that $M(K)|L$ is dense in $M(L)$. Since L is a peak set, $M(K)|L$ is closed [7, Theorem 2.4.3], and the proof is complete. $\square$

The next result is an immediate consequence of Theorem 3.1 and Theorem 3.2.

Corollary 3.11. *If K is a compact set in the open Riemann surface for which $A(K)$ and $M(K)$ coincide, and L is a peak set for this common algebra, then $A(L)$ and $M(L)$ also coincide.*

4. The main theorems

For $u \in C(K)$, let $A(K)\,[u]$ denote the norm-closed subalgebra of $C(K)$ generated by $A(K)$ and u. This section is devoted primarily to proving the following two theorems, which generalize the results of Izzo [13] to open Riemann surfaces.

Theorem 4.1. *Let K be a compact subset of an open Riemann surface. Let $u \in C(K)$ be real-valued and harmonic on the interior of K, but nonconstant on each component of interior of K. Then $A(K)\,[u] = C(K)$.*

Theorem 4.2. *If K is a compact subset of an open Riemann surface and E is a compact subset of K with no interior such that each component of E^c intersects K^c, then $A(K)|E$ is dense in $C(E)$.*

Using a reasoning similar to that used by S. Axler and A. Shields [2] and by Izzo [13] in the complex plane, we see that Theorem 4.1 follows from Theorem 4.2. However, Theorem 4.1 does not require the full strength of Theorem 4.2. To illustrate this, we first give an independent proof of Theorem 4.1. The general idea of the proof is taken from Axler and Shields [2]. We shall make use of the following lemmas.

Lemma 4.3. *Let K be a compact subset of an open Riemann surface. Let $u \in C(K)$ be real-valued and harmonic on the interior of K, but nonconstant on each component of K^0. Let E be a level set of u. Then $E \cap K^0$ has area measure zero.*

Proof. Assume, to get a contradiction, that $E \cap K^0$ has nonzero area measure. Because K is compact, there are finite open parametric disks Δ_j such that $K \subset \cup \Delta_j$. Hence, there is at least one parametric disk (which we may suppose to be Δ_1) such that $E \cap \Delta_1 \cap K^0$ has nonzero measure. Also, we notice that u is harmonic on $\Delta_1 \cap K^0$ and nonconstant on each component of $\Delta_1 \cap K^0$.

Let $\tau = (\rho|\Delta_1)^{-1}$. Then $u \circ \tau$ is harmonic on $\rho(\Delta_1 \cap K^0)$, and nonconstant on each component of $\rho(\Delta_1 \cap K^0)$. Moreover, $E_1 = \rho(E \cap \Delta_1 \cap K^0)$ is a subset, with nonzero planar area measure, of the level set of $u \circ \tau$.

Now we observe that $\rho(\Delta_1 \cap K^0)$ is a bounded open set in the complex plane and thus has at most countably many components. Hence there is at least one component (which we may suppose to be V) such that $E_1 \cap V$ has nonzero measure. Because $u \circ \tau$ is harmonic on V, this implies that $u \circ \tau$ is constant on V. This contradicts the fact that $u \circ \tau$ is nonconstant on each component. $\square$

Denote by $A_{\mathfrak{R}}(K)$ the uniform limits on K of functions holomorphic on $\mathfrak{R}$. E. Bishop obtained the following generalization of Mergelyan's theorem.

LEMMA 4.4 ([5, Corollary 2]). *Let K be compact in $\mathfrak{R}$. If $\mathfrak{R}^*\backslash K$ is connected, then $A(K) = A_{\mathfrak{R}}(K)$.*

Suppose A is a uniform algebra on a compact Hausdorff space X. A subset Y of X is said to be a *set of antisymmetry* for A if every function in A that is real-valued on Y is in fact constant on Y. We say that A is *antisymmetric* if every real-valued function in A is constant.

Proof Theorem 4.1. Let E be a maximal set of antisymmetry for $A(K)\,[u]$. The closure of every antisymmetric set is antisymmetric, so E is compact. Moreover, since u is a real-valued function in $A(K)\,[u]$, it must be constant on E, and Lemma 4.3 shows that $E \cap K^0$ has area measure zero.

We claim that each component of E^c intersects K^c. To prove this, let F be a component of E^c and assume, to get a contradiction, that F is contained in K. Then u is continuous on $\bar{F}$ and harmonic on F. Moreover, ∂F is contained in E, so u is constant on ∂F. Consequently, u is constant on F, which contradicts the hypothesis that u is nonconstant on each component of K^0. Thus each component of E^c intersects K^c.

Since $E \cap K^0$ has measure zero, Corollary 3.4 shows that the linear span of $A(K)|E$ and the functions $F(p, q_0)$, $q_0 \in K^0\backslash E$, is dense in $C(E)$. Since each component of E^c intersects K^c, Lemma 3.8 shows that each of the functions $F(p, q_0)$, $q_0 \in K^0\backslash E$, can be approximated by meromorphic functions with poles off K. Thus, $A(K)|E$ is dense in $C(E)$. Hence, $A(K)\,[u]\,|E$ is certainly dense in $C(E)$. Since E is a maximal set of antisymmetry for $A(K)\,[u]$, we know by [17, Theorem 12.1] that $A(K)\,[u]\,|E$ is closed in $C(E)$, and so $A(K)\,[u]\,|E = C(E)$. The Bishop antisymmetric decomposition (see [17, Theorem 12.1]) now implies that $A(K)\,[u] = C(K)$. $\square$

In view of Theorem A, it should be noted that the existence of compact sets K with no interior and with $M(K) \neq C(K)$ implies that Theorem 4.1 does not remain valid if $A(K)$ is replaced by $M(K)$.

Proof of Theorem 4.2. Let f in $C(E)$ be arbitrary and fix $\varepsilon > 0$. By Corollary 3.4, the linear span of $A(K)$ and the functions $F(p, q_0)$, $q_0 \in K^0$, is dense in $C(\partial K)$, and hence dense in $C(E \cap \partial K)$. Since E is has empty interior, each component of $(E \cap \partial K)^c$ contains a component of E^c, and hence intersects K^c. Therefore, by Lemma 3.8, each of the functions $F(p, q_0)$, $q_0 \in K^0$, can be approximated uniformly on $E \cap \partial K$ by meromorphic functions with poles off K. Consequently, $A(K)|(E \cap \partial K)$ is dense in $C(E \cap \partial K)$. So we can choose g in $A(K)$ such that

$$\|f - g\|_{(E\cap\partial K)} < \frac{\varepsilon}{2}.$$

By the Tietze extension theorem, there is a continuous function ϕ on E that agrees with $f - g$ on a neighborhood of $E \cap \partial K$ in E, and satisfies

$$\|\phi\|_E \leq \frac{\varepsilon}{2}.$$

Now $f - g - \phi$ is a continuous function on E that vanishes on a neighborhood of $E \cap \partial K$. For each point p lying in E but not on the boundary of K, choose an open parametric disk D_p that is contained in K. The hypothesis that each component of E^c intersects K^c implies that $E \cap \bar{D}_p$ has connected complement. Hence, recalling that E has empty interior, Lemma 4.4 shows that $M(E \cap \bar{D}_p) = C(E \cap \bar{D}_p)$. In particular, the restriction $f - g - \phi$ to $E \cap \bar{D}_p$ is in $M(E \cap \bar{D}_p)$. Since $f - g - \phi$ vanishes on a neighborhood of $E \cap \partial K$ in E, the localization theorem [14] now implies that $f - g - \phi$ is in $M(E)$. Therefore, since each component of E^c intersects K^c, Lemma 3.8 shows that $f - g - \phi$ can be approximated (on E) by meromorphic function with poles off K, and hence certainly by elements of $A(K)$. So we can choose h in $A(K)$ such that

$$\|(f - g - \phi) - h\|_E < \frac{\varepsilon}{2}.$$

Then we have

$$\|f - (g + h)\|_E \leq \|(f - g - \phi) - h\|_E + \|\phi\|_E < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since g and h are both in $A(K)$, we are done. □

As mentioned earlier, Theorem 4.2 enables us to give another proof of Theorem 4.1. One shows that if E is a maximal set of antisymmetry for $A(K)\,[u]$, then E satisfies the hypotheses of Theorem 4.2, so $A(K)|E$ is dense in $C(E)$. Consequently, $A(K)\,[u]\,|E = C(E)$, and the conclusion of Theorem 4.1 follows from the Bishop antisymmetric decomposition. Moreover, using now Theorem 4.2, we obtain the following characterization of those real-valued continuous functions on K that when adjoined with $A(K)$ generate all of $C(K)$.

Corollary 4.5. *Suppose that K is a compact subset of an open Riemann surface and $u \in C(K)$ is real-valued. Then $A(K)\,[u] = C(K)$ if and only if u is nonconstant on the boundary of each open set contained in K.*

Proof. The proof of the 'if' part is similar to the proof of Theorem 4.1 suggested above. To prove the 'only if' part, suppose u is constant on the boundary of an open set contained in K. There exists a connected open set U such that u is constant on ∂U. Since the maximum modulus principle implies that $A(\bar{U})|\partial U$ is closed in $C(\bar{U})$, it is easy to see that

$$A(K)\,[u]\,|(\partial U) \subset \overline{A(K)|(\partial U)} \subset A(\bar{U})|(\partial U).$$

Moreover, $A(\bar{U})|(\partial U)$ is not equal to $C(\partial U)$ since the connectedness of U implies that every real-valued function in $A(\bar{U})|(\partial U)$ is constant. So $A(K)\,[u] \neq C(K)$, and we are done. $\square$

Corollary 4.6. *Suppose that K is a compact subset of an open Riemann surface, and E is a compact subset of K. Then $A(K)|E$ is dense in $C(E)$ if and only if E has empty interior and each component of E^c intersects K^c.*

Proof. The 'if' statement is Theorem 4.2. To prove the 'only if' statement, suppose some component U of E^c does not intersect K^c. Then $\bar{U} \subset K$ and $\partial U \subset E$. Now, $A(K)|(\partial U) \subset A(\bar{U})|(\partial U)$, and $A(\bar{U})|(\partial U)$ is closed in $C(\partial U)$ (by the maximum modulus principle) and is not equal to $C(\partial U)$, since every real-valued function in $A(\bar{U})|(\partial U)$ is constant. Thus $A(K)|(\partial U)$ is not dense in $C(\partial U)$, so $A(K)|E$ is not dense in $C(E)$. $\square$

We can also use duality to prove Theorem 4.2, as we now illustrate.

Lemma 4.7. *Let K be a compact subset of an open Riemann surface, and E a compact subset of K such that each component of E^c intersects K^c. Suppose μ is a measure on E that annihilates $A(K)$. Then $\hat{\mu} = 0$ a.e. off E^0.*

In the proof we will need the following lemma.

Lemma 4.8 ([14, Lemma 7]). *Let μ be a complex measure supported on K and orthogonal to $M(K)$. For any covering $\{U_j\}$ of K by the coordinate neighborhoods, we can find measures μ_j, each supported on U_j and orthogonal to $M(K \cap \bar{U}_j)$, such that $\mu = \sum \mu_j$.*

Proof of Lemma 4.7. By Lemma 2.9, $\hat{\mu} = 0$ a.e. off K^0. The Cauchy transform of a measure is holomorphic off the closed support of the measure, so $\hat{\mu}$ is holomorphic off E. In view of the hypothesis on the components of E^c, it follows that $\hat{\mu} = 0$ off E. Thus it suffices to show that $\hat{\mu} = 0$ a.e. on $(\partial E) \cap K^0$.

For each $p \in (\partial E) \cap K^0$ choose an open parametric disk D_p centered at p and contained in K. Then $E \cap \bar{D}_p$ has connected complement, so by Lemma 4.4 we have $M(E \cap \bar{D}_p) = A(E \cap \bar{D}_p)$. Let D'_p be the parametric disk center at p with radius half that of D_p.

Now fix $q \in (\partial E) \cap K^0$, and let $U_1 = E \cap D_q$ and $U_2 = E \cap (\bar{D}'_q)^c$. Since $\hat{\mu} = 0$ off E, by Lemma 2.10 we have that $\mu \perp M(E)$. Hence, by Lemma 4.8, there exist measures μ_1 and μ_2 such that $\mu = \mu_1 + \mu_2$, $\mu_j \perp M(\bar{U}_j)$, and the closed support of μ_j is contained in U_j $(j = 1, 2)$. Now, $\mu_1 \perp M(E \cap \bar{D}_p) =$

$A(E\cap\bar{D}_p)$, so by Lemma 2.9, $\hat{\mu}_1 = 0$ a.e off $M\cap\bar{D}_q$. In particular, $\hat{\mu}_1 = 0$ a.e. on ∂E. Moreover, $\hat{\mu}_2 = 0$ off $\bar{U}_2$ (since $\mu_2 \perp M(\bar{U}_2)$), so $\hat{\mu}_2 = 0$ on D'_w. We conclude that $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2 = 0$ a.e. on $\partial E\cap D'_q$. Since there exists a countable collection $\mathcal{E}$ of points in $\partial E\cap K^0$ such that the parametric disks D'_p, $p\in\partial E$, cover $(\partial E)\cap K^0$, it follows that $\hat{\mu} = 0$ a.e. on $(\partial E)\cap K^0$, and the proof is complete. □

We can use this lemma to prove Theorem 4.2, because if L has no interior, the conclusion of the lemma is that $\hat{\mu} = 0$ a.e. whenever μ is a measure on L that annihilates $A(K)$. Thus, the zero functional is the only linear functional that annihilates $A(K)|L$, and hence $A(K)|L$ is dense in $C(L)$.

5. A generalization of a result of C. Bishop

THEOREM 5.1. *Suppose that Ω is a bounded open subset of an open Riemann surface $\mathfrak{R}$ and that $f\in H^\infty(\Omega)$ is nonconstant on each component of Ω. Then $C(\bar{\Omega})\subset H^\infty(\Omega)\left[\bar{f}\right]$.*

Following Bishop's original proof [4], we first obtain the following lemma.

LEMMA 5.2. *If Ω is a bounded open subset of an open Riemann surface and $g\in C(\bar{\Omega})$, then g can be approximated uniformly on $\bar{\Omega}$ by continuous functions on $\bar{\Omega}$ that are smooth on Ω and holomorphic on $\Omega\cap U$, for some neighborhood U of $\partial\Omega$.*

Proof. Fix $\varepsilon > 0$. By Corollary 3.4, there exist a function $h\in A(\Omega)$, points $q_1, q_2, \cdots, q_n$ in Ω, and complex numbers $a_1, a_2, \cdots, a_n$ such that

$$\left| g(p) - \left(h(p) + \sum_{i=1}^{N} a_i F(p, q_i) \right) \right| < \varepsilon, \quad \forall p\in\partial\Omega. \tag{5.1}$$

Let V be a neighborhood of $\partial\Omega$ such that (5.1) holds with $V\cap\bar{\Omega}$. Let ϕ, ψ, τ be smooth functions on the Riemann surface satisfying:

(i) $\operatorname{supp}\phi\subset V$, $\operatorname{supp}\psi\subset\Omega$, and $\operatorname{supp}\tau\subset\bar{\Omega}^c$;
(ii) $0\le\phi,\psi,\tau\le 1$;
(iii) $\phi+\psi+\tau = 1$.

Let ℓ be a smooth function on Ω such that

$$\sup_{p\in\Omega}|g(p) - \ell(p)| < \varepsilon.$$

Then the function

$$\lambda(p) = \phi(p)(h(p) + \sum_{i=1}^{n} a_i F(p, q_i)) + \psi(p)\ell(p)$$

is a continuous function on $\bar{\Omega}$ that is smooth on Ω, holomorphic on $\Omega \cap (\operatorname{supp}\psi)^c$, and satisfies

$$\|g(p) - \lambda(p)\| < \epsilon \quad \text{for} \quad p \in \bar{\Omega},$$

so we are done. $\square$

We are now ready to prove Theorem 5.1. Much of the proof is similar to Bishop's proof of the same theorem in the complex plane (see [4]) .

Proof of Theorem 5.1. Take $g \in C(\bar{\Omega})$ and suppose $f = u + iv \in H^\infty(\Omega)$ is nonconstant on each component of Ω. We wish to prove that $g \in H^\infty(\Omega)\left[\bar{f}\right]$. By Lemma 5.2, we can approximate g on $\bar{\Omega}$ by a function which is continuous on $\mathfrak{R}$ and of compact support, smooth on Ω and holomorphic on $\Omega \cap U$, for some neighborhood U of $\partial\Omega$. Hence we may assume that g has this form.

We may also assume that $\|f\|_\infty \le 1$. For each complex number λ with $|\lambda| < 1$, notice that

$$\{p \in \Omega : f(p) = \lambda\} \cap \operatorname{supp}(\bar{\partial} g)$$

is a finite subset of Ω. Let us modify g continuously to be constant in a neighborhood of each such point. Thus we obtain a function g_λ which approximates g and which is holomorphic on $\Omega \cap U$ and in a neighborhood of $\{f = \lambda\}$. Therefore,

$$\frac{\bar{\partial} g_\lambda}{f - \lambda}$$

is a smooth differential with compact support.

Let

$$h_\lambda(q) = \frac{1}{2\pi i} \int \frac{1}{f(p) - \lambda} F(p, q) \bar{\partial} g_\lambda(p) \wedge \partial\rho(p).$$

From Lemma 2.6 we conclude that h_λ is continuous on $\mathfrak{R}$. Moreover, if we also let μ be the "measure"

$$\frac{1}{f - \lambda} \bar{\partial} g_\lambda \wedge \partial\rho(p),$$

then $h_\lambda = \hat{\mu} \cdot (1/2\pi i)$, and (see [6, equation (8)])

$$\bar{\partial}(h_\lambda d\rho) = \partial(\frac{\hat{\mu}}{2\pi i} d\rho) = -\mu = -\frac{1}{f - \lambda} \bar{\partial} g_\lambda \wedge \partial\rho$$

in terms of currents. Therefore $g_\lambda + h_\lambda(f - \lambda)$ is a bounded holomorphic function on Ω and approximates g near the set $\{f = \lambda\}$.

Fix ε small. Let $D = \{(x, y) : x^2 + y^2 < 1\}$ be the unit disk in the complex plane. We choose a finite collection of points $\{\lambda_j\}$ such that the corresponding disks

$$B_j = \left\{(x, y) : \sqrt{(x - \operatorname{Re}(\lambda_j))^2 + (y - \operatorname{Im}(\lambda_j))^2} < \varepsilon / \|h_{\lambda_j}\|_{\infty, \bar{\Omega}}\right\}$$

cover $\bar{D}$, where $\|h_{\lambda_j}\|_{\infty,\bar{\Omega}}$ denotes the supremum of h_{λ_j} on $\bar{\Omega}$, and also choose continuously differentiable function ϕ_j with the following properties:

(i) ϕ_j is supported on B_j.
(ii) $\sum \phi_j(x,y) = 1$ for all $(x,y) \in \bar{D}$.
(iii) $0 \le \phi_j(x,y) \le 1$.
(iv) No point $z \in \bar{D}$ is contained in more than N of the disks B_j, where N is a universal constant.

Since $f \in H^\infty(\Omega)$, clearly $u = (f+\bar{f})/2$, $v = (f-\bar{f})/(2i) \in H^\infty(\Omega)\left[\bar{f}\right]$. Now, we define

$$\begin{aligned} G(p) &= \sum_j (g_{\lambda_j}(p) + h_{\lambda_j}(p)(f(p)-\lambda_j))\phi_j(u,v) \\ &= \sum_{\{j: f(p)\in B_j\}} (g_{\lambda_j}(p) + h_{\lambda_j}(p)(f(p)-\lambda_j))\phi_j(u,v). \end{aligned}$$

Since ϕ_j can be uniformly approximated by polynomials, we have that $G \in H^\infty(\Omega)\left[\bar{f}\right]$. To see that G approximates g on $\bar{\Omega}$, write

$$|g-G| \le \sum_{\{j: f(p)\in B_j\}} \left|g - g_{\lambda_j}\right| |\phi_j| + \sum_{\{j: f(p)\in B_j\}} \left|h_{\lambda_j}\right| |f-\lambda_j|\, |\phi_j|\,.$$

The first term is small since $g - g_\lambda$ was chosen small and $\sum_{\{j:f(p)\in B_j\}} |\phi_j| \le 1$. The second term is small because

$$\sum_{\{j: f(p)\in B_j\}} \left|h_{\lambda_j}\right| |f(p)-\lambda_j|\, |\phi_j| \le \sum_{\{j: f(p)\in B_j\}} \|h_{\lambda_j}\|_{\infty,\bar{\Omega}} \frac{\varepsilon}{\|h_{\lambda_j}\|_{\infty,\bar{\Omega}}} \le N\varepsilon.$$

This completes the proof of Theorem 5.1. □

The following result can be proved by making minor changes in the proof of Theorem 5.1 given above.

Theorem 5.3. *Suppose Ω is a bounded open set and $f \in A(\Omega)$ is nonconstant on each component of Ω. Then $C(\bar{\Omega}) = A(\Omega)\left[\bar{f}\right]$.*

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