Illinois Journal of Mathematics
Volume 45, Number 4, Winter 2001, Pages 1191–1245
S 0019-2082

ORBIT NONPROPER DYNAMICS ON LORENTZ MANIFOLDS

SCOT ADAMS

Abstract. An action of a topological group G on a topological space X is *orbit nonproper* if, for some $x \in X$, the map $g \mapsto gx : G \to X$ is non-proper. We describe the collection of connected, simply connected Lie groups admitting a locally faithful, orbit nonproper action by isometries of a connected Lorentz manifold.

Contents

Received September 19, 2000; received in final form December 29, 2000.
2000 *Mathematics Subject Classification.* 53C50, 54H15.
The author was supported in part by NSF grant DMS-9703480.

1. Introduction

In any kind of dynamics of groups, it is basic to determine the collection of groups that admit actions of the type under investigation. Once such a list is complete, a second problem is to determine, for each group in the list, all of its actions. In the rare situation where both of these problems can be solved, one can reasonably claim to have completed an area within dynamical systems.

This is the last in a series of papers including [AS99a], [AS99b], [Ad98a], [Ad98b], [Ad99a], [Ad99b] and [Ad99c], all of which were motivated by [Ko94] and [Ko96]. In this series, we have attempted to determine the collection of groups admitting an "interesting" smooth action by isometries of a Lorentz manifold. There is some information available limiting the possible actions of some of these groups, but we do not deal with that question here. (See §0.8.B, §5.4 and Corollary 5.4.A of [Gr88], Theorem 1.14 of [Ze98a], and Chapter 6 in [Ko94].)

We shall restrict ourselves to a very weak interpretation (described below) of the word "interesting". The surprising conclusion, observed by a number of researchers, is that, in Lorentz dynamics, even weak dynamical hypotheses result in strong restrictions on the list of allowable groups. (See Theorems 1 and 3 of [Zi84], [Zi86], §5.3.E of [Gr88], [Ko94], [Ze98a], [Ze98b], [AS99a], [AS99b], and [Ad98b].)

The isometry group of a Lorentz manifold is Lie, so we restrict our attention to real Lie groups. Discrete groups present many difficulties, so, as a first step, it is prudent to work with connected Lie groups. Since any group can act trivially, it seems reasonable to include faithfulness as part of the definition of "interesting". However, for technical reasons, we wish to be able to move from a group to its covering groups, so we require our actions only to be locally faithful; if we pull back a locally faithful action of a group to some covering group, the new action is still locally faithful.

Every Lie group admits a left-invariant Lorentz metric, so, if we impose no further dynamical conditions, then the list of groups is unrestricted. In [Ko96], N. Kowalsky considers only simple Lie groups and shows that even the most modest dynamical requirement causes a dramatic reduction in the list of groups: She shows that, if a connected simple Lie group G with finite center admits a nontrivial nonproper action on a connected Lorentz manifold, then, for some integer $n \geq 3$, G is locally isomorphic to $\mathrm{SO}(n-1,1)$ or to $\mathrm{SO}(n,2)$.

In moving beyond simple Lie groups with finite center, because of technical complications, it is helpful to make two minor modifications to the problem. *First*, we replace nonproperness by a slightly stronger condition: We say that an action of a locally compact topological group G on a locally compact topological space X is *orbit nonproper* if there exists $x \in X$ such that the map $g \mapsto gx : G \to X$ is nonproper. This condition is still very weak, compared with most dynamical conditions one might consider. For example, an action with an orbit that is not closed is *a fortiori* orbit nonproper. *Second*, we consider only connected Lie groups with simply connected nilradical. This class includes all connected, simply connected Lie groups. So, in particular, we have a classification of the Lie *algebras* of Lie groups admitting a locally faithful, orbit nonproper action on a connected Lorentz manifold.

Our main theorem (proved after Lemma 22.1) is:

THEOREM 1.1. *Let G be a connected Lie group with simply connected nilradical of N. Let L be a semisimple Levi factor of G. Then G admits a locally faithful, orbit nonproper action by isometries of a connected Lorentz manifold iff at least one of the following holds:*

(1) *The center $Z(G)$ of G is noncompact.*
(2) *The Adjoint image $\mathrm{Ad}_{\mathfrak{g}}(G)$ of G is not closed in $\mathrm{GL}(\mathfrak{g})$.*
(3) *For some integer $n \geq 2$, either $\mathfrak{so}(n,1)$ or $\mathfrak{so}(n,2)$ is a direct summand of $\mathfrak{g}$; that is, for some Lie algebra $\mathfrak{g}'$, we have that $\mathfrak{g}$ is isomorphic either to $\mathfrak{g}' \oplus \mathfrak{so}(n,1)$ or to $\mathfrak{g}' \oplus \mathfrak{so}(n,2)$.*
(4) *There exists a nonzero $(\mathrm{Ad}\ G)$-invariant subspace V_1 of $\mathfrak{z}(\mathfrak{n})$ such that $\mathrm{Ad}_{V_1}(L)$ is compact.*
(5) *There is an integer $n \geq 3$, there is an ideal $\mathfrak{l}_0$ of $\mathfrak{l}$ and there is an $(\mathrm{Ad}\ G)$-invariant subspace V_1 of $\mathfrak{z}(\mathfrak{n})$ such that the adjoint representation of $\mathfrak{l}_0$ on V_1 is isomorphic to the defining representation of $\mathfrak{so}(n-1,1)$ on $\mathbb{R}^{n\times 1}$.*

In (5), the statement that "the adjoint representation of $\mathfrak{l}_0$ on V_1 is isomorphic to the defining representation of $\mathfrak{so}(n-1,1)$ on $\mathbb{R}^{n\times 1}$" means that there are a Lie algebra isomorphism $F : \mathfrak{l}_0 \to \mathfrak{so}(n-1,1)$ and a vector space isomorphism $f : V_1 \to \mathbb{R}^{n\times 1}$ such that, for all $X \in \mathfrak{l}_0$, for all $Y \in V_1$, we have $f((\mathrm{ad}\ X)Y) = (F(X))(f(Y))$.

The conditions (1)–(5) are sufficiently structural in nature that, given any reasonable presentation of a Lie group, one may determine which of them it satisfies, if any. In particular, (4) and (5) can be effectively checked by decomposing the adjoint representation of a semisimple Levi factor on the center of the nilradical.

A more concise form of Theorem 1.1 is:

THEOREM 1.2. *Let G be a connected Lie group with simply connected nilradical. Then G admits a locally faithful, orbit nonproper action by isometries of a connected Lorentz manifold iff at least one of the following holds:*

(1) *The Adjoint homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is nonproper.*
(2) *For some integer $n \geq 2$, either $\mathfrak{so}(n,1)$ or $\mathfrak{so}(n,2)$ is a direct summand of $\mathfrak{g}$.*

(3) *Some nonzero Abelian ideal of* $\mathfrak{g}$ *has an* $(\mathrm{Ad}\, G)$*-conformal quadratic form that is either positive definite or Minkowski.*

By "$(\mathrm{Ad}\, G)$-conformal quadratic form" on an ideal, we mean that the Adjoint representation of G on the ideal is by linear transformations that are conformal with respect to the form.

Theorem 1.2 can be proved by a slight modification to the proof of Theorem 1.1. Alternatively, by basic Lie theoretic arguments, Theorem 1.2 and Theorem 1.1 are equivalent. While Theorem 1.2 is shorter than Theorem 1.1, it is (perhaps) not entirely obvious that (1) and (3) of Theorem 1.2 are easily checked, given a specific Lie group G.

Some of the work on this paper was done while visiting l'Université Henri Poincaré (Faculté des Sciences) in Nancy, France, and I appreciate very much the hospitality of L. Berard-Bergery, A. Besse and my other hosts. The basic collection of techniques used here were developed jointly with Garrett Stuck, in February, 1997, while participating in the Research-in-Pairs Program at Oberwolfach, sponsored by the Volkswagen-Stiftung. The research environment we found there was excellent. Over the last three years, many conversations with C. Leung, V. Reiner, J. Roberts, G. Stuck and D. Witte have been very helpful. The proofs of some of the lemmas appearing here were found only after a large amount of computation using various symbolic manipulators. Since my skill with this software is limited, I benefited greatly from C. Leung, V. Reiner and D. Witte who contributed significant amounts of time helping me with these computations. Finally, this entire line of research was inspired by the original insights of N. Kowalsky.

2. Global definitions

By a "manifold", we shall mean a smooth (Hausdorff, second countable, finite-dimensional) real manifold without boundary. By a "Lie group", we shall mean a smooth (Hausdorff, second countable, finite-dimensional) real Lie group. By a "connected Lie subgroup" of a Lie group, we mean a subgroup whose cosets form the leaves of a foliation of the Lie group. Such a subgroup need not be closed. We give it the Lie topology and manifold structure. The Lie topology may not agree with the inherited topology. By a "Lie algebra", we shall mean a finite-dimensional real Lie algebra, unless otherwise specified. By an "action" of a Lie group on a manifold, we shall mean a smooth action. By a "vector space", we shall mean a finite-dimensional real vector space, unless otherwise specified. A "root system" will not be assumed to be reduced. (That is, our convention is the opposite of [Hu72]. See the second sentence on p. 43 of [Hu72].)

Let G be a Lie group. By a "representation" of G, we mean a smooth representation on a finite-dimensional vector space. By a "real G-module" we mean a (real) vector space V together with a representation of G on V by real linear transformations. By a "complex G-module" we mean a complex vector space V together with a representation of G on V by complex linear transformations.

If $\mathfrak{g}$ is a Lie algebra, then we define real and complex $\mathfrak{g}$-modules in a similar way. Some authors (see [FH91], first paragraph of §26.3, p. 444) use the terms "real" and "complex" in a different way. If $\mathfrak{g}$ is a complex Lie algebra, then a "$\mathfrak{g}$-module" is a complex vector space together with a representation of $\mathfrak{g}$ on V.

Let $\mathfrak{g}$ be a Lie algebra. For any real $\mathfrak{g}$-module X, let $X^{\mathbb{C}}$ denote the complexification of X, so that $X^{\mathbb{C}}$ is a complex $\mathfrak{g}$-module. For any complex $\mathfrak{g}$-module $\mathcal{X}$, let $\mathcal{X}_{\mathbb{R}}$ denote the realization of $\mathcal{X}$, so that $\mathcal{X}_{\mathbb{R}}$ is a real $\mathfrak{g}$-module. That is, $\mathcal{X}_{\mathbb{R}}$ denotes the underlying real vector space of $\mathcal{X}$, with $\mathfrak{g}$ acting on $\mathcal{X}_{\mathbb{R}}$ by real linear transformations. For any complex $\mathfrak{g}$-module $\mathcal{X}$, let $\overline{\mathcal{X}}$ denote the conjugate module. That is, if $J : \mathcal{X} \to \mathcal{X}$ is the complex structure on $\mathcal{X}$, then the underlying real vector space of $\overline{\mathcal{X}}$ is the same as that of $\mathcal{X}$, but the complex structure on $\overline{\mathcal{X}}$ is $-J$.

Let a group G act on a set X. The action is said to be *faithful* if the intersection of the stabilizers is trivial. Assume G is a topological group. The action is said to be *locally faithful* if the intersection of the stabilizers is discrete. Assume that X is a locally compact topological space, assume that G is locally compact and assume that the G-action on X is continuous. The G-action on X is said to be *orbit nonproper* if, for some $x \in X$, the map $g \mapsto gx : G \to X$ is nonproper.

If V is a (real) vector space, then V^* denotes the dual of V, i.e., the vector space of homomorphisms $V \to \mathbb{R}$. Similarly, if V is a complex vector space then V^* is the vector space of homomorphisms $V \to \mathbb{C}$.

Let V be a vector space and let $T : V \to V$ be a linear transformation. We say that T is *real diagonalizable* if $T : V \to V$ is diagonalizable over $\mathbb{R}$. We shall say that T is *semisimple* if its complexification $T^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ is diagonalizable over $\mathbb{C}$. We shall say that T is *elliptic* if T is semisimple and if every characteristic root of T is pure imaginary. There exist unique linear transformations $T_D : V \to V$, $T_E : V \to V$ and $T_N : V \to V$ satisfying the following properties:

- T_D, T_E and T_N are pairwise commuting;
- T_D is real diagonalizable, T_E is elliptic and T_N is nilpotent; and
- $T = T_D + T_E + T_N$.

We shall say that T_D, T_E and T_N are, respectively, the *real diagonalizable, elliptic and nilpotent parts of* T. If $\mathfrak{g}$ is a semisimple Lie algebra and if $X \in \mathfrak{g}$, then we say that X is *real diagonalizable* (resp. *semisimple, elliptic, nilpotent*) if $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ is real diagonalizable (resp. *semisimple, elliptic, nilpotent*).

If G is a Lie group, then G^0 denotes the connected component of the identity in G. If G is a Lie group, then $Z(G)$ denotes the center of G and

$Z^0(G) := (Z(G))^0$. A Lie algebra will be said to be *compact* if it is either zero or semisimple with negative definite Killing form. It will be said to be *noncompact* otherwise.

If a group G acts on a set S and if $s \in S$, then we denote the stabilizer in G of s by $\mathrm{Stab}_G(s)$. If a Lie group G acts on a set S and if $s \in S$, then we define $\mathrm{Stab}^0_G(s) := (\mathrm{Stab}_G(s))^0$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let V and W be vector spaces. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\sigma : \mathfrak{h} \to \mathfrak{gl}(W)$ be representations. For $X \in \mathfrak{g}$, $v \in V$, $Y \in \mathfrak{h}$ and $w \in W$, we write $Xv := (\rho(X))(v)$ and $Yw := (\sigma(Y))(w)$. Following this notation, we say that ρ *is isomorphic to* σ if there is a Lie algebra isomorphism $F : \mathfrak{g} \to \mathfrak{h}$ and there is a vector space isomorphism $f : V \to W$ such that, for all $X \in \mathfrak{g}$, for all $v \in V$, we have $f(Xv) = (F(X))(f(v))$.

Let Q be a nondegenerate quadratic form on a real or complex vector space V. Then $\mathrm{O}(Q) \subseteq \mathrm{GL}(V)$ denotes the group of invertible linear transformations of V which preserve Q. We define

$$\mathrm{SO}(Q) := \{g \in \mathrm{O}(Q) \mid \det(g) = 1\} \quad \text{and} \quad \mathrm{SO}^0(Q) := (\mathrm{SO}(Q))^0.$$

The Lie algebra of $\mathrm{SO}^0(Q)$ is denoted by $\mathfrak{so}(Q)$. Let $I : V \to V$ be the identity transformation. Let $P := \{\lambda I \mid \lambda > 0\}$ be the collection of positive scalar transformations on V. We define $\mathrm{CO}^0(Q) := P(\mathrm{SO}^0(Q))$. The Lie algebra of $\mathrm{CO}^0(Q)$ is denoted by $\mathfrak{co}(Q)$.

Let $\mathfrak{g}$ be a Lie algebra. If $X, Y, T \in \mathfrak{g}$, then we say (X, Y, T) is a *standard* $\mathfrak{sl}_2(\mathbb{R})$ *basis of* $\mathfrak{g}$ if $\{X, Y, T\}$ forms a basis of $\mathfrak{g}$ and if

$$[T, X] = 2X, \qquad [T, Y] = -2Y \qquad \text{and} \qquad [X, Y] = T.$$

If $X, Y \in \mathfrak{g}$, then we say that (X, Y) is a *standard* $\mathfrak{sl}_2(\mathbb{R})$ *generating set in* $\mathfrak{g}$ if $(X, Y, [X, Y])$ is a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of some Lie subalgebra of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra, let V be a real $\mathfrak{g}$-module and let $n \geq 2$ be an integer. We say that V is *n-irreducible* if V is irreducible and if $\dim(V) = n$. We shall say that V is *stably n-irreducible* if there is an n-irreducible real $\mathfrak{g}$-submodule V_0 of V and a real $\mathfrak{g}$-submodule V_1 of V such that $V = V_0 + V_1$ and such that the representation of $\mathfrak{g}$ on V_1 is trivial.

Let $\mathfrak{s}$ be a Lie algebra and let V be a real $\mathfrak{s}$-module. Let U and U' be subspaces of V. We say that (U, U') is *almost $\mathfrak{s}$-invariant* if

- $U \cup (\mathfrak{s}U) \subseteq U'$; and
- the codimension in U' of U is ≤ 1.

We define *direct summand* and $\mathfrak{h} \,|\, \mathfrak{g}$ as in §2 of [Ad98b]. We define all of the following as in §2 of [Ad98b]: $\mathfrak{c}_{\mathfrak{g}}(X)$, $\mathfrak{c}_{\mathfrak{g}}(S)$, $\mathfrak{n}_{\mathfrak{g}}(S)$, $\mathcal{G}$, X_M, X_m, $\mathfrak{g}_m$, S_m, Q_d, ordered Q_d-basis, Minkowski vector space, $\mathrm{Tay}^s(\alpha)$, α^C, α^L, $X_{\mathcal{C}}$, $X^C_{\mathcal{C}}$, $X^L_{\mathcal{C}}$, $\mathcal{S}$. Warning: Some authors use G_m to denote the stabilizer in G of m and use $\mathfrak{g}_m$ to denote the Lie algebra of G_m; note that our conventions are different here. For all $\alpha \in \mathcal{G}$, let $\alpha^Q := \mathrm{Tay}^2(\alpha - \alpha^C - \alpha^L)$. For all $S \subseteq \mathcal{G}$, we define $S^C := \{\alpha^C \mid \alpha \in S\}$ and $S^L := \{\alpha^L \mid \alpha \in S\}$.

Let G be a Lie group acting smoothly on a manifold M preserving a smooth connection. Let $m_0 \in M$ and let $\mathcal{C}$ be an ordered basis of $T_{m_0}M$. For all $X \in \mathfrak{g}$, following the notation defined above, $X_{\mathcal{C}}^C$ and $X_{\mathcal{C}}^L$ are the first two terms in the Taylor expansion of $X_{\mathcal{C}}$; similarly, $X_{\mathcal{C}}^Q$ will denote the third term. For all $S \subseteq \mathfrak{g}$, we define $S_{\mathcal{C}}^C := \{X_{\mathcal{C}}^C \,|\, X \in S\}$, and $S_{\mathcal{C}}^L := \{X_{\mathcal{C}}^L \,|\, X \in S\}$,

Let V be a vector space. A quadratic form Q on V is said to be *Minkowski* if there is an integer $d \geq 2$ and an isomorphism $V \longleftrightarrow \mathbb{R}^d$ such that Q corresponds to Q_d. We denote the set of all Minkowski quadratic forms on V by $\mathrm{Mink}(V)$.

Fix an integer $d \geq 1$ for the rest of this section. Let $D := \{1, \ldots, d\}$. Let $x_1^0, \ldots, x_d^0 : \mathbb{R}^d \to \mathbb{R}$ be the coordinate projections. For all $i \in D$, let x_i be the germ at zero of x_i^0. Let $\partial_1^0, \ldots, \partial_d^0$ be the standard framing of $\mathbb{R}^d$, so, for all $i \in D$, we have $\partial_i^0 = \partial/\partial x_i^0$. For $i \in D$, let $\partial_i \in \mathcal{G}$ denote the germ at zero of ∂_i^0. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^{d\times 1}$. For all $i, j \in D$, let E_{ij} denote the $d \times d$ matrix with a one in the (i, j) entry, and with zeroes elsewhere. Define $\mathcal{F}^C : \mathcal{G}^C \to \mathbb{R}^{d\times 1}$ and $\mathcal{F}^L : \mathcal{G}^L \to \mathbb{R}^{d\times d}$ by

$$\mathcal{F}^C\left(\sum_j a_j\partial_j\right) = \sum_j a_j e_j, \qquad \mathcal{F}^L\left(\sum_{j,k} a_{jk}x_j\partial_k\right) = -\sum_{j,k} a_{jk}E_{kj}.$$

Then $\mathcal{F}^C : \mathcal{G}^C \to \mathbb{R}^{d\times 1}$ and $\mathcal{F}^L : \mathcal{G}^L \to \mathbb{R}^{d\times d}$ are both vector space isomorphisms. For $X \in \mathcal{G}$, let $X^{Cm} := \mathcal{F}^C(X^C)$ and $X^{Lm} := \mathcal{F}^L(X^L)$. For $S \subseteq \mathcal{G}$, set $S^{Cm} := \{X^{Cm} \,|\, X \in S\}$ and $S^{Lm} := \{X^{Lm} \,|\, X \in S\}$. The superscript "$m$" means "matrix form".

In the remainder of this section, the subscripts "E", "H" and "P" stand for the words "elliptic", "hyperbolic" and "parabolic", respectively. Assume, for the remainder of this section, that $d \geq 2$.

Let $\mathcal{N}_1 := E_{11} - E_{dd}$. Let $\mathcal{M}_E^1$ be the collection of all matrices $\sum a_{ij}E_{ij}$ in $\mathbb{R}^{d\times d}$ such that

- for all $i \in \{1, d\}$, for all $j \in \{1, \ldots, d\}$, we have $a_{ij} = 0$; and
- for all $i, j \in D$, we have $a_{ij} = -a_{ji}$.

Let $\mathcal{M}_H^1 := \mathcal{N}_1 + \mathcal{M}_E^1$.

If $d = 2$, then we define $\mathcal{M}_E^2 := \{0\}$, $\mathcal{M}_P^1 := \emptyset$, $\mathcal{M}_P^2 := \{0\}$. Assume, for the remainder of this section, that $d \geq 3$. For $j \in D\backslash\{1, d\}$, let $\mathcal{N}_j := E_{1j} - E_{jd}$. Let $\mathcal{M}_E^2$ be the collection of all matrices $\sum a_{ij}E_{ij}$ in $\mathbb{R}^{d\times d}$ such that

- for all $i \in \{1, 2, d\}$, for all $j \in D$, we have $a_{ij} = 0$; and
- for all $i, j \in D$, we have $a_{ij} = -a_{ji}$.

Let $\mathcal{M}_P^1 := \mathcal{N}_2 + \mathcal{M}_E^2$. Let $\mathcal{M}_P^2 := \mathbb{R}\mathcal{N}_2 + \cdots + \mathbb{R}\mathcal{N}_{d-1}$.

3. Basic facts

Lemma 3.1. *Let Q be a Minkowski form on a vector space V. Let $T \in \mathfrak{so}(Q)$. Let S be a nondegenerate subspace of (V,Q). Assume that $T(S) \subseteq S$ and that $T^2(S) = \{0\}$. Then $T(S) = \{0\}$.*

Proof. If $Q|S$ is positive definite, then the only nilpotent element of $\mathfrak{so}(Q|S)$ is zero, and so we are done. We therefore assume that $Q|S$ is not positive definite. Then, as $Q|S$ is nondegenerate, it follows that $Q|S$ is Minkowski. Replacing V by S, T by $T|S$ and Q by $Q|S$, we may assume that $V = S$. We have $T^2(V) = T^2(S) = \{0\}$, so (4) $\Longrightarrow$ (2) of Lemma 4.6 of [Ad99b] implies that $T = 0$. Then $T(S) = \{0\}$. $\square$

Lemma 3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{a}$ be a maximal $\mathbb{R}$-split torus in $\mathfrak{g}$. For all $\alpha \in \mathfrak{a}^*$, we define*

$$\mathfrak{g}_\alpha := \{W \in \mathfrak{g} \,|\, \forall T \in \mathfrak{a},\, [T,W] = (\alpha(T))W\}.$$

Let $\alpha_0 \in \mathfrak{a}^* \backslash \{0\}$. Assume that $\mathfrak{g}_{\alpha_0} \neq \{0\}$. Let $X \in \mathfrak{g}_{\alpha_0} \backslash \{0\}$. Then there exist $T \in \mathfrak{a}$ and $Y \in \mathfrak{g}_{-\alpha_0}$ such that (X,Y,T) is a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of some Lie subalgebra of $\mathfrak{g}$.

Proof. Choose $J \in \mathfrak{a}$ such that $\mathfrak{c}_{\mathfrak{g}}(J) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$. Then $[J,X] = (\alpha_0(J))X$. By Lemma 3.7, p. 622, of [Ko96] (with H replaced by T), choose $T \in \mathfrak{g}$ such that $[T,X] = 2X$, such that $T \in (\mathrm{ad}\, X)\mathfrak{g}$ and such that $[T,J] = 0$. By Lemma IX.7.6, p. 433, of [He78] (with H replaced by T and Y replaced by $\tilde{Y}$), choose $\tilde{Y} \in \mathfrak{g}$ such that $[T,\tilde{Y}] = -2\tilde{Y}$ and such that $[X,\tilde{Y}] = T$.

Let $\mathfrak{s} := \mathbb{R}X + \mathbb{R}\tilde{Y} + \mathbb{R}T$. Then $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Moreover, $\mathrm{ad}\, T : \mathfrak{s} \to \mathfrak{s}$ is real diagonalizable. By Lemma 7.6 of [Ad99b], we see that $\mathrm{ad}\, T : \mathfrak{g} \to \mathfrak{g}$ is real diagonalizable as well. We have $[T,J] = 0$, so $T \in \mathfrak{c}_{\mathfrak{g}}(J) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$. Then $\mathbb{R}T + \mathfrak{a}$ is an $\mathbb{R}$-split torus in $\mathfrak{g}$, so, by maximality of $\mathfrak{a}$, $T \in \mathfrak{a}$.

We have $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{a}^*} \mathfrak{g}_\alpha$. For all $\alpha \in \mathfrak{a}^*$, let $p_\alpha : \mathfrak{g} \to \mathfrak{g}_\alpha$ be the projection map. Let $\Psi := \{\alpha \in \mathfrak{a}^* \,|\, \mathfrak{g}_\alpha \neq \{0\}\}$. For all $\alpha \in \Psi$, define $\tilde{Y}_\alpha := p_\alpha(\tilde{Y})$. Then $\tilde{Y} = \sum_{\alpha \in \Psi} \tilde{Y}_\alpha$. As $X \in \mathfrak{g}_{\alpha_0} \backslash \{0\}$, we see that $\mathfrak{g}_{\alpha_0} \neq \{0\}$, so $\alpha_0 \in \Psi$.

We have $T = [X,\tilde{Y}] = \sum_{\alpha \in \Psi} [X,\tilde{Y}_\alpha]$ and $T \in \mathfrak{a} \subseteq \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{g}_0$. For all $\alpha \in \Psi$, we have $[X,\tilde{Y}_\alpha] \in [\mathfrak{g}_{\alpha_0}, \mathfrak{g}_\alpha] \subseteq \mathfrak{g}_{\alpha_0+\alpha}$. Thus, for all $\alpha \in \Psi$, we have $[X,\tilde{Y}_\alpha] = p_{\alpha_0+\alpha}(T) \in p_{\alpha_0+\alpha}(\mathfrak{g}_0)$. For all $\alpha \in \Psi \backslash \{-\alpha_0\}$, we have $p_{\alpha_0+\alpha}(\mathfrak{g}_0) = \{0\}$, so $[X,\tilde{Y}_\alpha] = 0$. Then

$$[X,\tilde{Y}_{-\alpha_0}] = [X,\tilde{Y}_{-\alpha_0}] + \sum_{\alpha \in \Psi \backslash \{-\alpha_0\}} [X,\tilde{Y}_\alpha] = \sum_{\alpha \in \Psi} [X,\tilde{Y}_\alpha] = [X,\tilde{Y}].$$

Let $Y := \tilde{Y}_{-\alpha_0} \in \mathfrak{g}_{-\alpha_0}$. Then $[X,Y] = [X,\tilde{Y}_{-\alpha_0}] = [X,\tilde{Y}] = T$.

Recall that $[T,X] = 2X$. Since $X \in \mathfrak{g}_{\alpha_0}$, we get $[T,X] = (\alpha_0(T))X$, so $2X = (\alpha_0(T))X$, so $\alpha_0(T) = 2$. Since $Y \in \mathfrak{g}_{-\alpha_0}$, we conclude that $[T,Y] =$

$(-\alpha_0(T))Y$, so $[T, Y] = -2Y$. It remains to show that X, Y, T are linearly independent.

We have $X \neq 0$ and $[T, X] = 2X$, so $T \neq 0$. We have $T \neq 0$ and $[X, Y] = T$, so $Y \neq 0$. Because X and Y are both nonzero and are elements of different eigenspaces of $\mathrm{ad}\, T : \mathfrak{g} \to \mathfrak{g}$, we see that X and Y are linearly independent. It remains to show that $T \notin \mathbb{R}X + \mathbb{R}Y$. Assume, for a contradiction, that $T \in \mathbb{R}X + \mathbb{R}Y$.

Then we have $2X = [T, X] \in [\mathbb{R}X + \mathbb{R}Y, X] = \mathbb{R}[Y, X] = \mathbb{R}T$ and $-2Y = [T, Y] \in [\mathbb{R}X + \mathbb{R}Y, Y] = \mathbb{R}[X, Y] = \mathbb{R}T$, so $\mathbb{R}X + \mathbb{R}Y \subseteq \mathbb{R}T$, so $\dim(\mathbb{R}X + \mathbb{R}Y) \leq 1$. Since X and Y are linearly independent, we have a contradiction. □

D. Witte pointed out to me that it suffices to prove Lemma 3.2 in the case where the $\mathbb{R}$-rank of $\mathfrak{g}$ is 1, because the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{\alpha_0}$ and $\mathfrak{g}_{-\alpha_0}$ has $\mathbb{R}$-rank 1. It is not be difficult to prove Lemma 3.2 case by case for Lie algebras of $\mathbb{R}$-rank one.

LEMMA 3.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra with no compact factors. Let $\mathcal{N}$ denote the set of nilpotent elements of $\mathfrak{g}$. Then there are an integer $k \geq 1$ and $X_1, \ldots, X_k, Y_1, \ldots, Y_k \in \mathfrak{g}$ such that*

(1) *no proper Lie subalgebra of $\mathfrak{g}$ contains $\{X_1, \ldots, X_k, Y_1, \ldots, Y_k\}$;*
(2) *for all i, (X_i, Y_i) is a standard $\mathfrak{sl}_2(\mathbb{R})$ generating set in $\mathfrak{g}$; and*
(3) *$\mathbb{R}X_1 + \cdots + \mathbb{R}X_k \subseteq \mathcal{N}$ and $\mathbb{R}Y_1 + \cdots + \mathbb{R}Y_k \subseteq \mathcal{N}$.*

Proof. We may assume that $\mathfrak{g}$ is simple and noncompact. Let $\mathfrak{a}$ be a maximal $\mathbb{R}$-split torus in $\mathfrak{g}$. For all $\alpha \in \mathfrak{a}^*$, let

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \,|\, \forall T \in \mathfrak{a},\, [T, X] = (\alpha(T))X\}.$$

Let $\Phi := \{\alpha \in \mathfrak{a}^* \backslash \{0\} \,|\, \mathfrak{g}_\alpha \neq \{0\}\}$. As $\mathfrak{g}$ is noncompact, $\mathfrak{a} \neq \{0\}$. Moreover, Φ is a root system in $\mathfrak{a}^*$. Let $\Psi := \Phi \cup \{0\} \subseteq \mathfrak{a}^*$.

Let Δ be a base of the root system Φ. Let Φ_+ (resp. Φ_-) denote the roots in Φ that are positive (resp. negative) with respect to Δ. Let $\mathfrak{n}_+ := \sum_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$ and let $\mathfrak{n}_- := \sum_{\alpha \in \Phi_-} \mathfrak{g}_\alpha$. Then $\mathfrak{n}_+ \subseteq \mathcal{N}$ and $\mathfrak{n}_- \subseteq \mathcal{N}$. Choose an integer $m \geq 1$ and $X_1, \ldots, X_m \in \bigcup_{\alpha \in \Phi_+} (\mathfrak{g}_\alpha \backslash \{0\})$ such that $\mathfrak{n}_+ = \mathbb{R}X_1 + \cdots + \mathbb{R}X_m$. Choose an integer $n \geq 1$ and $Y_1', \ldots, Y_n' \in \bigcup_{\alpha \in \Phi_-} (\mathfrak{g}_\alpha \backslash \{0\})$ such that $\mathfrak{n}_- = \mathbb{R}Y_1' + \cdots + \mathbb{R}Y_n'$.

By Lemma 3.2, choose $Y_1, \ldots, Y_m \in \mathfrak{n}_-$ such that, for $i \in \{1, \ldots, m\}$, we have that (X_i, Y_i) is a standard $\mathfrak{sl}_2(\mathbb{R})$ generating set in $\mathfrak{g}$. Using Lemma 3.2 again, choose $X_1', \ldots, X_n' \in \mathfrak{n}_+$ such that, for $i \in \{1, \ldots, n\}$, we have that (Y_i', X_i') is a standard $\mathfrak{sl}_2(\mathbb{R})$ generating set in $\mathfrak{g}$; then (X_i', Y_i') is a standard $\mathfrak{sl}_2(\mathbb{R})$ generating set in $\mathfrak{g}$.

Let $k := m + n$. For $i \in \{1, \ldots, n\}$, let $X_{m+i} := X_i'$ and $Y_{m+i} := Y_i'$. By construction, (2) holds. We have $\mathbb{R}X_1 + \cdots + \mathbb{R}X_k = \mathfrak{n}_+ \subseteq \mathcal{N}$ and

$\mathbb{R}Y_1 + \cdots + \mathbb{R}Y_k = \mathfrak{n}_- \subseteq \mathcal{N}$, proving (3). It remains to prove (1). Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}_+ + \mathfrak{n}_-$. We wish to show that $\mathfrak{h} = \mathfrak{g}$.

Choose $T_0 \in \mathfrak{a}$ such that, for all $\gamma \in \Delta$, we have $\gamma(T_0) > 0$. Let $\delta := \mathrm{ad}\, T_0 : \mathfrak{g} \to \mathfrak{g}$. Then $\delta(\mathfrak{g}) = \mathfrak{n}_+ + \mathfrak{n}_-$. Then, by Lemma 7.14 of [Ad99b], we see that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and since $\mathfrak{n}_+ + \mathfrak{n}_- \neq \{0\}$, we conclude that $\mathfrak{h} = \mathfrak{g}$. $\square$

Lemma 3.4. *Let E be a vector space and let $(\,\cdot\,,\,\cdot\,)$ be a positive definite symmetric bilinear form on E. Let Φ be a root system in E. For all $\omega \in E$, let $\omega^\perp := \{\omega' \in E \,|\, (\omega, \omega') = 0\}$. Let $\mathbf{W}$ be the Weyl group of Φ. Let $\nu \in E$ and let $\mathbf{W}' := \{f \in \mathbf{W} \,|\, f(\nu) = \nu\}$. Assume that $\nu^\perp$ is spanned by $\Phi \cap \nu^\perp$. Then the only $\mathbf{W}'$-fixpoint in $\nu^\perp$ is 0.*

Proof. Fix $\mu \in \nu^\perp \backslash \{0\}$. We wish to prove that there exists $f \in \mathbf{W}'$ such that $f(\mu) \neq \mu$.

As $\Phi \cap \nu^\perp$ spans $\nu^\perp$ and as $\nu^\perp \not\subseteq \mu^\perp$, we see that $\Phi \cap \nu^\perp \not\subseteq \mu^\perp$. Choose $\lambda \in \Phi \cap \nu^\perp$ such that $\lambda \notin \mu^\perp$. Let $f \in \mathbf{W}$ denote the orthogonal reflection through $\lambda^\perp$ defined by $f(\alpha) = \alpha - [2(\alpha, \lambda)/(\lambda, \lambda)]\lambda$. Since $\lambda \in \nu^\perp$, we have $\nu \in \lambda^\perp$, so $f(\nu) = \nu$, so $f \in \mathbf{W}'$. Since $\lambda \notin \mu^\perp$, we have $\mu \notin \lambda^\perp$, so $f(\mu) \neq \mu$. $\square$

Recall, from §2, the definitions of $X^{\mathbb{C}}$ and $\mathcal{X}_{\mathbb{R}}$.

Lemma 3.5. *Let $\mathfrak{g}$ be a Lie algebra. If X and Y are real $\mathfrak{g}$-modules, and if $X^{\mathbb{C}}$ and $Y^{\mathbb{C}}$ are isomorphic in the category of complex $\mathfrak{g}$-modules, then X and Y are isomorphic in the category of real $\mathfrak{g}$-modules.*

Proof. We have $(X^{\mathbb{C}})_{\mathbb{R}} \cong (Y^{\mathbb{C}})_{\mathbb{R}}$. We also have $(X^{\mathbb{C}})_{\mathbb{R}} \cong X \oplus X$ and $(Y^{\mathbb{C}})_{\mathbb{R}} \cong Y \oplus Y$. Then $X \oplus X \cong Y \oplus Y$. So, by the Krull-Schmidt Theorem, we get $X \cong Y$. $\square$

Lemma 3.6. *Let $\mathfrak{g}_0$ be a reductive Lie algebra. Let V be a vector space. Let $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ be a representation. Let $\mathfrak{l} := [\mathfrak{g}_0, \mathfrak{g}_0]$ be the semisimple Levi factor of $\mathfrak{g}_0$. Let $\mathfrak{l}_0$ be an ideal of $\mathfrak{l}$. Let $Q \in \mathrm{Mink}(V)$. Assume that $\rho(\mathfrak{l}_0) = \mathfrak{so}(Q)$. Then $\rho(\mathfrak{g}_0) \subseteq \mathfrak{co}(Q)$.*

Proof. Let $\mathfrak{g}_1$ be an ideal of $\mathfrak{g}_0$ such that $\mathfrak{l}_0 + \mathfrak{g}_1 = \mathfrak{g}_0$ and $[\mathfrak{g}_1, \mathfrak{l}_0] = \{0\}$. Let $I : V \to V$ be the identity transformation. Let $S := \{tI \,|\, t \in \mathbb{R}\}$ be the set of scalar transformations on V. We have $\mathfrak{so}(Q) \neq \{0\}$, so $\rho(\mathfrak{l}_0) \neq \{0\}$. Then $\rho(\mathfrak{l}_0)$ is semisimple, and so $\mathfrak{so}(Q)$ is semisimple. Then $\dim(V) \geq 3$, so the centralizer in $\mathfrak{gl}(V)$ of $\mathfrak{so}(Q)$ is S. So, since $\rho(\mathfrak{l}_0) = \mathfrak{so}(Q)$ and since $[\mathfrak{g}_1, \mathfrak{l}_0] = \{0\}$, we get $\rho(\mathfrak{g}_1) \subseteq S$. Then $\rho(\mathfrak{g}_0) = \rho(\mathfrak{l}_0 + \mathfrak{g}_1) \subseteq (\mathfrak{so}(Q)) + S = \mathfrak{co}(Q)$. $\square$

Lemma 3.7. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{l}$ be a semisimple Levi factor of $\mathfrak{g}$. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V_1)$ be a representation. Assume $\rho(\mathfrak{l}) \neq \{0\}$. Let*

$Q \in \mathrm{Mink}(V_1)$. *Assume* $\mathfrak{so}(Q) \subseteq \rho(\mathfrak{g}) \subseteq \mathfrak{co}(Q)$. *Then, for some integer* $n \geq 3$, *there is an ideal* $\mathfrak{l}_0$ *of* $\mathfrak{l}$ *such that* $\rho|\mathfrak{l}_0 : \mathfrak{l}_0 \to \mathfrak{gl}(V_1)$ *is isomorphic to the defining representation of* $\mathfrak{so}(n-1,1)$ *on* $\mathbb{R}^{n\times 1}$.

Proof. Let $n := \dim(V_1)$. We have $\rho(\mathfrak{l}) \neq \{0\}$, so $\rho(\mathfrak{l})$ is semisimple. As $\rho(\mathfrak{l}) \subseteq \rho(\mathfrak{g}) \subseteq \mathfrak{co}(Q)$, we see that $\mathfrak{co}(Q)$ contains a semisimple Lie subalgebra. Then $n \geq 3$.

Let $\mathfrak{h} := \rho(\mathfrak{g}) \subseteq \mathfrak{gl}(V_1)$. Then $\rho(\mathfrak{l})$ is a semisimple Levi factor of $\mathfrak{h}$. We have $\mathfrak{so}(Q) \subseteq \mathfrak{h} \subseteq \mathfrak{co}(Q)$. So, since the codimension in $\mathfrak{co}(Q)$ of $\mathfrak{so}(Q)$ is 1, we conclude either that $\mathfrak{h} = \mathfrak{so}(Q)$ or that $\mathfrak{h} = \mathfrak{co}(Q)$. In either case, we see that $\mathfrak{h}$ is reductive and that the unique semisimple Levi factor of $\mathfrak{h}$ is $\mathfrak{so}(Q)$. Then $\rho(\mathfrak{l}) = \mathfrak{so}(Q)$.

Fix a vector space isomorphism $f : V_1 \to \mathbb{R}^{n\times 1}$ such that $Q_n \circ f = Q$. Let $F_0 : \mathfrak{so}(Q) \to \mathfrak{so}(Q_n)$ be the corresponding Lie algebra isomorphism defined by $F_0(T) = f \circ T \circ f^{-1}$. For all $T \in \mathfrak{so}(Q)$, for all $v \in V_1$, we have $f(Tv) = (F_0(T))(f(v))$.

Let $F_1 := F_0 \circ (\rho|\mathfrak{l}) : \mathfrak{l} \to \mathfrak{so}(Q_n)$. Then

$$F(\mathfrak{l}) = F_0(\rho(\mathfrak{l})) = F_0(\mathfrak{so}(Q)) = \mathfrak{so}(Q_n).$$

Let $\mathfrak{l}_1$ be the kernel of F_1. Let $\mathfrak{l}_0$ be an ideal of $\mathfrak{l}$ such that $\mathfrak{l}_0$ is a vector space complement in $\mathfrak{l}$ to $\mathfrak{l}_1$. Let $F := F_1|\mathfrak{l}_0 : \mathfrak{l}_0 \to \mathfrak{so}(Q_n)$. Then $F : \mathfrak{l}_0 \to \mathfrak{so}(Q_n)$ is an isomorphism. For all $X \in \mathfrak{g}$, for all $v \in V_1$, let $Xv := (\rho(X))v$. Then for all $X \in \mathfrak{l}_0$, for all $v \in V_1$, we have $f(Xv) = (F(X))(f(v))$. $\square$

Recall, from §2, the definition of almost $\mathfrak{s}$-invariant.

LEMMA 3.8. *Let* $\mathfrak{s}$ *be a Lie algebra and let* V *be a real* $\mathfrak{s}$*-module. Let* U *and* U' *be subspaces of* V *and assume that* (U, U') *is almost* $\mathfrak{s}$*-invariant. Then both of the following are true:*

(1) *If* W *is a real* $\mathfrak{s}$*-submodule of* V, *then* $(U \cap W, U' \cap W)$ *is almost* $\mathfrak{s}$*-invariant.*
(2) *If* W *is a real* $\mathfrak{s}$*-module and if* $f : V \to W$ *is a* $\mathfrak{g}$*-equivariant linear transformation, then* $(f(U), f(U'))$ *is almost* $\mathfrak{s}$*-invariant.*

Proof. These both follow from the definition of almost $\mathfrak{s}$-invariant. $\square$

4. Structural results about $\mathfrak{so}(n,1)$, Part I

Let $\mathbb{R}_+ := (0,\infty)$. Let $d \geq 2$ be a positive integer. Let $\mathfrak{g} := \mathfrak{so}(Q_d)$.
Let $\mathcal{M}_E^1$, $\mathcal{M}_H^1$, $\mathcal{M}_P^1$, $\mathcal{M}_P^2$ and $\mathcal{N}_1, \ldots, \mathcal{N}_{d-1}$ be as in §2.

LEMMA 4.1. *Let* $T \in \mathfrak{g}$. *Assume that some characteristic root of* $\mathrm{ad}\, T : \mathfrak{g} \to \mathfrak{g}$ *is not pure imaginary. Then*

(1) T *is semisimple;*

(2) *for some* $a > 0$, *the set of real eigenvalues of* $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$ *is equal to* $\{-a, 0, a\}$; *and*
(3) *for all* $X \in \mathfrak{c}_{\mathfrak{g}}(T)$, *we have that* X *is semisimple.*

Proof. By Lemma 3.1 of [Ad99b], after a change of basis, we may assume that $T \in (\mathbb{R}_+ \mathcal{M}_H^1) \cup \mathcal{M}_P^1$. For any $A \in \mathcal{M}_P^1$, every characteristic root of $\operatorname{ad} A : \mathfrak{g} \to \mathfrak{g}$ is pure imaginary. So $T \in \mathbb{R}_+ \mathcal{M}_H^1$. In particular, T is semisimple, proving (1).

Choose $a > 0$ such that $T \in a\mathcal{M}_H^1$. Then the real diagonalizable part of T is $a\mathcal{N}_1$. Then the set of real eigenvalues of $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$ is the same as that of $\operatorname{ad}(a\mathcal{N}_1) : \mathfrak{g} \to \mathfrak{g}$. Since the set of eigenvalues of $\operatorname{ad} \mathcal{N}_1 : \mathfrak{g} \to \mathfrak{g}$ is $\{-1, 0, 1\}$, we see that (2) holds.

Moreover, because $a\mathcal{N}_1$ is the real diagonalizable part of T, we have $\mathfrak{c}_{\mathfrak{g}}(T) \subseteq \mathfrak{c}_{\mathfrak{g}}(a\mathcal{N}_1) = \mathfrak{c}_{\mathfrak{g}}(\mathcal{N}_1) = \mathbb{R}\mathcal{M}_H^1$. As every element of $\mathbb{R}\mathcal{M}_H^1$ is semisimple, we see that (3) holds. □

LEMMA 4.2. *Let* $T, A, B \in \mathfrak{g}$. *Assume that* $A \neq 0 \neq B$. *Assume that* $[T, A] = A$ *and that* $[T, B] = -B$. *Then* $[A, B] \neq 0$.

Proof. Let T_0 be the real diagonalizable part of T. Then $[T_0, A] = A$ and $[T_0, B] = -B$. Replacing T by T_0, we may assume that T is real diagonalizable. Then there exists $g \in \mathrm{SO}(Q_d)$ such that gTg^{-1} is a diagonal matrix. Conjugating T, A and B by g, we may assume that T is a diagonal matrix.

The set of diagonal matrices in $\mathfrak{g}$ is $\mathbb{R}\mathcal{N}_1$, so $T \in \mathbb{R}\mathcal{N}_1$. Choose $a \in \mathbb{R}$ such that $T = a\mathcal{N}_1$. The set of eigenvalues of $\operatorname{ad} \mathcal{N}_1 : \mathfrak{g} \to \mathfrak{g}$ is $\{-1, 0, 1\}$, so the set of eigenvalues of $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$ is $\{-a, 0, a\}$. As $(\operatorname{ad} T)A = A$, we see that $1 \in \{-a, 0, a\}$, so $a \in \{-1, 1\}$, so $T \in \{-\mathcal{N}_1, \mathcal{N}_1\}$. Replacing T by $-T$ and interchanging A and B, if necessary, we may assume that $T = \mathcal{N}_1$. Then $(\operatorname{ad} \mathcal{N}_1)A = A$ and $(\operatorname{ad} \mathcal{N}_1)B = -B$.

For $X \in \mathbb{R}^{d \times d}$, let X^t be the transpose of X. The $(+1)$-eigenspace and (-1)-eigenspace of $\operatorname{ad} \mathcal{N}_1 : \mathfrak{g} \to \mathfrak{g}$ are, respectively, $\mathcal{M}_P^2$ and $(\mathcal{M}_P^2)^t$, so $A \in \mathcal{M}_P^2$ and $B \in (\mathcal{M}_P^2)^t$. By matrix multiplication, for all $X, Y \in \mathcal{M}_P^2 \backslash \{0\}$, we have $[X, Y^t] \neq 0$. Thus $[A, B] \neq 0$. □

5. Structural results about $\mathfrak{so}(n, 1)$, Part II

Let $d \geq 3$ be an integer. For any quadratic form $R : \mathbb{R}^{d \times 1} \to \mathbb{R}$, let $R^{\mathbb{C}} : \mathbb{C}^{d \times 1} \to \mathbb{C}$ denote the unique extension of R to a complex quadratic form. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^{d \times 1}$. Define a quadratic form $Q : \mathbb{R}^{d \times 1} \to \mathbb{R}$ by

$$Q(x_1 e_1 + \cdots + x_d e_d) = x_1 x_d + x_2 x_{d-1} + \cdots + x_{d-1} x_2 + x_d x_1.$$

Let $\mathfrak{l}^{\mathbb{C}} := \mathfrak{so}(Q^{\mathbb{C}})$. Let $\mathfrak{c}$ denote the collection of diagonal matrices in $\mathfrak{l}^{\mathbb{C}}$. Then $\mathfrak{c}$ is a maximal $\mathbb{C}$-split torus in $\mathfrak{l}^{\mathbb{C}}$.

For all $g \in \mathrm{GL}_{d-2}(\mathbb{C})$, let $g^* \in \mathrm{GL}_d(\mathbb{C})$ denote the matrix whose $(1,1)$ entry is one, whose (d,d) entry is one, whose middle $(d-2) \times (d-2)$ block is g and whose other entries are all zero. Define a proper injective Lie group homomorphism $\iota : \mathrm{GL}_{d-2}(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C})$ by $\iota(g) = g^*$.

Let $Q_0 := Q_d$ and $\mathfrak{l}_0^{\mathbb{C}} := \mathfrak{so}(Q_0^{\mathbb{C}})$. Choose $f \in \iota(\mathrm{GL}_{d-2}(\mathbb{C}))$ such that $Q_0^{\mathbb{C}} \circ f = Q^{\mathbb{C}}$. Let $F : \mathfrak{l}^{\mathbb{C}} \to \mathfrak{l}_0^{\mathbb{C}}$ be the corresponding Lie algebra isomorphism defined by $F(X) = fXf^{-1}$. Let $\mathfrak{c}_0 := F(\mathfrak{c})$. Then $\mathfrak{c}_0$ is a maximal $\mathbb{C}$-split torus in $\mathfrak{l}_0^{\mathbb{C}}$. We define $\overline{F} := F|\mathfrak{c} : \mathfrak{c} \to \mathfrak{c}_0$. Then $\overline{F} : \mathfrak{c} \to \mathfrak{c}_0$ is a vector space isomorphism. Let $\overline{F}_* : \mathfrak{c}^* \to \mathfrak{c}_0^*$ be the vector space isomorphism defined by $\overline{F}_*(\mu) = \mu \circ (\overline{F}^{-1})$.

Let $\Phi \subseteq \mathfrak{c}^*$ be the set of roots of $\mathfrak{c}$ on $\mathfrak{l}^{\mathbb{C}}$. Let κ denote the Killing form on $\mathfrak{l}$. By Corollary 8.2, p. 36, of [Hu72] and Proposition 8.3, p. 36, of [Hu72], we find that $\kappa|\mathfrak{c}$ is nondegenerate. Thus $\kappa|\mathfrak{c}$ induces an isomorphism $\tilde{\kappa} : \mathfrak{c} \to \mathfrak{c}^*$ of complex vector spaces. Let κ^* be the symmetric bilinear form on $\mathfrak{c}^*$ corresponding to $\kappa|\mathfrak{c}$ under this isomorphism. Let $E \subseteq \mathfrak{c}^*$ be the real span of Φ. Let $(\,\cdot\,,\,\cdot\,)$ be the restriction of κ^* to E. By the two paragraphs preceding Theorem 8.5, p. 40, of [Hu72], we see that $\mathfrak{c}^* = E \oplus \sqrt{-1}E$ and that $(\,\cdot\,,\,\cdot\,)$ is positive definite.

In a similar way, from $\mathfrak{l}_0^{\mathbb{C}}$ and $\mathfrak{c}_0$, we define Φ_0, κ_0, $\tilde{\kappa}_0$, κ_0^*, E_0 and $(\,\cdot\,,\,\cdot\,)_0$. Under the isomorphism $F : \mathfrak{l}^{\mathbb{C}} \to \mathfrak{l}_0^{\mathbb{C}}$, we have: Φ corresponds to Φ_0, κ corresponds to κ_0, $\tilde{\kappa}$ corresponds to $\tilde{\kappa}_0$, κ^* corresponds to κ_0^*, E corresponds to E_0 and $(\,\cdot\,,\,\cdot\,)$ corresponds to $(\,\cdot\,,\,\cdot\,)_0$,

For all $\omega \in E$, let $\omega^\perp := \{\omega' \in E \,|\, (\omega, \omega') = 0\}$ denote the orthogonal complement in E to ω, with respect to $(\,\cdot\,,\,\cdot\,)$. For all $\omega \in E_0$, let $\omega^\perp := \{\omega' \in E_0 \,|\, (\omega, \omega')_0 = 0\}$ denote the orthogonal complement in E_0 to ω, with respect to $(\,\cdot\,,\,\cdot\,)_0$.

Let $I := \{1, \ldots, d\}$. For all $i, j \in I$, let $e_{ij} \in \mathbb{C}^{d \times d}$ be the matrix with a one in the (i,j) entry and with zeroes elsewhere. For all $i, j \in I$, define $e_{ij}^* : \mathbb{C}^{d \times d} \to \mathbb{C}$ by $e_{ij}^*(\sum a_{kl} e_{kl}) = a_{ij}$. For all $i \in I$, let $L_i := e_{ii}^*|\mathfrak{c} \in \mathfrak{c}^*$. Let $T := e_{11} - e_{dd} \in \mathfrak{c}$ and $T_0 := \overline{F}(T) \in \mathfrak{c}_0$. Let $\nu := L_1 \in \mathfrak{c}^*$ and $\nu_0 := \overline{F}_*(\nu) \in \mathfrak{c}_0^*$.

Let $\rho : \mathfrak{l}^{\mathbb{C}} \to \mathfrak{gl}_d(\mathbb{C})$ and $\rho_0 : \mathfrak{l}_0^{\mathbb{C}} \to \mathfrak{gl}_d(\mathbb{C})$ be the inclusion maps; these are both representations. Let $\Xi \subseteq \mathfrak{c}^*$ denote the set of weights of $\rho|\mathfrak{c} : \mathfrak{c} \to \mathfrak{gl}_d(\mathbb{C})$. Similarly, let $\Xi_0 \subseteq \mathfrak{c}_0^*$ denote the set of weights of $\rho_0|\mathfrak{c}_0 : \mathfrak{c}_0 \to \mathfrak{gl}_d(\mathbb{C})$. Then we have $\overline{F}_*(\Xi) = \Xi_0$.

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$. Given a vector space Z, a subset $S \subseteq Z$ and $m \in \mathbb{N}$, let

$$C_m(S, Z) := \left\{ \sum_{i=1}^m a_i s_i \,\middle|\, a_1, \ldots, a_m > 0,\, s_1, \ldots, s_m \in S \right\}.$$

For any vector space Z and any $S \subseteq Z$, let $C(S, Z) := \bigcup_{m \in \mathbb{N}} C_m(S, Z)$.

LEMMA 5.1. *Let $I_2 := \{(i,j) \in I^2 \,|\, i \neq j$ and $i+j \neq d+1\}$. All of the following are true:*

(1) *For all $i \in I$, we have $L_i = -L_{d-i+1}$.*
(2) *We have $\{L_i + L_j \,|\, (i,j) \in I_2\} = \Phi = \{L_i - L_j \,|\, (i,j) \in I_2\}$.*
(3) *For all $(i,j) \in I_2$, we have $(L_i, L_j) = \kappa^*(L_i, L_j) = 0$.*
(4) *We have $\Xi = \{L_1, \ldots, L_d\}$.*
(5) *We have $E = \mathbb{R}L_1 + \cdots + \mathbb{R}L_d$.*
(6) *We have $L_1^\perp = \mathbb{R}L_2 + \cdots + \mathbb{R}L_{d-1}$.*

Proof. Conclusions (1)–(4) are calculations and Conclusion (5) follows from Conclusion (2), so it remains to prove Conclusion (6).

By Conclusion (1), we have $L_1 = -L_d$, and so it follows from Conclusion (5) that the codimension in E of $\mathbb{R}L_2 + \cdots \mathbb{R}L_{d-1}$ is ≤ 1. As $L_1 \neq 0$, it follows that the codimension in E of $L_1^\perp$ is 1. By Conclusion (3), we have $\mathbb{R}L_2 + \cdots + \mathbb{R}L_{d-1} \subseteq L_1^\perp$. Conclusion (6) follows. ☐

LEMMA 5.2. *All of the following are true:*

(1) *We have $\nu \in E$.*
(2) *For all $\phi \in E$, we have $\phi(T) \in \mathbb{R}$.*
(3) *For all $\phi \in \nu^\perp$, we have $\phi(T) = 0$.*
(4) *We have $\{-\nu, \nu\} \subseteq \Xi \subseteq \{-\nu, \nu\} \cup \nu^\perp$.*
(5) *For some base Δ of Φ, we have $\nu \in C(\Delta, E)$.*
(6) *If $d \neq 4$, then $\nu^\perp$ is spanned by $\Phi \cap \nu^\perp$.*

Proof of (1). Since $\nu = L_1$, this follows from Conclusion (5) of Lemma 5.1.

Proof of (2). We have $T = e_{11} - e_{dd} \in \mathbb{R}^{d \times d} \cap \mathfrak{c}$. Therefore, for all $i \in I$, we get $L_i(T) \in L_i(\mathbb{R}^{d\times d} \cap \mathfrak{c}) \subseteq \mathbb{R}$. By Conclusion (5) of Lemma 5.1, we are done.

Proof of (3). Since $\nu = L_1$, Conclusion (6) of Lemma 5.1 asserts that $\nu^\perp = \mathbb{R}L_2 + \cdots + \mathbb{R}L_{d-1}$. Since $T = e_{11} - e_{dd}$, for all $i \in \{2, \ldots, d-1\}$, we have $L_i(T) = 0$. The result follows.

Proof of (4). By Conclusion (4) of Lemma 5.1, we have

$$\{L_1, L_d\} \quad \subseteq \quad \Xi \quad \subseteq \quad \{L_1, L_d\} \cup (\mathbb{R}L_2 + \cdots + \mathbb{R}L_{d-1}).$$

So, by Conclusion (6) of Lemma 5.1, we have

$$\{L_1, L_d\} \quad \subseteq \quad \Xi \quad \subseteq \quad \{L_1, L_d\} \cup L_1^\perp.$$

We have $\nu = L_1$. So, by Conclusion (1) of Lemma 5.1, $-\nu = L_d$. The result follows.

Proof of (5). Let $\mathcal{Q} := \bigcup_{\alpha \in \Phi} \alpha^\perp$ and let $\mathcal{R} := E \backslash \mathcal{Q}$. Then $\mathcal{R}$ is dense in E. Since $L_1 \neq 0$, by positive definiteness, we have $(L_1, L_1) > 0$. By (3) of Lemma 5.1, we have $(L_1, L_2) = 0$. Then $(L_1, L_1 + L_2) > 0$ and $(L_1, L_1 - L_2) > 0$. Choose $\eta \in \mathcal{R}$ sufficiently close to L_1 that $(\eta, L_1 + L_2) > 0$ and $(\eta, L_1 - L_2) > 0$. Let $H := \{\omega \in E \,|\, (\eta, \omega) > 0\}$. Let Δ be the set of indecomposable elements of $\Phi \cap H$. Let Φ_+ denote the set of roots in Φ that are positive with respect to Δ. Then we have $\Phi_+ \subseteq C(\Delta, E)$ and $\Phi_+ = \Phi \cap H$. Let $\sigma := L_1 + L_2$ and $\tau := L_1 - L_2$. By (2) of Lemma 5.1, $\sigma, \tau \in \Phi$. Then $\sigma, \tau \in \Phi \cap H = \Phi_+ \subseteq C(\Delta, E)$, so $\nu = L_1 = (1/2)(\sigma + \tau) \in C(\Delta, E)$.

Proof of (6). Let E' be the real span of $\Phi \cap \nu^\perp$. Then $E' \subseteq \nu^\perp$. We wish to show that $\nu^\perp \subseteq E'$. Since $\nu = L_1$, it follows from Conclusion (6) of Lemma 5.1 that $\nu^\perp = \mathbb{R}L_2 + \cdots + \mathbb{R}L_{d-1}$. Fix $i \in \{2, \ldots, d-1\}$. We wish to show that $L_i \in E'$.

Say, for this paragraph, that $d = 3$. Then $i \in \{2, \ldots, d-1\} = \{2\}$, so $i = 2$. By Conclusion (1) of Lemma 5.1, we have $L_2 = -L_{3-2+1}$, so $L_2 = -L_2$, so $L_2 = 0$. Then $L_i = L_2 = 0 \in E'$.

We may therefore assume that $d \neq 3$. By assumption, $d \geq 3$ and $d \neq 4$. Then $d \geq 5$. Then the cardinality of $\{2, \ldots, d-1\}$ is ≥ 3. Choose $j \in \{2, \ldots, d-1\} \backslash \{i, d-i+1\}$. Let $\sigma := L_i + L_j$ and let $\tau := L_i - L_j$. By Conclusion (2) of Lemma 5.1, we have $\sigma, \tau \in \Phi$. Moreover, $\sigma, \tau \in \mathbb{R}L_2 + \cdots + \mathbb{R}L_{d-1} = \nu^\perp$. Therefore $\sigma, \tau \in \Phi \cap \nu^\perp \subseteq E'$. Then $L_i = (1/2)(\sigma + \tau) \in E'$. $\square$

LEMMA 5.3. *All of the following are true:*

1. *We have* $\nu_0 \in E_0$.
2. *For all* $\phi \in E_0$, *we have* $\phi(T_0) \subseteq \mathbb{R}$.
3. *For all* $\phi \in \nu_0^\perp$, *we have* $\phi(T_0) = 0$.
4. *We have* $\{-\nu_0, \nu_0\} \subseteq \Xi_0 \subseteq \{-\nu_0, \nu_0\} \cup \nu_0^\perp$.
5. *For some base* Δ_0 *of* Φ_0, *we have* $\nu_0 \in C(\Delta_0, E_0)$.
6. *If* $d \neq 4$, *then* $\nu_0^\perp$ *is spanned by* $\Phi_0 \cap \nu_0^\perp$.
7. *We have* $T_0 = e_{11} - e_{dd}$ *and* $\nu_0 = e_{11}^*|\mathfrak{c}_0$.

Proof of (1)–(6). Conclusions (1)–(6) follow from Lemma 5.2 because $F : \mathfrak{l}^{\mathbb{C}} \to \mathfrak{l}_0^{\mathbb{C}}$ is a Lie algebra isomorphism, under which ν corresponds to ν_0, E corresponds to E_0, T corresponds to T_0, $(\,\cdot\,,\,\cdot\,)$ corresponds to $(\,\cdot\,,\,\cdot\,)_0$, Ξ corresponds to Ξ_0 and Φ corresponds to Φ_0.

Proof of (7). Since $f \in \iota(\mathrm{GL}_{d-2}(\mathbb{C}))$, we have $f e_{11} f^{-1} = e_{11}$ and $f e_{dd} f^{-1} = e_{dd}$; moreover, for all $X \in \mathbb{C}^{d \times d}$, $e_{11}^*(f^{-1} X f) = e_{11}^*(X)$.

Then we have $T_0 = F(T) = f(e_{11} - e_{dd}) f^{-1} = e_{11} - e_{dd}$. Moreover, for all $X \in \mathfrak{l}_0^{\mathbb{C}}$, we have $e_{11}^*(F^{-1}(X)) = e_{11}(f^{-1} X f) = e_{11}^*(X)$. We have $\nu_0 = \overline{F}_*(\nu)$ and $\nu = L_1 = e_{11}^*|\mathfrak{c}$. So, for all $X \in \mathfrak{c}_0$, we have $\nu_0(X) = \nu(\overline{F}^{-1}(X)) = e_{11}^*(F^{-1}(X)) = e_{11}^*(X)$. $\square$

6. Special modules

Recall, from §2, the definition of $\mathcal{X}_{\mathbb{R}}$ and $\overline{\mathcal{X}}$. Let $\mathfrak{g}$ be the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Let $\mathfrak{g}_{\mathbb{R}}$ be the real Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Let

$$X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad T := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$. For all $d \in \mathbb{N}$, let $\mathcal{X}_d$ be a d-dimensional irreducible $\mathfrak{g}$-module; then $\mathcal{X}_d$ is unique up to isomorphism of $\mathfrak{g}$-modules. For all $d \in \mathbb{N}$, let $\mathcal{Y}_d$ denote $\mathcal{X}_d$, as an irreducible complex $\mathfrak{g}_{\mathbb{R}}$-module. For all $d, e \in \mathbb{N}$, let $\mathcal{X}_{de} := \mathcal{X}_d \otimes_{\mathbb{C}} \mathcal{X}_e$, an object in the category of $(\mathfrak{g}\oplus\mathfrak{g})$-modules. For all $d, e \in \mathbb{N}$, let $\mathcal{Y}_{de} := \mathcal{Y}_d \otimes_{\mathbb{C}} \overline{\mathcal{Y}_e}$, an object in the category of complex $\mathfrak{g}_{\mathbb{R}}$-modules.

If $\mathfrak{l}$ is a semisimple Lie algebra and if $\mathcal{Z}$ is a complex $\mathfrak{l}$-module then we shall say that $\mathcal{Z}$ is *special* if all three of the following hold:

- $\mathcal{Z}$ is an irreducible complex $\mathfrak{l}$-module;
- $\mathcal{Z}_{\mathbb{R}}$ is a reducible real $\mathfrak{l}$-module; and
- for any real diagonalizable $W \in \mathfrak{l}\backslash\{0\}$, the map $z \mapsto Wz : \mathcal{Z} \to \mathcal{Z}$ has exactly one positive eigenvalue.

LEMMA 6.1. *If $\mathcal{Y}$ is an irreducible complex $\mathfrak{g}_{\mathbb{R}}$-module, then there exist $d, e \in \mathbb{N}$ such that $\mathcal{Y}$ is isomorphic to $\mathcal{Y}_{de}$ in the category of complex $\mathfrak{g}_{\mathbb{R}}$-modules.*

Proof. Let $\mathfrak{g}_0 := \{(W, \overline{W}) \,|\, W \in \mathfrak{g}\} \subseteq \mathfrak{g} \oplus \mathfrak{g}$, so $\mathfrak{g}_0$ is a real Lie subalgebra of the complex Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$. We have $\mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0 = \mathfrak{g} \oplus \mathfrak{g}$, which gives a natural correspondence between $(\mathfrak{g} \oplus \mathfrak{g})$-modules and complex $\mathfrak{g}_0$-modules. Moreover, $(W, \overline{W}) \mapsto W : \mathfrak{g}_0 \to \mathfrak{g}_{\mathbb{R}}$ is an isomorphism of (real) Lie algebras, which gives a natural correspondence between complex $\mathfrak{g}_0$-modules and complex $\mathfrak{g}_{\mathbb{R}}$-modules.

Under these correspondences, for all $d, e \in \mathbb{N}$, we have that the $(\mathfrak{g} \oplus \mathfrak{g})$-module $\mathcal{X}_{de}$ corresponds to the complex $\mathfrak{g}_{\mathbb{R}}$-module $\mathcal{Y}_{de}$. Let $\mathcal{X}$ be the $(\mathfrak{g}\oplus\mathfrak{g})$-module corresponding to the complex $\mathfrak{g}_{\mathbb{R}}$-module $\mathcal{Y}$. Then $\mathcal{X}$ is an irreducible $(\mathfrak{g} \oplus \mathfrak{g})$-module. By the representation theory of the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, we choose $d, e \in \mathbb{N}$ such that $\mathcal{X}$ is isomorphic to $\mathcal{X}_{de}$ in the category of $(\mathfrak{g} \oplus \mathfrak{g})$-modules. Then $\mathcal{Y}$ is isomorphic to $\mathcal{Y}_{de}$ in the category of complex $\mathfrak{g}_{\mathbb{R}}$-modules. $\square$

LEMMA 6.2. *Let $e \in \mathbb{N}$. Assume that $e \neq 1$. Then $(\mathcal{Y}_e)_{\mathbb{R}}$ is an irreducible real $\mathfrak{g}_{\mathbb{R}}$-module.*

Proof. Let $V := \mathcal{X}_e = \mathcal{Y}_e$. Let S be a $\mathfrak{g}$-invariant real subspace of V. Assume that $S \neq \{0\}$. We wish to show that $V = S$.

Let $I := \{1, \ldots, e\}$. For all $i \in I$, let $\lambda_i := e - 2i + 1$ and let $V_i := \{v \in V \,|\, Tv = \lambda_i v\}$. By the representation theory of the complex Lie algebra

$\mathfrak{sl}_2(\mathbb{C})$, the complex linear transformation $v \mapsto Tv : V \to V$ is diagonalizable, with eigenspaces $V_1, \ldots, V_e$. Then $V = V_1 + \cdots + V_e$.

Since $S \neq \{0\}$, since $TS \subseteq S$ and since $v \mapsto Tv : V \to V$ is diagonalizable, choose $i_0 \in I$ such that $V_{i_0} \cap S \neq \{0\}$. By the representation theory of the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we have $X^{i_0-1}V_{i_0} = V_1$ and, moreover, we have that the map $v \mapsto X^{i_0-1}v : V_{i_0} \to V_1$ is an isomorphism of complex vector spaces. Then, because $XS \subseteq S$, it follows that $V_1 \cap S \neq \{0\}$.

By the representation theory of the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we have $\dim_{\mathbb{C}}(V_1) = 1$, and it follows, for all $v \in V_1 \backslash \{0\}$, that the real span of v and $\sqrt{-1}v$ is V_1. Let $T' := \sqrt{-1}T \in \mathfrak{g}$. For all $v \in V_1$, we have $T'v = \sqrt{-1}\lambda_1 v$. Because $e \neq 1$, we have $\lambda_1 \neq 0$. So, for all $v \in V_1 \backslash \{0\}$, the real span of v and $T'v$ is V_1. So, because $V_1 \cap S \neq \{0\}$ and because $T'S \subseteq S$, we conclude that $V_1 \subseteq S$.

By the representation theory of the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, for all $i \in I$, we have $Y^{i-1}V_1 = V_i$. So, as $YS \subseteq S$, we conclude, for all $i \in I$, that $V_i \subseteq S$. Then $V = V_1 + \cdots + V_e \subseteq S \subseteq V$, so $V = S$. □

LEMMA 6.3. *Let $\mathcal{Y}$ be a special complex $\mathfrak{g}_{\mathbb{R}}$-module. Then $\mathcal{Y}$ is isomorphic to $\mathcal{Y}_{22}$ in the category of complex $\mathfrak{g}_{\mathbb{R}}$-modules.*

Proof. Since $\mathcal{Y}$ is special, it follows that $\mathcal{Y}$ is an irreducible complex $\mathfrak{g}_{\mathbb{R}}$-module. By Lemma 6.1, choose $d, e \in \mathbb{N}$ such that $\mathcal{Y}$ is isomorphic to $\mathcal{Y}_{de}$ as complex $\mathfrak{g}_{\mathbb{R}}$-modules. We wish to show that $d = 2 = e$.

Let $E := \{(1,2),(2,1),(1,3),(3,1),(2,2)\}$. Because $\mathcal{Y}$ is special, it follows that $\mathcal{Y}_{de}$ is special as well. Then $v \mapsto Tv : \mathcal{Y}_{de} \to \mathcal{Y}_{de}$ has exactly one positive eigenvalue. Then $(d,e) \in E$. We wish to show that $d \neq 1 \neq e$. We will show that $d \neq 1$; the proof that $1 \neq e$ is similar. Assume that $d = 1$. We aim for a contradiction.

Then $\mathcal{Y}_{1e}$ is special. Because $d = 1$ and because $(d,e) \in E$, we see that $e \neq 1$. We have $\mathcal{Y}_{1e} = \mathcal{Y}_1 \otimes_{\mathbb{C}} \overline{\mathcal{Y}_e}$. Since $\mathcal{Y}_1$ is one-dimensional and $\mathfrak{g}_{\mathbb{R}}$-trivial, it follows that $\mathcal{Y}_{1e}$ is isomorphic to $\overline{\mathcal{Y}_e}$ in the category of complex $\mathfrak{g}_{\mathbb{R}}$-modules. Then $\overline{\mathcal{Y}_e}$ is special. Since $(\overline{\mathcal{Y}_e})_{\mathbb{R}}$ is isomorphic to $(\mathcal{Y}_e)_{\mathbb{R}}$ in the category of real $\mathfrak{g}_{\mathbb{R}}$-modules, it follows from the definition of special that $(\mathcal{Y}_e)_{\mathbb{R}}$ is a reducible real $\mathfrak{g}_{\mathbb{R}}$-module. This contradicts Lemma 6.2. □

COROLLARY 6.4. *Let $\mathfrak{l}_0 := \mathfrak{so}(Q_4)$. Let $\mathcal{V}$ and $\mathcal{W}$ be special complex $\mathfrak{l}_0$-modules. Then $\mathcal{V}$ and $\mathcal{W}$ are isomorphic as complex $\mathfrak{l}_0$-modules.*

Proof. Since $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to $\mathfrak{so}(3,1)$ in the category of real Lie algebras, and since Q_4 has signature $(3,1)$, we see that $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to $\mathfrak{so}(Q_4)$. That is, $\mathfrak{g}_{\mathbb{R}}$ is isomorphic to $\mathfrak{l}_0$ in the category of real Lie algebras. By Lemma 6.3, any two special complex $\mathfrak{g}_{\mathbb{R}}$-modules are both isomorphic to $\mathcal{Y}_{22}$, and so are isomorphic to one another. Then any two special complex $\mathfrak{l}_0$-modules are isomorphic to one another. □

7. The defining representation of $\mathfrak{so}(n,1)$

Let $\mathfrak{l}_0$ be a semisimple Lie algebra. Let $\mathfrak{a}$ be a maximal $\mathbb{R}$-split torus in $\mathfrak{l}_0$. Let V be a vector space. Let $\rho : \mathfrak{l}_0 \to \mathfrak{gl}(V)$ be a representation. For all $\beta \in \mathfrak{a}^*$, if β is a weight of $\mathfrak{a}$ on V, then let V_β denote the β-weightspace of $\mathfrak{a}$ on V.

Recall, from §2, the definition of $X^{\mathbb{C}}$, $\mathcal{X}_{\mathbb{R}}$ and $\overline{\mathcal{X}}$.

Lemma 7.1. *Let $\alpha \in \mathfrak{a}^* \backslash \{0\}$. Assume that the set of roots of $\mathfrak{a}$ on $\mathfrak{l}_0$ is $\{-\alpha, \alpha\}$. Assume that the set of weights of $\mathfrak{a}$ on V is $\{-\alpha, 0, \alpha\}$. Assume that* $\dim(V_\alpha) = 1 = \dim(V_{-\alpha})$. *Then there exists $Q \in \mathrm{Mink}(V)$ such that $\rho(\mathfrak{l}_0) = \mathfrak{so}(Q)$.*

Proof. Because the set of roots of $\mathfrak{a}$ on $\mathfrak{l}_0$ is $\{-\alpha, \alpha\}$, we see that the root system of $\mathfrak{l}_0$ is reduced and has real rank 1. It follows, for some integer $d \geq 3$, that $\mathfrak{l}_0$ is Lie algebra isomorphic to $\mathfrak{so}(d-1,1)$. We may therefore assume that $d \geq 3$ is an integer and that $\mathfrak{l}_0 = \mathfrak{so}(Q_d)$.

Let Q_0, $\mathfrak{l}_0^{\mathbb{C}}$, $\mathfrak{c}_0$, Φ_0, E_0, $(\,\cdot\,,\,\cdot\,)_0$, $\omega^\perp$, e_{ij}^*, ν_0, T_0, ρ_0 and Ξ_0 all be defined as in §5. We have $\mathfrak{l}_0^{\mathbb{C}} = \mathfrak{l}_0 \oplus \sqrt{-1}\mathfrak{l}_0$. As $\mathbb{R}T_0$ is a maximal $\mathbb{R}$-split torus in $\mathfrak{l}_0$, by conjugacy of maximal $\mathbb{R}$-split tori, we may assume that $\mathfrak{a} = \mathbb{R}T_0$.

Let $W := \mathbb{R}^{d\times 1}$ be a real $\mathfrak{l}_0$-module, under the defining representation of $\mathfrak{so}(Q_d)$ on $\mathbb{R}^{d\times 1}$. It suffices to show that V is isomorphic to W in the category of real $\mathfrak{l}_0$-modules. Let $\mathcal{V} := V^{\mathbb{C}}$ and $\mathcal{W} := W^{\mathbb{C}}$. Because $\mathfrak{l}_0^{\mathbb{C}} = \mathfrak{l}_0 \oplus \sqrt{-1}\mathfrak{l}_0$, it follows that the complex representation of $\mathfrak{l}_0$ on $\mathcal{V}$ extends uniquely to a representation of $\mathfrak{l}_0^{\mathbb{C}}$ on $\mathcal{V}$. Similarly, the complex representation of $\mathfrak{l}_0$ on $\mathcal{W}$ extends uniquely to a representation of $\mathfrak{l}_0^{\mathbb{C}}$ on $\mathcal{W}$. Then $\mathcal{V}$ and $\mathcal{W}$ are complex $\mathfrak{l}_0$-modules, and, at the same time, they are $\mathfrak{l}_0^{\mathbb{C}}$-modules. Then $\Xi_0 \subseteq \mathfrak{c}_0^*$ is the set of weights of $\mathfrak{c}_0$ on $\mathcal{W}$. By Lemma 3.5, it suffices to show that $\mathcal{V}$ is isomorphic to $\mathcal{W}$ in the category of complex $\mathfrak{l}_0$-modules.

Let $\mathcal{U}$ be a nonzero irreducible complex $\mathfrak{l}_0$-submodule of $\mathcal{V}$. In the category of real $\mathfrak{l}_0$-modules, $\mathcal{V}_{\mathbb{R}}$ is isomorphic to $V \oplus V$, so, since $\mathcal{U}_{\mathbb{R}}$ is a nonzero real $\mathfrak{l}_0$-submodule of $\mathcal{V}_{\mathbb{R}}$, we conclude that $\mathcal{U}_{\mathbb{R}}$ is isomorphic either to V or to $V\oplus V$. Then V is a nonzero direct summand of $\mathcal{U}_{\mathbb{R}}$ in the category of real $\mathfrak{l}_0$-modules. Then $\mathcal{V}$ is a nonzero direct summand of $(\mathcal{U}_{\mathbb{R}})^{\mathbb{C}}$ in the category of complex $\mathfrak{l}_0$-modules.

If $\mathcal{X}$ is a complex $\mathfrak{l}_0$-module with complex structure $J : \mathcal{X} \to \mathcal{X}$, then every weightspace of $\mathfrak{a}$ on $\mathcal{X}_{\mathbb{R}}$ is J-invariant, and therefore has even dimension. In particular, the weightspace dimensions of $\mathfrak{a}$ on $\mathcal{U}_{\mathbb{R}}$ are all even. On the other hand, by hypothesis, the weightspace V_α of $\mathfrak{a}$ on V satisfies $\dim(V_\alpha) = 1$. We conclude that $V \not\cong \mathcal{U}_{\mathbb{R}}$. Then, by Lemma 3.5, we see that $\mathcal{V} \not\cong (\mathcal{U}_{\mathbb{R}})^{\mathbb{C}}$. So, because $\mathcal{V}$ is a nonzero complex $\mathfrak{l}_0$-submodule of $(\mathcal{U}_{\mathbb{R}})^{\mathbb{C}}$ and because $(\mathcal{U}_{\mathbb{R}})^{\mathbb{C}}$ is isomorphic to $\mathcal{U} \oplus \overline{\mathcal{U}}$ in the category of complex $\mathfrak{l}_0$-modules, it follows either that $\mathcal{V} \cong \mathcal{U}$ or that $\mathcal{V} \cong \overline{\mathcal{U}}$. In particular, $\mathcal{V}$ is an irreducible complex $\mathfrak{l}_0$-module.

Case A: $d = 4$. Define *special* as in §6. For any $W \in \mathfrak{a}$, the set of eigenvalues of $v \mapsto Wv : \mathcal{V} \to \mathcal{V}$ is $\{-\alpha(W), 0, \alpha(W)\}$. So, for any $W \in \mathfrak{a}\backslash\{0\}$, the map $v \mapsto Wv : \mathcal{V} \to \mathcal{V}$ has exactly one positive eigenvalue. By conjugacy of maximal $\mathbb{R}$-split tori, we see, for any real diagonalizable $W \in \mathfrak{l}_0\backslash\{0\}$, that the map $v \mapsto Wv : \mathcal{V} \to \mathcal{V}$ has exactly one positive eigenvalue.

As $\mathcal{V}_\mathbb{R}$ is isomorphic to $V \oplus V$, we see that $\mathcal{V}_\mathbb{R}$ is reducible in the category of real $\mathfrak{l}_0$-modules. Moreover, we have observed that $\mathcal{V}$ is an irreducible complex $\mathfrak{l}_0$-module. Then $\mathcal{V}$ is special. As $\mathcal{W}$ is also special, we conclude from Corollary 6.4 that $\mathcal{V}$ and $\mathcal{W}$ are isomorphic as complex $\mathfrak{l}_0$-modules.

Case B: $d \neq 4$. By (2) of Lemma 5.3, we define a restriction map $r : E_0 \to \mathfrak{a}^*$ by $r(\mu) = \mu|\mathfrak{a}$. Then $r(\nu_0) \neq 0$, so $r \neq 0$. By (1) of Lemma 5.3, we have $\nu_0 \in E_0$. By (7) of Lemma 5.3, $r(\nu_0) = e_{11}^*|\mathfrak{a}$.

We compute that the set of roots of $\mathfrak{a}$ on $\mathfrak{l}_0$ is $\{-e_{11}^*|\mathfrak{a}, e_{11}^*|\mathfrak{a}\}$. By assumption, the set of roots of $\mathfrak{a}$ on $\mathfrak{l}_0$ is $\{-\alpha, \alpha\}$. Then

$$\{-r(\nu_0), r(\nu_0)\} = \{-e_{11}^*|\mathfrak{a}, e_{11}^*|\mathfrak{a}\} = \{-\alpha, \alpha\}.$$

Replacing α by $-\alpha$ if necessary, we may assume that $r(\nu_0) = \alpha$. Let $\mathbf{W}$ be the Weyl group of Φ in E_0. Let $\mathbf{W}' := \{f \in \mathbf{W} \,|\, f(\nu_0) = \nu_0\}$.

Let $p : E_0 \to \mathbb{R}\nu_0$ be the orthogonal projection defined by the formula $p(\mu) = [(\mu, \nu_0)/(\nu_0, \nu_0)]\nu_0$. By (3) of Lemma 5.3, we have $r(\nu_0^\perp) = \{0\}$. Then $\nu_0^\perp \subseteq \ker(r)$. Since $\nu_0 \neq 0$, we see that the codimension in E_0 of $\nu_0^\perp$ is 1. Since $r \neq 0$, we see that the codimension in E_0 of $\ker(r)$ is 1. Then $\ker(r) = \nu_0^\perp$. Then, for all $\mu \in E_0$, for all $t \in \mathbb{R}$, we have:

$$(*) \qquad r(\mu) = t\alpha \text{ iff } \mu - t\nu_0 \in \ker(r) \text{ iff } \mu - t\nu_0 \in \nu_0^\perp \text{ iff } \mu \in t\nu_0 + \nu_0^\perp.$$

Let $\Lambda \subseteq \mathfrak{c}_0^*$ be the set of weights of $\mathfrak{c}_0$ on $\mathcal{V}$. By the representation theory of semisimple Lie algebras, we have $\Lambda \subseteq E_0$. For all $\mu \in \Lambda$, let $\mathcal{V}_\mu$ denote the μ-weightspace of $\mathfrak{c}_0$ on $\mathcal{V}$. Let $V_\alpha^\mathbb{C} \subseteq \mathcal{V}$ denote the complexification of V_α. Then $\dim_\mathbb{C}(V_\alpha^\mathbb{C}) = \dim(V_\alpha) = 1$. Since $\mathfrak{a}$ and $\mathfrak{c}_0$ centralize one another, we conclude that $V_\alpha^\mathbb{C}$ is $\mathfrak{c}_0$-invariant.

For all $\mu \in \Lambda$, we have

$$r(\mu) = \alpha \quad \Longrightarrow \quad \mathcal{V}_\mu \subseteq V_\alpha^\mathbb{C} \quad \Longrightarrow \quad \mathcal{V}_\mu \cap V_\alpha^\mathbb{C} \neq \{0\} \quad \Longrightarrow \quad r(\mu) = \alpha,$$

so

$$r(\mu) = \alpha \quad \Longleftrightarrow \quad \mathcal{V}_\mu \subseteq V_\alpha^\mathbb{C} \quad \Longleftrightarrow \quad \mathcal{V}_\mu \cap V_\alpha^\mathbb{C} \neq \{0\}.$$

Because $V_\alpha^\mathbb{C}$ is $\mathfrak{c}_0$-invariant, choose $\mu_+ \in \Lambda$ such that $\mathcal{V}_{\mu_+} \cap V_\alpha^\mathbb{C} \neq \{0\}$. Then $r(\mu_+) = \alpha$, so, by $(*)$, we have $\mu_+ \in \nu_0 + \nu_0^\perp$.

For all $\mu \in \Lambda$, we have: $\mathcal{V}_\mu \subseteq V_\alpha^\mathbb{C}$ iff $r(\mu) = \alpha$. So, by $(*)$, for all $\mu \in \Lambda$, we have: $\mathcal{V}_\mu \subseteq V_\alpha^\mathbb{C}$ iff $\mu \in \nu_0 + \nu_0^\perp$. So, since $\dim_\mathbb{C}(V_\alpha^\mathbb{C}) = 1$, it follows that $(\nu_0 + \nu_0^\perp) \cap \Lambda$ contains at most one element. Moreover, $\mu_+ \in (\nu_0 + \nu_0^\perp) \cap \Lambda$. Then $(\nu_0 + \nu_0^\perp) \cap \Lambda = \{\mu_+\}$. Since $\mathbf{W}'$ preserves both $\nu_0 + \nu_0^\perp$ and Λ, we conclude that μ_+ is a $\mathbf{W}'$-fixpoint. Similarly, there is some $\mu_- \in E_0$, such that $(-\nu_0 + \nu_0^\perp) \cap \Lambda = \{\mu_-\}$. Then μ_- is a $\mathbf{W}'$-fixpoint.

By (6) of Lemma 5.3, $\nu_0^\perp$ is spanned by $\Phi \cap \nu_0^\perp$, so, by Lemma 3.4, we see that the only $\mathbf{W}'$-fixpoint in $\nu_0^\perp$ is 0. Then the only $\mathbf{W}'$-fixpoint in $\nu_0 + \nu_0^\perp$ is ν_0 and the only $\mathbf{W}'$-fixpoint in $-\nu_0+\nu_0^\perp$ is $-\nu_0$. Then $\mu_+ = \nu_0$ and $\mu_- = -\nu_0$. Then $\{-\nu_0, \nu_0\} = \{\mu_-, \mu_+\} \subseteq \Lambda$.

The set of weights of $\mathfrak{a}$ on V is $r(\Lambda)$, so $r(\Lambda) = \{-\alpha, 0, \alpha\}$. Then, by $(*)$, we have $\Lambda \subseteq (-\nu_0 + \nu_0^\perp) \cup (\nu_0^\perp) \cup (\nu_0 + \nu_0^\perp)$. Then

$$\Lambda \subseteq [(-\nu_0 + \nu_0^\perp) \cap \Lambda] \cup [\nu_0^\perp] \cup [(\nu_0 + \nu_0^\perp) \cap \Lambda] = \{\mu_-\} \cup \nu_0^\perp \cup \{\mu_+\}.$$

Then $\Lambda \subseteq \{\mu_-, \mu_+\} \cup \nu_0^\perp = \{-\nu_0, \nu_0\} \cup \nu_0^\perp$.

Then $\{-\nu_0, \nu_0\} \subseteq \Lambda \subseteq \{-\nu_0, \nu_0\} \cup \nu_0^\perp$. Define $C(S, Z)$ as in §5. By (5) of Lemma 5.3, let Δ_0 be a base of Φ_0 such that $\nu_0 \in C(\Delta_0, E_0)$. Define a partial ordering $<$ on E_0 by:

$$\sigma < \tau \quad\iff\quad \forall \delta \in \Delta_0, \text{ we have } 0 < (\delta, \tau - \sigma).$$

So, as $\nu_0 \in C(\Delta_0, E_0)$, we see, for all $\sigma, \tau \in E_0$, that

$$\sigma < \tau \quad\Longrightarrow\quad 0 < (\nu_0, \tau - \sigma) \quad\Longrightarrow\quad (\nu_0, \sigma) < (\nu_0, \tau).$$

Setting $\sigma := \nu_0$, we see, for all $\tau \in E_0$, that

$$\nu_0 < \tau \Longrightarrow (\nu_0, \nu_0) < (\nu_0, \tau) \Longrightarrow 0 < (\nu_0, \tau) \Longrightarrow \tau \notin \{-\nu_0\} \cup \nu_0^\perp.$$

Then ν_0 is a maximal element in $\{-\nu_0, \nu_0\} \cup \nu_0^\perp$. By the representation theory of semisimple Lie algebras, we know that Λ has a unique maximal element. So, since $\{-\nu_0, \nu_0\} \subseteq \Lambda \subseteq \{-\nu_0, \nu_0\} \cup \nu_0^\perp$, we see that ν_0 is the unique maximal element in Λ.

Similarly, by (4) of Lemma 5.3, ν_0 is the unique maximal element in the set Ξ_0 of weights of $\mathfrak{c}_0$ on $\mathcal{W}$. As representations of complex semisimple Lie algebras are classified by highest weight, we conclude that $\mathcal{V}$ and $\mathcal{W}$ are isomorphic as $\mathfrak{l}_0^{\mathbb{C}}$-modules, and therefore as complex $\mathfrak{l}_0$-modules. ☐

8. Basic results about Lorentz dynamics

Let G be a Lie group acting locally faithfully by isometries of a Lorentz manifold M. Let $m_0 \in M$. Let $d := \dim(M)$.

If v_i is a sequence in a vector space V and if $v_\infty \in V$, then we write $v_i \rightharpoonup v_\infty$ if all three of the following are true:

- v_i leaves compact sets in V;
- $v_\infty \neq 0$; and
- $\mathbb{R}v_i \to \mathbb{R}v_\infty$ in the projectivization of V.

Define $\mathcal{S}$, $\mathcal{M}_P^2$ and $\mathcal{N}_2$ as in §2.

Lemma 8.1. *Let $\mathcal{C}'$ be an ordered Q_d-basis of $T_{m_0}M$. Let $A \in \mathfrak{g}\backslash\{0\}$. Assume that $A_{\mathcal{C}'} \in \mathcal{S}$. Then $d \geq 3$ and there exists an ordered Q_d-basis $\mathcal{C}$ of $T_{m_0}M$ such that $A_{\mathcal{C}}^{Lm} = \mathcal{N}_2$.*

Proof. By (1) of Lemma 3.6 of [Ad99a], we have $A_{\mathcal{C}'}^{Lm} \in \mathfrak{so}(Q_d)$. Since $A_{\mathcal{C}'} \in \mathcal{S}$, it follows that $A_{\mathcal{C}'}^{Lm} \in (\mathfrak{so}(Q_d)) \cap \mathcal{S}^{Lm} = \mathcal{M}_P^2$, so $A_{\mathcal{C}'}^{Lm}$ is nilpotent. Then Lemma 3.3 of [Ad99b] finishes the proof. □

LEMMA 8.2. *Let $X \in \mathfrak{g}$ and assume $X_{m_0} = 0$. Then there is an ordered Q_d-basis $\mathcal{C}$ of $T_{m_0}M$ such that, for all $Y \in ((\mathrm{ad}\ X)\mathfrak{g}) \cap (\mathfrak{c}_{\mathfrak{g}}(X))$, we have $Y_{\mathcal{C}} \in \mathcal{S}$.*

Proof. Let $H := \mathrm{Stab}_G^0(m_0)$. Then $X \in \mathfrak{h}$. Let t_i be a sequence in $(0, \infty)$ such that $t_i \to +\infty$. For all i, let $g_i := \exp(t_i X)$, let $m_i := m_0$ and let $m_i' := m_0$. For all i, we have $g_i \in H$, so $g_i m_0 = m_0$, so $g_i m_i = m_i'$. Choose $\mathcal{C}$ as in Lemma 8.1 of [Ad99b]. Fix $Y \in (\mathrm{ad}\ X)\mathfrak{g}$ such that $(\mathrm{ad}\ X)Y = 0$. We wish to show that $Y_{\mathcal{C}} \in \mathcal{S}$.

We may assume that $Y \neq 0$. Choose $W \in \mathfrak{g}$ such that $Y = (\mathrm{ad}\ X)W$. Then, for all i, we have $(\mathrm{Ad}\ g_i)W = W + t_i Y$. Then $(\mathrm{Ad}\ g_i)W \rightharpoonup Y$. By Lemma 8.1 of [Ad99b], we are done. □

LEMMA 8.3. *Let V be a normal Abelian connected Lie subgroup of G. Let $H := \mathrm{Stab}_V^0(m_0)$. Let $X \in \mathfrak{h}$. Then there is an ordered Q_d-basis $\mathcal{C}$ of $T_{m_0}M$ such that $((\mathrm{ad}\ X)\mathfrak{g})_{\mathcal{C}} \subseteq \mathcal{S}$.*

Proof. Since $X \in \mathfrak{h}$, we have $X_{m_0} = 0$. Let $\mathcal{C}$ be as in Lemma 8.2. Let $Y \in (\mathrm{ad}\ X)\mathfrak{g}$. We wish to show that $Y_{\mathcal{C}} \in \mathcal{S}$.

We have $Y \in [X, \mathfrak{g}] \subseteq [\mathfrak{v}, \mathfrak{g}] \subseteq \mathfrak{v}$. Then $(\mathrm{ad}\ X)Y = [X, Y] \in [\mathfrak{v}, \mathfrak{v}]$, so, since V is Abelian, we conclude that $(\mathrm{ad}\ X)Y = 0$. By Lemma 8.2, we have $Y_{\mathcal{C}} \in \mathcal{S}$. □

LEMMA 8.4. *Let V be an Abelian connected Lie subgroup of G and let $H := \mathrm{Stab}_V^0(m_0)$. Let $\mathcal{L}$ denote the light cone in $T_{m_0}M$ and let $\mathfrak{w}_1 := \{X \in \mathfrak{v} \,|\, X_{m_0} \in \mathcal{L}\}$. Then:*

(1) *$\mathfrak{h} \subseteq \mathfrak{w}_1$;*
(2) *$[\mathfrak{h}, \mathfrak{n}_{\mathfrak{g}}(\mathfrak{v})] \subseteq \mathfrak{w}_1$; and*
(3) *if $\mathfrak{w}_1$ is a subspace of $\mathfrak{v}$, then the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1.*

Proof of (1). For all $X \in \mathfrak{h}$, we have $X_{m_0} = 0 \in \mathcal{L}$, so $X \in \mathfrak{w}_1$, proving (1).

Proof of (2). Let $X \in \mathfrak{h}$, let $P \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{v})$ and let $Y = [X, P]$. We wish to show that $Y \in \mathfrak{w}_1$. That is, we wish to show that $Y_{m_0} \in \mathcal{L}$.

Let $\mathcal{L}'$ be the light cone in $(\mathbb{R}^{d \times 1}, Q_d)$. As $X \in \mathfrak{h}$, we get $X_{m_0} = 0$. We have $Y = (\mathrm{ad}\ X)P \in (\mathrm{ad}\ X)\mathfrak{g}$. Choose $\mathcal{C}$ as in Lemma 8.2. Then $Y_{\mathcal{C}} \in \mathcal{S}$, so $Y_{\mathcal{C}}^{Cm} \in \mathcal{S}^{Cm} \subseteq \mathcal{L}'$, so $Y_{m_0} \in \mathcal{L}$.

Proof of (3). Since $(\mathfrak{w}_1)_{m_0}$ is a lightlike subspace of $T_{m_0}M$, we see that $\dim((\mathfrak{w}_1)_{m_0}) \leq 1$. So, since $\mathfrak{h}$ is the kernel of

$$X \mapsto X_{m_0} : \mathfrak{w}_1 \to (\mathfrak{w}_1)_{m_0},$$

we see that the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1. □

Recall, from §2, the definition of almost $\mathfrak{s}$-invariant.

Corollary 8.5. *Let G_0 be a connected Lie subgroup of G. Let V be an Abelian connected Lie subgroup of G. Assume that G_0 normalizes V. Let $H := \mathrm{Stab}_V^0(m_0)$. Let $\mathcal{L}$ denote the light cone in $T_{m_0}M$. Let $\mathfrak{w}_1 := \{X \in \mathfrak{v} \,|\, X_{m_0} \in \mathcal{L}\}$. Assume that $\mathfrak{w}_1$ is a subspace of $\mathfrak{v}$. Then $(\mathfrak{h}, \mathfrak{w}_1)$ is almost* $(\mathrm{ad}\ \mathfrak{g}_0)$*-invariant.*

Proof. This follows from Lemma 8.4. □

Lemma 8.6. *Let $\lambda \in \mathbb{R}\backslash\{0\}$. Let $T, A \in \mathfrak{g}$ and assume $[T, A] = \lambda A$. Assume that $A \neq 0$ and that $A_{m_0} = 0$. Then $d \geq 3$ and there exists an ordered Q_d-basis $\mathcal{C}$ of $T_{m_0}M$ such that $A_{\mathcal{C}}^{Lm} = \mathcal{N}_2$.*

Proof. We have $(\mathrm{ad}\ A)T = -\lambda A$, so $A \in (\mathrm{ad}\ A)\mathfrak{g}$. Moreover, we have $(\mathrm{ad}\ A)A = 0$. Using Lemma 8.2 (with X replaced by A, Y replaced by A and $\mathcal{C}$ replaced by $\mathcal{C}'$), choose an ordered Q_d-basis $\mathcal{C}'$ of $T_{m_0}M$ such that $A_{\mathcal{C}'} \in \mathcal{S}$. By Lemma 8.1, we are done. □

9. Killing terms in binary forms

The results in this section were found with a good deal of help from C. Leung and D. Witte.

Let $d \geq 2$ be an integer. Let $I := \{0, \ldots, d\}$. Let $\mathcal{P}$ be the vector space of homogeneous polynomials $\mathbb{R}^2 \to \mathbb{R}$ of degree d. For each $\psi \in \mathcal{P}$, let $\alpha_0^\psi, \ldots, \alpha_d^\psi \in \mathbb{R}$ be defined as follows: for all $x, y \in \mathbb{R}$, we have $\psi(x, y) = \alpha_0^\psi x^d + \alpha_1^\psi x^{d-1}y + \cdots + \alpha_{d-1}^\psi xy^{d-1} + \alpha_d^\psi y^d$. For all $i \in I$, let $\alpha_i : \mathcal{P} \to \mathbb{R}$ be defined by $\alpha_i(\psi) = \alpha_i^\psi$. For each $\psi \in \mathcal{P}$, let $z(\psi)$ denote the cardinality of $\{i \in I \,|\, \alpha_i^\psi = 0\}$. Let $\mathcal{P}' := \{\psi \in \mathcal{P} \,|\, z(\psi) \geq 2\}$.

Let $S := \mathrm{SL}_2(\mathbb{R})$. Let S act on $\mathbb{R}^2$ by matrix multiplication, after identifying $\mathbb{R}^2$ with $\mathbb{R}^{2\times 1}$. Let S act on $\mathcal{P}$ by $(s\psi)(v) = \psi(s^{-1}v)$. For all $r > 0$, for all $t, u \in \mathbb{R}$, let

$$a_r := \begin{pmatrix} 1/r & 0 \\ 0 & r \end{pmatrix}, \qquad n_t := \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \qquad n'_u := \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}.$$

A function $f : \mathbb{R} \to \mathbb{R}$ will be said to be *global rational* if there exist polynomials $P, Q : \mathbb{R} \to \mathbb{R}$ such that, for all $u \in \mathbb{R}$, we have $Q(u) \neq 0$ and $f(u) = (P(u))/(Q(u))$.

Let $\mathcal{E} := \{\psi \in \mathcal{P} \,|\, \alpha_0^\psi = 0\}$. For all $\psi \in \mathcal{P}\backslash\mathcal{E}$, let $t_\psi := -\alpha_1^\psi/(d \cdot \alpha_0^\psi)$. Define $\pi : \mathcal{P}\backslash\mathcal{E} \to \mathcal{P}$ by $\pi(\psi) = n_{t_\psi}\psi$.

LEMMA 9.1. *Let $\psi \in \mathcal{P} \backslash \mathcal{E}$. For all $i \in I$, let $c_i := \alpha_i(\psi)$. Then*

(1) $\alpha_0(\pi(\psi)) = c_0$;
(2) $\alpha_1(\pi(\psi)) = 0$; *and*
(3) $\alpha_2(\pi(\psi)) = [(2d)c_0c_2 - (d-1)c_1^2]/[(2d)c_0]$.

Proof. We compute, for all $t \in \mathbb{R}$, that

- $\alpha_0(n_t\psi) = c_0$;
- $\alpha_1(n_t\psi) = c_1 + d \cdot c_0 t$; and
- $\alpha_2(n_t\psi) = c_2 + (d-1)c_1 t + (1/2)d(d-1)c_0 t^2$.

Substituting $t_\psi = -c_1/(d \cdot c_0)$ for t, we are done. □

LEMMA 9.2. *Let $\phi \in \mathcal{P}$. Assume that $(S\phi) \cap \mathcal{E} = \emptyset$. Let $\psi \in S\phi$. For all $i \in I$, let $c_i := \alpha_i(\psi)$. For all $u \in \mathbb{R}$, let $\psi_u := \pi(n'_u\psi)$. Then:*

(1) *for all $u \in \mathbb{R}$, we have $0 \neq \alpha_0(\psi_u) = c_0 + c_1 u + \cdots + c_d u^d$;*
(2) *for all $u \in \mathbb{R}$, we have $\alpha_1(\psi_u) = 0$;*
(3) *$u \mapsto \alpha_2(\psi_u) : \mathbb{R} \to \mathbb{R}$ is global rational; and*
(4) *if $c_1 = 0$, then $(d/du)_{u=0}(\alpha_2(\psi_u)) = 3c_3$.*

Proof. For all $u \in \mathbb{R}$, set $\psi^u := n'_u\psi$, so that $\psi_u = \pi(\psi^u)$. For all $i \in I$, for all $u \in \mathbb{R}$, let $c_i^u := \alpha_i(\psi^u)$. We then compute: For all $u \in \mathbb{R}$,

(A) $c_0^u = c_0 + c_1 u + \cdots + c_d u^d$;
(B) $c_1^u = c_1 + 2c_2 u + 3c_3 u^2 + \cdots + d \cdot c_d u^{d-1}$; and
(C) $c_2^u = [1/2][(2 \cdot 1)c_2 + (3 \cdot 2)c_3 u + \cdots + (d \cdot (d-1))c_d u^{d-2}]$.

From Lemma 9.1, for all $u \in \mathbb{R}$, we have:

(D) $\alpha_0(\pi(\psi^u)) = c_0^u$;
(E) $\alpha_1(\pi(\psi^u)) = 0$; and
(F) $\alpha_2(\pi(\psi^u)) = [(2d)c_0^u c_2^u - (d-1)(c_1^u)^2]/[(2d)c_0^u]$.

For all $u \in \mathbb{R}$, since $\psi_u \in S\phi$, it follows that $\alpha_0(\psi_u) \in \alpha_0(S\phi)$. So, since $0 \notin \alpha_0(S\phi)$, we conclude, for all $u \in \mathbb{R}$, that $\alpha_0(\psi_u) \neq 0$. Then, for all $u \in \mathbb{R}$, (A) and (D) imply

(G) $0 \neq \alpha_0(\psi_u) = \alpha_0(\pi(\psi^u)) = c_0^u = c_0 + c_1 u + \cdots + c_d u^d$,

verifying (1) of Lemma 9.2. Moreover, for all $u \in \mathbb{R}$, we have from (E) that $\alpha_1(\psi_u) = \alpha_1(\pi(\psi^u)) = 0$, verifying (2) of Lemma 9.2.

Define $P : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$ by

$$P(u) = (2d)c_0^u c_2^u - (d-1)(c_1^u)^2 \qquad \text{and} \qquad Q(u) = (2d)c_0^u.$$

Then, by (A), (B) and (C), we see that P and Q are both polynomials. By (G), for all $u \in \mathbb{R}$, we have $Q(u) \neq 0$. By (F), for all $u \in \mathbb{R}$, we have $\alpha_2(\psi_u) = \alpha_2(\pi(\psi^u)) = (P(u))/(Q(u))$. So $u \mapsto \alpha_2(\psi_u) : \mathbb{R} \to \mathbb{R}$ is global rational, proving (3) of Lemma 9.2. It remains to prove (4). Assume that $c_1 = 0$. We wish to show that $(d/du)_{u=0}(\alpha_2(\psi_u)) = 3c_3$.

By (A), we have $c_0^0 = c_0$ and $(d/du)_{u=0}(c_0^u) = c_1 = 0$. By (B), we have $c_1^0 = c_1 = 0$ and $(d/du)_{u=0}(c_1^u) = 2c_2$. By (C), we have $c_2^0 = c_2$ and $(d/du)_{u=0}(c_2^u) = 3c_3$. By substitution, we compute

$$P(0) = (2d)c_0c_2 \qquad \text{and} \qquad Q(0) = (2d)c_0.$$

By basic calculus and substitution, we compute

$$P'(0) = (6d)c_0c_3 \qquad \text{and} \qquad Q'(0) = 0.$$

From the Quotient Rule, we get

$$\left(\frac{d}{du}\right)_{u=0}(\alpha_2(\psi_u)) = \frac{(Q(0))\cdot(P'(0)) - (P(0))\cdot(Q'(0))}{(Q(0))^2}.$$

Then (4) of Lemma 9.2 follows by substitution. □

LEMMA 9.3. *Let $\phi \in \mathcal{P}$. Assume $0 \in \alpha_0(S\phi)$. Then $(S\phi) \cap \mathcal{P}' \neq \emptyset$.*

Proof. Choose $s \in S$ such that $\alpha_0(s\phi) = 0$. For $i \in I$, let $c_i := \alpha_i(s\phi)$. Then $c_0 = 0$. If $c_1 = 0$, then $s\phi \in \mathcal{P}'$, and we are done. We therefore assume that $c_1 \neq 0$.

For all $t \in \mathbb{R}$, since $c_0 = 0$, we calculate that $\alpha_0(n_ts\phi) = 0$ and that $\alpha_2(n_ts\phi) = c_2 + (d-1)c_1t$. Let $t_0 := c_2/[(1-d)c_1]$. Let $\psi := n_{t_0}s\phi$. Then $\alpha_0(\psi) = 0$ and $\alpha_2(\psi) = 0$. Then $\psi \in (S\phi) \cap \mathcal{P}'$. □

PROPOSITION 9.4. *Say $d \geq 3$. Let $\phi \in \mathcal{P}$. Then $(S\phi) \cap \mathcal{P}' \neq \emptyset$.*

Proof. By Lemma 9.3, we may assume that $0 \notin \alpha_0(S\phi)$. Then we have $(S\phi) \cap \mathcal{E} = \emptyset$. For all $\psi \in S\phi$, for all $u \in \mathbb{R}$, let $\psi_u := \pi(n'_u\psi)$.

Define $\beta : S\phi \to \mathbb{R}$ by $\beta(\psi) = [\alpha_0(\psi)]^{4-d}[\alpha_2(\psi)]^d$. For all $\psi \in \mathcal{P}$, for all $r > 0$, we compute $\alpha_0(a_r\psi) = r^d\psi$ and $\alpha_2(a_r\psi) = r^{d-4}\psi$, so $\beta(a_r\psi) = \psi$. Therefore β is A-invariant.

Let $\mathcal{P}_0 := \{\psi \in S\phi \,|\, \alpha_1(\psi) = 0\}$. Then, by Conclusion (2) of Lemma 9.1, $\pi(S\phi) \subseteq \mathcal{P}_0$. We compute that $\pi|\mathcal{P}_0 : \mathcal{P}_0 \to \mathcal{P}_0$ is the identity map. Then $\pi(S\phi) = \mathcal{P}_0$.

Fix $\psi \in S\phi$ for this paragraph. For all $u \in \mathbb{R}$, we have $\psi_u \in S\phi$, so, since $0 \notin \alpha_0(S\phi)$, we conclude that $\alpha_0(\psi_u) \neq 0$. From this and from Conclusion (1) of Lemma 9.2, we see that $u \mapsto \alpha_0(\psi_u) : \mathbb{R} \to \mathbb{R}$ is a nonvanishing polynomial. By Conclusion (3) of Lemma 9.2, we see that $u \mapsto \alpha_2(\psi_u) : \mathbb{R} \to \mathbb{R}$ is global rational. We conclude that the function $u \mapsto \beta(\psi_u) : \mathbb{R} \to \mathbb{R}$ is global rational.

In particular, $u \mapsto \beta(\phi_u) : \mathbb{R} \to \mathbb{R}$ is global rational.

By Conclusion (2) of Lemma 9.2 (with ψ replaced by ϕ), we see, for all $u \in \mathbb{R}$, that $\alpha_1(\phi_u) = 0$. Moreover, for all $u \in \mathbb{R}$, we have $\phi_u \in S\phi$. We may assume, for all $u \in \mathbb{R}$, that $\alpha_2(\phi_u) \neq 0$, since, otherwise, we have $\phi_u \in (S\phi) \cap \mathcal{P}'$, and we are done. For all $u \in \mathbb{R}$, we have $\alpha_0(\phi_u) \neq 0 \neq \alpha_2(\phi_u)$. It follows, for all $u \in \mathbb{R}$, that $\beta(\phi_u) \neq 0$.

If $f : \mathbb{R} \to \mathbb{R}$ is any global rational function and if $0 \notin f(\mathbb{R})$, then either or $-f$ or f attains an absolute maximum. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(u) = \beta(\phi_u)$. Then f is global rational and $0 \notin f(\mathbb{R})$. Choose $\gamma \in \{-\beta, \beta\}$ such that $u \mapsto \gamma(\phi_u) : \mathbb{R} \to \mathbb{R}$ attains an absolute maximum. Choose $u_0 \in \mathbb{R}$ such that $\gamma(\phi_{u_0}) = \sup\{\gamma(\phi_u)\}_{t\in\mathbb{R}}$. Let $\psi := \phi_{u_0}$. Since $\psi \in S\phi$, it suffices to prove that $\psi \in \mathcal{P}'$.

Let $\Phi := \{\phi_u\}_{u\in\mathbb{R}}$. By Conclusion (2) of Lemma 9.2 (with ψ replaced by ϕ), we have $\Phi \subseteq \mathcal{P}_0$. Calculation shows that $\mathcal{P}_0$ is A-invariant. Then $A\Phi \subseteq \mathcal{P}_0$. For all $n \in N$, for all $\rho \in \mathcal{P}_0$, we calculate that $\pi(n\rho) = \rho$. Then $\pi(NA\Phi) = A\Phi$.

For all $u \in \mathbb{R}$, we have $\phi_u = \pi(n'_u\phi) \in Nn'_u\phi$. Thus $N\Phi = NN'\phi$. So, as $NA = AN$, we get $NA\Phi = AN\Phi$. Then $NA\Phi = ANN'\phi$. Then, because ANN' is dense in S, we conclude that $NA\Phi$ is dense in $S\phi$, so $\pi(NA\Phi)$ is dense in $\pi(S\phi)$. Recall that $\pi(NA\Phi) = A\Phi$ and that $\pi(S\phi) = \mathcal{P}_0$. Then $A\Phi$ is dense in $\mathcal{P}_0$.

By Conclusion (2) of Lemma 9.2, we see that $\{\psi_u\}_{u\in\mathbb{R}} \subseteq \mathcal{P}_0$. We have $\gamma(\psi) = \gamma(\phi_{u_0}) = \sup\{\gamma(\phi_u)\}_{u\in\mathbb{R}} = \sup\gamma(\Phi)$. As β is A-invariant, it follows that γ is A-invariant. Then $\gamma(\psi) = \sup\gamma(A\Phi)$. So, as $\psi_0 = \psi$ and as $A\Phi$ is dense in $\mathcal{P}_0$, we get $\gamma(\psi_0) = \sup\gamma(\mathcal{P}_0)$.

So, since $\{\psi_u\}_{u\in\mathbb{R}} \subseteq \mathcal{P}_0$, we get $\gamma(\psi_0) = \sup\{\gamma(\psi_u)\}_{t\in\mathbb{R}}$. That is, $u \mapsto \gamma(\psi_u) : \mathbb{R} \to \mathbb{R}$ attains an absolute maximum at 0. The function $u \mapsto \beta(\psi_u) : \mathbb{R} \to \mathbb{R}$ is global rational, so $u \mapsto \gamma(\psi_u) : \mathbb{R} \to \mathbb{R}$ is global rational, and is therefore smooth. Then $(d/du)_{u=0}(\gamma(\psi_u)) = 0$, so $(d/du)_{u=0}(\beta(\psi_u)) = 0$.

For all $i \in I$, let $c_i := \alpha_i(\psi)$. Then $c_0 = \alpha_0(\psi) \in \alpha_0(S\phi)$. So, since $0 \notin \alpha_0(S\phi)$, we see that $c_0 \neq 0$. By Conclusion (2) of Lemma 9.2 (with ψ replaced by ϕ), we have $\alpha_1(\phi_{u_0}) = 0$, so $c_1 = \alpha_1(\psi) = \alpha_1(\phi_{u_0}) = 0$. It suffices to show, for some $i \in \{2, 3\}$, that $c_i = 0$.

Define $P : \mathbb{R} \to \mathbb{R}$ by $P(u) = \alpha_0(\psi_u)$. Define $Q : \mathbb{R} \to \mathbb{R}$ by $Q(u) = \alpha_2(\psi_u)$. By substitution, we have $P(0) = c_0$ and $Q(0) = c_2$. By (1) and (4) of Lemma 9.2, we have

$$P'(0) = c_1 = 0 \qquad \text{and} \qquad Q'(0) = (d/du)_{u=0}(\alpha_2(\psi_u)) = 3c_3.$$

For all $u \in \mathbb{R}$, we have $\beta(\psi_u) = [P(u)]^{4-d}[Q(u)]^d$. Moreover, we have $0 = (d/du)_{u=0}(\beta(\psi_u))$, so basic calculus yields

$$0 = (4-d)[P(0)]^{3-d}[P'(0)][Q(0)]^d + d[Q(0)]^{d-1}[Q'(0)][P(0)]^{4-d}.$$

Computing the right hand side, we get

$$0 = 0 + d(c_2^{d-1})(3c_3)c_0^{4-d} = (3d)c_0^{4-d}c_2^{d-1}c_3.$$

So, as $c_0 \neq 0$, we see either that $c_2 = 0$ or that $c_3 = 0$, as desired. □

10. Structural results about $\mathfrak{sl}_2(\mathbb{R})$

Let S be a connected Lie group. Assume that $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$.

LEMMA 10.1. *Let $\mathfrak{a} \subseteq \mathfrak{s}$ be a subspace and assume that $\dim(\mathfrak{a}) \geq 2$. Then there exists $t \in \mathbb{R}$ such that $\{T, X\} \cap [(\exp(t \text{ ad } X))\mathfrak{a}] \neq \emptyset$.*

Proof. Since $\dim(\mathfrak{a}) \geq 2$, it follows that $\mathfrak{a} \cap (\mathbb{R}X + \mathbb{R}T) \neq \{0\}$. Choose $a, b \in \mathbb{R}$ such that $0 \neq aX + bT \in \mathfrak{a}$. If $b = 0$, then $X \in \mathfrak{a}$ and, setting $t := 0$, we are done. We therefore assume that $b \neq 0$. Let $t := a/(2b)$, $A := 2tX + T$. Then $A = (1/b)(aX + bT)$, so $A \in \mathfrak{a}$. We have $(\text{ad } X)A = -2X$ and $(\text{ad } X)(-2X) = 0$, so $(\exp(t \text{ ad } X))A = A - 2tX$. Then $T = A - 2tX = (\exp(t \text{ ad } X))A \in (\exp(t \text{ ad } X))\mathfrak{a}$. $\square$

LEMMA 10.2. *Let V be a real $\mathfrak{s}$-module. Let $v \in V$. Assume that $Tv \in \mathbb{R}v$. Assume either that $XYv \in \mathbb{R}v$ or that $YXv \in \mathbb{R}v$. Then there exists an irreducible real $\mathfrak{s}$-submodule W of V such that $v \in W$.*

Proof. We may assume that $v \neq 0$. Replacing T by $-T$ and interchanging X and Y if necessary, we may assume that $XYv \in \mathbb{R}v$.

Choose $\lambda, \mu \in \mathbb{R}$ such that $Tv = \lambda v$ and $XYv = \mu v$. Choose an integer $l \geq 1$ and choose irreducible real $\mathfrak{s}$-submodules $V_1, \ldots, V_l \subseteq V$ such that $V = V_1 \oplus \cdots \oplus V_l$. Choose $v_1 \in V_1, \ldots, v_l \in V_l$ such that $v = v_1 + \cdots + v_l$. Reordering, let $k \geq 1$ be an integer such that $v_1 \neq 0, \ldots, v_k \neq 0$ and $v_{k+1} = \cdots = v_l = 0$. Let $K := \{1, \ldots, k\}$. For all $i \in K$, let $d_i := \dim(V_i)$.

Fix $i \in K$ for this paragraph. We have $Tv_i = \lambda v_i$. It therefore follows, from the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, that

$$4XYv_i = (d_i^2 - (\lambda - 1)^2)v_i.$$

On the other hand, we also have $4XYv_i = 4\mu v_i$. We conclude that $d_i^2 - (\lambda - 1)^2 = 4\mu$, so $d_i^2 = 4\mu + (\lambda - 1)^2$.

In particular, we have $d_1^2 = 4\mu + (\lambda - 1)^2$. So, for all $i \in K$, we have $d_i^2 = 4\mu + (\lambda - 1)^2 = d_1^2$, so $d_i^2 = d_1^2$, so $d_i = d_1$. Then, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, choose, for each $i \in K$, a real $\mathfrak{s}$-module isomorphism $f_i : V_1 \to V_i$.

Fix $i \in K$ for this paragraph. We have $Tv_i = \lambda v_i$ and we have $T(f_i(v_1)) = f_i(Tv_1) = f_i(\lambda v_1) = \lambda(f_i(v_1))$. So, as $v_i \neq 0 \neq f_i(v_1)$, it follows from the representation theory of $\mathfrak{sl}_2(\mathbb{R})$ that $\mathbb{R}v_i = \mathbb{R}(f_i(v_1))$. Choose $a_i \in \mathbb{R}\backslash\{0\}$ such that $v_i = a_i(f_i(v_1))$.

Define $f : V_1 \to V$ by $f(v) = a_1(f_1(v)) + \cdots + a_k(f_k(v))$. Then f is a nonzero $\mathfrak{g}$-equivariant linear transformation. So, since V_1 is an irreducible $\mathfrak{s}$-module, it follows that $f(V_1)$ an irreducible $\mathfrak{s}$-submodule of V. Let $W := f(V_1)$. Then $v = f(v_1) \in W$. $\square$

LEMMA 10.3. *Let V be an irreducible real S-module. Assume that* $\dim(V) \geq 4$. *Let V_0 be a subspace of V. Assume that the codimension in V of V_0 is ≤ 1. Then there exists $s \in S$ such that sV_0 contains two eigenvectors of $v \mapsto Tv : V \to V$ with different eigenvalues.*

Proof. Replacing V_0 by a smaller subspace, if necessary, we may assume that the codimension in V of V_0 is 1.

Let $d_0 := \dim(V)$. Then $d_0 \geq 4$. By the classification of irreducible representations of $\mathfrak{sl}_2(\mathbb{R})$, we know that, up to isomorphism, there is a unique real $\mathfrak{s}$-module of dimension d_0. We conclude that V and V^* are isomorphic as real $\mathfrak{s}$-modules, hence as real S-modules. Then V admits an S-invariant nondegenerate bilinear form.

Let $d := d_0 - 1$. Then $d \geq 3$. Let $I := \{0, \ldots, d\}$. Let $\mathcal{P}$ denote the vector space of homogeneous polynomials $\mathbb{R}^2 \to \mathbb{R}$ of degree d. For $i \in I$, define $\rho_i \in \mathcal{P}$ by $\rho_i(x,y) := x^i y^{d-i}$. Then $\{\rho_0, \ldots, \rho_d\}$ is a basis of $\mathcal{P}$. For each $\psi \in \mathcal{P}$, let $\alpha_0^\psi, \ldots, \alpha_d^\psi \in \mathbb{R}$ be defined by: for all $x, y \in \mathbb{R}$, we have $\psi(x,y) = \alpha_0^\psi x^d + \alpha_1^\psi x^{d-1}y + \cdots + \alpha_{d-1}^\psi xy^{d-1} + \alpha_d^\psi y^d$. Then, for all $\psi \in \mathcal{P}$, we have $\psi = \alpha_0^\psi \rho_d + \alpha_1^\psi \rho_{d-1} + \cdots + \alpha_d^\psi \rho_0$. For $\psi \in \mathcal{P}$, let $z(\psi)$ be the cardinality of $\{i \in I \,|\, \alpha_i^\psi = 0\}$.

Let $\mathrm{SL}_2(\mathbb{R})$ act on $\mathbb{R}^2$ by matrix multiplication, after identifying $\mathbb{R}^2$ with $\mathbb{R}^{2\times 1}$. Let $\mathrm{SL}_2(\mathbb{R})$ act on $\mathcal{P}$ by $(g\rho)(v) = \rho(g^{-1}v)$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we may assume that $S = \mathrm{SL}_2(\mathbb{R})$, that $V = \mathcal{P}$ and that

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $(\,\cdot\,,\,\cdot\,)$ be a nondegenerate S-invariant bilinear form on $\mathcal{P}$. For all $S \subseteq \mathcal{P}$, let $S^\perp := \{v \in \mathcal{P} \,|\, (v, S) = \{0\}\}$. Let a_r be defined as in §9. For all $i \in I$, for all $r > 0$, we have $a_r\rho_i = r^{d-2i}\rho_i$. So, for all $i, j \in I$, for all $r > 0$, we have $(\rho_i, \rho_j) = (a_r\rho_i, a_r\rho_j) = r^{2d-2i-2j}(\rho_i, \rho_j)$. From this and from the nondegeneracy of $(\,\cdot\,,\,\cdot\,)$, we conclude, for all $i, j \in I$, that $(\rho_i, \rho_j) = 0$ iff $i + j \neq d$.

Choose $\phi \in V_0^\perp \backslash \{0\}$. Then $V_0 = \{\phi\}^\perp$. By Proposition 9.4, choose $s \in S$ such that $z(s\phi) \geq 2$. Choose $m, n \in I$ such that $m \neq n$ and $\alpha_m^{s\phi} = \alpha_n^{s\phi} = 0$. Then $\rho_m, \rho_n \in \{s\phi\}^\perp = sV_0$. As ρ_m and ρ_n are eigen-vectors of $\nu \mapsto T\nu : \mathcal{P} \to \mathcal{P}$ with different eigenvalues, we are done. $\square$

Recall, from §2, the definition of almost $\mathfrak{s}$-invariant.

LEMMA 10.4. *Let V be a real $\mathfrak{s}$-module. Let U and U' be subspaces of V. Assume that (U, U') is almost $\mathfrak{s}$-invariant. Let $\hat{u}, \check{u} \in U' \backslash U$. Assume that $X\hat{u} \in U'$ and that $Y\check{u} \in U'$. Then $\mathfrak{s}U' \subseteq U'$.*

Proof. Since $\mathfrak{s}$ is generated by X and Y, it suffices to show both that $XU' \subseteq U'$ and that $YU' \subseteq U'$. We shall prove the former, the proof of the latter being similar. Let $v \in U'$. We wish to show that $Xv \in U'$

Since the codimension in U' of U is ≤ 1 and since $\hat{u} \in U' \backslash U$, choose $a \in \mathbb{R}$ such that $v + a\hat{u} \in U$. So, since $X\hat{u} \in U'$ and $XU \subseteq \mathfrak{s}U \subseteq U'$, we get $Xv = [X(v + a\hat{u})] - [a(X\hat{u})] \in XU - U' \subseteq U' - U' = U'$. □

11. Almost invariant pair of subspaces, Part I

Let S be a connected Lie group. Assume $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Let $\mathcal{R}$ denote the totality of real diagonalizable elements in $\mathfrak{s}\backslash\{0\}$. Let V be an irreducible real S-module. We define $d := \dim(V)$ and we define $D := \{1, \ldots, d\}$. For $i \in D$, let $\lambda_i := d - 2i + 1$.

Fix $T \in \mathcal{R}$ for this paragraph. For all $i \in D$, let

$$\mathcal{E}_i^T := \{v \in V \mid Tv = \lambda_i v\}.$$

By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, $V = \mathcal{E}_1^T \oplus \cdots \oplus \mathcal{E}_d^T$. For $i \in D$, let $q_i^T : V \to \mathcal{E}_i^T$ be the projection map. Define $\eta^T : V \to \{0\} \cup D$ by

$$\eta^T(v) := \begin{cases} \max\{i \in D \mid q_i^T(v) \neq 0\}, & \text{if } v \neq 0; \\ 0, & \text{if } v = 0. \end{cases}$$

Let U and U' be subspaces of V. Assume that $U \neq \{0\}$. Assume that (U, U') is almost $\mathfrak{s}$-invariant (see §2 for the definition).

Lemma 11.1. *Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. For all $t \in \mathbb{R}$, let $h_t := \exp(tX)$. Let $v \in V\backslash\{0\}$ and let $m := \eta^T(v)$. Then*
(1) *$\eta^T(Xv) < m$;*
(2) *for all $t \in \mathbb{R}$, we have $\eta^T(h_t v) = \eta^T(v)$; and*
(3) *$\eta^T(Tv - \lambda_m v) < m$.*

Proof. Conclusion (1) follows from the representation theory of $\mathfrak{sl}_2(\mathbb{R})$. For all $t \in \mathbb{R}$, we have $h_t v = v + Xv + (1/2!)(X^2 v) + (1/3!)(X^3 v) + \cdots$, so Conclusion (2) follows from Conclusion (1). Conclusion (3) follows from the definition of η^T. □

Lemma 11.2. *For some $T \in \mathcal{R}$, for some $u_0 \in U\backslash\{0\}$ we have $Tu_0 \in \mathbb{R}u_0$ and we have $\eta^T(u_0) = \min \eta^T(U\backslash\{0\})$.*

Proof. Let (X_0, Y_0, T_0) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. Then $T_0 \in \mathcal{R}$. Let $m := \min \eta^{T_0}(U\backslash\{0\})$. Choose $u_0 \in U\backslash\{0\}$ such that $\eta^{T_0}(u_0) = m$.

For all $t \in \mathbb{R}$, let $h_t := \exp(tX_0)$. Let $H := \{h_t\}_{t\in\mathbb{R}}$. By Conclusion (2) of Lemma 11.1, for all $t \in \mathbb{R}$, for all $v \in V$, we have that $\eta^{T_0}(h_t v) = \eta^{T_0}(v)$. That is, $\eta^{T_0} : V \to \{0\} \cup D$ is H-invariant. Then $m = \min \eta^{T_0}((HU)\backslash\{0\})$.

For $t \in \mathbb{R}$, let $u_t := h_t^{-1} u_0$, let $U_t := h_t^{-1} U$, let $T_t := (\mathrm{Ad}\ h_t^{-1})T_0$ and let $\mathfrak{a}_t := \{W \in \mathfrak{s} \mid Wu_t \in U_t\}$; then $\mathfrak{a}_t = (\mathrm{Ad}\ h_t)\mathfrak{a}_0 = (\exp(t \ \mathrm{ad}\ X_0))\mathfrak{a}_0$.

CLAIM 1. *For all $t \in \mathbb{R}$, we have $\eta^{T_0} = \eta^{T_t}$.*

Proof. Fix $t \in \mathbb{R}$ and $v \in V$. We wish to show that $\eta^{T_0}(v) = \eta^{T_t}(v)$.

For all $s \in S$, for all $R, R' \in \mathcal{R}$, for all $w, w' \in V$, we have:

$$(\mathrm{Ad}\ s)R = R' \quad \text{and} \quad sw = w' \qquad \Longrightarrow \qquad \eta^R(w) = \eta^{R'}(w').$$

In the case $s := h_t^{-1}$, $R := T_0$, $R' := T_t$, $w := h_t v$ and $w' := v$, this gives $\eta^{T_0}(h_t v) = \eta^{T_t}(v)$. So, by H-invariance of η^{T_0}, we have $\eta^{T_0}(v) = \eta^{T_t}(v)$. □

Define a linear transformation $f : \mathfrak{s} \to V$ by $f(W) = Wu_0$. Then we have $f(\mathfrak{s}) = \mathfrak{s}u_0 \subseteq \mathfrak{s}U \subseteq U'$. So, since the codimension in U' of U is ≤ 1 and since $\mathfrak{a}_0 = f^{-1}(U_0) = f^{-1}(U)$, we see that the codimension in $\mathfrak{s}$ of $\mathfrak{a}_0$ is ≤ 1. Thus $\dim(\mathfrak{a}_0) \geq 2$. By Lemma 10.1, choose $t_0 \in \mathbb{R}$ such that $\{X_0, T_0\} \cap [(\exp(t\ \mathrm{ad}\ X_0))\mathfrak{a}_0] \neq \emptyset$. Then $\{X_0, T_0\} \cap \mathfrak{a}_{t_0} \neq \emptyset$.

By H-invariance of η^{T_0}, we have $\eta^{T_0}(u_{t_0}) = \eta^{T_0}(u_0)$, so $\eta^{T_0}(u_{t_0}) = m$.

CLAIM 2. $T_0 u_{t_0} \in \mathbb{R}u_{t_0}$.

Proof. Because $\{X_0, T_0\} \cap \mathfrak{a}_{t_0} \neq \emptyset$, it follows either that $X_0 \in \mathfrak{a}_{t_0}$ or that $T_0 \in \mathfrak{a}_{t_0}$.

Case A: $X_0 \in \mathfrak{a}_{t_0}$. Then

$$X_0 u_{t_0} \in U_{t_0} = h_{t_0}^{-1} U \subseteq HU.$$

It follows from Conclusion (1) of Lemma 11.1 that $\eta^{T_0}(X_0 u_{t_0}) < m$. So, since $m = \min \eta^{T_0}((HU)\backslash\{0\})$, we conclude that $X_0 u_{t_0} = 0$. Then, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we conclude that $u_{t_0} \in \mathcal{E}_1^{T_0}$, so $T_0 u_{t_0} = \lambda_1 u_{t_0} \in \mathbb{R}u_{t_0}$.

Case B: $T_0 \in \mathfrak{a}_{t_0}$. Then $T_0 u_{t_0} \in U_{t_0}$. So, since $u_{t_0} \in U_{t_0}$, we have

$$T_0 u_{t_0} - \lambda_m u_{t_0} \in U_{t_0} = h_{t_0}^{-1} U \subseteq HU.$$

By Conclusion (3) of Lemma 11.1, $\eta^{T_0}(T_0 u_{t_0} - \lambda_m u_{t_0}) < m$. So, since $m = \min \eta^{T_0}((HU)\backslash\{0\})$, we conclude that $T_0 u_{t_0} - \lambda_m u_{t_0} = 0$, so $T_0 u_{t_0} = \lambda_m u_{t_0} \in \mathbb{R}u_{t_0}$. □

Let $T := T_{-t_0}$. By Claim 2, we have $T_0 u_{t_0} \in \mathbb{R}u_{t_0}$, so $Tu_0 \in \mathbb{R}u_0$. By Claim 1, $\eta^{T_0} = \eta^T$. So, since $\eta^{T_0}(u_0) = m = \min\ \eta^{T_0}(U\backslash\{0\})$, we conclude that $\eta^T(u_0) = \min\ \eta^T(U\backslash\{0\})$, □

LEMMA 11.3. *Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. Assume*

$$\{u \in U \,|\, Xu = 0\} = \{0\} = \{u \in U \,|\, Yu = 0\}.$$

Assume, for some $u_0 \in U\backslash\{0\}$, that $Tu_0 \in \mathbb{R}u_0$. Then $U' = V$.

Proof. Let $E := \{i \in D \,|\, \mathcal{E}_i^T \subseteq U\}$. From the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, for all $i \in D$, we have $\dim(\mathcal{E}_i^T) = 1$. From the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we also have

$$\{v \in V \,|\, Xv = 0\} = \mathcal{E}_1^T \qquad \text{and} \qquad \{v \in V \,|\, Yv = 0\} = \mathcal{E}_d^T.$$

So, by assumption, we get $\mathcal{E}_1^T \cap U = \{0\} = \mathcal{E}_d^T \cap U$. Thus $1 \notin E$ and $d \notin E$. Since $Tu_0 \in \mathbb{R}u_0$, choose $i_0 \in D$ such that $u_0 \in \mathcal{E}_{i_0}^T$. Then $0 \neq u_0 \in \mathcal{E}_{i_0}^T \cap U$, so, since $\dim(\mathcal{E}_{i_0}^T) = 1$, we have $\mathcal{E}_{i_0}^T \subseteq U$. Then $i_0 \in E$, so $E \neq \emptyset$. Let $j := \min E$ and $k := \max E$. Then $j, k \in E$ and $j-1, k+1 \notin E$.

Claim 1. *$j \in D\backslash\{1\}$ and $\mathcal{E}_j^T \subseteq U$ and $\mathcal{E}_{j-1}^T\backslash\{0\} \subseteq U'\backslash U$..*

Proof. Since $1 \notin E \subseteq D$, we see that $j \in D\backslash\{1\}$. As $j \in E$, we have $\mathcal{E}_j^T \subseteq U$.

Fix $v \in \mathcal{E}_{j-1}^T\backslash\{0\}$. We wish to show that $v \in U'$ and that $v \notin U$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $\mathcal{E}_{j-1}^T = X\mathcal{E}_j^T$. Then $v \in \mathcal{E}_{j-1}^T = X\mathcal{E}_j^T \subseteq \mathfrak{s}U \subseteq U'$. We have $j-1 \notin E$, so $\mathcal{E}_{j-1}^T \not\subseteq U$. So, since $\dim(\mathcal{E}_{j-1}^T) = 1$, we conclude that $\mathcal{E}_{j-1}^T \cap U = \{0\}$. Therefore, because $0 \neq v \in \mathcal{E}_{j-1}^T$, we get $v \notin U$. □

Claim 2. *$k \in D\backslash\{d\}$ and $\mathcal{E}_k^T \subseteq U$ and $\mathcal{E}_{k+1}^T\backslash\{0\} \subseteq U'\backslash U$.*

Proof. Similar to Claim 1, but use Y instead of X. □

Choose $\check{u} \in \mathcal{E}_{j-1}^T\backslash\{0\}$ and $\hat{u} \in \mathcal{E}_{k+1}^T\backslash\{0\}$. Then, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $X\hat{u} \in \mathcal{E}_k^T$ and $Y\check{u} \in \mathcal{E}_j^T$. Then, by Claim 1 and Claim 2, we get $\hat{u}, \check{u} \in U'\backslash U$ and $X\hat{u}, Y\check{u} \in U$. Since $U \subseteq U'$, we conclude that $X\hat{u}, Y\check{u} \in U'$. By Lemma 10.4, we get $\mathfrak{s}U' \subseteq U'$. We have $\{0\} \neq U \subseteq U'$, so $U' \neq \{0\}$. Since V is $\mathfrak{s}$-irreducible and since U' is nonzero and $\mathfrak{s}$-invariant, we conclude that $U' = V$. □

12. Almost invariant pair of subspaces, Part II

Let S be a connected Lie group. Assume that $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Let V be a real $\mathfrak{s}$-module. Let U and U' be subspaces of V. Assume that (U, U') is almost $\mathfrak{s}$-invariant (see §2 for the definition). In this section, we also assume:

$(*)$ For all real $\mathfrak{s}$-submodules $V_1 \subsetneq V$, we have $V_1 \cap U = \{0\}$.

Lemma 12.1. *Assume that V is reducible as a real $\mathfrak{s}$-module. Let $u \in U$. Assume that $Tu \in \mathbb{R}u$. Assume either that $XYu \in \mathbb{R}u$ or that $YXu \in \mathbb{R}u$. Then $u = 0$.*

Proof. By Lemma 10.2, choose an irreducible real $\mathfrak{s}$-submodule $W \subseteq V$ such that $u \in W$. Since V is reducible, while W is irreducible, it follows that $W \subsetneq V$. Then, by Assumption $(*)$, we have $W \cap U = \{0\}$. Then we have $u \in W \cap U = \{0\}$. □

Lemma 12.2. *Assume, for some real $\mathfrak{s}$-submodule $C \subsetneq V$, that we have $C \cap U' = \{0\}$. Then $U = \{0\}$.*

Proof. Assume that $U \neq \{0\}$. We aim for a contradiction.

Replacing C by a larger submodule, if necessary, we may assume that V/C is a nonzero irreducible real $\mathfrak{s}$-module. Let V_0 be an $\mathfrak{s}$-invariant vector space complement in V to C. Then V_0 is a nonzero irreducible real $\mathfrak{s}$-module. Moreover, $V = V_0 \oplus C$. Let $p : V \to V_0$ be the projection map. Then $\ker(p) = C$, so $p(C) = \{0\}$. Let $C' := C \cap U'$. Then

$$\{0\} \neq C' \subseteq C \subsetneq V \qquad \text{and} \qquad p(C') = \{0\}.$$

By Assumption $(*)$, we have $C \cap U = \{0\}$, so $C' \cap U = \{0\}$. Because $\{0\} \neq C \subsetneq V$ and because C is $\mathfrak{s}$-invariant, we conclude that V is reducible as a real $\mathfrak{s}$-module.

Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. For all $\lambda \in \mathbb{R}$, we define $\mathcal{F}_\lambda := \{v \in V \,|\, Tv = \lambda v\}$. Let $\mathcal{F}_+ := \bigoplus_{\lambda>0} \mathcal{F}_\lambda$ and $\mathcal{F}_- := \bigoplus_{\lambda<0} \mathcal{F}_\lambda$. Let $d := \dim(V_0)$. Let $\lambda^* := d-1$ and let $\lambda_* := 1-d$. Let $\mathcal{E}^* := \mathcal{F}_{\lambda^*} \cap V_0$ and let $\mathcal{E}_* := \mathcal{F}_{\lambda_*} \cap V_0$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $\mathcal{E}^* \neq \{0\} \neq \mathcal{E}_*$ and $X\mathcal{E}^* = \{0\} = Y\mathcal{E}_*$.

Claim 1. *$\dim(C') = 1$ and $C' + U = U'$.*

Proof. Since the codimension in U' of U is ≤ 1, since $\{0\} \neq C' \subseteq U'$ and since $C' \cap U = \{0\}$, the result follows. $\square$

Claim 2. *$p(U) = V_0$.*

Proof. Let $U_0 := p(U)$. We have $C \cap U = \{0\}$, so $p|U : U \to V_0$ is injective. So, since $U \neq \{0\}$, we see that $U_0 \neq \{0\}$. We have $\mathfrak{s}U_0 = p(\mathfrak{s}U) \subseteq p(U')$. By Claim 1, we have $U' = C' + U$. Then $p(U') = p(C' + U) = (p(C')) + (p(U)) = \{0\} + U_0 = U_0$.

So, $\mathfrak{s}U_0 \subseteq p(U') = U_0$. That is, U_0 is $\mathfrak{s}$-invariant. So, as V_0 is irreducible and as $\{0\} \neq U_0 \subseteq V_0$, we get $U_0 = V_0$. Then $p(U) = V_0$. $\square$

Fix $v^* \in \mathcal{E}^* \backslash \{0\} \subseteq V_0$. By Claim 2, let $u^* \in U$ satisfy $p(u^*) = v^*$.

Claim 3. *$Xu^* \in C'$ and $Tu^* - \lambda^* u^* \in C'$.*

Proof. Recall that $X\mathcal{E}^* = \{0\}$. Then $p(Xu^*) = Xv^* \in X\mathcal{E}^* = \{0\}$, so $Xu^* \in \ker(p) = C$. Also, $Xu^* \in \mathfrak{s}U \subseteq U'$. Then $Xu^* \in C \cap U' = C'$.

As $v^* \in \mathcal{E}^* \subseteq \mathcal{F}_{\lambda^*}$, we get $Tv^* = \lambda^* v^*$. Then $p(Tu^* - \lambda^* u^*) = 0$, so $Tu^* - \lambda^* u^* \in \ker(p) = C$. Also, $Tu^* - \lambda^* u^* \in \mathfrak{s}U - U \subseteq U' - U' = U'$. Then $Tu^* - \lambda^* u^* \in C \cap U' = C'$. $\square$

Claim 4. *If $Xu^* \neq 0$, then $C' \subseteq \mathcal{F}_+$.*

Proof. By Claim 3, we have $Xu^* \in C'$. By Claim 1, $\dim(C') = 1$. By assumption, $Xu^* \neq 0$. Then $C' = \mathbb{R}(Xu^*)$. It therefore suffices to show that $Xu^* \in \mathcal{F}_+$.

By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, the map $v \mapsto Tv : V \to V$ is real diagonalizable, i.e., we have that $V = \bigoplus_{\lambda \in \mathbb{R}} \mathcal{F}_\lambda$. For all $\lambda \in \mathbb{R}$, let $p_\lambda : V \to \mathcal{F}_\lambda$ be the projection map. For all $\lambda \in \mathbb{R}$, let $u^*_\lambda := p_\lambda(u^*)$. Let $\lambda_0 := \min\{\lambda \in \mathbb{R} \,|\, u^*_\lambda \neq 0\}$. Then $u^* \in \sum_{\lambda \geq \lambda_0} \mathcal{F}_\lambda$ and $u^*_{\lambda_0} \neq 0$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $Xu^* \in \sum_{\lambda \geq \lambda_0 + 2} \mathcal{F}_\lambda$. Then $p_{\lambda_0}(Xu^*) = 0$. As $C' = \mathbb{R}(Xu^*)$, we get $p_{\lambda_0}(C') = \{0\}$.

By Claim 3, $Tu^* - \lambda^* u^* \in C'$, so $p_{\lambda_0}(Tu^* - \lambda^* u^*) \in p_{\lambda_0}(C') = \{0\}$. For all $\lambda \in \mathbb{R}$, we have $p_\lambda(Tu^*) = \lambda u^*_\lambda$. Then

$$\lambda_0 u^*_{\lambda_0} - \lambda^* u^*_{\lambda_0} = p_{\lambda_0}(Tu^* - \lambda^* u^*) = 0.$$

Thus $(\lambda_0 - \lambda^*) u^*_{\lambda_0} = 0$. As $u^*_{\lambda_0} \neq 0$, we get $\lambda_0 = \lambda^* = d - 1 \geq 0$. Then $Xu^* \in \sum_{\lambda \geq \lambda_0 + 2} \mathcal{F}_\lambda \subseteq \sum_{\lambda \geq 2} \mathcal{F}_\lambda \subseteq \mathcal{F}_+$. ☐

CLAIM 5. *Either $XC' = \{0\}$ or $C' \subseteq \mathcal{F}_+$.*

Proof. Assume that $C' \not\subseteq \mathcal{F}_+$. We wish to show that $XC' = \{0\}$.

We have $XTu^* = TXu^* - [T, X]u^* = TXu^* - 2Xu^*$. By Claim 4, we have $Xu^* = 0$. Then $XTu^* = 0 - 0 = 0$. Therefore, we have $X(Tu^* - \lambda^* u^*) = XTu^* - \lambda^* Xu^* = 0 - 0 = 0$.

We have $p(u^*) = v^* \neq 0$, so $u^* \neq 0$. Because $Xu^* = 0$, it follows that $YXu^* = 0 \in \mathbb{R}u^*$. So, as $u^* \in U \backslash \{0\}$, by Lemma 12.1, we get $Tu^* \notin \mathbb{R}u^*$. In particular, we have $Tu^* - \lambda^* u^* \neq 0$. By Claim 3, $Tu^* - \lambda^* u^* \in C'$. By Claim 1, $\dim(C') = 1$. Then $C' = \mathbb{R}(Tu^* - \lambda^* u^*)$.

Then $XC' \subseteq \mathbb{R}(X(Tu^* - \lambda^* u^*)) = \{0\}$. ☐

Fix $v_* \in \mathcal{E}_* \backslash \{0\} \subseteq V_0$. By Claim 2, let $u_* \in U$ satisfy $p(u_*) = v_*$.

CLAIM 6. *$Yu_* \in C'$ and $Tu_* - \lambda_* u_* \in C'$.*

Proof. Similar to Claim 3, but use Y instead of X. ☐

CLAIM 7. *If $Yu_* \neq 0$, then $C' \subseteq \mathcal{F}_-$.*

Proof. Similar to Claim 4, but use Y instead of X. ☐

CLAIM 8. *Either $YC' = \{0\}$ or $C' \subseteq \mathcal{F}_-$.*

Proof. Similar to Claim 5, but use Y instead of X. ☐

CLAIM 9. *$C' \subseteq \mathcal{F}_+ + \mathcal{F}_0$.*

Proof. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, for all $v \in V$, we have: $Xv = 0 \Longrightarrow v \in \mathcal{F}_+ + \mathcal{F}_0$. Thus Claim 9 follows from Claim 5. ☐

CLAIM 10. *$C' \subseteq \mathcal{F}_- + \mathcal{F}_0$.*

Proof. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, for all $v \in V$, we have: $Yv = 0 \Longrightarrow v \in \mathcal{F}_- + \mathcal{F}_0$. Thus Claim 10 follows from Claim 8. $\square$

CLAIM 11. $XC' = \{0\}$.

Proof. Since $C' \neq \{0\}$, by Claim 10, we have $C' \not\subseteq \mathcal{F}_+$. Then, by Claim 5, we are done. $\square$

CLAIM 12. $YC' = \{0\}$.

Proof. Since $C' \neq \{0\}$, by Claim 9, we have $C' \not\subseteq \mathcal{F}_-$. Then, by Claim 8, we are done. $\square$

CLAIM 13. $\mathfrak{s}U' \subseteq U'$.

Proof. Fix $c \in C' \backslash \{0\}$. Then, as $C' \cap U = \{0\}$, we conclude that $c \notin U$. We have $c \in C' \subseteq U'$. By Claim 11, we have $Xc = 0$. By Claim 12, we have $Yc = 0$. Let $\hat{u} := c$ and $\check{u} := c$. Then $\hat{u}, \check{u} \in U' \backslash U$ and $X\hat{u} = 0 \in U'$ and $Y\check{u} = 0 \in U'$. Therefore Claim 13 follows from Lemma 10.4. $\square$

CLAIM 14. $\mathfrak{s}U' = \{0\}$.

Proof. By Claim 13, we conclude that U' is $\mathfrak{s}$-invariant. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, it suffices to show that any nonzero $\mathfrak{s}$-irreducible subspace U_1 of U' is one-dimensional.

As V is $\mathfrak{s}$-reducible, while U_1 is $\mathfrak{s}$-irreducible, it follows that $U_1 \subsetneq V$. By $(*)$, we have $U_1 \cap U \neq \{0\}$. So, since $U_1 \subseteq U'$ and since the codimension in U' of U is ≤ 1, we conclude that $\dim(U_1) \leq 1$. So, as $U_1 \neq \{0\}$, we have $\dim(U_1) = 1$. $\square$

Fix $u_1 \in U \backslash \{0\}$. Then $u_1 \in U \subseteq U'$. Then, by Claim 14, we conclude that $\mathfrak{s}u_1 = \{0\}$. Then $\mathbb{R}u_1$ is an irreducible real $\mathfrak{s}$-submodule of V. Because V is reducible, it follows that $\mathbb{R}u_1 \subsetneq V$. So, by $(*)$, we have $(\mathbb{R}u_1) \cap U = \{0\}$. However, $u_1 \in U \backslash \{0\}$, giving a contradiction. $\square$

LEMMA 12.3. *If $U \neq \{0\}$, then the real $\mathfrak{s}$-module V is irreducible.*

Proof. Assume that $U \neq \{0\}$ and that V is reducible as a real $\mathfrak{s}$-module. We aim for a contradiction.

Let V_0 be a nonzero irreducible real $\mathfrak{s}$-submodule of V. Let C be an $\mathfrak{s}$-invariant vector space complement in V to V_0. Then $V = V_0 \oplus C$. Let $p : V \to V_0$ be the projection map. Then $\ker(p) = C$. As $V_0 \neq \{0\}$, it follows that $C \subsetneq V$. By Lemma 12.2, we have $C \cap U' = \{0\}$.

Let $U_0 := p(U)$ and $U_0' := p(U')$. As (U, U') is almost $\mathfrak{s}$-invariant, we see by Conclusion (2) of Lemma 3.8 that (U_0, U_0') is almost $\mathfrak{s}$-invariant. Choose

T and u_0 as in Lemma 11.2 (with V replaced by V_0 and (U, U') replaced by $(U_0, U_0'))$. Then $0 \neq u_0 \in U_0 = p(U)$. Choose $u \in U$ such that $p(u) = u_0$. Since $u_0 \neq 0$, we conclude that $u \neq 0$.

By Lemma 11.2, we see that T is a real diagonalizable element of $\mathfrak{s}\backslash\{0\}$. Let $d := \dim(V_0)$. Let $D := \{1, \dots, d\}$. For all $i \in D$, we define $\lambda_i := d-2i+1$ and $\mathcal{E}_i := \{v \in V_0 \,|\, Tv = \lambda_i v\}$. Then, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $V_0 = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_d$ and, for all $i \in D$, we have $\dim(\mathcal{E}_i) = 1$. For all $i \in D$, let $q_i : V_0 \to \mathcal{E}_i$ be the projection map. Define $\eta : V_0 \to \{0\} \cup D$ by

$$\eta(v) := \begin{cases} \max\{i \in D \,|\, q_i(v) \neq 0\}, & \text{if } v \neq 0; \\ 0, & \text{if } v = 0. \end{cases}$$

Let $m := \min \eta(U_0\backslash\{0\})$. By Lemma 11.2, we have $Tu_0 \in \mathbb{R}u_0$ and we have $\eta(u_0) = m$. Choose $\lambda \in \mathbb{R}$ such that $Tu_0 = \lambda u_0$. We have $p(Tu - \lambda u) = Tu_0 - \lambda u_0 = 0$, so $Tu - \lambda u \in \ker(p) = C$. Moreover, $Tu - \lambda u \in \mathfrak{s}U - U \subseteq U' - U' = U'$. Then $Tu - \lambda u \in C \cap U' = \{0\}$. So $Tu = \lambda u \in \mathbb{R}u$.

Choose $X, Y \in \mathfrak{s}$ such that (X, Y, T) is a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$.

Claim 1. *$XYu \notin \mathbb{R}u$ and $YXu \notin \mathbb{R}u$.*

Proof. Since $u \neq 0$ and $Tu \in \mathbb{R}u$, this follows from Lemma 12.1. □

Claim 2. *$Xu_0 \neq 0$.*

Proof. Say, for a contradiction, that $Xu_0 = 0$.

Then $p(Xu) = 0$, so $Xu \in \ker(p) = C$. We have $Xu \in \mathfrak{s}U \subseteq U'$. Then $Xu \in C \cap U' = \{0\}$. Then $YXu = 0 \in \mathbb{R}u$, contradicting Claim 1. □

Claim 3. *$Xu \in U'\backslash U$.*

Proof. We have $u_0 \neq 0$, so, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $\eta(Xu_0) < \eta(u_0)$. By Claim 2, $Xu_0 \neq 0$. Since $\eta(Xu_0) < \eta(u_0) = m = \min \eta(U_0\backslash\{0\})$, we conclude that $Xu_0 \notin U_0$. So, since $Xu_0 = p(Xu)$ and since $U_0 = p(U)$, we get $Xu \notin U$. Moreover, $Xu \in \mathfrak{s}U \subseteq U'$. □

Claim 4. *$XYu \notin U$.*

Proof. Assume that $XYu \in U$. We aim for a contradiction.

Recall that $Tu_0 = \lambda u_0$. Let $\mu := [1/4][d^2 - (\lambda-1)^2]$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $XYu_0 = \mu u_0$. Then

$$p(XYu - \mu u) = XYu_0 - \mu u_0 = 0,$$

so $XYu - \mu u \in \ker(p) = C$. Moreover, $XYu \in U$ and $u \in U$, so $XYu - \mu u \in U$. Then $XYu - \mu u \in C \cap U \subseteq C \cap U' = \{0\}$. Therefore $XYu = \mu u \in \mathbb{R}u$, contradicting Claim 1. □

We have $Xu, Yu \in \mathfrak{s}U \subseteq U'$. The codimension in U' of U is ≤ 1. So choose $(a, b) \in \mathbb{R}^2\backslash\{(0,0)\}$ such that $a(Xu) + b(Yu) \in U$.

CLAIM 5. $b \neq 0$.

Proof. If $b = 0$, then, because $a(Xu) + b(Yu) \in U$, and because $(a, b) \neq (0,0)$, it follows that $Xu \in U$, which contradicts Claim 3. □

We have $a(X^2u) + b(XYu) = X(a(Xu) + b(Yu)) \in \mathfrak{s}U \subseteq U'$. By Claim 3, we have $Xu \in U'\backslash U$. So, since the codimension in U' of U is ≤ 1 and since $a(X^2u)+b(XYu) \in U'$, choose $c \in \mathbb{R}$ such that $a(X^2u)+b(XYu)+c(Xu) \in U$. Let $s := a(X^2u) + b(XYu) + c(Xu)$.

Let $s_0 := p(s)$. Then $s_0 = a(X^2u_0) + b(XYu_0) + c(Xu_0)$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have

$$\eta(X^2u_0) \leq \eta(u_0), \qquad \eta(XYu_0) \leq \eta(u_0) \qquad \text{and} \qquad \eta(Xu_0) \leq \eta(u_0).$$

Then $\eta(s_0) \leq \eta(u_0)$. Recall that, for all $i \in I$, we have $\dim(\mathcal{E}_i) = 1$. Then, by definition of η, we see, for all $x, y \in V_0$, that:

- if $y \neq 0$ and if $\eta(x) \leq \eta(y)$, then, for some $t \in \mathbb{R}$, we have $\eta(x + ty) < \eta(y)$.

So choose $t_0 \in \mathbb{R}$ such that $\eta(s_0 + t_0u_0) < \eta(u_0)$.

As $s, u \in U$, we get $s + t_0u \in U$, so $s_0 + t_0u_0 \in U_0$. So, as

$$\eta(s_0 + t_0u_0) < \eta(u_0) = m = \min \eta(U_0\backslash\{0\}),$$

we conclude that $s_0 + t_0u_0 = 0$, so $p(s + t_0u) = 0$, so $s + t_0u \in \ker(p) = C$. Then $s + t_0u \in C \cap U \subseteq C \cap U' = \{0\}$, so $s = -t_0u$.

For all $\mu \in \mathbb{R}$, let $\mathcal{F}_\mu := \{v \in V \,|\, Tv = \mu v\}$. Recall that

$$s = a(X^2u) + b(XYu) + c(Xu) \in U.$$

By Claim 4, we have $XYu \notin U$. By Claim 5, we have $b \neq 0$. Then $b(XY) \notin U$. By contrast, $s \in U$. Then $s \neq b(XYu)$. Then

$$a(X^2u) + c(Xu) = s - b(XYu) \neq 0.$$

Recall that $Tu = \lambda u$, so $u \in \mathcal{F}_\lambda$. Then, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have

$$a(X^2u) \in \mathcal{F}_{\lambda+4} \qquad \text{and} \qquad b(XYu) \in \mathcal{F}_\lambda \qquad \text{and} \qquad c(Xu) \in \mathcal{F}_{\lambda+2}.$$

Then $a(X^2u) + c(Xu) \in (\mathcal{F}_{\lambda+2} + \mathcal{F}_{\lambda+4})\backslash\{0\}$, so $a(X^2u) + c(Xu) \notin \mathcal{F}_\lambda$. So, since $b(XYu) \in \mathcal{F}_\lambda$, we conclude that $s \notin \mathcal{F}_\lambda$. On the other hand, $s = -t_0u$ and $u \in \mathcal{F}_\lambda$, so $s \in \mathcal{F}_\lambda$, a contradiction. □

13. Representations of $\mathfrak{sl}_2(\mathbb{R})$, Part I

The results in this section and the next were found with a good deal of help from V. Reiner.

Let S be a connected Lie group. Let S act locally faithfully by isometries of a connected Lorentz manifold M. Let $m_0 \in M$. Let g be the Lorentz metric on M. Let ∇ be the Levi-Civita connection of g.

Let $d := \dim(M)$. Let $x_1^0, \ldots, x_d^0 : \mathbb{R}^d \to \mathbb{R}$ be the coordinate projections. Let $I := \{1, \ldots, d\}$. For all $i \in I$, let x_i be the germ at 0 of x_i^0. Let $\partial_i^0, \ldots, \partial_d^0$ be the standard framing of $\mathbb{R}^d$; then, for all $i \in I$, we have $\partial_i^0 = \partial/\partial x_i^0$. For $i \in I$, let ∂_i be the germ at 0 of ∂_i^0. Let $\tilde{A} := -x_2\partial_1 + x_d\partial_2$.

Let $I_2 := \{(i,j) \in I^2 \,|\, i \neq j\}$. For $(i,j) \in I_2$, let $Q_{ij}^0 : \mathbb{R}^d \to \mathbb{R}$ be the quadratic form $Q_{ij}^0(t_1, \ldots, t_d) = 2t_it_j$. For all $i \in I$, let $Q_{ii}^0 : \mathbb{R}^d \to \mathbb{R}$ be the quadratic form $Q_{ii}^0(t_1, \ldots, t_d) = t_i^2$. For $i, j \in I$, let Q_{ij}^1 denote the translation-invariant quadratic differential on $\mathbb{R}^d$ corresponding to Q_{ij}^0, and let Q_{ij} denote the germ at 0 of Q_{ij}^1.

In this section, the abbreviations LVF, QVF, CP, LP, QP, RP will stand for "linear vector fields", "quadratic vector fields", "constant pairings", "linear pairings", "quadratic pairings", and "remainder pairings", respectively. (Polarization allows us to think of quadratic differentials as "pairings". We choose to say "pairing" instead of "quadratic differential" so that QP will stand for "quadratic pairing", thereby avoiding the awkward phrase "quadratic quadratic differential".)

Let LVF and QVF denote the real spans of

$$\{x_i\partial_j\}_{i,j\in I}, \qquad \{x_ix_j\partial_k \,|\, i \leq j\}_{i,j,k\in I},$$

respectively. Let CP denote the real span of $\{Q_{kl} \,|\, k \leq l\}_{k,l\in I}$. Let LP and QP denote the real spans of

$$\{x_iQ_{kl} \,|\, k \leq l\}_{i,k,l\in I}, \qquad \{x_ix_jQ_{kl} \,|\, i \leq j, k \leq l\}_{i,j,k,l\in I},$$

respectively. Since $\{x_i\partial_j\}_{i,j\in I}$ is a basis of LVF, there is a unique positive definite symmetric bilinear form σ on LVF with respect to which $\{x_i\partial_j\}_{i,j\in I}$ is orthonormal. For all $R \subseteq$ LVF, we let $R^\perp$ denote the orthogonal complement in LVF to R, with respect to σ. For $R \subseteq$ QVF or $R \subseteq$ CP or $R \subseteq$ LP or $R \subseteq$ QP, we define $R^\perp$ similarly. The notation $\alpha \perp \beta$ means $\alpha \in \{\beta\}^\perp$. Note, for example, that if $W \in$ LVF, then "$x_1\partial_2 \perp W$" is a formal way to express the statement that, on writing W in coordinates, we do not have a term involving $x_1\partial_2$.

Let $\mathcal{G}$ be as in §2 of [Ad99b]. In this section, we shall use $\mathbf{L}$ to denote Lie derivative. Let RP denote the collection of germs h at zero of quadratic differentials on $\mathbb{R}^d$ such that, for all $P, Q \in \mathcal{G}$, we have that h, $\mathbf{L}_Ph$ and $\mathbf{L}_P\mathbf{L}_Qh$ all vanish at zero.

For this paragraph, let $\mathcal{C}$ be an ordered basis of $T_{m_0}M$ and let h be a quadratic differential defined on a neighborhood of m_0 in M. Let $\iota : \mathbb{R}^d \to T_{m_0}M$ be the isomorphism which carries the standard ordered basis of $\mathbb{R}^d$ to $\mathcal{C}$. Let $U \subseteq \mathbb{R}^d$ be an open neighborhood of 0 such that $\exp_{m_0}^\nabla$ is defined on $\iota(U)$, such that $U_1 := \exp_{m_0}^\nabla(\iota(U))$ is open in M and such that $\exp_{m_0}^\nabla : \iota(U) \to U_1$ is a diffeomorphism. Define $e : U \to U_1$ by $e(u) = \exp_{m_0}^\nabla(\iota(u))$. Then $e : U \to U_1$ is a diffeomorphism. We shall denote by $h_{\mathcal{C}}$ the germ at 0 of $e^*(h|U_1)$. By Taylor's Theorem, choose $h_{\mathcal{C}}^C \in \mathrm{CP}$, $h_{\mathcal{C}}^L \in \mathrm{LP}$, $h_{\mathcal{C}}^Q \in \mathrm{QP}$ and $h_{\mathcal{C}}^R \in \mathrm{RP}$ such that $h_{\mathcal{C}} = h_{\mathcal{C}}^C + h_{\mathcal{C}}^L + h_{\mathcal{C}}^Q + h_{\mathcal{C}}^R$.

Let $\mathcal{F}^L$, $\mathcal{M}_E^2$, $\mathcal{M}_P^2$ be as in §2. Let $\mathcal{N}_2, \ldots, \mathcal{N}_{d-1}$ be as in §2.

LEMMA 13.1. *Assume that $d \geq 3$. Let $P \in \mathrm{QVF}$ and assume that $[\tilde{A}, P] = 0$. Then $P \perp x_1x_2\partial_1$.*

Proof. Let V denote the real span of

$$x_1x_2\partial_1, \qquad x_2x_2\partial_2, \qquad x_1x_d\partial_2, \qquad x_2x_d\partial_d$$

and let W denote the real span of

$$x_2x_2\partial_1, \qquad x_1x_d\partial_1, \qquad x_2x_d\partial_2, \qquad x_dx_d\partial_d.$$

Then computation shows that $(\mathrm{ad}\ \tilde{A})V = W$, that $\mathrm{ad}\ \tilde{A} : V \to W$ is a vector space isomorphism and that $(\mathrm{ad}\ \tilde{A})(V^\perp) \subseteq W^\perp$.

Choose $P' \in V$ and $P'' \in V^\perp$ such that $P = P' + P''$. Let

$$Q' := (\mathrm{ad}\ \tilde{A})P' \qquad \text{and} \qquad Q'' := (\mathrm{ad}\ \tilde{A})P''.$$

Then $Q' \in W$ and $Q'' \in W^\perp$ and $Q' + Q'' = (\mathrm{ad}\ \tilde{A})P = 0$. Then $Q' = 0$ and $Q'' = 0$. Since $(\mathrm{ad}\ \tilde{A})P' = Q' = 0$, and since $\mathrm{ad}\ \tilde{A} : V \to W$ is a vector space isomorphism, we have $P' = 0$. So $P = P''$. Since $P'' \in V^\perp$ and $x_1x_2\partial_1 \in V$, we get $P'' \perp x_1x_2\partial_1$. So $P = P'' \perp x_1x_2\partial_1$. □

LEMMA 13.2. *Assume that $d \geq 4$. Let $k \in \{3, \ldots, d-1\}$. Let $P \in \mathrm{QVF}$ and assume that $[\tilde{A}, P] = 0$. Then $P \perp x_2x_d\partial_k$.*

Proof. Let V be the real span of $x_2x_d\partial_k$. Let W be the real span of $x_dx_d\partial_k$. Computation shows that $(\mathrm{ad}\ \tilde{A})V = W$, that $\mathrm{ad}\ \tilde{A} : V \to W$ is a vector space isomorphism and that $(\mathrm{ad}\ \tilde{A})(V^\perp) \subseteq W^\perp$.

Choose $P' \in V$ and $P'' \in V^\perp$ such that $P = P' + P''$. Let

$$Q' := (\mathrm{ad}\ \tilde{A})P' \qquad \text{and} \qquad Q'' := (\mathrm{ad}\ \tilde{A})P''.$$

Then $Q' \in W_0$ and $Q'' \in W_0^\perp$ and $Q' + Q'' = (\mathrm{ad}\ \tilde{A})P = 0$. Then $Q' = 0$ and $Q'' = 0$. Since $(\mathrm{ad}\ \tilde{A})P' = Q' = 0$ and since $\mathrm{ad}\ \tilde{A} : V \to W$ is a vector space isomorphism, we have $P' = 0$. So $P = P''$. Since $P'' \in V^\perp$ and $x_2x_d\partial_k \in V$, we get $P'' \perp x_2x_d\partial_k$. So $P = P'' \perp x_2x_d\partial_k$. □

Recall that $\mathbf{L}$ denotes Lie derivative.

LEMMA 13.3. *Assume that* $d \geq 4$. *Let* $k \in \{3, \dots, d-1\}$. *If* $h \in \mathrm{QP}$ *and if* $\mathbf{L}_{\tilde{A}}(h) = 0$, *then* $h \perp x_1 x_2 Q_{kd}$.

Proof. Let V denote the real span of

$$x_1 x_2 Q_{kd}, \qquad x_2 x_2 Q_{2k}, \qquad x_2 x_d Q_{1k}, \qquad x_1 x_d Q_{2k}$$

and let W denote the real span of

$$x_d x_d Q_{1k}, \qquad x_2 x_d Q_{2k}, \qquad x_2 x_2 Q_{kd}, \qquad x_1 x_d Q_{kd}.$$

As $\tilde{A} = \tilde{A}^L$, it follows that $\mathbf{L}_{\tilde{A}}(\mathrm{QP}) \subseteq \mathrm{QP}$. Define $L : \mathrm{QP} \to \mathrm{QP}$ by $L(h) = \mathbf{L}_{\tilde{A}} h$. Then computation shows that $L(V) = W$, that $L|V : V \to W$ is a vector space isomorphism and that $L(V^\perp) \subseteq W^\perp$.

Choose $h' \in V$ and $h'' \in V^\perp$ such that $h = h' + h''$. Let

$$k' := L(h') \qquad \text{and} \qquad k'' := L(h'').$$

Then $k' \in W$ and $k'' \in W^\perp$ and $k' + k'' = L(h) = 0$. Then $k' = 0$ and $k'' = 0$. Since $L(h') = k' = 0$, and since $L|V : V \to W$ is a vector space isomorphism, it follows that $h' = 0$. So $h = h''$. Since $h'' \in V^\perp$ and $x_1 x_2 Q_{kd} \in V$, we get $h'' \perp x_1 x_2 Q_{kd}$. So $h = h'' \perp x_1 x_2 Q_{kd}$. $\square$

LEMMA 13.4. *Assume that* $d \geq 3$. *If* $\mathcal{C}$ *is an ordered* Q_d*-basis of* $T_{m_0} M$, *then* $g_{\mathcal{C}}^C = Q_{1d} + Q_{22} + \cdots + Q_{d-1,d-1}$ *and* $g_{\mathcal{C}}^L = 0$.

Proof. From the definition of "ordered Q_d-basis", we conclude that $g_{\mathcal{C}}^C = Q_{1d} + Q_{22} + \cdots + Q_{d-1,d-1}$. By Lemma 8.2 of [AS99a], $g_{\mathcal{C}}^L = 0$. $\square$

LEMMA 13.5. *Assume that* $d \geq 3$. *Let* $A, B, X \in \mathfrak{s}$. *Assume that* $[X, A] = 0$. *Let* $\mathcal{C}$ *be an ordered* Q_d*-basis of* $T_{m_0} M$. *Assume that* $A_{\mathcal{C}} = \tilde{A}$ *and that* $B_{\mathcal{C}}^{Lm} \in \mathcal{M}_P^2$. *Then* $[X_{\mathcal{C}}^L, B_{\mathcal{C}}^L] \perp x_2 \partial_1$.

Proof. We have $A_{\mathcal{C}} = \tilde{A}$, so $A_{\mathcal{C}}^{Lm} = \mathcal{N}_2$. By (1) of Lemma 3.6 of [Ad99a], we have $X_{\mathcal{C}}^{Lm} \in \mathfrak{so}(Q_d)$. We have $[X, A] = 0$, so

$$[X_{\mathcal{C}}^{Lm}, \mathcal{N}_2] = [X_{\mathcal{C}}^{Lm}, A_{\mathcal{C}}^{Lm}] = [X, A]_{\mathcal{C}}^{Lm} = 0.$$

The centralizer in $\mathfrak{so}(Q_d)$ of $\mathcal{N}_2$ is $\mathcal{M}_E^2 + \mathcal{M}_P^2$, so $X_{\mathcal{C}}^{Lm} \in \mathcal{M}_E^2 + \mathcal{M}_P^2$.

Then $\mathcal{F}^L([X_{\mathcal{C}}^L, B_{\mathcal{C}}^L]) = [X_{\mathcal{C}}^{Lm}, B_{\mathcal{C}}^{Lm}] \in [\mathcal{M}_E^2 + \mathcal{M}_P^2, \mathcal{M}_P^2]$. If $d = 3$, then $[\mathcal{M}_E^2 + \mathcal{M}_P^2, \mathcal{M}_P^2] = \{0\}$, so $[X_{\mathcal{C}}^L, B_{\mathcal{C}}^L] = 0 \perp x_2 \partial_1$, and we are done. We may therefore assume that $d \geq 4$.

A calculation shows that $[\mathcal{M}_E^2 + \mathcal{M}_P^2, \mathcal{M}_P^2] = \mathbb{R}\mathcal{N}_3 + \cdots + \mathbb{R}\mathcal{N}_{d-1}$, so $[X_{\mathcal{C}}^L, B_{\mathcal{C}}^L] \in (\mathcal{F}^L)^{-1}(\mathbb{R}\mathcal{N}_3 + \cdots + \mathbb{R}\mathcal{N}_{d-1})$. For all $j \in \{3, \dots, d-1\}$, we have $(\mathcal{F}^L)^{-1}(\mathcal{N}_j) = -x_j \partial_1 + x_d \partial_j \perp x_2 \partial_1$. Then $[X_{\mathcal{C}}^L, B_{\mathcal{C}}^L] \perp x_2 \partial_1$. $\square$

Recall that $I = \{1, \dots, d\}$.

LEMMA 13.6. *Assume that $d \geq 3$. Let $A, B, X \in \mathfrak{s}$. Assume that $[A, B] = 0$, that $[X, B] = A$ and that $[X, A] = 0$. Let $\mathcal{C}$ be an ordered Q_d-basis of $T_{m_0}M$. Assume that $A_{\mathcal{C}} = \tilde{A}$ and that $B_{\mathcal{C}}^C \in \mathbb{R}\partial_1$. Then $[X_{\mathcal{C}}^C, B_{\mathcal{C}}^Q] \perp x_2\partial_1$.*

Proof. We assume that $[X_{\mathcal{C}}^C, B_{\mathcal{C}}^Q] \not\perp x_2\partial_1$, and aim for a contradiction.

Let $K := I\backslash\{2, d\}$. Since $[X_{\mathcal{C}}^C, \tilde{A}] = [X_{\mathcal{C}}^C, A_{\mathcal{C}}] = [X, A]_{\mathcal{C}}^C = 0$, it follows that $X_{\mathcal{C}}^C \in \sum_{i\in K} \mathbb{R}\partial_i$. Then, because $[X_{\mathcal{C}}^C, B_{\mathcal{C}}^Q] \not\perp x_2\partial_1$, choose $k \in K$ such that $[\partial_k, B_{\mathcal{C}}^Q] \not\perp x_2\partial_1$, whence $B_{\mathcal{C}}^Q \not\perp x_2x_k\partial_1$.

We have $[\tilde{A}, B_{\mathcal{C}}^Q] = [A_{\mathcal{C}}, B_{\mathcal{C}}^Q] = [A, B]_{\mathcal{C}}^Q = 0$. We conclude from Lemma 13.1 that $B_{\mathcal{C}}^Q \perp x_1x_2\partial_1$. Therefore $k \neq 1$. So $k \in I\backslash\{1, 2, d\}$, so $I\backslash\{1, 2, d\} \neq \emptyset$. Therefore $d \geq 4$ and $k \in I\backslash\{1, 2, d\} = \{3, \dots, d-1\}$. Choose $\alpha \in \mathbb{R}$ and $Q'' \in \{x_2x_k\partial_1\}^\perp$ such that $B_{\mathcal{C}}^Q = \alpha(x_2x_k\partial_1) + Q''$. We have $B_{\mathcal{C}}^Q \not\perp x_2x_k\partial_1$, so $B_{\mathcal{C}}^Q \neq Q''$, so $\alpha \neq 0$. Let $Q' := \alpha(x_2x_k\partial_1)$.

By Lemma 13.2, $B_{\mathcal{C}}^Q \perp x_2x_d\partial_k$. So, since $Q' = \alpha(x_2x_k\partial_1) \perp x_2x_d\partial_k$, we have $Q'' = B_{\mathcal{C}}^Q - Q' \perp x_2x_d\partial_k$. Let $R := \{x_2x_k\partial_1, x_2x_d\partial_k\}$. Then $Q'' \in R^\perp$. By Lemma 13.4, we have $g_{\mathcal{C}}^C = Q_{1d} + Q_{22} + \cdots + Q_{d-1,d-1}$. Since $Q' = \alpha(x_2x_k\partial_1)$, we calculate that $\mathbf{L}_{Q'}(g_{\mathcal{C}}^C) = \alpha(x_2Q_{kd} + x_kQ_{2d})$. We also calculate, for all $W \in R^\perp$, that $\mathbf{L}_W(g_{\mathcal{C}}^C) \perp x_2Q_{kd}$. Then, because $\alpha \neq 0$ and because $Q'' \in R^\perp$, we get

$$\mathbf{L}_{Q'}(g_{\mathcal{C}}^C) \not\perp x_2Q_{kd} \qquad \text{and} \qquad \mathbf{L}_{Q''}(g_{\mathcal{C}}^C) \perp x_2Q_{kd}.$$

So, since $B_{\mathcal{C}}^Q = Q' + Q''$, we conclude that $\mathbf{L}_{B_{\mathcal{C}}^Q}(g_{\mathcal{C}}^C) \not\perp x_2Q_{kd}$.

As S acts by isometries of M, we get $\mathbf{L}_{A_{\mathcal{C}}}(g_{\mathcal{C}}) = 0 = \mathbf{L}_{B_{\mathcal{C}}}(g_{\mathcal{C}})$. Then $0 = (\mathbf{L}_{B_{\mathcal{C}}}(g_{\mathcal{C}}))^L = \mathbf{L}_{B_{\mathcal{C}}^C}(g_{\mathcal{C}}^Q) + \mathbf{L}_{B_{\mathcal{C}}^L}(g_{\mathcal{C}}^L) + \mathbf{L}_{B_{\mathcal{C}}^Q}(g_{\mathcal{C}}^C)$. By Lemma 13.4, we get $g_{\mathcal{C}}^L = 0$. Thus $\mathbf{L}_{B_{\mathcal{C}}^C}(g_{\mathcal{C}}^Q) = -\mathbf{L}_{B_{\mathcal{C}}^Q}(g_{\mathcal{C}}^C) \not\perp x_2Q_{kd}$. So, as $B_{\mathcal{C}}^C \in \mathbb{R}\partial_1$, we get $\mathbf{L}_{\partial_1}(g_{\mathcal{C}}^Q) \not\perp x_2Q_{kd}$. Then $g_{\mathcal{C}}^Q \not\perp x_1x_2Q_{kd}$. However, we have $\mathbf{L}_{\tilde{A}}(g_{\mathcal{C}}^Q) = \mathbf{L}_{A_{\mathcal{C}}}(g_{\mathcal{C}}^Q) = (\mathbf{L}_{A_{\mathcal{C}}}(g_{\mathcal{C}}))^Q = 0$, contradicting Lemma 13.3. $\square$

LEMMA 13.7. *Let $A, B, T, X \in \mathfrak{s}$. Assume, for some $\lambda \in \mathbb{R}\backslash\{0\}$, that $[T, A] = \lambda A$. Assume that $A \neq 0$, that $[A, B] = 0$, that $[X, B] = A$, that $[X, A] = 0$ and that $B \in (\mathrm{ad}\ A)\mathfrak{s}$. Then $A_{m_0} \neq 0$.*

Proof. Assume that $A_{m_0} = 0$. We aim for a contradiction.

Choose $Y \in \mathfrak{g}$ such that $B = (\mathrm{ad}\ A)Y$. By Lemma 8.6, we have $d \geq 3$. By Lemma 8.6, choose an ordered Q_d-basis $\mathcal{C}$ of $T_{m_0}M$ such that $A_{\mathcal{C}}^{Lm} = \mathcal{N}_2$. Then $A_{\mathcal{C}}^L = \tilde{A}$. By (1) of Remark 3.5 of [Ad99a], we get $A_{\mathcal{C}} = A_{\mathcal{C}}^L$. Then $A_{\mathcal{C}} = \tilde{A}$.

We have $\mathcal{N}_2(Y_{\mathcal{C}}^{Cm}) = (A_{\mathcal{C}}^{Lm})(Y_{\mathcal{C}}^{Cm}) = [A, Y]_{\mathcal{C}}^{Cm} = B_{\mathcal{C}}^{Cm}$ and

$$\mathcal{N}_2(B_{\mathcal{C}}^{Cm}) = (A_{\mathcal{C}}^{Lm})(B_{\mathcal{C}}^{Cm}) = [A, B]_{\mathcal{C}}^{Cm} = 0,$$

so $B_{\mathcal{C}}^{Cm}$ is in both the image and the kernel of $v \mapsto \mathcal{N}_2 v : \mathbb{R}^{d\times 1} \to \mathbb{R}^{d\times 1}$, so $B_{\mathcal{C}}^{Cm} \in \mathbb{R}e_1$, so $B_{\mathcal{C}}^C \in \mathbb{R}\partial_1$.

We have $[\mathcal{N}_2, Y_{\mathcal{C}}^{Lm}] = [A_{\mathcal{C}}^{Lm}, Y_{\mathcal{C}}^{Lm}] = [A, Y]_{\mathcal{C}}^{Lm} = B_{\mathcal{C}}^{Lm}$ and

$$[\mathcal{N}_2, B_{\mathcal{C}}^{Lm}] = [A_{\mathcal{C}}^{Lm}, B_{\mathcal{C}}^{Lm}] = [A, B]_{\mathcal{C}}^{Lm} = 0,$$

so B^{Lm} is in both the image and the kernel of ad $\mathcal{N}_2 : \mathfrak{so}(Q_d) \to \mathfrak{so}(Q_d)$, so $B_{\mathcal{C}}^{Lm} \in \mathcal{M}_P^2$.

We have $[\tilde{A}, X_{\mathcal{C}}^Q] = [A_{\mathcal{C}}, X_{\mathcal{C}}^Q] = [A, X]_{\mathcal{C}}^Q = 0$. So, from Lemma 13.1 we see that $X_{\mathcal{C}}^Q \perp x_1 x_2 \partial_1$, and therefore that $[X_{\mathcal{C}}^Q, \partial_1] \perp x_2 \partial_1$. Since $B_{\mathcal{C}}^C \in \mathbb{R}\partial_1$, we conclude that $[X_{\mathcal{C}}^Q, B_{\mathcal{C}}^C] \perp x_2 \partial_1$.

By Lemma 13.5, we get $[X_{\mathcal{C}}^L, B_{\mathcal{C}}^L] \perp x_2 \partial_1$. By Lemma 13.6, we get $[X_{\mathcal{C}}^C, B_{\mathcal{C}}^Q] \perp x_2 \partial_1$. Then

$$A_{\mathcal{C}}^L = [X_{\mathcal{C}}, B_{\mathcal{C}}]^L = [X_{\mathcal{C}}^Q, B_{\mathcal{C}}^C] + [X_{\mathcal{C}}^L, B_{\mathcal{C}}^L] + [X_{\mathcal{C}}^C, B_{\mathcal{C}}^Q] \perp x_2 \partial_1.$$

However, $A_{\mathcal{C}}^L = \tilde{A} = -x_2 \partial_1 + x_d \partial_2 \not\perp x_2 \partial_1$, a contradiction. ☐

14. Representations of $\mathfrak{sl}_2(\mathbb{R})$, Part II

Let G be a connected Lie group. Let S be a connected Lie subgroup of G. Assume that $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Let V be an Abelian connected Lie subgroup of G. Assume that $\dim(V) \geq 2$. Assume that $\mathfrak{s}$ normalizes $\mathfrak{v}$. Assume that the adjoint representation of $\mathfrak{s}$ on $\mathfrak{v}$ is irreducible.

Let G act locally faithfully by isometries of a connected Lorentz manifold M. Let $m_0 \in M$.

Lemma 14.1. *Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. Let $A \in \mathfrak{v}\backslash\{0\}$. Asume that $[X, A] = 0$. Then $A_{m_0} \neq 0$.*

Proof. Since $[X, A] = 0$ and since $\dim(V) \geq 2$, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, choose $\lambda, \mu \in \mathbb{R}\backslash\{0\}$ such that $[T, A] = \lambda A$ and $[X, [Y, A]] = \mu A$. Let $B := (1/\mu)[Y, A]$. Then $[X, B] = A$. Moreover, $B = (\text{ad } A)((-1/\mu)Y) \in (\text{ad } A)\mathfrak{s}$. Then $B \in (\text{ad } A)\mathfrak{s} \subseteq [\mathfrak{v}, \mathfrak{s}] \subseteq \mathfrak{v}$. Then $[A, B] \in [\mathfrak{v}, \mathfrak{v}] = \{0\}$. By Lemma 13.7, we are done. ☐

Lemma 14.2. *Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. Let $A \in \mathfrak{v}\backslash\{0\}$. Assume that $[Y, A] = 0$. Then $A_{m_0} \neq 0$.*

Proof. Let $X_0 := Y$, let $Y_0 := X$ and let $T_0 := -T$. Then (X_0, Y_0, T_0) is a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. By Lemma 14.1 (with (X, Y, T) replaced by (X_0, Y_0, T_0)), we are done. ☐

15. Representations of $\mathfrak{sl}_2(\mathbb{R})$, Part III

Let G be a connected Lie group. Let S be a connected Lie subgroup of G. Assume that $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Let V be an Abelian connected Lie subgroup of G. Assume that S normalizes V.

Let G act locally faithfully by isometries of a connected Lorentz manifold M. Let $m_0 \in M$. Let $H := \mathrm{Stab}_V(m_0)$. Let $\mathcal{L}$ be the light cone in $T_{m_0}M$. Let $\mathfrak{w}_1 := \{X \in \mathfrak{v} \,|\, X_{m_0} \in \mathcal{L}\}$. Assume that $\mathfrak{w}_1$ is a subspace of $\mathfrak{v}$.

Recall, from §2, the definition of almost $\mathfrak{s}$-invariant.

Lemma 15.1. *Let $\mathfrak{v}'$ be an* $(\mathrm{ad}\ \mathfrak{s})$*-invariant subspace of $\mathfrak{v}$. Then we have that* $(\mathfrak{h} \cap \mathfrak{v}', \mathfrak{w}_1 \cap \mathfrak{v}')$ *is almost $\mathfrak{s}$-invariant.*

Proof. By Corollary 8.5 (with G_0 replaced by S), we see that $(\mathfrak{h}, \mathfrak{w}_1)$ is almost $\mathfrak{s}$-invariant. By Conclusion (1) of Lemma 3.8, we are done. □

Lemma 15.2. *Let $\mathfrak{v}'$ be a nonzero* $(\mathrm{ad}\ \mathfrak{s})$*-irreducible subspace of $\mathfrak{v}$. Assume that $\mathfrak{v}'_{m_0} \subseteq \mathcal{L}$. Then either* $\dim(\mathfrak{v}') = 1$ *or* $\dim(\mathfrak{v}') = 3$.

Proof. Because $\mathfrak{v}'_{m_0} \subseteq \mathcal{L}$, we have $\mathfrak{v}' \subseteq \mathfrak{w}_1$, so $\mathfrak{w}_1 \cap \mathfrak{v}' = \mathfrak{v}'$. Let $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{v}'$. By Lemma 15.1, we see that $(\mathfrak{h}_0, \mathfrak{v}')$ is almost $\mathfrak{s}$-invariant. In particular, the codimension in $\mathfrak{v}'$ of $\mathfrak{h}_0$ is ≤ 1.

Let $d_0 := \dim(\mathfrak{v}')$. We wish to show that $d_0 \in \{1, 3\}$.

Claim 1. $d_0 \neq 2$.

Proof. Assume, for a contradiction, that $d_0 = 2$.

Let $(\tilde{X}, \tilde{Y}, \tilde{T})$ be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, choose $\tilde{A} \in \mathfrak{v}' \backslash \{0\}$ such that $[\tilde{X}, \tilde{A}] = 0$. The codimension in $\mathfrak{v}'$ of $\mathfrak{h}_0$ is ≤ 1 and $\dim(\mathfrak{v}') = d_0 = 2$, so $\mathfrak{h}_0 \neq \{0\}$. Choose $A \in \mathfrak{h}_0 \backslash \{0\}$. Since $d_0 = 2$, it follows from the representation theory of $\mathfrak{sl}_2(\mathbb{R})$ that the Adjoint action of S on $\mathfrak{v}' \backslash \{0\}$ is transitive. Choose $s \in S$ such that $(\mathrm{Ad}\ s)\tilde{A} = A$. Let $X := (\mathrm{Ad}\ s)\tilde{X}$ and $Y := (\mathrm{Ad}\ s)\tilde{Y}$ and $T := (\mathrm{Ad}\ s)\tilde{T}$. Then $[X, A] = 0$ and $A_{m_0} = 0$, contradicting Lemma 14.1. □

Claim 2. $d_0 \leq 3$.

Proof. Assume, for a contradiction, that $d_0 \geq 4$.

Let $(\tilde{X}, \tilde{Y}, \tilde{T})$ be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. By Lemma 10.3 (with (X, Y, T) replaced by $(\tilde{X}, \tilde{Y}, \tilde{T})$, V replaced by $\mathfrak{v}'$ and V_0 replaced by $\mathfrak{h}_0$), choose $s \in S$ such that $(\mathrm{Ad}\ s)\mathfrak{h}_0$ contains two eigenvectors of $\mathrm{ad}\ \tilde{T} : \mathfrak{v}' \to \mathfrak{v}'$ with different eigenvalues. Let

$$X := (\mathrm{Ad}\ s^{-1})\tilde{X}, \qquad Y := (\mathrm{Ad}\ s^{-1})\tilde{Y} \qquad \text{and} \qquad T = (\mathrm{Ad}\ s^{-1})\tilde{T}.$$

Then $\mathfrak{h}_0$ contains two eigenvectors of $\mathrm{ad}\ T : \mathfrak{v}' \to \mathfrak{v}'$, with different eigenvalues. Choose $A, B \in \mathfrak{h}_0 \backslash \{0\}$ and $\lambda, \mu \in \mathbb{R}$ such that $\lambda \neq \mu$, such that $[T, A] = \lambda A$ and such that $[T, B] = \mu B$. By interchanging A with B and λ with μ if necessary, we may assume that $\lambda \neq 0$.

Let $d := \dim(M)$. Let $\mathcal{C}$ be an ordered Q_d-basis of $T_{m_0}M$. By (3) of Remark 3.5 of [Ad99a], we have $A_{\mathcal{C}}^{Lm} \neq 0 \neq B_{\mathcal{C}}^{Lm}$. By (1) of Lemma 3.6 of [Ad99a], we have $X_{\mathcal{C}}^{Lm}, Y_{\mathcal{C}}^{Lm}, T_{\mathcal{C}}^{Lm}, A_{\mathcal{C}}^{Lm}, B_{\mathcal{C}}^{Lm} \in \mathfrak{so}(Q_d)$.

Case A: $\mu = 0$. Then $[T, B] = 0$. Let $T_0 := T_{\mathcal{C}}^{Lm}$. We have $[T_0, A_{\mathcal{C}}^{Lm}] = [T, A]_{\mathcal{C}}^{Lm} = \lambda A_{\mathcal{C}}^{Lm}$. Since $\lambda \in \mathbb{R}\backslash\{0\}$, it follows that λ is not pure imaginary. By (1) of Lemma 4.1 (with T replaced by T_0), we see that T_0 is semisimple.

Let $B_0 := B_{\mathcal{C}}^{Lm}$. Then $[T_0, B_0] = [T_{\mathcal{C}}^{Lm}, B_{\mathcal{C}}^{Lm}] = [T, B]_{\mathcal{C}}^{Lm} = 0$. By (3) of Lemma 4.1 (with T replaced by T_0 and X replaced by B_0), B_0 is semisimple. Define $f : \mathbb{R}^{d\times 1} \to \mathbb{R}^{d\times 1}$ and $F : \mathfrak{so}(Q_d) \to \mathfrak{so}(Q_d)$ by $f(v) = B_0 v$ and $F(R) = [B_0, R]$. Because B_0 is semisimple, we conclude that both f and F are semisimple linear transformations. Then $(\ker f) \cap (f(\mathbb{R}^{d\times 1})) = \{0\}$ and $(\ker F) \cap (F(\mathfrak{so}(Q_d))) = \{0\}$.

Let $C := (\operatorname{ad} X)B$. Since the adjoint representation of $\mathfrak{s}$ on $\mathfrak{v}'$ is irreducible, since $d_0 \geq 2$, since $(\operatorname{ad} T)B = 0$ and since $B \neq 0$, it follows from the representation theory of $\mathfrak{sl}_2(\mathbb{R})$ that $C \neq 0$.

Let $Z := -X$. Then $[B, Z] = C$, so $(B_{\mathcal{C}}^{Lm})(Z_{\mathcal{C}}^{Cm}) = [B, Z]_{\mathcal{C}}^{Cm} = C_{\mathcal{C}}^{Cm}$ and $[B_{\mathcal{C}}^{Lm}, Z_{\mathcal{C}}^{Lm}] = [B, Z]_{\mathcal{C}}^{Lm} = C_{\mathcal{C}}^{Lm}$. Then, as $B_0 = B_{\mathcal{C}}^{Lm}$, we have

$$f(Z_{\mathcal{C}}^{Cm}) = B_0(Z_{\mathcal{C}}^{Cm}) = C_{\mathcal{C}}^{Cm} \qquad \text{and} \qquad F(Z_{\mathcal{C}}^{Lm}) = [B_0, Z_{\mathcal{C}}^{Lm}] = C_{\mathcal{C}}^{Lm}.$$

Then $C_{\mathcal{C}}^{Cm} \in f(\mathbb{R}^{d\times 1})$ and $C_{\mathcal{C}}^{Lm} \in F(\mathfrak{so}(Q_d))$.

We have $B \in \mathfrak{h}_0 \subseteq \mathfrak{v}' \subseteq \mathfrak{v}$, so $C = [X, B] \in [\mathfrak{s}, \mathfrak{v}] \subseteq \mathfrak{v}$. Therefore, we have $[B, C] \in [\mathfrak{v}, \mathfrak{v}] = \{0\}$. Then $(B_{\mathcal{C}}^{Lm})(C_{\mathcal{C}}^{Cm}) = [B, C]_{\mathcal{C}}^{Cm} = 0$ and $[B_{\mathcal{C}}^{Lm}, C_{\mathcal{C}}^{Lm}] = [B, C]_{\mathcal{C}}^{Lm} = 0$. Then, as $B_0 = B_{\mathcal{C}}^{Lm}$, we have

$$f(C_{\mathcal{C}}^{Cm}) = B_0 C_{\mathcal{C}}^{Cm} = 0 \qquad \text{and} \qquad F(C_{\mathcal{C}}^{Lm}) = [B_0, C_{\mathcal{C}}^{Lm}] = 0.$$

Then $C_{\mathcal{C}}^{Cm} \in \ker(f)$ and $C_{\mathcal{C}}^{Lm} \in \ker(F)$.

Then $C_{\mathcal{C}}^{Cm} \in (\ker f) \cap (f(\mathbb{R}^{d\times 1}))$ and $C_{\mathcal{C}}^{Lm} \in (\ker F) \cap (F(\mathfrak{so}(Q_d)))$, so $C_{\mathcal{C}}^{Cm} = 0$ and $C_{\mathcal{C}}^{Lm} = 0$. So, by (3) of Remark 3.5 of [Ad99a], we have $C = 0$, a contradiction.

Case B: $\mu \neq 0$. Recall that $A_{\mathcal{C}}^{Lm} \neq 0 \neq B_{\mathcal{C}}^{Lm}$. We have

$$[T_{\mathcal{C}}^{Lm}, A_{\mathcal{C}}^{Lm}] = [T, A]_{\mathcal{C}}^{Lm} = \lambda A_{\mathcal{C}}^{Lm}, \qquad [T_{\mathcal{C}}^{Lm}, B_{\mathcal{C}}^{Lm}] = [T, B]_{\mathcal{C}}^{Lm} = \mu B_{\mathcal{C}}^{Lm}.$$

As $\lambda \in \mathbb{R}\backslash\{0\}$, we see that λ is not pure imaginary. So, by (2) of Lemma 4.1 (with T replaced by $T_{\mathcal{C}}^{Lm}$), we choose $a > 0$ such that $\lambda, \mu \in \{-a, 0, a\}$. So, as $\lambda \neq 0 \neq \mu \neq \lambda$, we conclude that $\lambda = -\mu$.

Let $T_1 := (1/\lambda)T_{\mathcal{C}}^{Lm}$, let $A_1 := A_{\mathcal{C}}^{Lm}$ and let $B_1 := B_{\mathcal{C}}^{Lm}$. We have $T_1, A_1, B_1 \in \mathfrak{so}(Q_d)$. We have $[T_1, A_1] = A_1$ and $[T_1, B_1] = -B_1$ and $A_1 \neq 0 \neq B_1$. By Lemma 4.2, we have $[A_1, B_1] \neq 0$. On the other hand, since we have $[A, B] \in [\mathfrak{h}_0, \mathfrak{h}_0] \subseteq [\mathfrak{v}', \mathfrak{v}'] \subseteq [\mathfrak{v}, \mathfrak{v}] = \{0\}$ and since we have $[A_1, B_1] = [A_{\mathcal{C}}^{Lm}, B_{\mathcal{C}}^{Lm}] = [A, B]_{\mathcal{C}}^{Lm}$, it follows that $[A_1, B_1] = 0$, a contradiction. □

Since $\mathfrak{v}' \neq \{0\}$, we conclude that $d_0 \geq 1$. So, by Claim 2, we have $d_0 \in \{1, 2, 3\}$. So, by Claim 1, we have $d_0 \in \{1, 3\}$. □

LEMMA 15.3. *Let* $\mathfrak{v}'$ *be an* (ad $\mathfrak{s}$)*-irreducible subspace of* $\mathfrak{v}$. *Assume that* $\mathfrak{h} \cap \mathfrak{v}' \neq \{0\}$. *Then* $\mathfrak{v}' \subseteq \mathfrak{w}_1$.

Proof. If $\dim(\mathfrak{v}') = 1$, then, because $\mathfrak{h} \cap \mathfrak{v}' = \{0\}$, we get $\mathfrak{v}' \subseteq \mathfrak{h} \subseteq \mathfrak{w}_1$, and we are done. We may therefore assume that $\dim(\mathfrak{v}') \neq 1$. Since $\mathfrak{h} \cap \mathfrak{v}' \neq \{0\}$, we conclude that $\mathfrak{v}' \neq \{0\}$. Then $\dim(\mathfrak{v}') \geq 2$.

Let $U := \mathfrak{h} \cap \mathfrak{v}'$ and let $U' := \mathfrak{w}_1 \cap \mathfrak{v}'$. By Lemma 15.1, we see that (U, U') is almost (ad $\mathfrak{s}$)-invariant. By Lemma 11.2 (with V replaced by $\mathfrak{v}'$), choose $T \in \mathfrak{s}\backslash\{0\}$ and choose $u_0 \in U\backslash\{0\}$ such that T is real diagonalizable and such that $(\text{ad } T)u_0 \in \mathbb{R}u_0$. Choose $X, Y \in \mathfrak{s}$ such that (X, Y, T) is a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$.

By Lemma 14.1 and Lemma 14.2 (with $\mathfrak{v}$ replaced by $\mathfrak{v}'$), we see, for all $A \in U\backslash\{0\}$, that $(\text{ad } X)A \neq 0 \neq (\text{ad } Y)A$. By Lemma 11.3 (with V replaced by $\mathfrak{v}'$), we get $U' = \mathfrak{v}'$. Then $\mathfrak{v}' = U' = \mathfrak{w}_1 \cap \mathfrak{v}' \subseteq \mathfrak{w}_1$. □

LEMMA 15.4. *Assume that* (ad $\mathfrak{s}$)$\mathfrak{w}_1 \subseteq \mathfrak{w}_1$. *Then the adjoint representation of* $\mathfrak{s}$ *on* $\mathfrak{w}_1$ *is either trivial or stably* 3*-irreducible.*

Proof. Assume that the adjoint representation of $\mathfrak{s}$ on $\mathfrak{w}_1$ is nontrivial. We wish to show that it is stably 3-irreducible.

By (3) of Lemma 8.4 we see that the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1. Let $\mathfrak{f}$ denote the set of (Ad S)-fixpoints in $\mathfrak{w}_1$. Let $\mathfrak{c}$ be an (ad $\mathfrak{s}$)-invariant vector space complement in $\mathfrak{w}_1$ to $\mathfrak{f}$. Since the adjoint representation of $\mathfrak{s}$ on $\mathfrak{w}_1$ is nontrivial, it follows that $\mathfrak{c} \neq \{0\}$. We wish to show that the adjoint representation of $\mathfrak{s}$ on $\mathfrak{c}$ is 3-irreducible.

Choose $k \geq 1$ and choose (ad $\mathfrak{s}$)-irreducible subspaces $\mathfrak{c}_1, \ldots, \mathfrak{c}_k \subseteq \mathfrak{c}$ such that $\mathfrak{c} = \mathfrak{c}_1 \oplus \cdots \oplus \mathfrak{c}_k$. Let $K := \{1, \ldots, k\}$. Because $\mathfrak{f} \cap \mathfrak{c} = \{0\}$ and because $\mathfrak{s}$ is semisimple, we see, for all $i \in K$, that $\dim(\mathfrak{c}_i) \geq 2$. So, since the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1, we conclude, for all $i \in K$, that $\mathfrak{h} \cap \mathfrak{c}_i \neq \{0\}$.

For all $i \in K$, by Lemma 15.3 (with $\mathfrak{v}'$ replaced by $\mathfrak{c}_i$), we have $\mathfrak{c}_i \subseteq \mathfrak{w}_1$, so, by Lemma 15.2 (with $\mathfrak{v}'$ replaced by $\mathfrak{c}_i$), we see that $\dim(\mathfrak{c}_i) \in \{1, 3\}$, so, since $\dim(\mathfrak{c}_i) \geq 2$, we conclude that $\dim(\mathfrak{c}_i) = 3$. We wish to show that $k = 1$. Assume, for a contradiction, that $k \geq 2$.

By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, choose $A_1 \in \mathfrak{c}_1\backslash\{0\}$ and choose $A_2 \in \mathfrak{c}_2\backslash\{0\}$ such that $(\text{ad } X)A_1 = 0$ and $(\text{ad } X)A_2 = 0$. Since $\dim(\mathfrak{c}_1) = 3$, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $(\text{ad } T)A_1 = 2A_1$ and $(\text{ad } T)A_2 = 2A_2$. The codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1, so $(\mathbb{R}A_1 + \mathbb{R}A_2) \cap \mathfrak{h} \neq \{0\}$. Choose $A \in (\mathbb{R}A_1 + \mathbb{R}A_2)\backslash\{0\}$ such that $A \in \mathfrak{h}$. Then $(\text{ad } X)A = 0$ and $(\text{ad } T)A = 2A$. Then

$$(\text{ad } T)A = 2A \in \mathbb{R}A \qquad \text{and} \qquad (\text{ad } Y)(\text{ad } X)A = 0 \in \mathbb{R}A.$$

Then, by Lemma 10.2 (with V replaced by $\mathfrak{w}_1$ and v replaced by A), choose an (ad $\mathfrak{s}$)-irreducible subspace $\mathfrak{v}'$ of $\mathfrak{w}_1$ such that $A \in \mathfrak{v}'$. We have $(\text{ad } T)A = 2A \neq 0$, so $(\text{ad } T)\mathfrak{v}' \neq \{0\}$. So, since $\mathfrak{s}$ is semisimple, it follows that $\dim(\mathfrak{v}') \geq$

2. We have $[X, A] = 0$ and $A_{m_0} = 0$, contradicting Lemma 14.1 (with $\mathfrak{v}$ replaced by $\mathfrak{v}'$). $\square$

LEMMA 15.5. *Assume* $\mathfrak{h} \neq \{0\}$. *Then there is an* $(\mathrm{ad}\ \mathfrak{s})$*-irreducible subspace* $\mathfrak{v}' \subseteq \mathfrak{v}$ *such that* $\mathfrak{h} \cap \mathfrak{v}' \neq \{0\}$.

Proof. Let $\mathcal{W}$ be the collection of all $(\mathrm{ad}\ \mathfrak{s})$-invariant subspaces $\mathfrak{w}$ of $\mathfrak{v}$ satisfying $\mathfrak{h} \cap \mathfrak{w} \neq \{0\}$. Then $\mathfrak{v} \in \mathcal{W}$, so $\mathcal{W} \neq \emptyset$. Choose $\mathfrak{v}' \in \mathcal{W}$ such that $\dim(\mathfrak{v}') = \min\{\dim(\mathfrak{w}) \,|\, \mathfrak{w} \in \mathcal{W}\}$. Let $U := \mathfrak{h} \cap \mathfrak{v}'$ and let $U' := \mathfrak{w}_1 \cap \mathfrak{v}'$. By Lemma 15.1, (U, U') is almost $(\mathrm{ad}\ \mathfrak{s})$-invariant.

Since $\mathfrak{v}' \in \mathcal{W}$, it follows that $U \neq \{0\}$. By minimality of $\dim(\mathfrak{v}')$, for any $(\mathrm{ad}\ \mathfrak{s})$-invariant subspace $V_1 \subsetneq \mathfrak{v}'$, we have $V_1 \cap \mathfrak{h} = \{0\}$, whence $V_1 \cap U = \{0\}$. Then, by Lemma 12.3 (with V replaced by $\mathfrak{v}'$), the adjoint representation of $\mathfrak{s}$ on $\mathfrak{v}'$ is irreducible. $\square$

LEMMA 15.6. *Assume that* $(\mathrm{ad}\ \mathfrak{s})\mathfrak{h} \not\subseteq \mathfrak{h}$. *Then* $(\mathrm{ad}\ \mathfrak{s})\mathfrak{w}_1 \subseteq \mathfrak{w}_1$ *and the adjoint representation of* $\mathfrak{s}$ *on* $\mathfrak{w}_1$ *is stably* 3*-irreducible.*

Proof. Let $\mathfrak{f}$ denote the set of all $(\mathrm{Ad}\ S)$-fixpoints in $\mathfrak{v}$. Every subspace of $\mathfrak{f}$ is $(\mathrm{ad}\ \mathfrak{s})$-invariant, so, in particular, $\mathfrak{f} \cap \mathfrak{h}$ is an $(\mathrm{ad}\ \mathfrak{s})$-invariant subspace of $\mathfrak{v}$. Let $\mathfrak{c}$ be an $(\mathrm{ad}\ \mathfrak{s})$-invariant vector space complement in $\mathfrak{v}$ to $\mathfrak{f} \cap \mathfrak{h}$. We have $\mathfrak{h} = (\mathfrak{f} \cap \mathfrak{h}) + (\mathfrak{c} \cap \mathfrak{h})$ and $\mathfrak{w}_1 = (\mathfrak{f} \cap \mathfrak{h}) + (\mathfrak{c} \cap \mathfrak{w}_1)$ and $(\mathfrak{c} \cap \mathfrak{f}) \cap (\mathfrak{c} \cap \mathfrak{h}) = \{0\}$. Replacing $\mathfrak{v}$ by $\mathfrak{c}$, $\mathfrak{h}$ by $\mathfrak{c} \cap \mathfrak{h}$, $\mathfrak{w}_1$ by $\mathfrak{c} \cap \mathfrak{w}_1$ and $\mathfrak{f}$ by $\mathfrak{c} \cap \mathfrak{f}$, we may assume that $\mathfrak{f} \cap \mathfrak{h} = \{0\}$.

Since $(\mathrm{ad}\ \mathfrak{s})\mathfrak{h} \not\subseteq \mathfrak{h}$, we see that $\mathfrak{h} \neq \{0\}$. By Lemma 15.5, choose an $(\mathrm{ad}\ \mathfrak{s})$-irreducible subspace $\mathfrak{v}' \subseteq \mathfrak{v}$ such that $\mathfrak{h} \cap \mathfrak{v}' \neq \{0\}$. Then, as $\mathfrak{f} \cap \mathfrak{h} = \{0\}$, we get $\mathfrak{v}' \not\subseteq \mathfrak{f}$. So, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we get $\dim(\mathfrak{v}') \geq 2$. By Lemma 15.3, we have $\mathfrak{v}' \subseteq \mathfrak{w}_1$.

Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, choose $A, B \in \mathfrak{v}' \backslash \{0\}$ such that $[X, A] = 0$ and $[Y, B] = 0$. By Lemma 14.1 and Lemma 14.2, we have $A_{m_0} \neq 0 \neq B_{m_0}$, so $A, B \notin \mathfrak{h}$. On the other hand, we have $A, B \in \mathfrak{v}' \subseteq \mathfrak{w}_1$. Let $U := \mathfrak{h}$ and $U' := \mathfrak{w}_1$. By Lemma 15.1 (with $\mathfrak{v}'$ replaced by $\mathfrak{v}$), we see that (U, U') is almost $(\mathrm{ad}\ \mathfrak{s})$-invariant. Let $\hat{u} := A$ and $\check{u} := B$. Then $\hat{u}, \check{u} \in U' \backslash U$ and $(\mathrm{ad}\ X)\hat{u} = 0 \in U'$ and $(\mathrm{ad}\ Y)\check{u} = 0 \in U'$. By Lemma 10.4 (with V replaced by $\mathfrak{v}$), we get $(\mathrm{ad}\ \mathfrak{s})\mathfrak{w}_1 \subseteq \mathfrak{w}_1$.

Since $\mathfrak{h} \subseteq \mathfrak{w}_1$ and since $(\mathrm{ad}\ \mathfrak{s})\mathfrak{h} \not\subseteq \mathfrak{h}$, we conclude that the adjoint representation of $\mathfrak{s}$ on $\mathfrak{w}_1$ is nontrivial. Therefore, by Lemma 15.4, we see that the adjoint representation of $\mathfrak{s}$ on $\mathfrak{w}_1$ is stably 3-irreducible. $\square$

LEMMA 15.7. *Let* (X, Y, T) *be a standard* $\mathfrak{sl}_2(\mathbb{R})$ *basis of* $\mathfrak{s}$. *Assume that* $(\mathrm{ad}\ \mathfrak{s})\mathfrak{h} \not\subseteq \mathfrak{h}$. *Then* $(\mathrm{ad}\ X)\mathfrak{h} \not\subseteq \mathfrak{h}$ *and* $(\mathrm{ad}\ Y)\mathfrak{h} \not\subseteq \mathfrak{h}$.

Proof. By (3) of Lemma 8.4, we see that the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1. By Lemma 15.6, choose an $(\mathrm{ad}\ \mathfrak{s})$-invariant subspace $\mathfrak{v}'$ of $\mathfrak{w}_1$ such that the adjoint representation of $\mathfrak{s}$ on $\mathfrak{v}'$ is 3-irreducible. Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$.

By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, because the adjoint representation of $\mathfrak{s}$ on $\mathfrak{v}'$ is 3-irreducible, choose $B \in \mathfrak{v}' \backslash \{0\}$ such that $[T, B] = 0$. Let $A := [X, B]$ and let $C := [Y, B]$. Then, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, because the adjoint representation of $\mathfrak{s}$ on $\mathfrak{v}'$ is 3-irreducible, it follows that $[X, A] = 0$ and $[Y, C] = 0$. We have $A, B, C \in \mathfrak{v}' \subseteq \mathfrak{w}_1$.

By Lemma 14.1 (with $\mathfrak{v}$ replaced by $\mathfrak{v}'$), we have $A_{m_0} \neq 0$, so $A \notin \mathfrak{h}$. We have $B \in \mathfrak{w}_1$ and $A \in \mathfrak{w}_1 \backslash \mathfrak{h}$. So, since the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1, choose $r \in \mathbb{R}$ such that $B + rA \in \mathfrak{h}$. Then, because we have $(\mathrm{ad}\ X)(B + rA) = A \notin \mathfrak{h}$, it follows that $(\mathrm{ad}\ X)\mathfrak{h} \not\subseteq \mathfrak{h}$.

By Lemma 14.2 (with $\mathfrak{v}$ replaced by $\mathfrak{v}'$), we have $C_{m_0} \neq 0$, so $C \notin \mathfrak{h}$. We have $B \in \mathfrak{w}_1$ and $C \in \mathfrak{w}_1 \backslash \mathfrak{h}$. So, since the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1, choose $t \in \mathbb{R}$ such that $B + tC \in \mathfrak{h}$. Then, because we have $(\mathrm{ad}\ Y)(B + tC) = C \notin \mathfrak{h}$, it follows that $(\mathrm{ad}\ Y)\mathfrak{h} \not\subseteq \mathfrak{h}$. $\square$

16. Moving from nilpotent element to nilpotent element

Let G be a connected Lie group. Let G_1 be a semisimple connected Lie subgroup of G. Let V be an Abelian connected Lie subgroup of G. Assume that G_1 normalizes V.

Let G act locally faithfully by isometries of a connected Lorentz manifold M. Let $m_0 \in M$. Let $H := \mathrm{Stab}_V^0(m_0)$. Let $\mathcal{L}$ denote the light cone in $T_{m_0}M$. Let $\mathfrak{w}_1 := \{X \in \mathfrak{v} \,|\, X_{m_0} \in \mathcal{L}\}$. Assume that $\mathfrak{w}_1$ is a subspace of $\mathfrak{v}$.

LEMMA 16.1. *Let $X \in \mathfrak{g}_1$ be nilpotent. Then either* $(\mathrm{ad}\ X)\mathfrak{h} \subseteq \mathfrak{h}$ *or* $(\mathrm{ad}\ X)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$.

Proof. By Jacobson-Morozov (Theorem IX.7.4, p. 432, of [He78]), choose $Y, T \in \mathfrak{g}$ such that (X, Y, T) is a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of some Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. By Lemma 15.6, we conclude either that $(\mathrm{ad}\ \mathfrak{s})\mathfrak{h} \subseteq \mathfrak{h}$ or that $(\mathrm{ad}\ \mathfrak{s})\mathfrak{w}_1 \subseteq \mathfrak{w}_1$. Since $X \in \mathfrak{s}$, we are done. $\square$

LEMMA 16.2. *Let $\mathcal{N}$ denote the set nilpotent elements of $\mathfrak{g}$. Let $\mathcal{U}$ be a subspace of $\mathfrak{g}_1$ such that $\mathcal{U} \subseteq \mathcal{N}$. Assume, for some $X_0 \in \mathcal{U}$, that* $(\mathrm{ad}\ X_0)\mathfrak{h} \not\subseteq \mathfrak{h}$. *Then, for all $X \in \mathcal{U}$, we have* $(\mathrm{ad}\ X)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$.

Proof. Let $X \in \mathcal{U}$. Assume, for a contradiction, that $(\mathrm{ad}\ X)\mathfrak{w}_1 \not\subseteq \mathfrak{w}_1$.

Using Lemma 16.1, we have $(\mathrm{ad}\ X_0)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$ and $(\mathrm{ad}\ X)\mathfrak{h} \subseteq \mathfrak{h}$. Let $Y := (X_0 + X)/2$. Because $(\mathrm{ad}\ X)\mathfrak{h} \subseteq \mathfrak{h}$ and $X_0 \in \mathbb{R}X + \mathbb{R}Y$ and $(\mathrm{ad}\ X_0)\mathfrak{h} \not\subseteq \mathfrak{h}$, we see that $(\mathrm{ad}\ Y)\mathfrak{h} \not\subseteq \mathfrak{h}$. Then, by Lemma 16.1, we have $(\mathrm{ad}\ Y)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$. Then, because $(\mathrm{ad}\ X_0)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$ and $X \in \mathbb{R}X_0 + \mathbb{R}Y$, we see that $(\mathrm{ad}\ X)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$, a contradiction. $\square$

LEMMA 16.3. *Assume that* $\mathfrak{g}_1$ *has no compact factors. Then either* $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{h} \subseteq \mathfrak{h}$ *or* $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$.

Proof. Assume $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{h} \not\subseteq \mathfrak{h}$. We wish to show that $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$.

Choose $k \geq 1$ and $X_1, \ldots, X_k, Y_1, \ldots, Y_k$ as in Lemma 3.3 (with $\mathfrak{g}$ replaced by $\mathfrak{g}_1$). Let $K := \{1, \ldots, k\}$. For all $i \in K$, we define $\mathfrak{s}_i := \mathbb{R}X_i + \mathbb{R}Y_i + \mathbb{R}[X_i, Y_i]$. By (2) of Lemma 3.3, for all $i \in K$, $\mathfrak{s}_i$ is a Lie subalgebra of $\mathfrak{g}_1$ and $\mathfrak{s}_i$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. For $i \in K$, let S_i be the connected Lie subgroup of G_1 corresponding to $\mathfrak{s}_i$. As $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{h} \not\subseteq \mathfrak{h}$, by (1) of Lemma 3.3, choose $i_0 \in K$ such that $(\mathrm{ad}\ \mathfrak{s}_{i_0})\mathfrak{h} \not\subseteq \mathfrak{h}$. Then, by Lemma 15.7 (with S replaced by S_{i_0}), we see both that $(\mathrm{ad}\ X_{i_0})\mathfrak{h} \not\subseteq \mathfrak{h}$ and that $(\mathrm{ad}\ Y_{i_0})\mathfrak{h} \not\subseteq \mathfrak{h}$.

By Lemma 16.2 (with $\mathcal{U}$ replaced by $\mathbb{R}X_1 + \cdots + \mathbb{R}X_k$), we see, for all $i \in K$, that $(\mathrm{ad}\ X_i)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$. Similarly, by Lemma 16.2 (with $\mathcal{U}$ replaced by $\mathbb{R}Y_1 + \cdots + \mathbb{R}Y_k$), we see, for all $i \in K$ that $(\mathrm{ad}\ Y_i)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$. Then, by (1) of Lemma 3.3, we conclude that $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$. □

17. A fact about rank two root systems

Let $(\cdot, \cdot)$ be a positive definite symmetric bilinear form on a vector space E. Let Φ be an irreducible root system in E.

For all $\alpha \in \Phi$, let $p_\alpha : E \to \mathbb{R}\alpha$ be the orthogonal projection defined by $p_\alpha(\beta) = [(\alpha, \beta)/(\alpha, \alpha)]\alpha$. Let $\mathbb{N} := \{1, 2, 3, \ldots\}$. Let $F_0 \subseteq E$ be a finite set. Let $\chi : F_0 \to \mathbb{N}$ be a function. Let $d := \sum_{f \in F_0} \chi(f)$. For all $\alpha \in \Phi_0$, define $\chi_\alpha : \{-\alpha, 0, \alpha\} \to \mathbb{Z}$ by

$$\chi_\alpha(-\alpha) = 1, \qquad \chi_\alpha(0) = d - 2, \qquad \chi_\alpha(\alpha) = 1.$$

For all finite $F \subseteq \mathfrak{a}^*$, for all functions $p : E \to E$, for all $\lambda \in E$, we define $S(F, p, \lambda) := (p^{-1}(\lambda)) \cap F$. For all finite $F \subseteq \mathfrak{a}^*$, for all functions $\phi : F \to \mathbb{Z}$, for all functions $p : E \to E$, we define a function $p(\phi) : p(F) \to \mathbb{Z}$ by $(p(\phi))(\lambda) = \sum_{\mu \in S(F,p,\lambda)} \phi(\mu)$.

LEMMA 17.1. *Assume that* $\dim(E) \geq 2$. *Then there exists* $\alpha \in \Phi_0$ *such that* $p_\alpha(\chi) \neq \chi_\alpha$.

Proof. Choose $\beta, \gamma \in \Phi$ such that $\mathbb{R}\beta \neq \mathbb{R}\gamma$ and such that $(\beta, \gamma) \neq 0$. Let $E_0 := \mathbb{R}\beta + \mathbb{R}\gamma$. Let $\Phi_0 := E \cap \Phi$. Then Φ_0 is a root system in E_0. Because $\dim(E_0) = 2$, because $\beta, \gamma \in \Phi_0$, because $(\beta, \gamma) \neq 0$ and because $\mathbb{R}\beta \neq \mathbb{R}\gamma$, we conclude that Φ_0 is irreducible.

Let $q : E \to E_0$ be the orthogonal projection map. For all $\alpha \in \Phi_0$, we have $p_\alpha \circ q = p_\alpha$, so $p_\alpha(q(\chi)) = p_\alpha(\chi)$. Let Φ_0' be a reduced root system such that $\Phi_0' \subseteq \Phi_0$ and such that the real span of Φ_0' is E_0. Replacing Φ with Φ_0', χ with $q(\chi)$ and E with E_0, we may assume that Φ is irreducible and reduced and that the rank of Φ is two.

By the classification of irreducible reduced root systems of rank two, we see that the type of Φ_0 is A_2, B_2 or G_2. (See Figure 1 on p. 44 of [Hu72], but keep in mind that $A_1 \times A_1$ is reducible.) For each of these three types, basic plane geometry yields the result. $\square$

18. Representations of noncompact simple groups, Part I

Let $\mathfrak{l}_0$ be a noncompact simple Lie algebra. Let $\mathfrak{a}$ be a maximal $\mathbb{R}$-split torus in $\mathfrak{l}_0$. Let κ be the Killing form on $\mathfrak{l}_0$. Then $\kappa|\mathfrak{a}$ is positive definite, and so induces an isomorphism $\mathfrak{a}^* \longleftrightarrow \mathfrak{a}$. Let $(\,\cdot\,,\,\cdot\,)$ be the positive definite symmetric bilinear form on $\mathfrak{a}^*$ corresponding to $\kappa|\mathfrak{a}$. Let $E := \mathfrak{a}^*$.

Let $\Phi \subseteq E$ be the set of roots of $\mathfrak{a}$ on $\mathfrak{l}_0$. For $\alpha \in E$, let $p_\alpha : E \to \mathbb{R}\alpha$ be the orthogonal projection defined by $p_\alpha(\beta) = [(\alpha,\beta)/(\alpha,\alpha)]\alpha$. For all $\alpha \in \Phi$, let $\alpha^\perp := p_\alpha^{-1}(0)$. For any $\alpha, \beta \in \Phi$, we define the *α-rootstring through β* to be the set $(\mathbb{R}\alpha + \beta) \cap \Phi$. The *center* of a rootstring is the average of its elements.

Let $\rho : \mathfrak{l}_0 \to \mathfrak{gl}(V)$ be a representation. In this section, we assume

(∗∗) For any Lie subalgebra $\mathfrak{s}$ of $\mathfrak{l}_0$, if $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, then $\rho|\mathfrak{s} : \mathfrak{s} \to \mathfrak{gl}(V)$ is stably 3-irreducible.

Let $\Lambda \subseteq E$ be the set of weights of $\mathfrak{a}$ on V. For all $\lambda \in \Lambda$, let V_λ denote the λ-weightspace of V. Let $\mathbb{N} := \{1, 2, 3, \ldots\}$. Let $\chi : \Lambda \to \mathbb{N}$ be defined by $\chi(\lambda) = \dim(V_\lambda)$.

Let $d := \dim(V)$. For all $\alpha \in \Phi$, and let $\chi_\alpha : \{-\alpha, 0, \alpha\} \to \mathbb{Z}$ be defined as in §17. For all finite $F \subseteq E$, for all functions $\phi : F \to \mathbb{Z}$, for all functions $p : E \to E$, define the function $p(\phi) : p(F) \to \mathbb{Z}$ as in §17.

LEMMA 18.1. *For all $\alpha \in \Phi$, we have $p_\alpha(\chi) = \chi_\alpha$.*

Proof. For all $\gamma \in \Phi$, let $\mathfrak{l}_0^\gamma$ denote the γ-rootspace of $\mathfrak{l}_0$.

Fix $X \in \mathfrak{l}_0^\alpha \backslash \{0\}$. By Lemma 3.2 (with $\mathfrak{g}$ replaced by $\mathfrak{l}_0$), choose $T \in \mathfrak{a}$ and $Y \in \mathfrak{l}_0^{-\alpha}$ such that (X, Y, T) is a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{l}_0$. Then $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

CLAIM 1. *For all $\beta \in \alpha^\perp$, $\beta(T) = 0$.*

Proof. Let $q : E \to \alpha^\perp$ be the orthogonal projection defined by $q(\beta) = \beta - p_\alpha(\beta)$. Let $r : E \to E$ be the orthogonal reflection through $\alpha^\perp$ defined by $r(\beta) = \beta - 2(p_\alpha(\beta))$.

By Weyl-invariance of Φ, any α-rootstring is invariant under the reflection $r : E \to E$. Thus, for all $\beta \in \Phi$, the center of the α-rootstring through β is $q(\beta)$. For all $\gamma \in E$, if γ is the center of an α-rootstring in Φ, then, by the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, we have $\gamma(T) = 0$. Thus, for all $\gamma \in q(\Phi)$, we have $\gamma(T) = 0$. Since Φ spans E, it follows that $q(\Phi)$ spans $\alpha^\perp$. Thus, for all $\gamma \in \alpha^\perp$, we have $\gamma(T) = 0$. $\square$

CLAIM 2. *For all $\lambda \in E$, $t \in \mathbb{R}$, if $p_\alpha(\lambda) = t\alpha$, then $\lambda(T) = 2t$.*

Proof. Let $\beta := \lambda - t\alpha$. Then $p_\alpha(\beta) = 0$, so $\beta \in \alpha^\perp$. Then, by Claim 1, we have $\beta(T) = 0$. Then $\lambda(T) = t(\alpha(T))$. We have $(\alpha(T))X = [T, X] = 2X$, so $\alpha(T) = 2$. We conclude that $\lambda(T) = 2t$. □

For all $t \in \mathbb{R}$, let $\Lambda_t := \Lambda \cap (p_\alpha^{-1}(t\alpha))$. Let $B := \{t \in \mathbb{R} \,|\, \Lambda_t \neq \emptyset\}$. For $t \in B$, let $\mathcal{D}_t := \bigoplus_{\lambda \in \Lambda_t} V_\lambda$. For $t \in \mathbb{R}$, let $\mathcal{E}_t := \{v \in V \,|\, Tv = 2tv\}$. Let $C := \{t \in \mathbb{R} \,|\, \mathcal{E}_t \neq \{0\}\}$. By Claim 2, we see, for all $t \in B$, for all $\lambda \in \Lambda_t$, that $V_\lambda \subseteq \mathcal{E}_t$. Thus $B \subseteq C$ and, for all $t \in B$, we have $\mathcal{D}_t \subseteq \mathcal{E}_t$. So, since $\bigoplus_{t \in B} \mathcal{D}_t = \bigoplus_{\lambda \in \Lambda} V_\lambda = V = \bigoplus_{t \in C} \mathcal{E}_t$, we conclude that $B = C$, and we also conclude, for all $t \in C$, that $\mathcal{D}_t = \mathcal{E}_t$.

Since $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$, by Assumption $(**)$, we see that $\rho|\mathfrak{s} : \mathfrak{s} \to \mathfrak{gl}(V)$ is stably 3-irreducible. Choose real $\mathfrak{s}$-submodules V' and V'' of V such that V' is three-dimensional and $\mathfrak{s}$-irreducible, such that V'' is $\mathfrak{s}$-trivial and such that $V = V' + V''$. Then $V' \cap V'' = \{0\}$, so $V = V' \oplus V''$, so $\dim(V'') = d - 3$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, choose a basis $\{P, Q, R\}$ of V' such that $[T, P] = 2P$, $[T, Q] = 0$, $[T, R] = -2R$. Since V'' is $\mathfrak{s}$-trivial, we conclude that $[T, V''] = \{0\}$. Then $C = \{-1, 0, 1\}$, $\mathcal{E}_1 = \mathbb{R}P$, $\mathcal{E}_{-1} = \mathbb{R}R$ and $\mathcal{E}_0 = \mathbb{R}Q + V''$.

We have $p_\alpha(\Lambda) = \{t\alpha \,|\, \Lambda_t \neq \emptyset\} = \{t\alpha \,|\, t \in B\} = \{t\alpha \,|\, t \in C\}$. Then $p_\alpha(\Lambda) = \{-\alpha, 0, \alpha\}$. It remains to show that $(p_\alpha(\chi))(\alpha) = 1$, that $(p_\alpha(\chi))(-\alpha) = 1$ and that $(p_\alpha(\chi))(0) = d - 2$.

We have $\bigoplus_{\lambda \in \Lambda_1} V_\lambda = \mathcal{D}_1 = \mathcal{E}_1 = \mathbb{R}P$, so $\sum_{\lambda \in \Lambda_1} \dim(V_\lambda) = \dim(\mathbb{R}P) = 1$. By the definition of $p_\alpha(\chi)$, we have $(p_\alpha(\chi))(\alpha) = \sum_{\lambda \in \Lambda_1} \chi(\lambda)$. Then $(p_\alpha(\chi))(\alpha) = \sum_{\lambda \in \Lambda_1} \dim(V_\lambda) = 1$. Because $\bigoplus_{\lambda \in \Lambda_{-1}} V_\lambda = \mathcal{D}_{-1} = \mathcal{E}_{-1} = \mathbb{R}R$, a similar argument shows that $(p_\alpha(\chi))(-\alpha) = \dim(\mathbb{R}R) = 1$. Finally, because we have $\bigoplus_{\lambda \in \Lambda_0} V_\lambda = \mathcal{D}_0 = \mathcal{E}_0 = \mathbb{R}Q + V''$, a similar argument shows that $(p_\alpha(\chi))(0) = \dim(\mathbb{R}Q + V'') = 1 + (d - 3) = d - 2$. □

Lemma 18.2. *The root system Φ is reduced.*

Proof. Let $\alpha, \beta \in \Phi$ satisfy $\mathbb{R}\alpha = \mathbb{R}\beta$. We wish to show $\alpha \in \{-\beta, \beta\}$.

By Lemma 18.1, $p_\alpha(\chi) = \chi_\alpha$ and $p_\beta(\chi) = \chi_\beta$. We have $\mathbb{R}\alpha = \mathbb{R}\beta$, so $p_\alpha = p_\beta$, so $p_\alpha(\chi) = p_\beta(\chi)$. Then $\chi_\alpha = \chi_\beta$. Then $\alpha \in \{-\beta, \beta\}$. □

Lemma 18.3. *We have* $\dim(\mathfrak{a}) = 1$.

Proof. Since $\mathfrak{l}_0$ is noncompact, it follows that $\mathfrak{a} \neq \{0\}$, so $\dim(\mathfrak{a}) \geq 1$. By Lemma 18.1, for all $\alpha \in \Phi$, we have $p_\alpha(\chi) = \chi_\alpha$. So, by Lemma 17.1, we have $\dim(E) \leq 1$. Then $\dim(\mathfrak{a}) = \dim(\mathfrak{a}^*) = \dim(E) \leq 1$. □

Lemma 18.4. *There exists $Q \in \mathrm{Mink}(V)$ such that $\rho(\mathfrak{l}_0) = \mathfrak{so}(Q)$.*

Proof. By Lemma 18.3, we have $\dim(\mathfrak{a}) = 1$. By Lemma 18.2, the root system of $\mathfrak{l}_0$ is reduced. Choose $\alpha \in E \backslash \{0\}$ such that $\Phi = \{-\alpha, \alpha\}$. Because $\dim(\mathfrak{a}) = 1$, we conclude that $p_\alpha : E \to \mathbb{R}\alpha$ is the identity map, so $\chi = p_\alpha(\chi)$. By Lemma 18.1, $p_\alpha(\chi) = \chi_\alpha$. Then $\chi = \chi_\alpha$.

Then χ and χ_α have the same domain. That is, $\Lambda = \{-\alpha, 0, \alpha\}$. Moreover, we have $\dim(V_\alpha) = \chi(\alpha) = \chi_\alpha(\alpha) = 1$. Similarly, we have $\dim(V_{-\alpha}) = \chi(-\alpha) = \chi_\alpha(-\alpha) = 1$. Therefore, Lemma 18.4 follows from Lemma 7.1. $\square$

19. Representations of noncompact simple groups, Part II

Let G be a connected Lie group. Let L_0 be a simple connected Lie subgroup of G. Assume that L_0 is noncompact. Let V be an Abelian connected Lie subgroup of G. Assume that L_0 normalizes V.

Let G act locally faithfully by isometries of a connected Lorentz manifold M. Let $m_0 \in M$. Let $\mathcal{L}$ denote the light cone in $T_{m_0}M$.

Lemma 19.1. *Assume that* $\mathfrak{v}_{m_0} \subseteq \mathcal{L}$. *Assume* $(\mathrm{ad}\ \mathfrak{l}_0)\mathfrak{v} \neq \{0\}$. *Then there exists* $Q \in \mathrm{Mink}(\mathfrak{v})$ *such that* $\mathrm{ad}_{\mathfrak{v}}(\mathfrak{l}_0) = \mathfrak{so}(Q)$.

Proof. Let $\rho := \mathrm{ad}_{\mathfrak{v}} : \mathfrak{l}_0 \to \mathfrak{gl}(\mathfrak{v})$. By Lemma 18.4 (with V replaced by $\mathfrak{v}$), it suffices to prove Assumption $(**)$ of §18 (with V replaced by $\mathfrak{v}$). Let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{l}_0$ such that $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Let $\rho_1 := \rho|\mathfrak{s} : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{v})$. We wish to show that ρ_1 is stably 3-irreducible.

Let S be the connected Lie subgroup of L_0 corresponding to $\mathfrak{s}$. Let $\mathfrak{w}_1 := \{X \in \mathfrak{v} \,|\, X_{m_0} \in \mathcal{L}\}$. As $\mathfrak{v}_{m_0} \subseteq \mathcal{L}$, we conclude that $\mathfrak{w}_1 = \mathfrak{v}$. Then $(\mathrm{ad}\ \mathfrak{s})\mathfrak{w}_1 = (\mathrm{ad}\ \mathfrak{s})\mathfrak{v} \subseteq \mathfrak{v} = \mathfrak{w}_1$. By Lemma 15.4, we conclude that ρ_1 is either trivial or stably 3-irreducible. As $(\mathrm{ad}\ \mathfrak{l}_0)\mathfrak{v} \neq \{0\}$, it follows that $\rho(\mathfrak{l}_0) \neq \{0\}$. Therefore, by simplicity of $\mathfrak{l}_0$, we have $\ker(\rho) = \{0\}$, so $\rho(\mathfrak{s}) \neq \{0\}$. Thus ρ_1 is nontrivial, and is therefore stably 3-irreducible. $\square$

20. Representations of reductive groups, Part I

Let G be a connected Lie group. Let G_0 be a reductive connected Lie subgroup of G. Let V be an Abelian connected Lie subgroup of G. Assume that G_0 normalizes V. Let $\mathfrak{z} := \mathfrak{z}(\mathfrak{g}_0)$ be the solvable radical of $\mathfrak{g}_0$. Let $\mathfrak{l} := [\mathfrak{g}_0, \mathfrak{g}_0]$ be the semisimple Levi factor of $\mathfrak{g}_0$. Let $\mathfrak{k}$ and $\mathfrak{g}_1$ be ideals of $\mathfrak{l}$. Assume that $\mathfrak{k}$ is compact, that $\mathfrak{g}_1$ has no compact factors and that $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{g}_1$. Let G_1 be the connected Lie subgroup of G_0 corresponding to $\mathfrak{g}_1$.

Let G act locally faithfully by isometries of a connected Lorentz manifold M. Let $m_0 \in M$. Let $H := \mathrm{Stab}^0_V(m_0)$. Let $\mathcal{L}$ be the light cone in $T_{m_0}M$.

Recall, from §2, the definition of almost $\mathfrak{s}$-invariant.

Lemma 20.1. *Let* $\mathfrak{w}_1 := \{X \in \mathfrak{v} \,|\, X_{m_0} \in \mathcal{L}\}$. *Assume that* $\mathfrak{w}_1$ *is a subspace of* $\mathfrak{v}$. *Assume that no nonzero vector in* $\mathfrak{v}$ *is* $(\mathrm{Ad}\ G_1)$*-fixed. Then either* $(\mathrm{ad}\ \mathfrak{g}_0)\mathfrak{h} \subseteq \mathfrak{h}$ *or* $(\mathrm{ad}\ \mathfrak{g}_0)\mathfrak{w}_1 \subseteq \mathfrak{w}_1$.

Proof. By Lemma 16.3, choose $\mathfrak{w} \in \{\mathfrak{h}, \mathfrak{w}_1\}$ such that $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{w} \subseteq \mathfrak{w}$. It suffices to show that $(\mathrm{ad}(\mathfrak{k} + \mathfrak{z}))\mathfrak{w} \subseteq \mathfrak{w}$. Fix $X_0 \in \mathfrak{k} \oplus \mathfrak{z}$. Let $\mathfrak{w}' := (\mathrm{ad}\ X_0)\mathfrak{w}$. We wish to show that $\mathfrak{w}' \subseteq \mathfrak{w}$, i.e., that $\mathfrak{w}' = \mathfrak{w}' \cap \mathfrak{w}$.

By Corollary 8.5, we see that $(\mathfrak{h}, \mathfrak{w}_1)$ is almost (ad $\mathfrak{g}_0$)-invariant. Then, for all $X \in \mathfrak{g}_0$, we have $(\mathrm{ad}\ X)\mathfrak{h} \subseteq \mathfrak{w}_1$. Also, the codimension in $\mathfrak{w}_1$ of $\mathfrak{h}$ is ≤ 1. So, as $\mathfrak{h} \subseteq \mathfrak{w} \subseteq \mathfrak{w}_1$, it follows that the codimension in $\mathfrak{w}$ of $\mathfrak{h}$ is ≤ 1 and that the codimension in $\mathfrak{w}_1$ of $\mathfrak{w}$ is ≤ 1.

Claim 1. *The codimension in* $\mathfrak{w}'$ *of* $\mathfrak{w}' \cap \mathfrak{w}$ *is* ≤ 1.

Proof. We know either that $\mathfrak{w} = \mathfrak{h}$ or that $\mathfrak{w} = \mathfrak{w}_1$.

Case A: $\mathfrak{w} = \mathfrak{h}$. By almost invariance, we have $(\mathrm{ad}\ X_0)\mathfrak{h} \subseteq \mathfrak{w}_1$. Then $\mathfrak{w}' = (\mathrm{ad}\ X_0)\mathfrak{w} = (\mathrm{ad}\ X_0)\mathfrak{h} \subseteq \mathfrak{w}_1$, so $\mathfrak{w}' = \mathfrak{w}' \cap \mathfrak{w}_1$. The codimension in $\mathfrak{w}_1$ of $\mathfrak{w}$ is ≤ 1, so the codimension in $\mathfrak{w}' \cap \mathfrak{w}_1$ of $\mathfrak{w}' \cap \mathfrak{w}$ is ≤ 1. That is, the codimension in $\mathfrak{w}'$ of $\mathfrak{w}' \cap \mathfrak{w}$ is ≤ 1.

Case B: $\mathfrak{w} = \mathfrak{w}_1$. As the codimension in $\mathfrak{w}$ of $\mathfrak{h}$ is ≤ 1, we see that the codimension in $(\mathrm{ad}\ X_0)\mathfrak{w}$ of $(\mathrm{ad}\ X_0)\mathfrak{h}$ is ≤ 1. That is, the codimension in $\mathfrak{w}'$ of $(\mathrm{ad}\ X_0)\mathfrak{h}$ is ≤ 1. As $\mathfrak{h} \subseteq \mathfrak{w}$, we have $(\mathrm{ad}\ X_0)\mathfrak{h} \subseteq (\mathrm{ad}\ X_0)\mathfrak{w} = \mathfrak{w}'$. By almost invariance, we have $(\mathrm{ad}\ X_0)\mathfrak{h} \subseteq \mathfrak{w}_1 = \mathfrak{w}$. Then

$$(\mathrm{ad}\ X_0)\mathfrak{h} \subseteq \mathfrak{w}' \cap \mathfrak{w} \subseteq \mathfrak{w}'.$$

Let $\mathfrak{p} := (\mathrm{ad}\ X_0)\mathfrak{h}$, let $\mathfrak{q} := \mathfrak{w}' \cap \mathfrak{w}$ and let $\mathfrak{r} := \mathfrak{w}'$. We have $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{r}$ and we know that the codimension in $\mathfrak{r}$ of $\mathfrak{p}$ is ≤ 1. It follows that the codimension in $\mathfrak{r}$ of $\mathfrak{q}$ is ≤ 1. $\square$

As $\mathfrak{w}$ is (ad $\mathfrak{g}_1$)-invariant and as $\mathfrak{g}_1$ centralizes $\mathfrak{k} \oplus \mathfrak{z}$, it follows that $\mathfrak{w}'$ is (ad $\mathfrak{g}_1$)-invariant. Then $\mathfrak{w}' \cap \mathfrak{w}$ is (ad $\mathfrak{g}_1$)-invariant, as well. Let $\mathfrak{c}$ be an (ad $\mathfrak{g}_1$)-invariant vector space complement in $\mathfrak{w}'$ to $\mathfrak{w}' \cap \mathfrak{w}$. We wish to show that $\mathfrak{c} = \{0\}$.

By Claim 1, we know that $\dim(\mathfrak{c}) \leq 1$. Because G_1 is semisimple and because there are no nonzero (Ad G_1)-fixed vectors in $\mathfrak{v}$, it follows that there are no (ad $\mathfrak{g}_1$)-invariant lines in $\mathfrak{v}$. So, in particular, we see that $\dim(\mathfrak{c}) \neq 1$. Then $\dim(\mathfrak{c}) = 0$, so $\mathfrak{c} = \{0\}$. $\square$

Lemma 20.2. *Assume that* $\mathfrak{v}_{m_0} \subseteq \mathcal{L}$. *Assume that the adjoint representation of* $\mathfrak{g}_0$ *on* $\mathfrak{v}$ *is irreducible. Assume that* $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{v} \neq \{0\}$. *Then there exists* $Q \in \mathrm{Mink}(\mathfrak{v})$ *such that* $\mathfrak{so}(Q) \subseteq \mathrm{ad}_{\mathfrak{v}}(\mathfrak{g}_0) \subseteq \mathfrak{co}(Q)$.

Proof. Let $\mathfrak{l}_0$ be a simple ideal of $\mathfrak{g}_1$ such that $(\mathrm{ad}\ \mathfrak{l}_0)\mathfrak{v} \neq \{0\}$. Since $\mathfrak{g}_1$ has no compact factors, it follows that $\mathfrak{l}_0$ is noncompact. Let L_0 be the connected Lie subgroup of G_1 corresponding to $\mathfrak{l}_0$. Then L_0 is noncompact. By Lemma 19.1, we choose $Q \in \mathrm{Mink}(\mathfrak{v})$ such that $\mathrm{ad}_{\mathfrak{v}}(\mathfrak{l}_0) = \mathfrak{so}(Q)$.

Then $\mathfrak{so}(Q) = \mathrm{ad}_{\mathfrak{v}}(\mathfrak{l}_0) \subseteq \mathrm{ad}_{\mathfrak{v}}(\mathfrak{g}_0)$. Let $\rho := \mathrm{ad}_{\mathfrak{v}} : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{v})$. By Lemma 3.6 (with V replaced by $\mathfrak{v}$), we are done. $\square$

21. Representations of reductive groups, Part II

Let G be a connected Lie group. Let G_0 be a reductive connected Lie subgroup of G. Let L be the semisimple Levi factor of G_0. Let V be an Abelian connected Lie subgroup of G. Assume G_0 normalizes V.

Let G act locally faithfully by isometries of a connected Lorentz manifold M. Let $m_0 \in M$. Let $H := \mathrm{Stab}_V^0(m_0)$.

LEMMA 21.1. *Assume that* $(\mathrm{ad}\ \mathfrak{g}_0)\mathfrak{h} \subseteq \mathfrak{h}$. *Then* $\mathrm{Ad}_{\mathfrak{h}}(L)$ *is compact.*

Proof. Assume, for a contradiction, that $\mathrm{Ad}_{\mathfrak{h}}(L)$ is noncompact.

Since $\mathrm{ad}_{\mathfrak{h}}(\mathfrak{l})$ is noncompact and semisimple, choose a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{l}$ such that $\mathfrak{s}$ is Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ and such that $\mathrm{ad}_{\mathfrak{h}}(\mathfrak{s}) \neq \{0\}$. Choose an $(\mathrm{ad}\ \mathfrak{s})$-irreducible subspace $\mathfrak{v}_0$ in $\mathfrak{h}$ such that $\dim(\mathfrak{v}_0) \geq 2$. Let (X, Y, T) be a standard $\mathfrak{sl}_2(\mathbb{R})$ basis of $\mathfrak{s}$. By the representation theory of $\mathfrak{sl}_2(\mathbb{R})$, choose $A \in \mathfrak{v}_0$ such that $[X, A] = 0$. We have $A \in \mathfrak{v}_0 \subseteq \mathfrak{h}$, so $A_{m_0} = 0$. By Lemma 14.1 (with $\mathfrak{v}$ replaced by $\mathfrak{v}_0$), we have a contradiction. $\square$

THEOREM 21.2. *Assume that* $\mathfrak{h} \neq \{0\}$. *Then at least one of the following is true:*

(1) *There exists a nonzero* $(\mathrm{ad}\ \mathfrak{g}_0)$*-invariant subspace* $\mathfrak{v}_1$ *of* $\mathfrak{v}$ *such that* $\mathrm{Ad}_{\mathfrak{v}_1}(L)$ *is compact.*
(2) *There is an* $(\mathrm{ad}\ \mathfrak{g}_0)$*-irreducible subspace* $\mathfrak{v}_1$ *of* $\mathfrak{v}$ *and there is some* $Q \in \mathrm{Mink}(\mathfrak{v}_1)$ *such that* $\mathfrak{so}(Q) \subseteq \mathrm{ad}_{\mathfrak{v}_1}(\mathfrak{g}_0) \subseteq \mathfrak{co}(Q)$.

Proof. Replacing G by G_0V, we may assume that V is normal in G.

Case A: $\mathfrak{g}_{m_0}$ *is nondegenerate.* For all $X \in \mathfrak{h}$, we have

$$(\mathrm{ad}\ X)\mathfrak{g} \subseteq [\mathfrak{h}, \mathfrak{g}] \subseteq [\mathfrak{v}, \mathfrak{g}] \subseteq \mathfrak{v},$$

so $(\mathrm{ad}\ X)^2\mathfrak{g} \subseteq (\mathrm{ad}\ X)\mathfrak{v} \subseteq [\mathfrak{h}, \mathfrak{v}] \subseteq [\mathfrak{v}, \mathfrak{v}] = \{0\}$. Let $\mathcal{C}$ be an ordered Q_d-basis of $T_{m_0}M$.

Fix $X \in \mathfrak{h}$ for this paragraph. Let $T := X_{\mathcal{C}}^{Lm}$ and $S := \mathfrak{g}_{\mathcal{C}}^{Cm}$. Then S is Q_d-nondegenerate and $T^2(S) = ((\mathrm{ad}\ X)^2\mathfrak{g})_{\mathcal{C}}^{Cm} = \{0\}$. By Lemma 3.1 (with (V, Q) replaced by $(\mathbb{R}^d, Q_d)$), we have $T(S) = \{0\}$. Then $((\mathrm{ad}\ X)\mathfrak{g})_{\mathcal{C}}^{Cm} = T(S) = \{0\}$, so $(\mathrm{ad}\ X)\mathfrak{g} \subseteq \mathfrak{h}$.

We conclude that $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. Then $(\mathrm{ad}\ \mathfrak{g}_0)\mathfrak{h} = [\mathfrak{g}_0, \mathfrak{h}] \subseteq [\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$. By Lemma 21.1, we see that (1) of Theorem 21.2 (with $\mathfrak{v}_1$ replaced by $\mathfrak{h}$) holds.

Case B: $\mathfrak{g}_{m_0}$ *is degenerate.* Let $\mathcal{L}$ be the light cone in $T_{m_0}M$. Let $\mathfrak{w}_1 := \{X \in \mathfrak{v} \,|\, X_{m_0} \in \mathcal{L}\}$. As $\mathfrak{g}_{m_0}$ is degenerate, it follows that $\mathcal{L} \cap \mathfrak{g}_{m_0}$ is a subspace of $\mathfrak{g}_{m_0}$, so $\mathcal{L} \cap \mathfrak{v}_{m_0}$ is a subspace of $\mathfrak{v}_{m_0}$. Then $\mathfrak{w}_1$ is a subspace of $\mathfrak{v}$.

Let L be the semisimple Levi factor of G_0. Let $\mathfrak{k}$ and $\mathfrak{g}_1$ be ideals of $\mathfrak{l}$ such that $\mathfrak{k}$ is compact, such that $\mathfrak{g}_1$ has no compact factors and such that $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{g}_1$. Let G_1 and K be the connected Lie subgroups of G_0 corresponding to $\mathfrak{g}_1$ and $\mathfrak{k}$, respectively. Then $L = G_1K$ and G_1 is a normal subgroup of G_0. Moreover, K is compact.

Let $\mathfrak{f}$ denote the set of (Ad G_1)-fixpoints in $\mathfrak{v}$. Since G_1 is a normal subgroup of G_0, we conclude that $\mathfrak{f}$ is (Ad G_0)-invariant. Since $\mathrm{Ad}_{\mathfrak{f}}(G_1)$ is trivial, we see that $\mathrm{Ad}_{\mathfrak{f}}(L) = \mathrm{Ad}_{\mathfrak{f}}(G_1K) = \mathrm{Ad}_{\mathfrak{f}}(K)$. Then $\mathrm{Ad}_{\mathfrak{f}}(L)$ is compact. So, if $\mathfrak{f} \neq \{0\}$, then (1) of Theorem 21.2 (with $\mathfrak{v}_1$ replaced by $\mathfrak{f}$) holds. We therefore assume that $\mathfrak{f} = \{0\}$, i.e., that no nonzero vector in $\mathfrak{v}$ is (Ad G_1)-fixed.

By Lemma 20.1, choose $\mathfrak{w} \in \{\mathfrak{h}, \mathfrak{w}_1\}$ such that $(\mathrm{ad}\ \mathfrak{g}_0)\mathfrak{w} \subseteq \mathfrak{w}$. Let $\mathfrak{v}_1$ be a nonzero (ad $\mathfrak{g}_0$)-irreducible subspace of $\mathfrak{w}$. Then $\mathfrak{v}_1 \subseteq \mathfrak{w} \subseteq \mathfrak{w}_1$, so $(\mathfrak{v}_1)_{m_0} \subseteq \mathcal{L}$. Because $\mathfrak{f} = \{0\}$, we see that $(\mathrm{ad}\ \mathfrak{g}_1)\mathfrak{v}_1 \neq \{0\}$. By Lemma 20.2 (with $\mathfrak{v}$ replaced by $\mathfrak{v}_1$), we see that (2) of Theorem 21.2 holds. □

22. Proof of Theorem 1.1

If G is a Lie group and if G_0 is a connected Lie subgroup of G, then we shall say that (G, G_0) is a *nonproper pair* if there exists a locally faithful action of G by isometries of a connected Lorentz manifold M such that the action of G_0 on M is orbit nonproper.

LEMMA 22.1. *Let G be a connected Lie group with simply connected nilradical. Let V_1 be an Abelian ideal of $\mathfrak{g}$. Let $S \subseteq \mathrm{GL}(V_1)$ be a connected Lie subgroup. Assume $\mathrm{Ad}_{V_1}(G) \subseteq S$. Let $G' := S \ltimes V_1$. Assume (G', V_1) is a nonproper pair. Then there exists a locally faithful, orbit nonproper action of G by isometries of a connected Lorentz manifold.*

Proof. Let $N := \exp(V_1)$ be the connected Lie subgroup of G corresponding to V_1. Let $e := \exp : V_1 \to N$. Because G has simply connected nilradical, and because V_1 is an Abelian ideal of $\mathfrak{g}$, it follows that e is an isomorphism of Lie groups. Define $E : \mathrm{GL}(V_1) \to \mathrm{Aut}(N)$ by $E(g) = e \circ g \circ e^{-1}$. Let $R := E(S)$. Let $H := R \ltimes N$. Since (G', V_1) is a nonproper pair, it follows that (H, N) is a nonproper pair. Define $\phi : G \to \mathrm{Aut}(N)$ by $(\phi(g))(n) = gng^{-1}$. Define $\psi : H \to \mathrm{Aut}(N)$ by $(\psi(h))(n) = hnh^{-1}$. Then $\phi(G) = E(\mathrm{Ad}_{V_1}(G)) \subseteq E(S) = R = \psi(H)$. In the notation of [Ad99c], $\mathrm{Int}_N(G) = \phi(G) \subseteq \psi(H) = \mathrm{Int}_N(H)$.

Let H act locally faithfully by isometries of a connected Lorentz manifold M such that the action of N on M is orbit nonproper. We define $G \times_N M$ as in the first paragraph of §1 of [Ad99c] and we let $M' := G \times_N M$. By (2) of Lemma 3.6 in [Ad99c], the G-action on M' is orbit nonproper. By (4) of Lemma 3.6 in [Ad99c], the G-action on M' is locally faithful. By Corollary 4.4 in [Ad99c], the G-action on M' preserves a Lorentz metric. □

Proof of "if" part of Theorem 1.1. For (1), (2) and (3) we use the "if" part of Theorem 1.3 of [Ad99b].

To prove (4), we let V_1' be a nonzero (Ad G)-irreducible subspace of V_1. Replacing V_1 by V_1', we may assume that the Adjoint representation of G on V_1 is irreducible.

Let $I : V_1 \to V_1$ denote the identity map. Let $P := \{\lambda I \,|\, \lambda > 0\}$ be the set of positive scalar transformations of V_1. Let $Q := \mathrm{GL}(V_1)$. Then $Q^0 = \{q \in Q \,|\, \det(q) > 0\}$. Let $\pi : Q^0 \to Q^0/P$ be the canonical homomorphism.

Let $L_1 := \mathrm{Ad}_{V_1}(L)$. By assumption, L_1 is compact. Let R denote the solvable radical of G. Let $R_1 := \mathrm{Ad}_{V_1}(R)$. Let $G_1 := \mathrm{Ad}_{V_1}(G)$. Then $G_1 = L_1R_1 \subseteq Q^0$. We have $\mathrm{Ad}_{V_1}(N) = \{I\}$. Therefore, by (iii) of Theorem 3.8.3, p. 206, of [Va74] we conclude that $R_1 \subseteq Z(G_1)$. Then R_1 is Abelian. Moreover, L_1 and R_1 centralize one another.

Since the representation of G_1 on V_1 is irreducible, since L_1 and R_1 centralize one another and since $G_1 = L_1R_1$, it follows that the representation of R_1 on V_1 is isotypic. By the isotypic representation theory of connected Abelian Lie groups, we conclude that $\pi(R_1)$ is compact. So, as L_1 is compact, and as $G_1 = L_1R_1$, we see that $\pi(G_1)$ is compact. The map $\pi|\,\mathrm{SL}(V_1) : \mathrm{SL}(V_1) \to Q^0/P$ is an isomorphism, so choose a compact subgroup K of $\mathrm{SL}(V_1)$ such that $\pi(G_1) = \pi(K)$. Then $G_1 \subseteq KP$. Let Q be a positive definite symmetric bilinear form on V_1 such that $K \subseteq \mathrm{SO}^0(Q)$. Let $S := \mathrm{CO}^0(Q) \subseteq \mathrm{GL}(V_1)$. Then

$$\mathrm{Ad}_{V_1}(G) = G_1 \subseteq KP \subseteq (\mathrm{SO}^0(Q))P = S.$$

Let $G' := S \ltimes V_1$ and $n := \dim(V_1)$. As $\mathrm{CO}^0(n) \ltimes \mathbb{R}^n$ is isomorphic to a subgroup of $\mathrm{SO}^0(Q_{n+2})$, we see that G' admits a smooth isometric action on flat $(n+2)$-dimensional Minkowski space, fixing the origin. Then (G', V_1) is a nonproper pair. So, by Lemma 22.1, we are done.

To prove (5), choose $Q \in \mathrm{Mink}(V_1)$ such that $\mathrm{ad}_{V_1}(\mathfrak{l}_0) = \mathfrak{so}(Q)$. Let $\mathfrak{g}_1 := \mathrm{ad}_{V_1}(\mathfrak{g}) \subseteq \mathfrak{gl}(V_1)$. As $V_1 \subseteq \mathfrak{z}(\mathfrak{n})$, it follows that $\mathfrak{n}$ is contained in the kernel of the surjective Lie algebra homomorphism $\mathrm{ad}_{V_1} : \mathfrak{g} \to \mathfrak{g}_1$, so $\mathfrak{g}_1$ is reductive. Let $\rho : \mathfrak{g}_1 \to \mathfrak{gl}(V_1)$ be inclusion. By Lemma 3.6 (with $\mathfrak{g}_0$ replaced by $\mathfrak{g}_1$, V replaced by V_1, $\mathfrak{l}_0$ replaced by $\mathrm{ad}_{V_1}(\mathfrak{l}_0)$ and $\mathfrak{l}$ replaced by $\mathrm{ad}_{V_1}(\mathfrak{l})$), we have $\rho(\mathfrak{g}_1) \subseteq \mathfrak{co}(Q)$. Let $S := \mathrm{CO}^0(Q)$. Then $\mathrm{ad}_{V_1}(\mathfrak{g}) = \mathfrak{g}_1 = \rho(\mathfrak{g}_1) \subseteq \mathfrak{s}$, so $\mathrm{Ad}_{V_1}(G) \subseteq S$. Let $G' := S \ltimes V_1$. Let $d := \dim(V_1)$. Then $d = n \geq 3$. By Lemma 10.4 of [Ad99b] we see that (G', V_1) is a nonproper pair. So, by Lemma 22.1, we are done. $\square$

Proof of "only if" part of Theorem 1.1. Assume that (1), (2), (3) and (4) of Theorem 1.1 are all false. We wish to show that (5) of Theorem 1.1 is true.

Let $V := \mathfrak{z}(\mathfrak{n})$. Let $G_0 := \mathrm{Ad}_V(G)$. Then G_0 is reductive. Define $G' := G_0 \ltimes V$. Then V is a normal subgroup of G' and, at the same time, V is an ideal of $\mathfrak{g}$.

If W is a vector space and if $S \subseteq \mathrm{GL}(W)$ is a connected Lie subgroup, then we shall say that S is *admissible* if one of the following occurs:

- S has compact semisimple Levi factor; or
- there exists $Q \in \mathrm{Mink}(W)$ such that $\mathfrak{so}(Q) \subseteq \mathfrak{s} \subseteq \mathfrak{co}(Q)$.

Since (1), (2) and (3) of Theorem 1.1 are all false, by the "only if" part of Theorem 1.3 of [Ad99b], we see that there exists a locally faithful action of G' by isometries of a connected Lorentz manifold M such that some noncompact closed connected subgroup of V fixes a point $m_0 \in M$. Let $H := \mathrm{Stab}_V^0(m_0)$. Then $\mathfrak{h} \neq \{0\}$.

By Theorem 21.2 (with G replaced by G'), we let $\mathfrak{v}_1$ be a nonzero ($\mathrm{ad}\, \mathfrak{g}_0$)-invariant subspace of $\mathfrak{v}$ such that $\mathrm{Ad}_{\mathfrak{v}_1}(G')$ is an admissible subgroup of $\mathrm{GL}(\mathfrak{v}_1)$. Let $V_1 := \exp(\mathfrak{v}_1)$ be the connected Lie subgroup of V corresponding to $\mathfrak{v}_1$. Since (4) of Theorem 1.1 is false, we conclude that $\mathrm{Ad}_{V_1}(L)$ is noncompact.

Let $e := \exp : \mathfrak{v}_1 \to V_1$. Then V_1 is a vector subspace of V and $e : \mathfrak{v}_1 \to V_1$ is a vector space isomorphism. Let $E : \mathrm{GL}(\mathfrak{v}_1) \to \mathrm{GL}(V_1)$ be the corresponding isomorphism of Lie groups, which is defined by $E(g) = e \circ g \circ e^{-1}$.

Then $E(\mathrm{Ad}_{\mathfrak{v}_1}(G')) = \mathrm{Ad}_{V_1}(G)$. So, since $\mathrm{Ad}_{\mathfrak{v}_1}(G')$ is an admissible subgroup of $\mathrm{GL}(\mathfrak{v}_1)$, we see that $\mathrm{Ad}_{V_1}(G)$ is an admissible subgroup of $\mathrm{GL}(V_1)$. So, as $\mathrm{Ad}_{V_1}(L)$ is noncompact, by definition of "admissible", we may choose $Q \in \mathrm{Mink}(W)$ such that $\mathfrak{so}(Q) \subseteq \mathrm{ad}_{V_1}(\mathfrak{g}) \subseteq \mathfrak{co}(Q)$. Let $\rho := \mathrm{ad}_{V_1} : \mathfrak{g} \to \mathfrak{gl}(V_1)$. As $\rho(\mathfrak{l}) = \mathrm{ad}_{V_1}(\mathfrak{l})$, we see that $\rho(\mathfrak{l})$ is noncompact. In particular, we have $\rho(\mathfrak{l}) \neq \{0\}$. By Lemma 3.7, we see that (5) of Theorem 1.1 is true. $\square$

References

[Ad98a] S. Adams, *Isometric actions of* $\mathrm{SO}(2) \ltimes \mathbb{R}^2$, Preprint, 1998.

[Ad98b] ———, *Locally free actions on Lorentz manifolds*, Geom. Funct. Anal. **10** (2000), 453–515.

[Ad99a] ———, *Orbit nonproper actions on Lorentz manifolds*, Geom. Funct. Anal. **11** (2001), 201–243.

[Ad99b] ———, *Transitive actions on Lorentz manifolds with noncompact stabilizer*, Preprint, 1999.

[Ad99c] ———, *Induction of geometric actions*, Geom. Dedicata **88** (2001), 91–112.

[AS99a] S. Adams and G. Stuck, *Isometric actions of* $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}_n$ *on Lorentz manifolds*, Israel J. Math. **121** (2001), 93–111.

[AS99b] ———, *Conformal actions of* $\mathrm{SL}_n(\mathbb{R})$ *and* $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ *on Lorentz manifolds*, Trans. Amer. Math. Soc. **352** (2000), 3913–3936.

[FH91] W. Fulton and J. Harris, *Representation theory. A first course*, Springer-Verlag, New York, 1991.

[Gr88] M. Gromov, *Rigid transformation groups*, Géometrie différentielle, Travaux en cours, vol. 33, Hermann, Paris, 1988.

[He78] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.

[Hu72] J. E. Humprheys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972.

[KN63] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Volume I*, Interscience Publishers, a division of John Wiley & Sons, New York, 1963.

[Ko94] N. Kowalsky, *Actions of non-compact simple groups on Lorentz manifolds and other geometric manifolds*, Thesis, University of Chicago, 1994.

[Ko96] ———, *Noncompact simple automorphism groups of Lorentz manifolds and other geometric manifolds*, *Ann. of Math.* **144** (1996), 611–640.

[Va74] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer-Verlag, New York, 1974.

[Ze98a] A. Zeghib, *Sur les espaces-temps homogènes*, The Epstein birthday schrift, Geom. Topol. Monogr. **1**, Coventry, 1998, pp. 551–576 (electronic).

[Ze98b] ———, *The identity component of the isometry group of a compact Lorentz manifold*, Duke Math. J. **92** (1998), 321–333.

[Zi84] R. J. Zimmer, *Kazhdan groups acting on compact manifolds*, Invent. Math. **75** (1984), 425–436.

[Zi86] ———, *On the automorphism group of a compact Lorentz manifold and other geometric manifolds*, Invent. Math. **83** (1986), 411–424.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: adams@math.umn.edu