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# ISOMETRIES IN ALEKSANDROV SPACES OF CURVATURE BOUNDED ABOVE

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Abstract. We study the isometry groups of Aleksandrov spaces with curvature bounded above. We prove that the metric of any finitely compact geodesically complete CAT($K$)-space ($K < 0$), all of whose spheres are arcwise connected, can be recovered from the family of all closed balls of a given positive radius. As a corollary, we obtain that every bijection of such a space onto itself which preserves this family is an isometry. In particular, these results hold for any simply connected Riemannian space with sectional curvature at most $K$, where $K < 0$, and of dimension greater than 1.

## 1. Introduction and main results

In this paper we prove some of the results on Aleksandrov spaces of curvature at most $K$ (see [1], [2], [7], [9]) announced in our earlier paper [6].

We first introduce some necessary definitions and notation. The distance between two points $x, y$ of a metric space $M$ is denoted by $xy$. A *(locally) inner* (or *(locally) interior* or *(locally) length*) *metric space* is a metric space in which (locally) any two points $x, y$ can be joined by a path with length arbitrarily close to $xy$. A path joining the points $x, y$ in $M$ is called *shortest arc* or *segment* (with ends $x, y$) if its length is equal to $xy$. A *(locally) geodesic space* is a metric space in which (locally) any two points can be joined by shortest arc.

A point $y$ lies between two points $x$ and $z$ if $xz = xy + yz$ and $y \neq x, y \neq z$; we denote this relation by $(xyz)$. An open (respectively closed) ball in $M$ of radius $r$ with center $x$ is denoted by $U(x,r)$ (respectively $B(x,r)$). $S_K$ denotes the complete simply connected two-dimensional Riemannian manifold of constant sectional curvature $K$.

We study the properties of isometry groups of a geodesic space $M$ with inner metric of curvature at most $K$.

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Theorem 1.1. *Let $M$ be a locally compact complete geodesically complete* $\mathrm{CAT}(K)$*-space with curvature at most $K$, where $K < 0$, in which all spheres are arcwise connected. Then every bijection $f$ of the space $M$ onto itself such that $f$ and $f^{-1}$ map any closed ball of some fixed radius $r > 0$ onto some closed ball of radius $r$, is an isometry.*

This theorem generalizes the corresponding result of Guc [12] for Lobachevskii spaces and is also connected with the Beckman–Quarles characterization of isometries of finite-dimensional Euclidean spaces given in [4] (see also [8]). Kuzminykh later generalized both of these results in his paper [13].

We recall some natural concepts that arise in connection with Theorem 1.1 and its relatives.

Let $M$ be a metric space and $r > 0$ a fixed real number. The set $V \subset M \times M$ of all ordered pairs $(x, y)$ of points in $M$ satisfying $xy \leq r$ forms a *diagonal tube* about the diagonal $\Delta \subset M \times M$ and defines a reflexive and symmetric binary relation on $M$. Given a natural number $n$, we denote by $nV$ the $n$th self-composition of the relation $V$, defined inductively by the condition that $(x, z) \in (n + 1)V \subset M \times M$ if and only if there is a point $y \in M$ such that $(x, y) \in nV$ and $(y, z) \in V$. By the triangle inequality in $M$, we have $nV \subset W$, where $W$ is the diagonal tube corresponding to the number $nr$. The reverse inclusion is satisfied if $M$ is a geodesic space, but it does not hold in general. We set $0V = \Delta$. The union of the sets $nV$ over all natural numbers $n$ provides a transitive closure of the relation $V$. This closure coincides with the set $M \times M$ if $M$ is connected.

We consider here only one diagonal tube $V$. An *automorphism of a space* $(M, V)$ is a bijection $f$ of a set $M$ onto itself such that $f$ and $f^{-1}$ preserve the given relation $V$, i.e., $(x, y) \in V$ implies that $(f(x), f(y)) \in V$ and $(f^{-1}(x), f^{-1}(y)) \in V$.

The following questions arise naturally:

- Can one recover the initial metric on $M$ from $V$?
- Is every automorphism of $(M, V)$ an isometry of $M$?

It is clear that a positive answer to the first question implies a positive answer to the second question.

The second question is closely connected with the following problem, suggested by A.D. Aleksandrov in 1960s:

- Under what conditions is a map of a metric space into itself preserving a fixed distance (for example, unit distance) an isometry of this metric space?

In 1953 Beckman and Quarles proved (see [4]) that any map $f$ of an Euclidean $n$-space ($n \geq 2$) onto itself which preserves unit distance (i.e., satisfies $f(x)f(y) = 1$ if $xy = 1$) is an isometry.

Closely connected with this problem are the investigations of a student of Aleksandrov, Kuzminykh, who in [13] proved the following remarkable result:

A map $f$ of an $n$-dimensional Lobachevskii space ($n \geq 2$) into itself is an isometry if, for any fixed positive numbers $a, a'$, the following condition holds: If $xy = a$, then $f(x)f(y) = a'$. In particular, this implies that $a = a'$. Kuzminykh also proved many generalizations of this result, which we will not discuss here. In the same paper [13] a weak characterization of isometries of Euclidean $n$-space $E^n$ ($n \geq 2$) is given that generalizes the result of Beckman and Quarles and uses only a subset of measure zero of an open ball in $E^n$. All previous generalizations involved weakening the requirements on the map $f$.

In this paper we obtain a different generalization by weakening the requirements of the space itself. As far as we know, this is the first time that such a generalization had been considered.

Note that a bijection $f$ of a metric space $M$ that preserves the relation $V$ above is in general not an isometry. An example of such a bijection for the Euclidean space $E^n$ is self-similarity with a coefficient $\lambda < 1$. For the Lobachevski space with a labelled point $O$ and a number $\lambda < 1$, we can define $f$ by the relation $f(x) = x'$, where $(Ox'x)$ and $Ox' = \lambda Ox$. Clearly, this bijection preserves the relation $V$, but it is not an isometry. There are many other maps of this type.

## 2. Isometries of Aleksandrov spaces with curvature at most $K$, when $K$ is nonnegative

In this section we study isometries in CAT($K$)-spaces for nonnegative $K$.

Definition 2.1. *The radius of uniqueness of shortest arcs at a point $x$* in a (locally geodesic) metric space $M$ is defined as the least upper bound of all numbers $r$ such that for any two points $y, z$ in the open ball $U(x, r)$ there is exactly one shortest arc $[yz]$ joining these points. We denote this radius by $u(x)$. *The radius of uniqueness of shortest arcs of a space $M$* is defined as

$$u(M) := \inf\{u(x), x \in M\}.$$

The following lemma easily follows from known properties of Aleksandrov spaces with curvature (locally) bounded above, the triangle inequality, and the above definition.

Lemma 2.2. *The radius of uniqueness of shortest arcs for a locally geodesic metric space $M$ is a positive Lipshitz function* (*with constant* 1) *$u(x)$ of a point $x \in M$. If, in addition, $M$ is compact, then the number $u(M)$ is positive. In particular, this holds for any Aleksandrov space $M$ with curvature bounded above.*

Definition 2.3. *The displacement function* of an isometry $\phi$ of a metric space $M$ at a point $x$ is defined as $d(\phi, x) := x\phi(x)$.

One can also easily prove the following lemma.

LEMMA 2.4. *The displacement function $d(\phi, x)$ of an isometry $\phi$ of a metric space $M$ is a Lipshitz function (with constant 2) of a point $x \in M$. If, in addition, $M$ is compact, then this function attains its minimum $d(\phi)$.*

PROPOSITION 2.5. *Let $\phi$ be an isometry of a locally geodesic metric space $M$ whose displacement function $d(\phi, x)$ attains a positive minimum $d(\phi) := d_0$ at a point $x_0$ and such that $d_0 < u(x_0)$ (see Definition 2.1). Then $x_0$ is contained in a unique nontrivial geodesic line $\gamma(t)$, $t \in R$, which is invariant under the isometry group $\{\phi^k\}$, $k \in Z$. Moreover, $d(\phi, x) = d_0$ for every point on the geodesic $\gamma$.*

*Proof.* By the hypotheses of the proposition, the points $x_0$ and $\phi(x_0)$ are joined by a unique shortest arc $L$. Then $\phi(L)$ is the unique shortest arc joining the points $\phi(x_0)$ and $\phi^2(x_0)$. If $(x_0 x \phi(x_0))$, then $x$ lies on $L$, and by the triangle inequality,

$$x\phi(x) \leq x\phi(x_0) + \phi(x_0)\phi(x) = x_0\phi(x_0).$$

Since $x_0$ is a point of minimal displacement for $\phi$, the inequality above must be an equality (and $d(\phi, x) = d_0$). Thus, $[x\phi(x_0)] \cup [\phi(x_0)\phi(x)] = [x\phi(x)]$ is the (unique) shortest arc. Hence the union of the shortest arcs $\phi^k(L)$ over all integers $k$ forms a geodesic line $\gamma$ in $M$, which is evidently invariant under the isometry group $\{\phi^k\}$, $k \in Z$. We can parametrize $\gamma$ by an arclength parameter $t$. The last assertion of the proposition has been proved previously. $\square$

DEFINITION 2.6. We say that a one-parameter group of isometries $\{\phi_t, t \in R\}$ of a metric space $M$ *acts locally freely* on $M$ if for every point $x_0 \in M$ there exist a neighborhood $U(x_0)$ of the point $x_0$ and a number $\varepsilon > 0$, such that for any number $t$ with $0 < |t| < \varepsilon$ and any point $x \in U(x_0)$ we have $\phi_t(x) \neq x$.

PROPOSITION 2.7. *Let $\{\phi_t, t \in R\}$ be a one-parameter (continuous) group of isometries of a compact Aleksandrov space $M$ with curvature bounded above acting locally freely on $M$. Then some orbit of $\{\phi_t, t \in R\}$ in $M$ is a nontrivial geodesic $\gamma$. Moreover, the geodesic $\gamma$ is closed if and only if it is closed as a subset of $M$. More generally, the orbit of every point in the closure $\overline{\gamma}$ is geodesic, all these geodesic orbits are pairwise isometric, and $\overline{\gamma}$ is invariant under the group $\{\phi_t, t \in R\}$ which acts as a group of Clifford-Wolf translations on $\overline{\gamma}$. The set of all points whose orbit under $\{\phi_t, t \in R\}$ is geodesic is closed in $M$.*

*Proof.* Since $M$ is compact, there exists a number $\varepsilon > 0$ such that for any number $t$ with $0 < |t| < \varepsilon$ and any point $x \in M$ we have $\phi_t(x) \neq x$. Furthermore, we can assume that for the same values of $\varepsilon$ and $t$ and all points

$x \in M$ the inequality $\phi_t(x)x < u(M)$ holds (see Lemma 2.2). By Lemma 2.4, for any $t_0$ with $0 < |t_0| < \varepsilon$ there is a point $x_0$ of minimal positive displacement for the isometry $\phi_{t_0}$. By Proposition 2.5 it follows that the point $x_0$ lies on a unique geodesic $\gamma_0$ which is invariant under the isometry group $\phi_{t_0}^k$, $k \in Z$, and starts at the point $x_0$. In the same way, given any natural number $n$ and setting $t_n := t_0/2^n$, we can choose a geodesic $\gamma_n$ which starts at some point $x_n$ and is invariant under the isometry group $\phi_{t_n}^k$, $k \in Z$. Then $\gamma_n$ is also invariant under the groups $\phi_{t_m}^k$, $k \in Z$, for all natural numbers $m$ satisfying $1 \le m \le n$, since $\phi_{t_m}^k = \phi_{kt_m} = \phi_{2^{n-m}kt_n} = \phi_{t_n}^{2^{n-m}k}$.

Since $M$ is a compact Aleksandrov space with curvature bounded above, one can easily prove that any geodesic $\gamma(s)$, $s \in R$, in $M$ that is parametrized by the arclength is a shortest arc on any $s$-segment of length less than some fixed positive number $r_0$ which depends only on $M$. Evidently, a pointwise limit of a sequence of shortest arcs in any metric space is again a shortest arc. It follows from these observations that a pointwise limit of a sequence of nontrivial (parametrized) geodesics $\gamma_n(s)$, $s \in R$, in $M$ is again a nontrivial parametrized geodesic.

By the compactness of $M$, we can choose a subsequence of the sequence $x_n$ (which we will also denote by $x_n$) that converges to a point $x_\infty$. Since the map $\phi : R \times M \to M$, $\phi(t,x) = \phi_t(x)$, is continuous, it follows from the above arguments that, for all fixed numbers of the form $t = kt_m = t_0/2^m$, the points $\phi_t(x_\infty) = \lim \phi_t(x_n)$, lie on some nontrivial geodesic $\gamma$. Hence the orbit of the one-parameter isometry group $\phi_t(x_\infty)$, $t \in R$, forms an invariant nontrivial geodesic $\gamma$, for which $t$ is a parameter proportional to arclength on the geodesic $\gamma$ starting at the point $x_\infty$. The remaining assertions of the proposition are evident. □

Remark 2.8. The example of an irrational winding of the torus shows that the geodesic orbit in the above proposition is not necessarily closed and may even be dense in $M$. The next theorem shows that the situation described in this proposition is cannot occur in the case of a compact geodesically complete Aleksandrov space with negative curvature.

Theorem 2.9. *Let $M$ be a compact geodesically complete Aleksandrov space with curvature at most $K < 0$ and of topological dimension at least two. Then the full isometry group $I(M)$ of $M$ is finite, $\Gamma = \pi_1 M$ is infinite, and the centralizer $C(\Gamma)$ of subgroup $\Gamma$ in $I(\tilde{M})$ is trivial.*

Remark 2.10. We announced this theorem in [6]. We omit the proof since it follows directly from Theorem II.6.17, part (4), in the book of Bridson and Haefliger [9]. André Haefliger informed the author about their then unpublished results at the ICM 1994. Unfortunately, it was difficult to change the text of announcement [6].

## 3. Recovering the metric for some $\mathrm{CAT}(K)$-spaces, when $K$ is negative, from closed balls of given positive radius

A metric space $M$ is called *connected at infinity* if for any closed ball $B(x,r)$, $r \geq 0$, its complement with respect to $M$ is connected. We will prove the following theorem.

Theorem 3.1. *Let $M$ be a locally compact complete geodesically complete $\mathrm{CAT}(K)$-space with curvature at most $K < 0$ which is connected at infinity, and let $V \subset M \times M$ be the diagonal tube corresponding to a number $r > 0$. Then the metric of the space $M$ is uniquely determined by $V$.*

Remark 3.2. As a corollary of the so-called Cartan–Hadamard–Aleksandrov theorem, which was conjectured by M. Gromov in [11] and proved by Alexander and Bishop in [3] (see also Theorem II.4.1 in [9]), and Aleksandrov's patchwork (see [1], [2], or Proposition II.4.9 in [9]), any locally compact complete simply connected Aleksandrov space of curvature at most $K < 0$ is a $\mathrm{CAT}(K)$-space.

The shortest arcs in $M$ depend continuously on their ends, and the space $M$ is contractible. One can easily prove that any geodesic in $M$ is a shortest arc and can be extended (not necessarily uniquely) to a *line* in $M$, i.e., an isometric embedding of the whole line (with the usual metric) into $M$. In the sequel we will use these properties of the space $M$ as well as the assumptions of Theorem 3.1 without explicit reference to them.

We need the following proposition.

Proposition 3.3. *Under the assumption of the other conditions in Theorem 3.1, the condition of connectedness at infinity is equivalent to the condition of arcwise connectedness of all spheres $S(x,r')$, $r' > 0$, $x \in M$.*

*Proof.* Note that $M$ is locally arcwise connected. Thus any open subset $U \subset M$ is connected if and only if it is arcwise connected.

Let $M$ be connected at infinity. Then the complement of any closed ball $B(x,r')$ is arcwise connected. Let $x_1$ and $x_2$ be arbitrary points in $S(x,r')$. Extend the shortest arcs $[xx_1]$ and $[xx_2]$ to longer shortest arcs $[xx_1']$ and $[xx_2']$ and join the points $x_1'$ and $x_2'$ with a path $x(t)$, $0 \leq t \leq 1$, in the complement of $B(x,r')$. The shortest arc $[xx(t)]$, $0 \leq t \leq 1$, intersects $S(x,r')$ at a unique point $y(t)$, $0 \leq t \leq 1$. As a corollary of the global $K$-concavity condition (see [7]), $y(t)$, $0 \leq t \leq 1$, is a (continuous) path in $S(x,r')$ that joins the given points $x_1, x_2 \in S(x,r')$. Thus $S(x,r')$ is (arcwise) connected, as required.

The proof of the converse statement is much simpler. □

Lemma 3.4. *Let $[xy]$ be a shortest arc in $M$ of length $r_1 > 0$, and let $r_2$ and $r_3$ be positive numbers such that the largest of the three numbers $r_1$, $r_2$,* *and $r_3$ does not exceed the sum of the other two numbers. Then there exists a point $z \in M$ such that $xz = r_2$ and $yz = r_3$.*

*Proof.* We can assume that the maximum of the numbers $r_1, r_2, r_3$ is strictly less than the sum of the other two. (Otherwise the assertion is obvious.) In this case some inner point $x_0$ of the shortest arc $[xy]$ lies inside the intersection

$$C := B(x, r_2) \cap B(y, r_3),$$

and there exists an extension $[x'y']$ of the shortest arc $[xy]$ (note that $x'$ or $y'$ may coincide with $x$ or $y$) such that

$$y' \in B(y, r_3) - B(x, r_2), \quad x' \in B(x, r_2) - B(y, r_3).$$

As a corollary of the global $K$-concavity condition, every closed ball $B$ in $M$ with a positive radius is strictly convex, i.e., for any two distinct points $a, b \in B$, the (relative) interior of the shortest arc $[ab]$ is contained in $B$. Hence the set $C$ above is also strictly convex.

Take any point $w$ outside of $B(x, r_2) \cup B(y, r_3)$. Since $M$ is connected at infinity, the points $x'$ and $w$ can be joined by a path $p_1$ outside of $B(y, r_3)$, while the points $w$ and $y'$ can be joined by a path $p_2$ outside of $B(x, r_2)$. The concatenation $p$ of paths $p_1, p_2$ (at the point $w$) goes outside of $C$.

It follows from the strict convexity of the set $C$ that for any point $a$ in the image of $p$ the shortest arc $[x_0 a]$ intersects the boundary of the set $C$ at a unique point $a'$. By the $K$-concavity, the point $a'$ depends continuously on $a$. As a result, we get a (continuous) path $p'$ in the boundary of the set $C$. It follows from the choice of $x'$ and $y'$ that the path $p'$ starts at $S(y, r_3)$ and ends at $S(x, r_2)$. Hence some point $z$ in the image of $p'$ is contained in the intersection $S(x, r_2) \cap S(y, r_3)$. This point $z$ has the desired properties. □

We will later need a special description of the topological completion $\overline{M} = M \cup hb(M)$ of a complete CAT(0)-space $M$ in terms of Busemann (or ray) functions which induces on $M$ the initial metric topology. We denote by $\mathcal{C}(M)$ the topological vector space of continuous real-valued functions on $M$ equipped with the topology of uniform convergence on bounded subsets. This topology can be defined by a sequence of sup-norms, whose $n$th member is the sup-norm on the closed ball of radius $n$ with a fixed center $p \in M$. In particular, if $M$ is finitely compact, then this is the more familiar topology of uniform convergence on compact subsets. Let $p$ be a given fixed point in $M$. There is a natural topological embedding $i_p : M \to \mathcal{C}(M)$ obtained by associating to each $x \in M$ the shifted distance function $d_{p,x} : y \to xy - xp$. Then $\overline{M}$ can be identified with the closure of $i_p(M)$ in $\mathcal{C}(M)$. This definition does not depend on the choice of the fixed point $p$. The complement to the set $i_p(M)$ in $\overline{M}$ is called the *ideal* or *hyperbolic* boundary, and we denote it by $hb(M)$. Any ideal element in $hb(M)$ is the *Busemann* or *ray* function, corresponding to exactly one *ray* $\gamma(t)$, $t \in [0, +\infty)$, with origin $p$ (i.e., we have $\gamma(t_1)\gamma(t_2) = |t_1 - t_2|$ and $\gamma(0) = p$). The Busemann function of such a ray is defined on $M$ by the equation

$$f_\gamma(x) := \lim_{t\to+\infty}(t - x\gamma(t)), \tag{3.1}$$

where the limit on the right side is uniform on every bounded subset in $M$ (see [11] or [9]). (We have changed the sign here, as this will be convenient later.)

A level set of a function $f_\gamma$ is called a *horosphere*. Any two geodesic rays in $M$ have the same limit (ideal) points $\lim_{t\to+\infty}\gamma(t)$ if and only if the corresponding Busemann functions differ by a constant (see [9]) and thus have the same family of horospheres if one ignores their levels. If we change the fixed point $p$, then the ray corresponding to a given ideal element in $hb(M)$ will change, but the corresponding Busemann functions will only change by a constant depending on this ideal element. If $M$ is finitely compact, then the space $\overline{M}$ is compact.

In [9] it was shown that this definition is equivalent to another definition which involves the cone topology. If $M$ is a CAT$(K)$-space for some $K < 0$, then this definition of $\overline{M}$, which is mainly due to M. Gromov, coincides up to a homeomorphism with Gromov's definition using a scalar product (see [11]).

Proposition 3.5. *Let $M$ be a locally compact complete geodesically complete* CAT(0)*-space. Then the metric topology $\tau_m$ on $M$ is equal to the initial topology $\tau_f$ relative to the family of all ray functions on $M$.*

*Proof.* Since any ray function is continuous on $M$, we have $\tau_f \subset \tau_m$. To prove the reverse inclusion, we need to show that every open ball $U(x,s)$, where $x \in M$ and $s > 0$, contains a neighborhood of the point $x$ which is a finite intersection of preimages of intervals for ray functions.

Consider the family $\Gamma$ of all Busemann functions for rays with origin $x$. Let $u$, $0 < u < s$, be an arbitrary real number. Any shortest arc with origin $x$ can be extended to a ray (or even line) because $M$ is geodesically complete. Then it is not hard to prove that

$$B(x,u) = \bigcap_{\gamma\in\Gamma} F(\gamma,u) \subset U(x,s), \tag{3.2}$$

where $F(\gamma,u) := f_\gamma^{-1}([-u,u])$. The space $M$ is finitely compact by the Cohn-Vossen Theorem. Thus $\overline{M} = hb(M) \cup M$ is compact. Define $G(\gamma,u)$ to be the closure of $F(\gamma,u)$ in $\overline{M}$. Then $G(\gamma,u)$ is compact, and $G(\gamma,u) - F(\gamma,u) \subset hb(M)$. Also $a \notin G(\gamma_-,u)$ if a ray $\gamma$ ends at $a \in hb(M)$, where $\gamma_-$ is a ray with origin $x$ opposite to $\gamma$, in the sense that $\gamma \cup \gamma_-$ gives a line in $M$. Since any point $a \in hb(M)$ is an end point of a ray $\gamma \in \Gamma$, it follows from these observations that the same relation (3.2) holds with $G(\gamma,u)$ in place of $F(\gamma,u)$.

From the above argument it follows that there is a finite set $\{\gamma_1, ..., \gamma_k\} \subset \Gamma$ such that

$$G(\gamma_1, u) \cap ... \cap G(\gamma_k, u) \subset U(x, s),$$

because the sets $G(\gamma, u)$ are compact and $U(x, s)$ is open (see Corollary 3.15 in [10]). Thus

$$x \in f_{\gamma_1}^{-1}(-u, u) \cap ... \cap f_{\gamma_k}^{-1}(-u, u) \subset U(x, s),$$

which gives the desired neighborhood of $x$. □

*Proof of Theorem 3.1.* Any ordered triple of points $(x, y, w)$ in $M$ satisfying

$$xy = yw = r, \quad xw = 2r \tag{3.3}$$

is characterized by the property that for each pair $x, w$ there exists a unique point $y$ such that $(x, y) \in V$ and $(y, w) \in V$. This follows from the fact that any two points in $M$ are joined by a unique shortest arc. This implies that the relation $V$ determines all spheres and open balls in $M$ of radius $r$.

Consider all bilaterally infinite sequences of points $\{x_z\}$, $z \in Z$, indexed by the integers, such that for every $z \in Z$ the triple $(x, y, w) = (x_{z-1}, x_z, x_{z+1})$ satisfies condition (3.3). We will call such a sequence an *$r$-sequence*. Clearly, $r$-sequences can be defined by means of the relation $V$ alone.

Any $r$-sequence $\{x_z\}$ together with the shortest arcs $[x_z x_{z+1}]$, $z \in Z$, gives a unique oriented line $L$. We say an $r$-sequence $\{x_z\}$ is *line-equivalent* to another $r$-sequence $\{x'_z\}$ if the oriented lines defined by these sequences coincide. This holds if and only if $\{x_z\}$ and $\{x'_z\}$ have the same limit points $x_{+\infty} = x'_{+\infty} \in hb(M)$ and $x_{-\infty} = x'_{-\infty} \in hb(M)$, both when $z \to +\infty$ and $z \to -\infty$. Note that all four of these limit points exist. The necessity is evident. The sufficiency follows from the Flat Strip Theorem (Theorem II.2.13 in [9]).

Also, any two $r$-sequences $x_z, x'_z$ are line-equivalent if and only if there exists an integer $k$ such that

$$(x_z, x'_{z+k}) \in V \tag{3.4}$$

for all integers $z$. The necessity is again evident. The sufficiency follows from the above considerations because any two $r$-sequences with this condition have the same limit points in $hb(M)$, both at $+\infty$ and at $-\infty$.

Any oriented line $L \subset M$ is a union of all points in all $r$-sequences from some line-equivalence class. The orientation of $L$ is defined by the order of any $r$-sequence from this class. Thus we can define any oriented line $L$ using only the relation $V$. Note also that the (induced) topology on any oriented line $L$ can be defined by means of $V$. Indeed, suppose that the conditions $x < y < w$ and $xw < r$ are satisfied for an ordered triple of points on an oriented line $L$. These conditions can be expressed by means of $V$ in the following form: $(x, y), (y, w), (x, w) \in V$. There is no $r$-sequence which contains both $x$ and

$w$; for any $r$-sequences $x_z$, $y_z$, $w_z$ satisfying $x_0 = x, y_0 = y, w_0 = w$ from the line-equivalence class defined by $L$ we must have

$$(y_{-1}, x_0) \in V, \quad (y_{-1}, w_0) \notin V, \quad (y_1, x_0) \notin V, \quad (y_1, w_0) \in V.$$

It remains to prove that one can find the midpoint of the segment $[x_z x_{z+1}]$ for some $r$-sequence $\{x_z, z \in Z\}$ in a line-equivalence class defined by an oriented line in $M$.

Let $\gamma(t)$, $t \in R$, be a parametrization of an oriented line $L$ by the arclength, i.e., $\gamma(t_1)\gamma(t_2) = |t_1 - t_2|$ and $\gamma(t_1) < \gamma(t_2)$ if $t_1 < t_2$. Then the formula (3.1) defines the corresponding Busemann function $f_\gamma$, which we will also denote by $f_L$, and the corresponding horospheres.

We can define the horosphere $h(L, x_0)$ defined by an oriented line $L$ and passing through a given point $x_0 \in L$ using only the relation $V$. There exists a unique $r$-sequence $\{x_z, z \in Z\}$ in the line-equivalence class of $L$ with initial term $x_0$. Then, for any natural number $n$, the open ball $U(x_n, nr)$ is the set of all points $y$ in $M$ such that $(x_n, y) \in nU$, where $(x, y) \in U$ if and only if $xy < r$. If $\gamma(t)$, $t \in R$, is the parametrization of $L$ by arclength such that $\gamma(0) = x_0$, then $f_\gamma^{-1}(0, +\infty)$ is an open set in $M$, which is equal to the union of all balls $U(x_n, nr)$. We will use the description of $h(L, x_0)$ as the boundary of this set to show that $V$ determines $h(L, x_0)$.

Consider the family of all oriented geodesics $L'$ which have a common (right) end $a \in hb(M)$ with $L$. Any such geodesic $L'$ is characterized by the property that for any $r$-sequence $\{x'_z, z \in Z\}$ in the line-equivalence class of $L'$ there exist integers $k, m$ such that relation (3.4) is satisfied for all $z \geq m$ (see [9]). Thus this family of oriented geodesics can be defined by $V$. Any point $x \in M$ lies on some line $L'$ of this type. Then the ray in $L'$ which starts at $x$ and ends at $a$ is the unique ray in $M$ with these ends (see [9]). We have

$$|f_\gamma(x) - f_\gamma(x')| = xx'; \quad x, x' \in L', \tag{3.5}$$

and $f_\gamma(x) < f_\gamma(x')$ if $x < x'$. Thus $x \in L'$ lies on $h(L, x_0)$ if and only if $x$ is the unique boundary point of the open infinite interval (ray) $f_\gamma^{-1}(0, +\infty) \cap L'$. Since, as we showed above, the induced topology on $L'$ is defined entirely by the relation $V$, any point $x \in h(L, x_0)$ can be defined by $V$. Thus the horosphere itself is defined by $V$. Therefore, by Proposition 3.5, the relation $V$ defines the metric topology on $M$.

By equation (3.5), if $\{x_z, z \in Z\}$ is an $r$-sequence in $L$ and $x'_z$ is the unique point in $L' \cap h(L, x_z)$, then $\{x'_z, z \in Z\}$ is also an $r$-sequence (in $L'$).

Let now $\{x_z, z \in Z\}$ be any $r$-sequence in $L$. By Lemma 3.4, corresponding to the points $x = x_{-n}$ and $y = x_n$ there exists a point $z = z_n \in M$ such that $z_n x_{-n} = z_n x_n = x_{-n} x_n$, i.e., there exists an equilateral triangle $\Delta = \Delta_n$ with sides $2nr$. For some number $\delta > 0$, the space $M$ is $\delta$-hyperbolic. Hence the so-called *insize* of $\Delta$ is at most $\delta$ (see [11]). Therefore we have the inequalities

$$w_{-n} x_0 \leq \delta, \quad w_n x_0 \leq \delta, \tag{3.6}$$

where $w_{-n}$ (respectively $w_n$) is the midpoint of the side $[z_n x_{-n}]$ (respectively $[z_n x_n]$). Since $M$ is finitely compact and the inequalities (3.6) are satisfied, we can suppose that a subsequence $w_{-n_k}$ (respectively $w_{n_k}$) converges to a point $x_0^- \in M$ (respectively $x_0^+ \in M$) and the oriented geodesic segment $[z_n x_{-n}]$ (respectively $[z_n x_n]$) converges in the compact-open topology (relative to $R$ and $M$) to some oriented geodesic line $L_-$ (respectively $L_+$). This convergence can be described via the metric topology on $M$, and hence via the relation $V$, as we have shown previously.

As a result, the oriented line $L_+$ has a common end point $a \in hb(M)$ with $L$, and $L_-$ has a common starting point $b \in hb(M)$ with $L_+$, while the end point $c \in hb(M)$ of the line $L_-$ coincides with the origin of the line $L$. We can assume that the oriented geodesic $L_-$ (respectively $L_+$) starts at the point $x_0^-$ (respectively $x_0^+$).

Construct the horospheres $h(-L, x_z)$, $z \in Z$, where $-L$ denotes the oriented line $L$ endowed with the opposite orientation. The intersections of these horospheres with the line $L_-$ define a unique $r$-sequence $x_{-z}^-$ of the oriented line $L_-$, as we have shown above. Similarly, the intersections of the horospheres $h(L, x_z)$ with the line $L_+$ define a unique $r$-sequence $x_z^+$ of the oriented line $L_+$. It follows from the construction of $L_-$ and $L_+$ and Proposition II.8.19 in [9] (or, rather, by the definition of $\overline{M}$ which we assume here) that the sequence of functions $-(d_{z_{n_k}} - n_k r)$ converges, uniformly on any compact subset in $M$, simultaneously to the function $f_{-L_-}$ and to the function $f_{-L_+}$. Thus we have $x_z^- \in h(-L_+, x_z^+)$.

Take any point $x \in L$ such that $x_0 < x < x_1$ and then define the points

$$x^+ := h(L, x) \cap L_+, \quad x^- := h(-L_+, x^+) \cap L_-.$$

Evidently, we have $x_0^+ < x^+ < x_1^+$ on the line $L_+$ and $x_0^- < x^- < x_1^-$ on the line $L_-$. Thus, $x^- \in h(-L, y)$ for some point $y$, $x_{-1} < y < x_0$ on $L$. Clearly,

$$yx = yx_0 + x_0 x, \quad yx_0 = x_0 x.$$

It follows from this that a point $x$ is the midpoint of the segment $[x_0 x_1]$ if and only if the corresponding point $y$, which is defined only by means of $V$ and $x$, lies in the $r$-sequence in $L$ with point $x$. The last condition can also be defined by $V$.

We define a new binary relation $W$ in terms of $V$ by setting $2W = V$. Then $(x, y) \in W$ if and only if $xy \leq r/2$. By induction, for any natural number $n$ we define a binary relation $U$ by setting $2^n U = V$, so that $(x, y) \in U$ if and only if $xy \leq r/2^n$. Since we can define, by means of $V$, the induced topology on any (oriented) line in $M$, we can define, using $V$, the original distance on the metric space $M$. This completes the proof of the theorem. □

*Proof of Theorem 1.1.* The theorem follows immediately from Theorem 3.1. □

Remark 3.6. We can replace the relation $V$ in Theorem 3.1 by the binary relation $S = \{(x, y) \in M \times M : xy = r\}$, since in the presence of the other conditions on $M$, the relation $V$ defines $S$, while $2S = 2V = \{(x, y) \in M \times M : xy = 2r\}$.

The following is a natural question:

Question. *Are Theorems 3.1 and 1.1 (as well as the theorem of Beckman–Quarles [4]) true in the case when $K = 0$?*

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