Illinois Journal of Mathematics
Volume 46, Number 2, Summer 2002, Pages 405–419
S 0019-2082

# INTERPOLATION OF WEIGHTED $L^1$ SPACES—A NEW PROOF OF THE SEDAEV-SEMENOV THEOREM

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ABSTRACT. A new simpler proof is given of the theorem of Sedaev-Semenov that the couple $(L^1_{w_0}, L^1_{w_1})$ of weighted $L^1$ spaces on an arbitrary measure space is a *Calderón couple*, i.e., all interpolation spaces with respect to this couple can be described in terms of the $K$-functional. This theorem has other important consequences. It is a component in an alternative proof of the Brudnyi-Krugljak $K$-divisibility theorem. Also, as shown by Dmitriev, it leads readily to a proof of Sparr's more general result that $(L^p_{w_0}, L^q_{w_1})$ is a Calderón couple.

## 1. Introduction

Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space. We shall refer to all measurable functions $w\colon \Omega \to (0,\infty)$ as *weight functions*. For each $p \in [1,\infty)$ and each weight function $w$, the space $L^p_w = L^p_w(\Omega)$ is defined to consist of all (equivalence classes) of measurable functions $f\colon \Omega \to \mathbb{C}$ for which the norm $\|f\|_{L^p_w} := (\int_\Omega (|f|w)^p d\mu)^{1/p}$ is finite. If for $j = 0,1$ the functions $w_j : \Omega \to (0,\infty)$ are weight functions, then the couple $\mathbf{L}^p = (L^p_{w_0}, L^p_{w_1})$ forms a Banach couple in the sense of interpolation theory (see, e.g., [7], [5], [3], or [26]). The couples $\mathbf{L}^p$ have been studied extensively; see, e.g., [15], [18], [19], [20], [22], [23], [24]. It is easy to find an equivalent expression for the $K$-functional for the couple $\mathbf{L}^p$, and in the case $p = 1$ this becomes the following (well known) exact formula:

$$K(t, f; \mathbf{L}^1) = \int_\Omega \min(w_0, tw_1)|f|d\mu$$

for each $f \in L^1_{w_0} + L^1_{w_1}$ and each $t > 0$.

The following theorem was proved by Sedaev and Semenov [23].

Received June 27, 2001; received in final form November 5, 2001.
2000 *Mathematics Subject Classification.* Primary 46B70, 46E30.
Research supported by the Technion V.P.R. Fund – R. and M. Rochlin Research Fund.

Theorem 1.1. *Suppose that $f$ and $g$ are functions in $L^1_{w_0} + L^1_{w_1}$ which satisfy*

$$K(t, g; \mathbf{L}^1) \le K(t, f; \mathbf{L}^1)$$

*for all $t > 0$. Then for each $\epsilon > 0$ there exists a bounded linear operator $T\colon L^1_{w_0} + L^1_{w_1} \to L^1_{w_0} + L^1_{w_1}$ whose restriction to $L^1_{w_j}$ maps into $L^1_{w_j}$ with norm not exceeding $1 + \epsilon$ for $j = 0, 1$, and such that $Tf = g$.*

Theorem 1.1 plays a rather more central rôle in interpolation theory than was realized at first. This has prompted us to seek an alternative simpler proof, which is the main contribution of this paper.

Initially Theorem 1.1 was proved in order to enable all interpolation spaces of the couple $\mathbf{L}^1$ to be described in a relatively simple way. An analogous theorem enabling an analogous description of the interpolation spaces of the couple $(L^1, L^\infty)$ had been obtained a few years earlier by Calderón [8]. A related result was obtained by Mityagin [17]. For details we refer to [23]; general discussions of couples which satisfy theorems similar to Theorem 1.1 can be found in, e.g., [7] or [13]. These couples are often called *Calderón couples*. (Many authors use alternative names, such as *Calderón-Mityagin couples*, *$K$-adequate couples*, *$K$-monotone couples*, or *$C$-couples*.)

However, Theorem 1.1 has turned out to have consequences beyond the study of the particular couple $\mathbf{L}^1$. It is one of the components in an alternative proof given in [11] of the important $K$-divisibility theorem of Brudnyi-Krugljak [6], [7]. We mention that Bennett and Sharpley have presented another variant of this approach to proving $K$-divisibility (see [3, p. 326-328] or [4]) in which the rôle of the Sedaev-Semenov theorem is played instead by a theorem of Lorentz and Shimogaki (which is, in fact, closely related to the analogue of the Sedaev-Semenov theorem for the couple $(L^1, L^\infty)$ obtained by Calderón [8]).

Theorem 1.1 has been generalized to many other Banach couples $\mathbf{A} = (A_0, A_1)$. However, in many of these generalizations, while the norms $\|T\|_{A_j \to A_j}$ for $j = 0, 1$ can be bounded by absolute constants depending only on the couple, it has not been shown that these constants can be taken arbitrarily close to 1.

Among these generalizations let us mention the cases of the couples $\mathbf{L}^p$ for $1 \le p \le \infty$ which were treated by Sedaev [22]. In turn, Sedaev's results were generalized by Gunnar Sparr [24] who treated the couples $(L^{p_0}_{w_0}, L^{p_1}_{w_1})$ for all $p_0, p_1 \in [1, \infty]$. Sparr's theorem is, in a certain sense, the strongest possible result of this kind for couples of Banach lattices. More specifically, as shown in [13], if $X$ and $Y$ are Banach lattices of measurable functions on some measure space and if, for *all* choices of weight functions $w_0$ and $w_1$, the couple of weighted lattices $(X_{w_0}, Y_{w_1})$ is a Calderón couple, then $X$ and $Y$ must both be (weighted) $L^p$ spaces.

Here again Theorem 1.1 turns out to be unexpectedly important. As shown by V. I. Dmitriev [14], there is a relatively simple argument which enables the result of Sparr to be deduced from Theorem 1.1 for all $p_0$ and $p_1$ in $[1, \infty)$.

In fact, the above-mentioned results from [8], [22] and [24], and some generalizations of these results, can also be obtained via different $K$-divisibility type arguments, as shown in [11]. However, this approach gives weaker estimates for the norms of operators $T$ appearing in the analogues of Theorem 1.1.

Another alternative proof of Sparr's theorem for the couple $(L^{p_0}_{w_0}, L^{p_1}_{w_1})$ can be found in [2, pp. 255–264]; see the remark on p. 256 of [2]. However, this proof does not apply to the cases where $p_0 = p_1$. Thus yet another reason for presenting a new proof is to provide the analogue of the alternative and perhaps simpler approach used in [2] for the missing cases $p_0 = p_1 = p$. Here we only deal explicitly with the case where $p = 1$, but this case indicates how to handle the remaining cases. As remarked by Sedaev [22], the case where $p = \infty$ is a simple exercise. The result for $p \in (1, \infty)$, also originally due to Sedaev, can be obtained by straightforward variants of our approach here. It can also be deduced from the case $p = 1$, either by the method of [14] mentioned above, or by a different method to be presented in [12]. Other approaches to proving Theorem 1.1 and its generalization in [22] can be found in [9] and in [16]. But they also apparently give weaker estimates for the norms of the operator. When $1 < p < \infty$, none of these approaches shows that the norms of the operator $T$ can be bounded by constants arbitrarily close to 1. However, recently, Yacin Ameur [1] has shown, using very different methods, that this is true when $p = 2$. Thus we now know that such "exact" results hold for the couple $(L^{p_0}_{w_0}, L^{p_1}_{w_1})$ when $p_0 = p_1$ equals 1 or 2 or $\infty$, and also in the case when $p_0 = 1$ and $p_1 = \infty$. In view of Sparr's examples [24, pp. 254–256], analogues of these "exact" results do not hold when $p_0 = 1$ and $p_1 \in (1, \infty)$.

One advantage of Sparr's proofs over the alternative ones given here and in [14] and [2] is that they also apply to the quasi-Banach case; i.e., in [24] $p_0$ and $p_1$ can take values in the extended range $(0, \infty]$, provided certain assumptions are made on the underlying measure space. (For an analogous result for the couple of sequence spaces $(\ell^p, \ell^\infty)$ for $p < 1$ see also [10, pp. 129–132].)

Remark 1.1. In [2] the weight functions $w_0$ and $w_1$ are both taken to be identically 1, but this is not really a restriction: When $p_0 \neq p_1$, a very simple "change of variables", which was originally introduced for other purposes by Stein and Weiss [25], immediately extends the result to the case of general weights; cf. [9, Corollary 2, p. 234].

Sections 2 and 3 of this paper contain our new proof of Theorem 1.1. In the original proof in [23] the first step was to obtain an operator in the special case where the measure space consists of finitely many atoms. This was done by solving various systems of linear equations and identifying the extreme points of certain convex sets. The second step used a compactness argument based on results of Sedaev [21] to generalize to the case where the measure space consists of countably many atoms. As Boris Begun informed us in a private communication, it is possible to give an alternative proof of this second step using Banach limits (somewhat analogously to the proofs in [8]). The straightforward third step of the proof was to extend from the case of countably many atoms to an arbitrary measure space.

In our proof here we give a "graphical" argument in Section 2, which simultaneously supplies the above-mentioned first and second steps without requiring the compactness techniques of [21] and without using Banach limits. It also gives better (and, in fact, optimal) estimates for the norms of the operator. In Section 3, for completeness and for the reader's convenience, we also present a proof for the third step.

We observe (this was apparently also clear from the original proof) that if the functions $f$ and $g$ are non-negative, then the operator $T$ appearing in Theorem 1.1 can be chosen to be a *positive operator*, i.e., $Th$ is a non-negative function whenever $h \in L^1_{w_0} + L^1_{w_1}$ is non-negative. This positivity property is needed if one wishes to use Dmitriev's method [14] to extend Theorem 1.1 to other couples.

Finally, in Section 4, we present a refinement of Theorem 1.1 showing that we can, in fact, take $\epsilon = 0$, i.e., we can assume that the operator $T$ has norm not exceeding 1 on $L^1_{w_j}$ for $j = 0, 1$. Here we do need to use Banach limits, and we also need the positivity property of $T$ mentioned above.

The first author to use a graphical approach in such problems was apparently Sparr [24]. His matrix lemmata, which are important tools for obtaining his wide generalizations of the Sedaev-Semenov theorem, are proved using an examination of the graphs of certain piecewise linear (i.e., piecewise affine) functions to guide the construction of the required matrices or operators (see pp. 260–270 of [24]). Something rather like this is also done in certain steps of the proofs given here and in [2]. These similarities suggest that it is perhaps possible to further simplify or shorten the proofs here or in [2] by directly incorporating some parts of Sparr's arguments. However, we have not been able to do this.

## 2. The main step of the proof: the case of an atomic measure space

We shall use a special measure space $(\Omega^*, \Sigma^*, \mu^*)$ defined by setting $\Omega^* = (0, \infty)$, letting $\Sigma^*$ consist of *all* subsets of $\Omega^*$ and taking $\mu^*$ to be counting measure $\mu^*(E) = \mathrm{card}(E)$. We shall also use the special weight functions $w_0^*(x) = x$ and $w_1^*(x) = 1$. Let $\mathbf{L}^1_*$ denote the couple $(L^1_{w_0^*}(\Omega^*), L^1_{w_1^*}(\Omega^*))$.

We present the main step of our proof of Theorem 1.1 separately as the following theorem.

THEOREM 2.1. *For each fixed number $r > 1$, Theorem 1.1 holds for the case of the measure space $(\Omega^*, \Sigma^*, \mu^*)$ and the weights $w_0^*$ and $w_1^*$, and for all real-valued non-negative functions $f$ and $g$ which are zero except possibly at the points $r^n$, $n \in \mathbb{Z}$. Furthermore, the operator $T$ mapping $f$ to $g$ can be constructed so that for $j = 0, 1$ its norm as a map from $L^1_{w_j^*}(\Omega^*)$ into itself does not exceed 1.*

*Proof.* We first need to establish a few elementary properties of the couple $\mathbf{L}^1_*$ and its associated $K$-functional. Let $u$ be an arbitrary function in $L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*) = L^1_{\min(w_0^*, w_1^*)}(\Omega^*)$. Then the set $\{t > 0 : u(t) \neq 0\}$ is finite or countable. We denote this set by $\{\tau_n\}_{n \in E}$ for some subset $E$ of $\mathbb{Z}$. The $K$-functional of $u$ is given by

$$(1)\quad K(t, u; \mathbf{L}^1_*) = \sum_{\tau_n < t} \tau_n |u(\tau_n)| + t \sum_{\tau_n \geq t} |u(\tau_n)| = \sum_{\tau_n \leq t} \tau_n |u(\tau_n)| + t \sum_{\tau_n > t} |u(\tau_n)|$$

for each $t > 0$. Like all $K$-functionals it is a continuous concave function of $t$.

We claim that for each open interval $I = (\alpha, \beta) \subset (0, \infty)$ and each $u \in L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$ the function $U(t) := K(t, u; \mathbf{L}^1_*)$ is affine on $I$ if and only if $u(t) = 0$ for all $t \in I$.

If $u$ vanishes on $I$, then the fact that $U$ is affine on $I$ is obvious from (1). Intuitively, the converse seems almost as obvious. To prove it rigorously we use the inequality

$$(2)\qquad \sum_{\tau_n \geq t} |u(\tau_n)| \leq \frac{U(t) - U(s)}{t - s} \leq \sum_{\tau_n \geq s} |u(\tau_n)|,$$

which holds whenever $0 < s < t$. (To establish (2) simply observe that

$$U(t) - U(s) = \sum_{s \leq \tau_n < t} \tau_n |u(\tau_n)| + t \sum_{\tau_n \geq t} |u(\tau_n)| - s \sum_{\tau_n \geq s} |u(\tau_n)|,$$

and estimate this expression from above (respectively below) by replacing all coefficients $\tau_n$ in the first sum by $t$ (respectively $s$).) Let $a$, $b$, $c$ and $d$ be arbitrary numbers in $I$ such that $\alpha < a < b < c < d < \beta$. Then from (2) we obtain that

$$(3)\qquad \frac{U(d) - U(c)}{d - c} \leq \sum_{\tau_n \geq c} |u(\tau_n)| \leq \sum_{\tau_n \geq b} |u(\tau_n)| \leq \frac{U(b) - U(a)}{b - a}.$$

If $U$ is affine on $I$ then the four expressions compared in (3) are all equal. This implies that $u$ vanishes on $[b, c)$, and consequently on all of $I$, proving our claim.

It will be convenient to state the following property of the $K$-functional for the couple $\mathbf{L}^1_*$ as a separate lemma.

Lemma 2.2. *Let* $u \in L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$ *and let* $U\colon (0,\infty) \to (0,\infty)$. *Let* $I \subset (0,\infty)$ *be an open interval such that* $U(t) = K(t, u; \mathbf{L}^1_*)$ *for all* $t \in I$. *Suppose further that* $I$ *is the union of finitely many non-overlapping intervals, on each of which* $U$ *is an affine function. Then*

$$|u(t)| = U'(t-) - U'(t+) \text{ for all } t \in I. \tag{4}$$

The proof of this lemma is immediate since, by the above claim, $u(t)$ must vanish in $I$ except at those points $t$ where the graph of $U$ changes its slope. It is clear from (1) that (4) holds at these points and indeed at all points of $I$.

We next describe some rather simple operators $S_{\alpha B\gamma}$, which will play an important rôle in our proof.

Let $0 \le \alpha < \gamma \le \infty$. Let $B = \{\beta_m\}_{m\in M}$ be a finite or countable set of points in $(\alpha, \gamma)$. For each $m \in M$, let $\delta_m = (\gamma - \beta_m)/(\gamma - \alpha)$ if $\gamma < \infty$, and $\delta_m = 1$ if $\gamma = \infty$. For each function $h$ in $L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$ we define $S_{\alpha B\gamma}h$ by

$$\begin{aligned}
S_{\alpha B\gamma}h(\beta_m) &= 0 \text{ for all } m \in M,\\
S_{\alpha B\gamma}h(\alpha) &= h(\alpha) + \sum_{m\in M} \delta_m h(\beta_m) \text{ if } \alpha > 0,\\
S_{\alpha B\gamma}h(\gamma) &= h(\gamma) + \sum_{m\in M} (1-\delta_m) h(\beta_m) \text{ if } \gamma < \infty,\\
S_{\alpha B\gamma}h(t) &= h(t) \text{ for all } t \in (0,\infty)\backslash\{\alpha\}\backslash\{\gamma\}\backslash B.
\end{aligned}$$

It is straightforward to check that the linear operator $\mathrm{S}_{\alpha B\gamma}$ has the following properties:

(i) For $j = 0, 1$, if $h \in L^1_{w_j^*}(\Omega^*)$ then $S_{\alpha B\gamma}h \in L^1_{w_j^*}(\Omega^*)$ and

$$\|S_{\alpha B\gamma}h\|_{L^1_{w_j^*}(\Omega^*)} \le \|h\|_{L^1_{w_j^*}(\Omega^*)}.$$

(ii) If $h$ is non-negative then $S_{\alpha B\gamma}h$ is also non-negative and

$$K(t, S_{\alpha B\gamma}h, \mathbf{L}^1_*) = K(t, h, \mathbf{L}^1_*) \quad \text{for all } t \in (0,\infty)\backslash[\alpha,\gamma]. \tag{5}$$

Let us now fix two arbitrary non-negative functions $f$ and $g$ in $L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$, which are zero except possibly at the points $r^n$, $n \in \mathbb{Z}$, and which satisfy $K(t, g; \mathbf{L}^1_*) \le K(t, f; \mathbf{L}^1_*)$ for all $t > 0$. It will be convenient to use the notation $F(t) := K(t, f; \mathbf{L}^1_*)$ and $G(t) := K(t, g; \mathbf{L}^1_*)$.

We can now make our remarks in the introduction about using a "graphical" approach more explicit: We shall, much like in [2], and similarly to some steps in [24], use a sequence of operators of the form $S_{\alpha B\gamma}$ to, in some sense, successively "slice off" segments of the graph of $F$ until it coincides with the graph of $G$ . It will turn out that the composition of this sequence of operators also defines the required operator $T$ which satisfies $Tf = g$.

We consider the collection $\mathcal{G}$ of all intervals $[c, d]$ such that

(6a) $c, d \in [0, \infty]$,

(6b) $g(t) = 0$ for all $t \in (c, d)$,

(6c) either $g(c) > 0$ or $c = 0$,

(6d) either $g(d) > 0$ or $d = \infty$.

Of course each such $c$ and each such $d$ is either of the form $r^n$ for some $n \in \mathbb{Z}$ or is 0 or $\infty$. Clearly we can list all the elements of $\mathcal{G}$ as a finite or infinite sequence $\{[c_k, d_k]\}_{1 \leq k < \nu}$, for some $\nu \in [2, \infty]$. They are of course non-overlapping and their union contains or equals $(0, \infty)$. More precisely, for each $n \in \mathbb{N}$ there exists an integer $\kappa(n) \in [1, \nu)$ such that

$$[r^{-n}, r^n] \subset \bigcup_{k=1}^{\kappa(n)} [c_k, d_k]. \tag{7}$$

It will be convenient to use the notation $J_n = [r^{-n}, r^n]$ for these intervals.

For each integer $k \in [1, \nu)$ it follows from (6b) that $G(t)$ coincides with an affine function on $[c_k, d_k] \cap (0, \infty)$. Let us denote this function by $L_k(t) = a_k t + b_k$. By (6c) and (6d) and the concavity of $G$, we have $L_k(t) > G(t)$ for all $t \in (0, \infty) \backslash [c_k, d_k]$, from which it follows that

$$G(t) \leq L_k(t) \quad \text{for all } t > 0 \text{ and each } k \in [1, \nu). \tag{8}$$

Let us define a sequence $\{F_k\}_{0 \leq k < \nu}$ of non-negative concave functions on $(0, \infty)$ by $F_0 := F$ and $F_k := \min(F_{k-1}, L_k)$. (They will, in fact, all be strictly positive except in the trivial case where $g$ is identically zero.) From (8) and the fact that $G \leq F$ it is evident that

$$G(t) \leq F_k(t) \text{ for all } t > 0 \text{ and each } k \in [1, \nu).$$

We observe further that

$$F_{k'}(t) = G(t) \quad \text{for all } t \in [c_k, d_k] \cap (0, \infty) \text{ and all } k', k \leq k' < \nu, \tag{9}$$

since clearly for all such $t$ and $k'$ we have $G(t) = L_k(t) \geq F_k(t) \geq F_{k'}(t) \geq G(t)$. Thus we deduce from (9) and (7) that, for each $n \in \mathbb{N}$,

$$F_k(t) = G(t) \quad \text{for each } t \in J_n \text{ and each } k \text{ such that } \kappa(n) \leq k < \nu. \tag{10}$$

We now construct a sequence of linear operators $\{U_k\}_{1 \leq k < \nu}$ and an associated sequence of functions $\{f_k\}_{0 \leq k < \nu}$ with the following properties:

(11a) $U_k : L^1_{w_j^*}(\Omega^*) \to L^1_{w_j^*}(\Omega^*)$ with norm not exceeding 1 for $j = 0, 1$.

(11b) The functions $f_k$ are non-negative and are defined iteratively by $f_0 := f$ and $f_k := U_k f_{k-1}$ for $k \geq 1$.

(11c) $K(t, f_k; \mathbf{L}^1_*) = F_k(t)$ for each $k$ and for all $t > 0$.

We remark that, by construction, each function $F_k$ is piecewise affine. More precisely, each compact subinterval of $(0,\infty)$ is the union of finitely many intervals, on each of which $F_k$ is affine. Thus, by applying Lemma 2.2 on each interval $J_n$ we will be able to deduce from (11c) that $f_k(t) = F_k'(t-) - F_k'(t+)$ for all $t > 0$. Lemma 2.2 and the properties of $g$ also imply that $g(t) = G'(t-) - G'(t+)$ for all $t > 0$, so it will follow from our construction and (10) that

$$f_k(t) = g(t) \text{ for each } t \in J_{n-1} \text{ and each } k \text{ such that } \kappa(n) \le k < \nu. \tag{12}$$

We construct the $U_k$'s and$f_k$'s iteratively: We of course have (11a) and (11b) for $k = 0$, so let $\kappa$ be an integer in $[1,\nu)$ and suppose we have already obtained functions $f_k$ for each integer $k \in [0,\kappa)$ and operators $U_k$ for each integer $k \in [1,\kappa)$ which satisfy (11a), (11b) and (11c).

If $L_\kappa(t) \ge F_{\kappa-1}(t)$ for all $t > 0$ then of course $F_\kappa = F_{\kappa-1}$, so we may take $U_\kappa$ simply as the identity operator, and $f_\kappa = f_{\kappa-1}$. Otherwise the set $\{t \in (0,\infty) : L_\kappa(t) < F_{\kappa-1}(t)\}$ is nonempty and it must necessarily be an open interval $(\alpha_\kappa, \gamma_\kappa)$ for some suitable $\alpha_\kappa$ and $\gamma_\kappa$ in $[0,\infty]$. Let us remark here, for later use, that since $G(t) \le L_\kappa(t) \le F_{\kappa-1}(t)$ for all $t \in (\alpha_\kappa, \gamma_\kappa)$, it follows from (10) that

$$J_n \cap (\alpha_\kappa, \gamma_\kappa) = \emptyset \text{ whenever } \kappa \ge \kappa(n) + 1. \tag{13}$$

We choose $U_\kappa = S_{\alpha_\kappa B_\kappa \gamma_\kappa}$, where $B_\kappa = \{\beta_m\}_{m\in M}$ is the sequence of points $s \in (\alpha_\kappa, \gamma_\kappa)$ such that $f_{\kappa-1}(s) = F'_{\kappa-1}(s-) - F'_{\kappa-1}(s+) > 0$. Clearly (cf. (5)) we have $K(t, U_\kappa f_{\kappa-1}; \mathbf{L}^1_*) = \min(F_{\kappa-1}(t), L_\kappa(t)) := F_\kappa(t)$ for all $t > 0$. Moreover, since $f_{\kappa-1}$ is non-negative, $f_\kappa := U_\kappa f_{\kappa-1}$ must also be non-negative, so we have established that properties (11a), (11b) and (11c) also hold for $k = \kappa$. The construction can thus be completed for all integers $k \in [1,\nu)$.

Now we define another sequence of operators $T_k$ by $T_1 = U_1$ and $T_k = U_k T_{k-1}$ for $2 \le k < \nu$. We claim that for each $n \in \mathbb{N}$ and each $h \in L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$, whenever $k > \kappa(n)$ and $t \in J_{n-1}$, we have

$$T_k h(t) = T_{\kappa(n)} h(t) \tag{14}$$

and

$$T_k f(t) = g(t). \tag{15}$$

We obtain (14) from (13), which implies that, for each $k$ as above, $J_{n-1} \cap [\alpha_k, \gamma_k] = \emptyset$. This ensures that the operator $U_k$ does not change the values of functions at any $t \in J_{n-1}$, and so

$$T_k h(t) = U_k U_{k-1} \ldots . U_{\kappa(n)+1} T_{\kappa(n)} h(t) = T_{\kappa(n)} h(t).$$

To obtain (15) we simply observe that $T_k f = f_k$ and use (12).

Finally, we define the operator $T$ required for the proof of the theorem. For each $h \in L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$ let $Th$ be the function defined by $Th(t) =$

$T_{\nu-1}h(t)$ if $\nu < \infty$, and $Th(t) = \lim_{k\to\infty} T_k h(t)$ if $\nu = \infty$. It follows immediately from (15) that $Tf = g$. The fact that $T\colon L^1_{w_j^*}(\Omega^*) \to L^1_{w_j^*}(\Omega^*)$ with norm not exceeding 1 for $j = 0, 1$ follows immediately from (14) together with the fact that the operators $T_k$ have the same property, since $\|Th\|_{L^1_{w_0^*}(\Omega^*)} = \sup_{n\in\mathbb{N}} \|Th \cdot \chi_{J_n}\|_{L^1_{w_0^*}(\Omega^*)}$. $\square$

REMARK 2.1. Since each $U_k = S_{\alpha_k B_k \gamma_k}$ is a positive operator, it follows immediately that the same is true for each $T_k$, and so also for $T$.

## 3. The proof that Theorem 1.1 follows from Theorem 2.1

Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space and $w_0$ and $w_1$ arbitrary weight functions on $\Omega$. As before, let $\mathbf{L}^1 = (L^1_{w_0}(\Omega), L^1_{w_1}(\Omega))$.

This part of the proof is similar to arguments presented in [23] and also rather similar to the proof of the well known fact that the couple $\mathbf{L}^1_*$ is a *retract* of the couple $\mathbf{L}^1$ (cf., e.g., [7, p. 160, Example 2.3.22(c)]).

Choose $\epsilon > 0$ and let $f$ and $g$ be arbitrary functions in $L^1_{w_0} + L^1_{w_1}$ satisfying

$$K(t, g; \mathbf{L}^1) \le K(t, f; \mathbf{L}^1) \quad \text{for all } t > 0. \tag{16}$$

Let $M_f$ and $N_g$ be operators of pointwise multiplication, defined by $M_f h = \overline{\operatorname{sgn}(f)} h$ and $N_g h = \operatorname{sgn}(g) h$ for each $h \in L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$. These operators both obviously map $L^1_{w_j}(\Omega)$ into itself with norm 1 for $j = 0, 1$. Since $M_f f = |f|$ and $N_g|g| = g$ our main task will be to find an operator $V$ such that $V\colon L^1_{w_j}(\Omega) \to L^1_{w_j}(\Omega)$ with norm not exceeding $1+\epsilon$ and $V|f| = |g|$. Then we will simply choose $T = N_g V M_f$ to obtain $Tf = g$.

We fix a number $r > 1$ such that $r^2 < 1 + \epsilon$. For each $n \in \mathbb{Z}$ let

$$\Omega_n = \{x \in \Omega : r^{n-1} \le \frac{w_0(x)}{w_1(x)} < r^n\}.$$

We now define an operator $P\colon L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega) \to L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$. For each $h \in L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$ we set

$$Ph(r^n) = \int_{\Omega_n} h w_1 d\mu \text{ for each } n \in \mathbb{Z}$$

and $Ph(t) = 0$ for all other values of $t \in (0, \infty)$. We observe that for $j = 0, 1$ we have $|Ph(r^n)| w_j^*(r^n) \le r^{1-j} \int_{\Omega_n} |h| w_j d\mu$ for each $n \in \mathbb{Z}$ and so $P\colon L^1_{w_j}(\Omega) \to L^1_{w_j^*}(\Omega^*)$ with norm not exceeding $r^{1-j}$. Furthermore,

$$\begin{aligned} K(t, P|f|; \mathbf{L}^1_*) &= \sum_{n\in\mathbb{Z}} \int_{\Omega_n} |f| w_1 d\mu \cdot \min(r^n, t) \\ &\ge \sum_{n\in\mathbb{Z}} \int_{\Omega_n} |f| \min(w_0, t w_1) d\mu = K(t, f; \mathbf{L}^1) \end{aligned} \tag{17}$$

for all $t > 0$.

Next, we define an operator $Q\colon L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*) \to L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$. For each $u \in L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$ we set

$$Qu = |g| \cdot \sum_{n\in\mathbb{Z}^*} \frac{u(r^n)}{\int_{\Omega_n} |g| w_1 d\mu} \chi_{\Omega_n} \ ,$$

where $\mathbb{Z}^*$ is the set of all integers $n$ such that $g$ does not vanish a.e. on $\Omega_n$. Then for each $n \in \mathbb{Z}^*$ we have that

$$\int_{\Omega_n} |Qu| w_j d\mu = \frac{\int_{\Omega_n} |g| w_j d\mu}{\int_{\Omega_n} |g| w_1 d\mu} |u(r^n)| \le w_j^*(r^n)|u(r^n)|,$$

and so $Q\colon L^1_{w_j^*}(\Omega^*) \to L^1_{w_j}(\Omega)$ with norm not exceeding 1 for $j = 0, 1$. Furthermore, if we define the function $g^* : (0,\infty) \to (0,\infty)$ by $g^*(r^n) = \int_{\Omega_n} |g| w_1 d\mu$ for each $n \in \mathbb{Z}$ and $g^*(t) = 0$ for all other values of $t \in (0,\infty)$, then $Qg^* = |g|$. We also have that

$$\begin{aligned} K(t, g^*; \mathbf{L}^1_*) &= \sum_{n\in\mathbb{Z}} \int_{\Omega_n} |g| w_1 d\mu \cdot \min(r^n, t) \\ &\le r \sum_{n\in\mathbb{Z}} \int_{\Omega_n} |g| \min(w_0, t w_1) d\mu = rK(t, g; \mathbf{L}^1) \end{aligned}$$

for all $t > 0$. From this estimate and (16) and (17) we get that

$$K(t, \frac{1}{r} g^*; \mathbf{L}^1_*) \le K(t, P|f|; \mathbf{L}^1_*) \text{ for all } t > 0.$$

This enables us to apply Theorem 2.1 to obtain an operator $A\colon L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*) \to L^1_{w_0^*}(\Omega^*) + L^1_{w_1^*}(\Omega^*)$ such that $A\colon L^1_{w_j^*}(\Omega^*) \to L^1_{w_j^*}(\Omega^*)$ with norm not exceeding 1 for $j = 0, 1$ and $A(P|f|) = (1/r)g^*$. Then it is clear that the operator $V := rQAP$ satisfies $V|f| = |g|$ and $V\colon L^1_{w_j}(\Omega) \to L^1_{w_j}(\Omega)$ for $j = 0, 1$ with norm not exceeding $r^{2-j}$. Since we have chosen $r$ so that $r^2 < 1 + \epsilon$, the proof is complete.

Remark 3.1. By Remark 2.1, $A$ can be chosen to be a positive operator. Since $P$ and $Q$ are clearly positive operators, we deduce that $V$ is also positive.

## 4. A refinement of Theorem 1.1: operators with norm 1

In this section we shall prove that Theorem 1.1 also holds for $\epsilon = 0$.

Let the measure spaces and weight functions and the functions $f$ and $g$ satisfying (16) be as in Section 3. For each $n \in \mathbb{N}$ let $V_n$ be a *positive* operator, constructed as in Section 3, such that $V_n|f| = |g|$ and $V_n : L^1_{w_j}(\Omega) \to L^1_{w_j}(\Omega)$ for $j = 0, 1$, with norm not exceeding $1 + 1/n$. We use an approach suggested by the proofs of analogous results in [8]. Let $\lambda$ be a *Banach limit*, i.e., a linear functional on $\ell^\infty(\mathbb{N})$ whose existence follows from the Hahn-Banach theorem, such that $|\lambda(\alpha)| \le \limsup_{n\to\infty} |\alpha_n|$ for all sequences $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ in $\ell^\infty(\mathbb{N})$ and $\lambda(\alpha) = \lim_{n\to\infty} \alpha_n$ if $\{\alpha_n\}_{n\in\mathbb{N}}$ is convergent. For each $h \in L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$ and for each measurable set $E \subset \Omega$ let

$$\Psi(E,h) := \lambda\left(\left\{\int_E \min(w_0,w_1)V_n h\, d\mu\right\}_{n\in\mathbb{N}}\right).$$

$\Psi(E,h)$ is well defined since for each $n$,

$$\begin{aligned}\left|\int_E \min(w_0,w_1)V_n h\, d\mu\right| &\le \int_\Omega \min(w_0,w_1)|V_n h|\, d\mu \\ &= K(1,V_n h;\mathbf{L}^1) \le \left(1+\frac{1}{n}\right)K(1,h;\mathbf{L}^1)\end{aligned}$$

(cf., e.g., [5, p. 41]). Observe that for each fixed $E$ the functional $\Psi(E,h)$ depends linearly on $h$. On the other hand, for each fixed $h$, $\Psi(E,h)$ is a finitely additive set function on the $\sigma$-algebra $\Sigma$.

For each $k \in \mathbb{N}$ define

$$\Xi_k = \left\{x \in \Omega : |f(x)| \ge \frac{1}{k}, \frac{1}{k} \le w_0(x) \le k, \frac{1}{k} \le w_1(x) \le k\right\}.$$

Note that $\mu(\Xi_k) \le \int_{\Xi_n} k^2 |f| \min(w_0,w_1) d\mu < \infty$ and also that $\chi_{\Xi_k} \in L^1_{w_0}(\Omega) \cap L^1_{w_1}(\Omega)$. Let us fix $k$ and a measurable set $B \subset \Xi_k$. Then for each measurable set $E \in \Sigma$ we have (since each $V_n$ is positive and $k|f| - \chi_B \ge 0$)

$$\begin{aligned}\Psi(E,\chi_B) &\le \limsup_{n\to\infty} \int_E \min(w_0,w_1)V_n\chi_B\, d\mu \\ &\le \limsup_{n\to\infty}\ k\int_E \min(w_0,w_1)V_n|f|\, d\mu \\ &= k\int_E \min(w_0,w_1)|g|\, d\mu.\end{aligned}$$

From these estimates and the finite additivity of $\Psi(\cdot,\chi_B)$ we deduce that $\Psi(\cdot,\chi_B)$ is, in fact, countably additive, i.e., it is a measure on $\Sigma$. (Its total variation does not exceed $k\int_\Omega |g| \min(w_0,w_1) d\mu$.) It is of course absolutely continuous with respect to the measure $\mu$.

If $\mu$ is $\sigma$-finite we can apply the Radon-Nikodým theorem to show that there exists a $\mu$-integrable function $\rho : \Omega \to \mathbb{C}$ such that $\Psi(E,\chi_B) = \int_E \rho d\mu$ for all $E \in \Sigma$. But, in fact, we can obtain the existence of a function $\rho$ with these properties even when $\mu$ is not $\sigma$-finite by considering the set $\Gamma := \{x \in \Omega : g(x) \ne 0\}$. Because of the factor $|g|$ that appears in the formula for each operator which plays the rôle of the operator $Q$ in the proof of Section 3, it follows that $V_n h = \chi_\Gamma V_n h$ for each $n$ and all $h \in L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$. Thus $\Psi(E,h) = \Psi(E\cap\Gamma,h)$ for all $E \in \Sigma$. The measure $\mu_\Gamma$ defined by

$\mu_\Gamma(E) = \mu(E \cap \Gamma)$ is of course $\sigma$-finite. Hence we obtain $\rho$ as required; $\rho$ will of course be supported on $\Gamma$.

Since $\rho$ depends on $B$ and, in fact, is determined uniquely $\mu$-a.e. by the choice of $B$, we shall use the more explicit notation $\rho_B$ for $\rho$. Obviously, for disjoint measurable $B_1$ and $B_2$ contained in some $\Xi_k$ we have

$$\rho_{B_1 \cup B_2} = \rho_{B_1} + \rho_{B_2}. \tag{18}$$

We now define an operator $V$ which acts on the space $\mathcal{S}$ of all (complex-valued) simple functions $h$ of the form $h = \sum_{m=1}^M \beta_m \chi_{B_m}$, where the measurable sets $B_m$ are pairwise disjoint and are each contained in $\Xi_k$ for some $k \in \mathbb{N}$. For each such $h$ we set

$$Vh = \frac{\sum_{m=1}^M \beta_m \rho_{B_m}}{\min(w_0, w_1)}.$$

It follows from (18) that this definition is independent of the representation of $h$ and that $V$ is linear. It also follows from the $\mu$-integrability of each $\rho_{B_m}$ that

$$\int_\Omega |Vh| \min(w_0, w_1) d\mu < \infty. \tag{19}$$

We see that

$$\int_E Vh \cdot \min(w_0, w_1) d\mu = \sum_{m=1}^M \beta_m \int_E \rho_{B_m} d\mu = \sum_{m=1}^M \beta_m \Psi(E, \chi_{B_m}) = \Psi(E, h)$$

for each measurable $E$. Thus, for each (complex-valued) simple function $z = \sum_{i=1}^N \zeta_i \chi_{E_i}$ we obtain

$$\begin{aligned} \int_\Omega zVh \cdot \min(w_0, w_1) d\mu &= \sum_{i=1}^N \zeta_i \Psi(E_i, h) \\ &= \lambda\left(\left\{\int_\Omega zV_n h \cdot \min(w_0, w_1) d\mu\right\}_{n \in \mathbb{N}}\right). \end{aligned}$$

Consequently, for $j = 0, 1$,

$$\begin{aligned} &\left|\int_\Omega zVh \cdot \min(w_0, w_1) d\mu\right| \leq \limsup_{n\to\infty} \int_\Omega |zV_n h| \cdot \min(w_0, w_1) d\mu \\ &\quad\leq \limsup_{n\to\infty} \int_\Omega |V_n h| w_j d\mu \cdot \text{ess sup} \frac{|z| \min(w_0, w_1)}{w_j} \\ &\quad\leq \limsup_{n\to\infty} \left(1 + \frac{1}{n}\right) \int_\Omega |h| w_j d\mu \cdot \text{ess sup} \frac{|z| \min(w_0, w_1)}{w_j} \\ &\quad= \int_\Omega |h| w_j d\mu \cdot \text{ess sup} \frac{|z| \min(w_0, w_1)}{w_j}. \end{aligned}$$

It follows that if $\phi$ is a second simple function satisfying $|\phi| \le 1$ a.e., then

$$\left|\int_\Omega z\phi Vh \cdot \min(w_0, w_1) d\mu\right| \le \int_\Omega |h| w_j d\mu \cdot \operatorname*{ess\,sup}_\Omega \frac{|z| \min(w_0, w_1)}{w_j}.$$

Hence, using a sequence of simple functions $\phi_n$ with $|\phi_n| \le 1$ such that $\phi_n$ converges pointwise to $\overline{\operatorname{sgn}(Vh)}$, we obtain by dominated convergence (recalling (19)) that

$$\left|\int_\Omega z|Vh| \min(w_0, w_1) d\mu\right| \le \int_\Omega |h| w_j d\mu \cdot \operatorname*{ess\,sup}_\Omega \frac{|z| \min(w_0, w_1)}{w_j}.$$

Now consider a monotone increasing sequence of non-negative simple functions $z_n$ whose pointwise limit is $w_j / \min(w_0, w_1)$. Substituting these functions $z_n$ into the preceding estimate and applying monotone convergence, we deduce that

$$\int_\Omega |Vh| w_j d\mu \le \int_\Omega |h| w_j d\mu \text{ for } j = 0, 1 \text{ and all } h \in \mathcal{S}. \tag{20}$$

Let $\Xi = \bigcup_{n=1}^\infty \Xi_n$. Of course $\Xi = \{x \in \Omega : f(x) \neq 0\}$. Let $L^1_{w_j}(\Xi)$ be the subspace of $L^1_{w_j}(\Omega)$ consisting of those functions which vanish a.e. on $\Omega \backslash \Xi$. Clearly $\mathcal{S}$ is a dense subspace of $L^1_{w_j}(\Xi)$ for $j = 0, 1$ and also of $L^1_{w_0}(\Xi) \cap L^1_{w_1}(\Xi)$. So by (20) the operator $V$ has a unique extension, which we will still denote by $V$, to an operator which maps $L^1_{w_0}(\Xi) \cap L^1_{w_1}(\Xi)$ into $L^1_{w_0}(\Omega) \cap L^1_{w_1}(\Omega)$ with norm not exceeding 1. (We define the norms on these intersection spaces in the usual way; cf., e.g., [5].) Using (20) again, we can, for $j = 0, 1$, further extend $V$ uniquely to an operator mapping $L^1_{w_j}(\Xi)$ into $L^1_{w_j}(\Omega)$ with norm not exceeding 1. Since the extension for $j = 0$ and the extension for $j = 1$ coincide on functions in $L^1_{w_0}(\Xi) \cap L^1_{w_1}(\Xi)$ it follows that they define a unique operator from $L^1_{w_0}(\Xi) + L^1_{w_1}(\Xi)$ into $L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$. We again permit ourselves to denote this extension of $V$ to $L^1_{w_0}(\Xi) + L^1_{w_1}(\Xi)$ by $V$.

The formula

$$\int_E \min(w_0, w_1) Vh \, d\mu = \Psi(E, h) := \lambda\left(\left\{\int_E \min(w_0, w_1) V_n h \, d\mu\right\}_{n \in \mathbb{N}}\right) \tag{21}$$

was obtained above for all $h \in \mathcal{S}$. We now show that it holds in fact for all $h \in L^1_{w_0}(\Xi) + L^1_{w_1}(\Xi)$. For each $h \in L^1_{w_0}(\Xi) + L^1_{w_1}(\Xi)$ there exists a sequence $\{h_k\}_{k \in \mathbb{N}}$ of functions in $\mathcal{S}$ such that

$$\lim_{k \to \infty} \int_\Xi |h - h_k| \min(w_0, w_1) d\mu = \lim_{k \to \infty} K(1, h - h_k; \mathbf{L}^1) = 0. \tag{22}$$

By standard $K$-functional estimates we also have that

$$\begin{aligned} \int_\Xi |Vh - Vh_k| \min(w_0, w_1) d\mu &= K(1, V(h - h_k); \mathbf{L}^1) \\ &\le K(1, h - h_k; \mathbf{L}^1) \end{aligned} \tag{23}$$

and

$$\int_{\Xi} |V_n h - V_n h_k| \min(w_0, w_1) d\mu = K(1, V_n(h - h_k); \mathbf{L}^1) \leq \left(1 + \frac{1}{n}\right) K(1, h - h_k; \mathbf{L}^1). \tag{24}$$

The estimates (22), (23) and (24) can now be used in an obvious way to extend (21) as required.

Finally let us substitute $h = |f|$ in (21). Since $V_n|f| = |g|$ for each $n$ we obtain

$$\int_E V|f| \min(w_0, w_1) d\mu = \int_E |g| \min(w_0, w_1) d\mu.$$

Since this formula holds for every $E \in \Sigma$, it follows that $V|f| = |g|$ almost everywhere. Thus, to get an operator $T$ which maps $L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$ into $L^1_{w_0}(\Omega) + L^1_{w_1}(\Omega)$, and $L^1_{w_j}(\Omega)$ into $L^1_{w_j}(\Omega)$ with norm not exceeding 1, and for which $Tf = g$, we can take $Th = N_g V(\chi_\Xi \cdot M_f h)$.

This completes the proof that Theorem 1.1 holds for $\epsilon = 0$.

Remark 4.1. We observe that the functional $\lambda$ introduced above must be positive, i.e., $\lambda(\alpha) \geq 0$ whenever the elements of the sequence $\alpha$ are all non-negative. This is perhaps most easily seen by using the fact that $\lambda$ is representable as a finitely additive measure $\lambda_*$ on the $\sigma$-algebra of all subsets of $\mathbb{N}$. Since $\lambda_*(\mathbb{N}) = \lambda(\chi_{\mathbb{N}}) = 1$ and since the total variation of $\lambda_*$ equals $\|\lambda\| = 1$, it follows that $\lambda_*(Y) \geq 0$ for all $Y \subset \mathbb{N}$. This immediately implies the positivity of $\lambda$. We can now use (21) for each $E \in \Sigma$ and the positivity of each $V_n$ to deduce that $V$ is a positive operator. Then obviously, if $f$ and $g$ are non-negative, $T$ will also be a positive operator, justifying our claim in the introduction.

**Acknowledgement.** We thank Boris Begun for some helpful discussions on the paper of Sedaev and Semenov. We also thank Yacin Ameur and Yuri Brudnyi for some helpful remarks.

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