Illinois Journal of Mathematics
Volume 48, Number 1, Spring 2004, Pages 319–337
S 0019-2082

NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS

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Abstract. Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let k be a positive integer; let h be a positive number; and let a be a function holomorphic in D such that $a(z) \neq 0$ for $z \in D$. For $k \neq 2$ we show that if, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, $f(z) = 0 \Longrightarrow f^{(k)}(z) = a(z)$, and $f^{(k)}(z) = a(z) \Longrightarrow |f^{(k+1)}(z)| \leq h$, then $\mathcal{F}$ is normal in D. For $k = 2$ we prove the following result: Let $s \geq 4$ be an even integer. If, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least 2, $f(z) = 0 \Longrightarrow f''(z) = a(z)$, and $f''(z) = a(z) \Longrightarrow |f'''(z)| + |f^{(s)}(z)| \leq h$, then $\mathcal{F}$ is normal in D. This improves the well-known normality criterion of Miranda.

1. Introduction

Let $\mathcal{F}$ be a family of holomorphic functions on a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains either a subsequence which converges to an analytic function f uniformly on each compact subset of D or a subsequence which converges to ∞ uniformly on each compact subset of D.

In 1912, Montel [10] proved:

Theorem A. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D; and let a, b be distinct complex numbers. If, for every $f \in \mathcal{F}$, $f \neq a, b$, then $\mathcal{F}$ is normal in D.*

Later (see [13, p. 125]), he made the following conjecture.

Received March 25, 2003; received in final form August 11, 2003.
2000 *Mathematics Subject Classification.* 30D45.
The research of the second author was supported by the NNSF of China (Grant No. 10071038), by the Fred and Barbara Kort Sino-Israel Post Doctoral Fellowship Program at Bar-Ilan University, and by the German-Israeli Foundation for Scientific Research and Development, G.I.F. Grant No. G-643-117.6/1999. The research of the third author was supported by the German-Israeli Foundation for Scientific Research and Development, G.I.F. Grant No. G-643-117.6/1999.

Conjecture. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D, and let a, b be complex numbers with $b \neq 0$. If, for every $f \in \mathcal{F}$, $f \neq a$, and $f' \neq b$, then $\mathcal{F}$ is normal in D.*

In 1935, Miranda [9] confirmed this conjecture and proved the following more general result.

Theorem B. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D; let a, b be complex numbers with $b \neq 0$; and let k be a positive integer. If, for every $f \in \mathcal{F}$, $f \neq a$, and $f^{(k)} \neq b$, then $\mathcal{F}$ is normal in D.*

In this paper, we extend Theorem B as follows.

Theorem 1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $k \neq 2$ be a positive integer; let h be a positive number; and let a be a function holomorphic in D such that $a(z) \neq 0$ for $z \in D$. If, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, $f(z) = 0 \Longrightarrow f^{(k)}(z) = a(z)$, and $f^{(k)}(z) = a(z) \Longrightarrow |f^{(k+1)}(z)| \leq h$, then $\mathcal{F}$ is normal in D.*

Remark 1. Theorem 1 is not valid for $k = 2$.

Example 1. ([12]) Let $\mathcal{F} = \{f_n\}$ on the unit disc Δ, where

$$f_n(z) = \frac{1}{n^2}(e^{nz} + e^{-nz} - 2) = \frac{1}{n^2}e^{-nz}(e^{nz} - 1)^2,$$

so that

$$f_n^{(j)}(z) = n^{j-2}[e^{nz} + (-1)^j e^{-nz}],\ j = 1, 2, \ldots.$$

Clearly, all zeros of f_n are double, $f_n(z) = 0 \Longrightarrow f_n''(z) = 2$, and $f_n''(z) = 2 \Longrightarrow f_n'''(z) = 0$ for any $f_n \in \mathcal{F}$, but $\mathcal{F}$ is not normal in Δ.

For $k = 2$, using the method of [12], we get the following result.

Theorem 2. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let h be a positive number; and let a be a nonzero complex number. If, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least 2, $f(z) = 0 \Longrightarrow f''(z) = a$, and $f''(z) = a \Longrightarrow 0 < |f'''(z)| \leq h$, then $\mathcal{F}$ is normal in D.*

In view of Theorems 1 and 2, it is natural to ask whether Theorem 2 is valid if the nonzero complex number a is replaced by a holomorphic function $a(z)$ in D with $a(z) \neq 0$ for $z \in D$. The following example shows that the answer is negative.

EXAMPLE 2. Let $\mathcal{F} = \{f_n : n = 2, 3, \dots\}$ on the unit disc Δ, where

$$f_n(z) = \frac{n^2-1}{2n^2}\left(\frac{e^{(n+1)z}}{(n+1)^2} + \frac{e^{-(n-1)z}}{(n-1)^2} - \frac{2e^z}{n^2-1}\right) \tag{1.1}$$
$$= \frac{n^2-1}{2n^2}e^{-(n-1)z}\left(\frac{e^{nz}}{n+1} - \frac{1}{n-1}\right)^2,$$

and $a(z) = e^z$, $h = 3e$. Then

$$f_n''(z) = \frac{n^2-1}{2n^2}\left(e^{(n+1)z} + e^{-(n-1)z} - \frac{2e^z}{n^2-1}\right), \tag{1.2}$$

$$f_n'''(z) = \frac{n^2-1}{2n^2}\left((n+1)e^{(n+1)z} - (n-1)e^{-(n-1)z} - \frac{2e^z}{n^2-1}\right). \tag{1.3}$$

Obviously, all zeros of f_n are double. If $f_n(z) = 0$, then by (1.1) we have

$$e^{nz} = \frac{n+1}{n-1};$$

so by (1.2), we get

$$f_n''(z) = \frac{n^2-1}{2n^2}\left(\frac{n+1}{n-1} + \frac{n-1}{n+1} - \frac{2}{n^2-1}\right)e^z$$
$$= e^z.$$

Thus $f_n(z) = 0 \Longrightarrow f_n''(z) = e^z$.

Now let $f_n''(z) = e^z$. Then by (1.2), we have

$$e^{nz} + e^{-nz} - \frac{2}{n^2-1} = \frac{2n^2}{n^2-1}.$$

Solving the above equation, we get either $e^{nz} = (n+1)/(n-1)$ or $e^{nz} = (n-1)/(n+1)$. If $e^{nz} = (n+1)/(n-1)$, then by (1.3),

$$f_n'''(z) = \frac{n^2-1}{2n^2}\left((n+1)\frac{n+1}{n-1} - (n-1)\frac{n-1}{n+1} - \frac{2}{n^2-1}\right)e^z \tag{1.4}$$
$$= \frac{n^2-1}{2n^2}\frac{(n+1)^3-(n-1)^3-2}{n^2-1}e^z$$
$$= 3e^z.$$

If $e^{nz} = (n-1)/(n+1)$, then by (1.3),

$$f_n'''(z) = \frac{n^2-1}{2n^2}\left((n+1)\frac{n-1}{n+1} - (n-1)\frac{n+1}{n-1} - \frac{2}{n^2-1}\right)e^z \tag{1.5}$$
$$= \frac{n^2-1}{2n^2}\left(-2 - \frac{2}{n^2-1}\right)e^z$$
$$= -e^z.$$

Thus by (1.4) and (1.5), we find that $f_n''(z) = e^z \Longrightarrow 0 < |f_n'''(z)| \le 3e$ on Δ. But $\mathcal{F}$ is not normal in Δ.

For $k = 2$ and a holomorphic function a, we have the following result.

Theorem 3. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let h be a positive value and $s \geq 4$ an even integer; and let a be a function holomorphic in D such that $a(z) \neq 0$ for $z \in D$. If, for every $f \in \mathcal{F}$, all zeros of f have multiplicity at least 2, $f(z) = 0 \Longrightarrow f''(z) = a(z)$, and $f''(z) = a(z) \Longrightarrow |f'''(z)| + |f^{(s)}(z)| \leq h$, then $\mathcal{F}$ is normal in D.*

Remark 2. Example 1 also shows that $f''(z) = a(z) \Longrightarrow |f^{(s)}(z)| \leq h$ is necessary and that one cannot replace even s by odd s in Theorem 3.

Theorem 4. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $k \geq 2$ be a positive integer; and let a be a function holomorphic in D such that $a(z) \neq 0$ for $z \in D$. If, for every $f \in \mathcal{F}$, $f(z) = 0 \Longrightarrow f'(z) = a(z)$, and $f'(z) = a(z) \Longrightarrow |f^{(k)}(z)| \leq h$, then $\mathcal{F}$ is normal in D.*

Theorem 4 improves results of Chen and Hua [2, Theorem 1], Pang [11, Theorem 1], and Fang and Xu [6, Theorem 3].

Remark 3. In Theorems 1, 3 and 4, the condition $a(z) \neq 0$ is necessary, and cannot be replaced by $a(z) \not\equiv 0$.

Example 3. For $k \neq 2$, let $\mathcal{F} = \{n^{k+2}z^{k+2} : n = 1, 2, 3, \dots\}$; let $a(z) = z^2$, $h = 1$; and let $D = \{z : |z| < 1\}$. Then, for any $f \in \mathcal{F}$, all zeros of f are of multiplicity at least k; $f(z) = 0 \Longrightarrow f^{(k)}(z) = a(z)$; and $f^{(k)}(z) = a(z) \Longrightarrow |f^{(k+1)}(z)| \leq h$ for $z \in D$, but $\mathcal{F}$ is not normal in D.

Example 4. For $s \geq 6$, let $\mathcal{F} = \{n^4 z^4 : n = 1, 2, \dots\}$ and $a(z) = z^2$; for $s = 4$, let $\mathcal{F} = \{n^4(z^4 - 1/n^4)^2 : n = 1, 2, \dots\}$ and $a(z) = 32z^2$. Let $D = \{z : |z| < 1\}$. Then for any $f \in \mathcal{F}$, all zeros of f are of multiplicity ≥ 2; $f(z) = 0 \Longrightarrow f''(z) = a(z)$; and $f''(z) = a(z) \Longrightarrow |f'''(z)| + |f^{(s)}(z)| \leq 1920$ for any $z \in D$, but $\mathcal{F}$ is not normal in D.

Example 5. For $l \geq 3$, let $\mathcal{F} = \{n^2 z^2 : n = 1, 2, \dots\}$; for $l = 2$, let $\mathcal{F} = \{(nz - 1)z^2 : n = 1, 2, \dots\}$. Let $a(z) = z$ and $D = \{z : |z| < 1\}$. Then for any $f \in \mathcal{F}$, $f(z) = 0 \Longrightarrow f'(z) = a(z)$; and $f'(z) = a(z) \Longrightarrow |f^{(l)}(z)| \leq 4$ for any $z \in D$, but $\mathcal{F}$ is not normal in D.

Remark 4. Theorems 1, 3 and 4 do not hold for meromorphic a.

Example 6. Let $\mathcal{F} = \{(nz - 1)^k : n = 1, 2, 3, \dots\}$; let $a(z) = k!/z^k$, $h = 1$; and let $D = \{z : |z| < 1\}$. Then, for any $f \in \mathcal{F}$, $f(z) = 0 \Longrightarrow f^{(k)}(z) = a(z)$, and $f^{(k)}(z) = a(z) \Longrightarrow |f^{(k+1)}(z)| \leq h$ for any $z \in D$, but $\mathcal{F}$ is not normal in D.

2. Some lemmas

In order to prove our theorems, we require the following results. We assume the standard notation of value distribution theory, as presented and used in [7].

LEMMA 1 ([12, Lemma 2]). *Let $\mathcal{F}$ be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$;*
(b) *points z_n, $|z_n| < r$;*
(c) *functions $f_n \in \mathcal{F}$; and*
(d) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly, where g is a nonconstant entire function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

LEMMA 2 ([5]). *Let f be an entire function, and let M be a positive number. If $f^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, then $\rho(f) \leq 1$.*

Here and in the sequel, $\rho(f)$ is the order of f.

LEMMA 3 (see [1, Theorem 1], [3, Lemma 4]). *Let P be a nonzero polynomial; let k be a positive integer; and let $g \not\equiv 0$ be a solution of the equation*

$$g^{(k)} = Pg. \tag{2.1}$$

Then $\rho(g) = 1 + d/k$, where $d = \deg P$.

LEMMA 4 (see [8]). *Let f be meromorphic in $|z| < \infty$. If $f(0) \neq 0, \infty$, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) \leq C_k \left\{1 + \log^+ \log^+ \frac{1}{|f(0)|} + \log^+ \frac{1}{r} + \log^+ r + \log^+ T(2r, f)\right\},$$

where k is a positive integer, and C_k depends only on k. In particular, when f is of finite order,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r), \ \text{as } r \to \infty. \tag{2.2}$$

LEMMA 5. *Let g be a nonconstant entire function with $\rho(g) \leq 1$ whose zeros have multiplicity at least k, and let a be a nonzero value. If $g(z) = 0 \Longrightarrow g^{(k)}(z) = a$ and $g^{(k)}(z) = a \Longrightarrow g^{(k+1)}(z) = 0$, then*

(i) $g(z) = \frac{a}{k!}(z - z_0)^k$, *for* $k \neq 2$;

(ii) *either* $g(z) = \frac{a}{2}(z - z_0)^2$ *or* $g(z) = (Ae^{\lambda z} - \frac{a}{8A\lambda^2}e^{-\lambda z})^2$, *for* $k = 2$.

Proof . Since $g(z) = 0 \Longrightarrow g^{(k)}(z) = a \neq 0$ and the multiplicities of the zeros of $g(z)$ are at least k, the multiplicity of the zeros of $g(z)$ is exactly k. Since g is entire, there exists a nonconstant entire function h, all of whose zeros are simple, such that

$$g(z) = h^k(z). \tag{2.3}$$

Let $z = z_0$ be a zero of h. We have (near z_0)

$$h(z) = a_1(z - z_0) + a_2(z - z_0)^2 + O((z - z_0)^3), \quad (a_1 \neq 0). \tag{2.4}$$

Thus

$$g(z) = (h(z))^k = a_1^k(z - z_0)^k + ka_1^{k-1}a_2(z - z_0)^{k+1} + O((z - z_0)^{k+2}),$$

so

$$g^{(k+1)}(z_0) = (k + 1)!ka_1^{k-1}a_2. \tag{2.5}$$

Since $g(z) = 0 \Longrightarrow g^{(k+1)}(z) = 0$, we get $a_2 = 0$. This means $h''(z_0) = 0$. Thus we have shown that

$$h(z) = 0 \Longrightarrow h''(z) = 0. \tag{2.6}$$

Set

$$P = \frac{h''}{h}. \tag{2.7}$$

Since the zeros of h are all simple, P is an entire function. Moreover, since $\rho(g) \leq 1$, it is clear from (2.3) that $\rho(h) \leq 1$. By Lemma 4, we have

$$T(r, P) = T\left(r, \frac{h''}{h}\right) = m\left(r, \frac{h''}{h}\right) = O(\log r), \text{ as } r \to \infty.$$

So P is a polynomial. Now we consider two cases.

Case 1. $P \equiv 0$. Then by (2.7), $h'' \equiv 0$. Thus $h(z) = cz + d$, where $c(\neq 0), d$ are constants. Hence

$$g(z) = (cz + d)^k,$$

and

$$g^{(k)}(z) \equiv k!c^k.$$

By the condition, $k!c^k = a$. Thus

$$g(z) = \frac{a}{k!}(z - z_0)^k.$$

Case 2. $P \not\equiv 0$. By (2.7), h is a transcendental entire function. Thus by Lemma 3, the order of h is $1 + \deg P/2$. Since $\rho(h) \leq 1$, $\deg P = 0$. Thus P is a nonzero constant. Solving the equation (2.7), we obtain

$$h = Ae^{\lambda z} + Be^{-\lambda z},$$

where A, B are two constants and $\lambda(\neq 0)$ is a solution of the equation $z^2 = P$.

Obviously, from the assumptions of the lemma, $A \neq 0$ and $B \neq 0$. Thus by (2.3), we have

$$g(z) = \left(Ae^{\lambda z} + Be^{-\lambda z}\right)^k = \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j} e^{(2j-k)\lambda z}. \tag{2.8}$$

Hence

$$g^{(k)}(z) = \lambda^k \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j} (2j-k)^k e^{(2j-k)\lambda z} \tag{2.9}$$

and

$$g^{(k+1)}(z) = \lambda^{k+1} \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j} (2j-k)^{k+1} e^{(2j-k)\lambda z}. \tag{2.10}$$

Let z_0 be a zero of g. Then by (2.8), we have

$$e^{2\lambda z_0} = -\frac{B}{A}.$$

Now we consider two subcases.

Case 2.1. $k = 2m+1$. Let $e^{\lambda z_0} = K$ and $e^{\lambda z_1} = -K$, where K is a constant satisfying $K^2 = -B/A$. Then by (2.8), $g(z_0) = 0$ and $g(z_1) = 0$. So by $g(z) = 0 \Longrightarrow g^{(k)}(z) = a$, we get $a = g^{(k)}(z_0) = g^{(k)}(z_1)$. Thus by (2.9), we have

$$\begin{aligned} 2a &= g^{(k)}(z_0) + g^{(k)}(z_1) \\ &= \lambda^{2m+1} \sum_{j=0}^{2m+1} \binom{2m+1}{j} A^j B^{2m+1-j} (2j-2m-1)^{2m+1} \\ &\qquad\qquad \times \left[K^{2j-2m-1} + (-K)^{2j-2m-1}\right] \\ &= 0, \end{aligned} \tag{2.11}$$

which contradicts $a \neq 0$.

Case 2.2. $k = 2m$. Then by (2.9), we get

$$a = \lambda^{2m} A^m B^m \sum_{j=0}^{2m} (-1)^{j-m} \binom{2m}{j} (2j-2m)^{2m}. \tag{2.12}$$

By (2.9)–(2.10), we have

$$g^{(2m)}(z) = \lambda^{2m} \sum_{j=0}^{2m} \binom{2m}{j} A^j B^{2m-j} (2j-2m)^{2m} e^{2(j-m)\lambda z}, \tag{2.13}$$

$$g^{(2m+1)}(z) = \lambda^{2m+1} \sum_{j=0}^{2m} \binom{2m}{j} A^j B^{2m-j} (2j-2m)^{2m+1} e^{2(j-m)\lambda z}. \tag{2.14}$$

If $m = 1$, then

$$a = -8AB\lambda^2;$$

and it follows from (2.8) that

$$g = \left(Ae^{\lambda z} - \frac{a}{8A\lambda^2} e^{-\lambda z}\right)^2.$$

Assume now that $m \geq 2$.

By (2.12)–(2.14), we have

(2.15)

$$g^{(2m)}(z) - a = (2\lambda)^{2m} B^{2m} e^{-2m\lambda z} \left\{ \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m} \left(-\frac{A}{B} e^{2\lambda z}\right)^j - \left(-\frac{A}{B} e^{2\lambda z}\right)^m \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m} \right\}$$

and

(2.16)

$$g^{(2m+1)}(z) = (2\lambda)^{2m+1} B^{2m} e^{-2m\lambda z} \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m+1} \left(-\frac{A}{B} e^{2\lambda z}\right)^j.$$

Let

$$\omega = -\frac{A}{B} e^{2\lambda z}.$$

Since $g^{(2m)}(z) = a \Longrightarrow g^{(2m+1)}(z) = 0$, every solution of the equation

$$\sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m} \omega^j = \omega^m \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m} \tag{2.17}$$

is also a solution of the equation

$$\sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m+1} \omega^j = 0. \tag{2.18}$$

By (2.18) and (2.17), for every solution $\omega = \omega_0$ of (2.17), we have

$$\begin{aligned} \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m} j \omega_0^j &= m \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m} \omega_0^j \\ &= m \omega_0^m \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (j-m)^{2m}. \end{aligned}$$

Thus, since $\omega = 0$ is not a solution of (2.17), every solution of the equation (2.17) is multiple. Equation (2.17) can be rewritten as

$$\sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}(j-m)^{2m}(\omega^j+\omega^{2m-j}-2\omega^m)=0. \tag{2.19}$$

Denote the left side of (2.19) by $Q(\omega)$. Then $Q(\omega)$ is a polynomial with integer coefficients. It is easy to see that

$$Q(\omega)=(\omega-1)^2\sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}(j-m)^{2m}\omega^j\left(\sum_{s=0}^{m-1-j}\omega^s\right)^2. \tag{2.20}$$

By the factorization theorem for polynomials in $\mathbb{Z}[\omega]$ (see [4, pp. 134,167]), we have

$$Q(\omega)=N_0(\omega-1)^{p_0}Q_1^{p_1}(\omega)Q_2^{p_2}(\omega)\cdot\cdots\cdot Q_n^{p_n}(\omega), \tag{2.21}$$

where $Q_j(\omega)$ $(1 \le j \le n)$ are distinct primitive irreducible polynomials in $\mathbb{Z}[\omega]$, p_j $(\ge 2, 0 \le j \le n)$ are integers, and N_0 is the greatest common divisor of the coefficients of $Q(\omega)$ and hence also of the coefficients of $Q(\omega)/(\omega-1)^2$.

Now we discuss two subcases.

Case 2.2.1. $m \ge 2$ is even. Let

$$a_j=(-1)^j\frac{1}{2m}\binom{2m}{j}(j-m)^{2m}\ (0\le j\le m-1).$$

Then a_j are integers for $j = 0, 1, \ldots, m-1$, and

$$a_0=\frac{1}{2}m^{2m-1}=2k_1,\quad a_1=-(m-1)^{2m}=2k_2+1,$$

where k_1 and k_2 are integers.

Then $N_0 = 2m(2l + 1)$, where l is an integer; and $R(\omega) = Q(\omega)/(2m)$ has integer coefficients. By (2.20), we have

$$\begin{aligned} R(\omega)&=(\omega-1)^2\sum_{j=0}^{m-1}a_j\omega^j\left(\sum_{s=0}^{m-1-j}\omega^s\right)^2 \\ &=2k_1(\omega-1)^2\left(\sum_{s=0}^{m-1}\omega^s\right)^2+\omega\left[(2k_2+1)(\omega-1)^2\left(\sum_{s=0}^{m-2}\omega^s\right)^2\right. \\ &\qquad\left.+\sum_{j=2}^{m-1}a_j\omega^{j-1}(\omega-1)^2\left(\sum_{s=0}^{m-1-j}\omega^s\right)^2\right] \\ &=2k_1A(\omega)+\omega\left[(2k_2+1)B(\omega)+C(\omega)\right], \end{aligned} \tag{2.22}$$

where

$$A(\omega) = (\omega-1)^2\left(\sum_{s=0}^{m-1}\omega^s\right)^2,$$
$$B(\omega) = (\omega-1)^2\left(\sum_{s=0}^{m-2}\omega^s\right)^2,$$
$$C(\omega) = \sum_{j=2}^{m-1}a_j\omega^{j-1}(\omega-1)^2\left(\sum_{s=0}^{m-1-j}\omega^s\right)^2.$$

Hence by (2.21), we get

$$\begin{aligned}(2l+1)(\omega-1)^{p_0}Q_1^{p_1}(\omega)Q_2^{p_2}(\omega)\cdot\dots\cdot Q_n^{p_n}(\omega)\\ = 2k_1A(\omega)+\omega[(2k_2+1)B(\omega)+C(\omega)].\end{aligned} \tag{2.23}$$

Let $\omega = 0$. Then we have

$$(2l+1)(-1)^{p_0}Q_1^{p_1}(0)Q_2^{p_2}(0)\cdot\dots\cdot Q_n^{p_n}(0) = 2k_1.$$

Hence there exists j such that $Q_j^{p_j}(0)$ is an even number. Without loss of generality, we may assume $j=1$. Thus $Q_1(0)$ is an even number, say $Q_1(0) = 2k_3$, where k_3 is an integer. Hence

$$Q_1(\omega) = \omega Q_{11}(\omega)+Q_1(0) = \omega Q_{11}(\omega)+2k_3. \tag{2.24}$$

Thus by (2.23) and (2.24),

$$\begin{aligned}(2l+1)(\omega-1)^{p_0}[\omega^{p_1}Q_{11}^{p_1}(\omega)+2k_3D(\omega)]Q_2^{p_2}(\omega)\cdot\dots\cdot Q_n^{p_n}(\omega)\\ = 2k_1A(\omega)+\omega[(2k_2+1)B(\omega)+C(\omega)],\end{aligned}$$

where $D(\omega)$ is a polynomial with integer coefficients. Hence

$$\begin{aligned}(2l+1)(\omega-1)^{p_0}\omega^{p_1}Q_{11}^{p_1}(\omega)Q_2^{p_2}(\omega)\cdot\dots\cdot Q_n^{p_n}(\omega)\\ +2(2l+1)k_3D(\omega)(\omega-1)^{p_0}Q_2^{p_2}(\omega)\cdot\dots\cdot Q_n^{p_n}(\omega)\\ = 2k_1A(\omega)+\omega[(2k_2+1)B(\omega)+C(\omega)].\end{aligned} \tag{2.25}$$

Differentiating the two sides of (2.25) yields

$$\begin{aligned}(2l+1)p_1\omega^{p_1-1}(\omega-1)^{p_0}Q_{11}^{p_1}(\omega)Q_2^{p_2}(\omega)\cdot\dots\cdot Q_n^{p_n}(\omega)\\ +(2l+1)\omega^{p_1}[(\omega-1)^{p_0}Q_{11}^{p_1}(\omega)Q_2^{p_2}(\omega)\cdot\dots\cdot Q_n^{p_n}(\omega)]'\\ +2(2l+1)k_3[D(\omega)(\omega-1)^{p_0}Q_2^{p_2}(\omega)\cdot\dots\cdot Q_n^{p_n}(\omega)]'\\ = 2k_1A'(\omega)+[(2k_2+1)B(\omega)+C(\omega)]\\ +\omega[(2k_2+1)B'(\omega)+C'(\omega)].\end{aligned} \tag{2.26}$$

Setting $\omega = 0$ in (2.26), we see that $2k_2+1$ must be even, a contradiction.

Case 2.2.2. $m \geq 3$ is odd. Let p be a prime divisor of m, and set

$$b_j = (-1)^j \frac{1}{m} \binom{2m}{j} (j-m)^{2m} \quad (0 \leq j \leq m-1).$$

Then b_j are integers for $j = 0, 1, \ldots, m-1$, and

$$b_0 = m^{2m-1} = k_1 p, \quad b_1 = -2(m-1)^{2m} = k_2 p - 2,$$

where k_1 and k_2 are integers. Then $N_0 = m(lp+q)$, where l, q are integers and $1 \leq q \leq p-1$; and $S(\omega) = Q(\omega)/m$ has integer coefficients. By (2.20), we have

$$\begin{aligned}
(2.27) \quad S(\omega) &= (\omega-1)^2 \sum_{j=0}^{m-1} b_j \omega^j \left(\sum_{s=0}^{m-1-j} \omega^s \right)^2 \\
&= k_1 p (\omega-1)^2 \left(\sum_{s=0}^{m-1} \omega^s \right)^2 + \omega \left[(k_2 p + 2)(\omega-1)^2 \left(\sum_{s=0}^{m-2} \omega^s \right)^2 \right. \\
&\qquad \left. + \sum_{j=2}^{m-1} b_j \omega^{j-1} (\omega-1)^2 \left(\sum_{s=0}^{m-1-j} \omega^s \right)^2 \right] \\
&= k_1 p A(\omega) + \omega[(k_2 p - 2) B(\omega) + C(\omega)],
\end{aligned}$$

where $A(\omega)$, $B(\omega)$, and $C(\omega)$ are as in (2.22).

Using an argument similar to that in Case 2.2.1, we obtain the contradiction that $k_2 p - 2 = \lambda p$, where k_2, λ are integers and $p \geq 3$ is a prime number. We omit the details. This completes the proof of Lemma 5. □

In a similar way, we can prove the following result.

LEMMA 6. *Let g be a nonconstant entire function with $\rho(g) \leq 1$ whose zeros are of multiplicity at least 2; let a be a nonzero finite value; and let $s \geq 4$ be an even integer. If $g(z) = 0 \Longrightarrow g''(z) = a$ and $g''(z) = a \Longrightarrow g'''(z) = g^{(s)}(z) = 0$, then $g(z) = a(z-z_0)^2/2$, where z_0 is a constant.*

LEMMA 7 ([7, Corollary to Theorem 3.5]). *Let f be a transcendental meromorphic function, and let a be a non-zero value. Then, for each positive integer k, either f or $f^{(k)} - a$ has infinitely many zeros.*

LEMMA 8. *Let g be a nonconstant entire function with $\rho(g) \leq 1$; let $k \geq 2$ be an integer; and let a be a nonzero finite value. If $g(z) = 0 \Longrightarrow g'(z) = a$, and $g'(z) = a \Longrightarrow g^{(k)}(z) = 0$, then*

$$(2.28) \qquad g(z) = a(z-z_0),$$

where z_0 is a constant.

Proof. Suppose that g is a nonconstant polynomial. Since $g(z) = 0 \Longrightarrow g'(z) = a$, all zeros of g are simple. Let

$$g(z) = a_l z^l + a_{l-1} z^{l-1} + \cdots + a_0, \quad \text{where} \quad a_l \neq 0.$$

Then there exist $z_1, z_2, \ldots, z_l$ such that $g(z_j) = 0$ $(j = 1, 2, \ldots, l)$ and $z_i \neq z_j$. Hence $g'(z_j) = a$ for $j = 1, 2, \ldots, l$, so $g'(z) \equiv a$, and $l = 1$. Thus we get (2.28).

Assume now that g is transcendental. Using the same reasoning as in Lemma 5, we see that

$$P = \frac{g^{(k)}}{g} \tag{2.29}$$

is a nonzero constant. Let $c^k = 1/P$ and $f(z) = g(cz)$. Then, by (2.29), we have

$$f^{(k)} \equiv f, \tag{2.30}$$

and

$$f(z) = 0 \Longleftrightarrow f'(z) = ac. \tag{2.31}$$

By (2.30), we have

$$f(z) = \sum_{j=0}^{k-1} C_j \exp(\omega^j z), \tag{2.32}$$

where $\omega = \exp(2\pi i/k)$ and C_j are constants.

Since f is transcendental, there exists $C_j \in \{C_1, C_2, \ldots, C_{k-1}\}$ such that $C_j \neq 0$. We denote the nonzero constants in $\{C_j\}$ by C_{j_m} $(0 \leq j_m \leq k-1$, $m = 0, 1, \cdots, s,\ s \leq k-1)$. Thus we have

$$f(z) = \sum_{m=0}^{s} C_{j_m} \exp(\omega^{j_m} z). \tag{2.33}$$

By Lemma 7, f has infinitely many zeros $z_n = r_n e^{i\theta_n} (n = 1, 2, \cdots)$, where $0 \leq \theta_n < 2\pi$. Without loss of generality, we may assume that $\theta_n \to \theta_0$ and $r_n \to +\infty$ as $n \to \infty$.

Let

$$L = \max_{0 \leq m \leq s} \cos\left(\theta_0 + \frac{2 j_m \pi}{k}\right). \tag{2.34}$$

Then, either there exists an index m_0 such that $\cos(\theta_0 + 2j_{m_0}\pi/k) = L$ or there exist two indices m_1, m_2 $(m_1 \neq m_2)$ such that $\cos(\theta_0 + 2j_{m_1}\pi/k) = \cos(\theta_0 + 2j_{m_2}\pi/k) = L$.

We consider these cases separately.

Case 1. There exists an index m_0 such that

$$\cos\left(\theta_0 + \frac{2j_{m_0}\pi}{k}\right) = L > \cos\left(\theta_0 + \frac{2j_m\pi}{k}\right)$$

for $m \neq m_0$. Then there exists $\delta > 0$ such that for n sufficiently large,

$$\cos\left(\theta_n + \frac{2j_{m_0}\pi}{k}\right) - \cos\left(\theta_n + \frac{2j_m\pi}{k}\right) \geq \delta, \quad \text{for} \quad m \neq m_0. \tag{2.35}$$

Since

$$\sum_{m=0}^{s} C_{j_m} \exp(\omega^{j_m} z_n) = 0,$$

we have

$$C_{j_{m_0}} + \sum_{m \neq m_0} C_{j_m} \exp(\omega^{j_m} z_n - \omega^{j_{m_0}} z_n) = 0. \tag{2.36}$$

By (2.35),

$$\begin{aligned} &|\exp(\omega^{j_m} z_n - \omega^{j_{m_0}} z_n)| \\ &\qquad = \exp\left\{ r_n \left(\cos\left(\theta_n + \frac{2j_m\pi}{k}\right) - \cos\left(\theta_n + \frac{2j_{m_0}\pi}{k}\right)\right)\right\} \\ &\qquad \leq e^{-\delta r_n} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.37}$$

Thus from (2.36) and (2.37), we obtain $C_{j_{m_0}} = 0$, which contradicts our assumption.

Case 2. There exist two indices m_1, m_2 $(m_1 \neq m_2)$ such that

$$\cos\left(\theta_0 + \frac{2j_{m_1}\pi}{k}\right) = \cos\left(\theta_0 + \frac{2j_{m_2}\pi}{k}\right) = L > \cos\left(\theta_0 + \frac{2j_m\pi}{k}\right) \tag{2.38}$$

for $m \neq m_1, m_2$. Thus there exists $\delta > 0$ such that, for n sufficiently large,

$$\cos\left(\theta_n + \frac{2j_{m_1}\pi}{k}\right) - \cos\left(\theta_n + \frac{2j_m\pi}{k}\right) \geq \delta \quad (m \neq m_1,\ m_2). \tag{2.39}$$

Since $f(z_n) = 0$ and $f'(z_n) = ac$, we have

$$C_{j_{m_1}} \exp(\omega^{j_{m_1}} z_n) + C_{j_{m_2}} \exp(\omega^{j_{m_2}} z_n) + \sum_{m \neq m_1, m_2} C_{j_m} \exp(\omega^{j_m} z_n) = 0 \tag{2.40}$$

and

$$\begin{aligned} &C_{j_{m_1}} \omega^{j_{m_1}} \exp(\omega^{j_{m_1}} z_n) + C_{j_{m_2}} \omega^{j_{m_2}} \exp(\omega^{j_{m_2}} z_n) \\ &\qquad + \sum_{m \neq m_1, m_2} C_{j_m} \omega^{j_m} \exp(\omega^{j_m} z_n) = ac. \end{aligned} \tag{2.41}$$

Thus we get

$$C_{j_{m_1}}(\omega^{j_{m_1}} - \omega^{j_{m_2}})\exp(\omega^{j_{m_1}} z_n) + \sum_{m\neq m_1,m_2} C_{j_m}(\omega^{j_m} - \omega^{j_{m_2}})\exp(\omega^{j_m} z_n) = ac. \tag{2.42}$$

Using the same reasoning as that used in proving $C_{j_{m_0}} = 0$ above and the fact that $\omega^j \neq \omega^l$ $(j \neq l, 0 \leq j, l \leq k-1)$, we obtain

$$\exp(\omega^{j_{m_1}} z_n) \to c_0 \quad (n \to \infty), \tag{2.43}$$

where $c_0 \neq 0$ is a constant.

It follows that

$$\cos\left(\theta_0 + \frac{2j_{m_1}\pi}{k}\right) = \lim_{n\to\infty} \cos\left(\theta_n + \frac{2j_{m_1}\pi}{k}\right) = 0, \tag{2.44}$$

so by (2.38),

$$\cos\left(\theta_0 + \frac{2j_{m_2}\pi}{k}\right) = 0. \tag{2.45}$$

Thus, by (2.44)–(2.45), we have

$$\left|\frac{2j_{m_1}\pi}{k} - \frac{2j_{m_2}\pi}{k}\right| = \pi, \tag{2.46}$$

that is, $|j_{m_1} - j_{m_2}| = k/2$. Hence k is an even integer.

Without loss of generality, we may assume that

$$j_{m_2} = j_{m_1} + \frac{k}{2},\ \theta_0 + \frac{2j_{m_1}\pi}{k} = \frac{\pi}{2}. \tag{2.47}$$

Thus, by (2.38), (2.44), and (2.47), we have

$$\begin{aligned} 0 > \cos\left(\theta_0 + \frac{2j_m\pi}{k}\right) &= \cos\left[\left(\theta_0 + \frac{2j_{m_1}\pi}{k}\right) + \frac{2(j_m - j_{m_1})\pi}{k}\right] \\ &= \cos\left[\frac{\pi}{2} + \frac{2(j_m - j_{m_1})\pi}{k}\right] \\ &= -\sin\frac{2(j_m - j_{m_1})\pi}{k}, \end{aligned}$$

whence

$$\sin\frac{2(j_m - j_{m_1})\pi}{k} > 0,\ \text{for } m \neq m_1, m_2. \tag{2.48}$$

Also,

$$f(z) = C_{j_{m_1}} \exp(\omega^{j_{m_1}} z) + C_{j_{m_2}} \exp(-\omega^{j_{m_1}} z) + \sum_{m \neq m_1, m_2} C_{j_m} \exp(\omega^{j_m} z) = A \left\{ \exp[\omega^{j_{m_1}}(z+z_0)] - \exp[-\omega^{j_{m_1}}(z+z_0)] \right\} + \sum_{m \neq m_1, m_2} C_{j_m} \exp(\omega^{j_m} z), \tag{2.49}$$

where A and z_0 are constants satisfying

$$\exp(2\omega^{j_{m_1}} z_0) = -\frac{C_{j_{m_1}}}{C_{j_{m_2}}}, \; A = C_{j_{m_1}} \exp(-\omega^{j_{m_1}} z_0).$$

Set

$$F(z) = A \left\{ \exp[\omega^{j_{m_1}}(z+z_0)] - \exp[-\omega^{j_{m_1}}(z+z_0)] \right\}, \tag{2.50}$$

$$\phi(z) = \sum_{m \neq m_1, m_2} C_{j_m} \exp(\omega^{j_m} z). \tag{2.51}$$

Fix δ such that $0 < \delta < 1/2$. Then by (2.50), for any zero $z_n^* = -z_0 + n\pi i \omega^{-j_{m_1}}$ $(n = 1, 2, 3, \dots)$ of F, we have for $z = z_n^* + \delta e^{i\theta}$

$$|F(z)| = |A| \sqrt{\exp(2\delta c) + \exp(-2\delta c) - 2\cos(2\delta\sqrt{1-c^2})},$$

where $c = \cos(\theta + 2j_{m_1}\pi/k)$. Thus, for $z = z_n^* + \delta e^{i\theta}$, we have

$$\begin{aligned} |F(z)| &\geq |A| \sqrt{\exp(2\delta c) + \exp(-2\delta c) - 2\cos(2\delta)} \\ &\geq |A| \sqrt{2 - 2\cos(2\delta)} \\ &\geq 2|A| \sin\delta \geq |A|\delta. \end{aligned} \tag{2.52}$$

On the other hand, by (2.51) and (2.48),

$$\begin{aligned} |\phi(z)| &\leq \sum_{m \neq m_1, m_2} |C_{j_m}| |\exp(\omega^{j_m}(z - z_n^*))| |\exp(\omega^{j_m} z_n^*)| \\ &= \sum_{m \neq m_1, m_2} |C_{j_m}| \; |\exp(\omega^{j_m} \delta e^{i\theta})| \exp(\omega^{j_m}(-z_0 + n\pi i \omega^{-j_{m_1}})| \\ &\leq e \sum_{m \neq m_1, m_2} |C_{j_m}| \exp(-n\pi \sin \frac{2(j_m - j_{m_1})\pi}{k}) | \exp(-\omega^{j_m} z_0)| \\ &\to 0 \;\; (n \to +\infty, \; z = z_n^* + \delta e^{i\theta}). \end{aligned} \tag{2.53}$$

Hence, by Rouché's Theorem, for every large positive integer n, there exists $z_n^{(1)} \in \Delta_\delta = \{z : |z| < \delta\}$ such that $z_n = z_n^* + z_n^{(1)}$ is a zero of f, that is,

$$f(z_n) = 0. \tag{2.54}$$

Without loss of generality, we may assume that

$$z_{2n}^{(1)} \to z_0^{(1)} \in \Delta_\delta, \quad (n \to \infty), \tag{2.55}$$

$$z_{2n+1}^{(1)} \to z_1^{(1)} \in \Delta_\delta, \quad (n \to \infty). \tag{2.56}$$

By (2.54), (2.43) and (2.47), we have

$$\exp(\omega^{j_{m_1}} z_{2n}^{(1)}) = \exp(\omega^{j_{m_1}} z_{2n}) \exp(\omega^{j_{m_1}} z_0) \to c_0 \exp(\omega^{j_{m_1}} z_0) \tag{2.57}$$

and

$$-\exp(\omega^{j_{m_1}} z_{2n+1}^{(1)}) = \exp(\omega^{j_{m_1}} z_{2n+1}) \exp(\omega^{j_{m_1}} z_0) \to c_0 \exp(\omega^{j_{m_1}} z_0). \tag{2.58}$$

It follows from (2.55)–(2.58) that

$$\exp(\omega^{j_{m_1}} z_0^{(1)}) + \exp(\omega^{j_{m_1}} z_1^{(1)}) = 0,$$

which leads to the contradiction $\pi \le |z_0^{(1)} - z_1^{(1)}| \le 2\delta \le 1$. The proof of Lemma 8 is complete. □

3. Proofs of Theorems 1–4

Proof of Theorem 1. It suffices to show that $\mathcal{F}$ is normal on each disc Δ contained, with its closure, in D. We may assume that Δ is the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1, we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, $|z_n| < r < 1$, and $\rho_n \to 0^+$ such that $g_n(\zeta) = \rho_n^{-k} f_n(z_n + \rho_n \zeta)$ converges locally uniformly to a nonconstant entire function g on $\mathbb{C}$, which satisfies $g^\#(\zeta) \le g^\#(0) = k(|d| + 1) + 1$, where $d = \max\{|a(z)| : |z| \le 1\}$, and the zeros of g are of multiplicity at least k. By Lemma 2, $\rho(g) \le 1$. Taking a subsequence and renumbering, we may assume that $z_n \to z_0 \in \Delta$.

We claim

(i) $g(\zeta) = 0 \Longrightarrow g^{(k)}(\zeta) = a(z_0)$; and
(ii) $g^{(k)}(\zeta) = a(z_0) \Longrightarrow g^{(k+1)}(\zeta) = 0$.

Suppose that $g(\zeta_0) = 0$. Then by Hurwitz's Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$g_n(\zeta_n) = \rho_n^{-k} f_n(z_n + \rho_n \zeta_n) = 0.$$

Thus $f_n(z_n + \rho_n \zeta_n) = 0$. Since $f_n(\zeta) = 0 \Longrightarrow f_n^{(k)}(\zeta) = a(\zeta)$, we have

$$g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a(z_n + \rho_n \zeta_n).$$

Hence $g^{(k)}(\zeta_0) = \lim_{n\to\infty} g_n^{(k)}(\zeta_n) = a(z_0)$. Thus $g(\zeta) = 0 \Longrightarrow g^{(k)}(\zeta) = a(z_0)$. This proves (i).

Next we prove (ii). Suppose that $g^{(k)}(\zeta_0) = a(z_0)$. Then $g(\zeta_0) \ne \infty$. Further, $g^{(k)}(\zeta) \not\equiv a(z_0)$, for otherwise $g(\zeta) = \frac{a(z_0)}{k!}(\zeta - \zeta_1)^k$. A simple calculation then shows that

$$g^\#(0) \le \begin{cases} k/2 & \text{if } |\zeta_1| \ge 1, \\ |a(z_0)| & \text{if } |\zeta_1| < 1, \end{cases}$$

so that $g^{\#}(0) < k(|d|+1)+1$, a contradiction. Since $g^{(k)}(\zeta_0) - a(z_0) = 0$ and $g_n^{(k)}(z_n+\rho_n\zeta) - a(z_n+\rho_n\zeta) \to g^{(k)}(\zeta) - a(z_0)$ on a neighborhood of ζ_0, by Hurwitz's Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large) $f_n^{(k)}(z_n+\rho_n\zeta_n) = g_n^{(k)}(\zeta_n) = a(z_n+\rho_n\zeta_n)$. It follows that $|f_n^{(k+1)}(z_n+\rho_n\zeta_n)| \le h$, so that $|g_n^{(k+1)}(\zeta_n)| = |\rho_n f_n^{(k+1)}(z_n+\rho_n\zeta_n)| \le \rho_n h$. Thus $g^{(k+1)}(\zeta_0) = \lim_{n\to\infty} g_n^{(k+1)}(\zeta_n) = 0$. This proves (ii).

Thus, by Lemma 5, $g(\zeta) = (a(z_0)/k!)(\zeta-\zeta_1)^k$. It follows that $g^{\#}(0) < k(|d|+1)+1$, which is a contradiction. Thus $\mathcal{F}$ is normal on Δ and hence on D. □

Proof of Theorem 2. We may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1, we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, and $\rho_n \to 0^+$ such that $g_n(\zeta) = \rho_n^{-2} f_n(z_n+\rho_n\zeta)$ converges locally uniformly to a nonconstant entire function g, all of whose zeros are multiple, which satisfies $g^{\#}(\zeta) \le g^{\#}(0) = 2(|a|+1)+1$. By Lemma 2, $\rho(g) \le 1$.

As in the proof of Theorem 1, we have

(i) $g(\zeta) = 0 \Longrightarrow g''(\zeta) = a$; and
(ii) $g''(\zeta) = a \Longrightarrow g'''(\zeta) = 0$.

If $g \neq 0$, then $g(\zeta) = e^{A\zeta+B}$, where $A \neq 0$, B are constants. Thus

$$g''(\zeta) = A^2 e^{Az+B}, \text{ and } g'''(\zeta) = A^3 e^{A\zeta+B}.$$

Let $g''(\zeta_0) = a$. Then $A^3 e^{A\zeta_0+B} = g'''(\zeta_0) = 0$, which is impossible. Hence, there exists ζ_0 such that $g(\zeta_0) = 0$. Now $g'' \not\equiv a$, for otherwise $g(\zeta) = \frac{a}{2}(\zeta-\zeta_1)^2$ which, as in the proof of Theorem 1, would contradict $g^{\#}(0) = 2(|a|+1)+1$. Thus by (i) and (ii), ζ_0 is a zero of $g''(\zeta) - a$ with multiplicity $m \ge 2$. Hence $g^{(2+m)}(\zeta_0) \neq 0$, and there exists $\delta > 0$ such that for $|\zeta-\zeta_0| < \delta$,

$$g^{(2+m)}(\zeta) \neq 0. \tag{3.1}$$

So, by Hurwitz's theorem, there exist m sequences $\{\zeta_{in}\}$, $i = 1,2,\ldots,m$, such that $\lim_{n\to\infty} \zeta_{in} = \zeta_0$, and for large n,

$$g_n''(\zeta_{in}) = a, \quad i = 1,2,\ldots,m. \tag{3.2}$$

Hence, by $f_n''(z) = a \Longrightarrow f_n'''(z) \neq 0$, we have

$$g_n'''(\zeta_{in}) = \rho_n f_n'''(z_n+\rho_n\zeta_{in}) \neq 0, \ (i = 1,2,\ldots,m). \tag{3.3}$$

Thus

$$\zeta_{in} \neq \zeta_{jn}, \quad 1 \le i < j \le m. \tag{3.4}$$

Hence by (3.2) and (3.4), $g^{(2+m)}(\zeta_0) = 0$, which contradicts (3.1).

Hence $\mathcal{F}$ is normal in D. This proves Theorem 2. □

Proof of Theorem 3. As in the proof of Theorem 1, we show that $\mathcal{F}$ is normal on each disc Δ contained, with its closure, in D. We may assume that Δ is the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1, we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, $|z_n| < r < 1$, and $\rho_n \to 0^+$ such that $g_n(\zeta) = \rho_n^{-2} f_n(z_n + \rho_n\zeta)$ converges locally uniformly to a nonconstant entire function g, which satisfies $g^{\#}(\zeta) \le g^{\#}(0) = 2(|d| + 1) + 1$, where $d = \max\{|a(z)| : |z| \le 1\}$. As before, we may also assume that $z_n \to z_0 \in \Delta$.

As in the proof of Theorem 1, we have

(i) $g(\zeta) = 0 \Longrightarrow g''(\zeta) = a(z_0)$; and

(ii) $g''(\zeta) = a(z_0) \Longrightarrow g'''(\zeta) = g^{(s)}(\zeta) = 0$.

Thus by Lemma 6, $g(\zeta) = a(z_0)(\zeta - \zeta_1)^2/2$. But then $g^{\#}(0) < 2(|d| + 1) + 1$, which is a contradiction.

Thus $\mathcal{F}$ is normal on Δ and hence on D. This proves Theorem 3. □

Proof of Theorem 4. Again we prove that $\mathcal{F}$ is normal on each disc Δ contained, with its closure, in D. As before, we may assume that Δ is the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1, we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, $|z_n| < r < 1$, and $\rho_n \to 0^+$ such that $g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n\zeta)$ converges locally uniformly to a nonconstant entire function g on $\mathbb{C}$ which satisfies $g^{\#}(\zeta) \le g^{\#}(0) = |d| + 2$, where $d = \max\{|a(z)| : |z| \le 1\}$. Moreover, g is of order at most one. Again, we may assume that $z_n \to z_0 \in \Delta$.

As in the proof of Theorem 1, we have

(i) $g(\zeta) = 0 \Longrightarrow g'(\zeta) = a(z_0)$; and

(ii) $g'(\zeta) = a(z_0) \Longrightarrow g^{(k)}(\zeta) = 0$.

Thus by Lemma 8, $g(\zeta) = a(z_0)(\zeta - \zeta_1)$. So $g^{\#}(0) \le |a(z_0)| < |d| + 2$, a contradiction.

Thus $\mathcal{F}$ is normal on Δ and hence on D. This completes the proof of Theorem 4. □

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