Illinois Journal of Mathematics
Volume 48, Number 1, Spring 2004, Pages 273–294
S 0019-2082

MINIMAL RELATIVE HILBERT-KUNZ MULTIPLICITY

KEI-ICHI WATANABE AND KEN-ICHI YOSHIDA

Abstract. In this paper we ask the following question: What is the minimal value of the difference $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I')$ for ideals $I' \supseteq I$ with $l_A(I'/I) = 1$? In order to answer to this question, we define the notion of *minimal relative Hilbert-Kunz multiplicity* for strongly F-regular rings. We calculate this invariant for quotient singularities and for the coordinate rings of Segre embeddings: $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \hookrightarrow \mathbb{P}^{rs-1}$.

Introduction

Throughout this paper, let A be a Noetherian ring containing a field of characteristic $p > 0$. The purpose of this paper is to introduce the notion of minimal relative Hilbert-Kunz multiplicity, which is a new invariant of local rings in positive characteristic.

The notion of Hilbert-Kunz multiplicity has been introduced by Kunz [Ku1] in 1969, and has been studied in detail by Monsky [Mo]; see also, e.g., [BC], [BCP], [Co], [HaM], [Se], [WaY1], [WaY2], [WaY3].

Further, Hochster and Huneke [HH2] have pointed out that the tight closure I^* of I is the largest ideal containing I having the same Hilbert-Kunz multiplicity as I; see Lemma 1.3. Thus it seems to be important to understand Hilbert-Kunz multiplicities well. For example, the authors [WaY1] have proved that an unmixed local ring whose Hilbert-Kunz multiplicity is one is regular. Also, they [WaY3] have given a formula for $e_{\mathrm{HK}}(I)$ for any integrally closed ideal I in a two-dimensional F-rational double point using McKay correspondence and the Riemann–Roch formula.

One of the most important conjectures about Hilbert-Kunz multiplicities is that it is always a rational number. Let A be a local ring and I, J be $\mathfrak{m}$-primary ideals in A. Also, suppose that J is a parameter ideal. Then it is known that $e_{\mathrm{HK}}(J) = e(J)$, the usual multiplicity (and hence $e_{\mathrm{HK}}(J)$ is an

Received March 19, 2003; received in final form December 27, 2003.
2000 *Mathematics Subject Classification.* Primary 13D40, 13A35. Secondary 13H10, 13H15.
The first author was supported in part by Grant Aid in Scientific Research, # 13440015 and # 13874006. The second author was supported in part by NSF Grant #14540020.

integer). In order to investigate the value of $e_{\mathrm{HK}}(I)$, we study the difference "$e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(I)$". Then it is natural to ask the following question.

QUESTION. What is the minimal value of the difference $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I')$ for $\mathfrak{m}$-primary ideals $I' \supseteq I$ with $l_A(I'/I) = 1$?

To answer to this question, we introduce the notion of *minimal relative Hilbert-Kunz multiplicity* $m_{\mathrm{HK}}(A)$ as follows:

$$m_{\mathrm{HK}}(A) = \liminf_{e\to\infty} \frac{l_A(A/\operatorname{ann}_A z^{p^e})}{p^{ed}},$$

where z is a generator of the socle of the injective hull $E_A(A/\mathfrak{m})$. Then we can show that $m_{\mathrm{HK}}(A) \le e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I')$ for ($\mathfrak{m}$-primary) ideals $I \subseteq I'$ with $l_A(I'/I) = 1$. Also, we believe that equality holds for some pair (I, I'). This is true if A is a Gorenstein local ring. Namely, if A is a Gorenstein local ring, then

$$e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(J : \mathfrak{m}) = m_{\mathrm{HK}}(A)$$

for any parameter ideal J of A; see Theorem 2.1 for details.

In general, if A is not weakly F-regular, then $m_{\mathrm{HK}}(A) = 0$. Thus it suffices to consider weakly F-regular local rings in our context.

In Section 3, we will give a formula for minimal relative Hilbert-Kunz multiplicities of the canonical cover of $\mathbb{Q}$–Gorenstein F-regular local rings:

THEOREM 1 (see Theorem 3.1). *Let A be a $\mathbb{Q}$–Gorenstein strongly F-regular local ring of characteristic $p > 0$. Also, let $B = A \oplus K_A t \oplus K_A^{(2)} t^2 \oplus \cdots \oplus K_A^{(r-1)} t^{r-1}$, the canonical cover of A, where $r = \operatorname{ord}(\operatorname{cl}(K_A))$, $K_A^{(r)} = fA$ and $ft^r = 1$. Also, suppose that $(r, p) = 1$. Then we have*

$$m_{\mathrm{HK}}(B) = r \cdot m_{\mathrm{HK}}(A).$$

In Section 4, as an application of Theorem 3.1, we will give a formula for minimal relative Hilbert-Kunz multiplicities of quotient singularities.

THEOREM 2 (see Theorem 4.2). *Let k be a field of characteristic $p > 0$, and let $A = k[x_1, \dots, x_d]^G$ be the invariant subring by a finite subgroup G of $GL(d, k)$ with $(p, |G|) = 1$. Also, assume that G contains no pseudo-reflections. Then $m_{\mathrm{HK}}(A) = 1/|G|$.*

In Section 5, we will give a formula for minimal relative Hilbert-Kunz multiplicities of normal toric rings and Segre products.

THEOREM 3 (see Theorem 5.8). *Let $A = k[x_1, \dots, x_r]\#k[y_1, \dots, y_s]$, where $2 \le r \le s$, and put $d = r + s - 1$. Then*

$$m_{\mathrm{HK}}(A) = \frac{r!}{d!} S(d, r) + \frac{1}{d!} \sum_{k=1}^{r-1} \sum_{j=1}^{r-k} \binom{r}{k+j} \binom{s}{j} (-1)^{r+k} k^d,$$

where $S(n,k)$ denotes the Stirling number of the second kind (see Section 5).

In particular,

$$e_{\mathrm{HK}}(A)+m_{\mathrm{HK}}(A)=\frac{r!\cdot S(d,r)+s!\cdot S(d,s)}{d!}.$$

Huneke and Leuschke [HuL] (see also [AL]) defined the notion of "F-signature" as follows: Let $(A,\mathfrak{m},k)$ be an F-finite reduced local ring of characteristic $p>0$. Put $\alpha(A)=\log_p[k:k^p]$. For each $q=p^e$, decompose $A^{1/q}$ as a direct sum of finitely generated A-modules $A^{a_q}\oplus M_q$, where M_q has no nonzero free direct summands. The *F-signature* $s(A)$ of A is

$$s(A)=\lim_{q\to\infty}\frac{a_q}{q^{d+\alpha(A)}}$$

provided the limit exists.

The referee pointed out that Yao [Ya] recently proved that the F-signature coincides with our minimal relative Hilbert-Kunz multiplicity.

Acknowledgement. The authors wish to thank the referee for many valuable comments.

1. Minimal relative Hilbert-Kunz multiplicity

In this section, we define the notion of *minimal relative Hilbert-Kunz multiplicity* and give its fundamental properties. In the following, we let $(A,\mathfrak{m},k)$ be a Noetherian excellent reduced local ring containing an infinite field of characteristic $p>0$, unless specified. We let E_A denote the injective hull of the residue field $k=A/\mathfrak{m}$, and $H^i_{\mathfrak{m}}(A)$ the ith local cohomology module of A with support in $\{\mathfrak{m}\}$. We always suppose that A is a homomorphic image of a Gorenstein local ring, and we let K_A denote a canonical module of A.

1.1. Peskine-Szpiro functor. First, let us recall the definition of the Peskine–Szpiro functor. Let ${}^eA$ denote the ring A viewed as an A-algebra via $F^e\colon A\to A$ $(a\mapsto a^{p^e})$. Then $\mathbb{F}^e_A(-)={}^eA\otimes_A -$ is a covariant functor from the category of A-modules to itself. Since ${}^eA$ is isomorphic to A as rings (via F^e), we can regard $\mathbb{F}^e_A$ as a covariant functor from A-modules to themselves. We call this functor $\mathbb{F}^e_A$ *the Peskine–Szpiro functor* of A. The A-module structure on $\mathbb{F}^e_A(M)$ is such that $a'(a\otimes m)=a'a\otimes m$. On the other hand, $a'\otimes am=a'a^q\otimes m$; see, e.g., [PS], [Hu]. Suppose that an A-module M has a finite presentation $A^m\xrightarrow{\phi}A^n\to M\to 0$, where the map ϕ is defined by a matrix (a_{ij}). Then $\mathbb{F}^e_A(M)$ has a finite presentation $A^m\xrightarrow{\phi_q}A^n\to\mathbb{F}^e_A(M)\to 0$, where the map ϕ_q is defined by the matrix (a_{ij}^q). For example, $\mathbb{F}^e_A(A/I)=A/I^{[p^e]}$, where $I^{[p^e]}$ is the ideal generated by $\{a^{p^e}:a\in I\}$.

Also, one can identify the Frobenius map $F^e\colon A \to {}^eA$ with the embedding $A \hookrightarrow A^{1/q}$ $(q = p^e)$.

1.2. Tight closure, Hilbert-Kunz multiplicity. Using the Peskine–Szpiro functor, we define the notion of tight closure.

Definition 1.1 ([HH1], [HH2], [Hu]).

(1) Let M be an A-module, and let N be an A-submodule of M. Put $N_M^{[p^e]} = \mathrm{Ker}(\mathbb{F}_A^e(M) \to \mathbb{F}_A^e(M/N))$, and denote by x^q $(q = p^e)$ the image of x under the Frobenius map $M \to \mathbb{F}_A^e(M)$ $(x \mapsto 1 \otimes x)$. Then the *tight closure* N_M^* of N (in M) is the submodule generated by elements for which there exists an element $c \in A^0 := A \setminus \bigcup_{P \in \mathrm{Min}(A)} P$ such that for all sufficiently large $q = p^e$, $cx^q \in N_M^{[q]}$. By definition, we put $I^* = I_A^*$. Also, we say that N is *tightly closed* (in M) if $N_M^* = N$.

(2) A local ring A in which every ideal is tightly closed is called *weakly F-regular*. A ring whose localization is always weakly F-regular is called *F-regular*.

(3) Suppose that A is F-finite, that is, ${}^1A$ is finitely generated as an A-module. A is said to be *strongly F-regular* if for any element $c \in A^0$ there exists $q = p^e$ such that the A-linear map $A \to A^{1/q}$ defined by $a \to c^{1/q}a$ is split injective.

(4) A Noetherian ring R is F-regular (resp. weakly F-regular, strongly F-regular) if and only if so is $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$.

Remark 1. Strongly F-regular rings are F-regular. In general, it is not known whether the converse is true, but it is known that F-finite $\mathbb{Q}$–Gorenstein weakly F-regular rings are always strongly F-regular; see [AM], [Mc], [Wi].

The notion of Hilbert-Kunz multiplicity plays the central role in this paper.

Definition 1.2 ([Ku2], [Mo]). Let I be an $\mathfrak{m}$-primary ideal in A and M a finite A-module. Then we define the *Hilbert-Kunz multiplicity* $e_{\mathrm{HK}}(I, M)$ of I with respect to M as

$$e_{\mathrm{HK}}(I,\, M) := \lim_{e\to\infty} \frac{l_A(M/I^{[p^e]}M)}{p^{de}}.$$

By definition, we put $e_{\mathrm{HK}}(I) := e_{\mathrm{HK}}(I, A)$ and $e_{\mathrm{HK}}(A) := e_{\mathrm{HK}}(\mathfrak{m})$.

Also, the *multiplicity* $e(I)$ of I is defined as

$$e(I) = \lim_{n\to\infty} \frac{d! \cdot l_A(A/I^n)}{n^d}.$$

Let $I \subseteq I'$ be $\mathfrak{m}$-primary ideals in A. Then it is known that I' and I have the same integral closure (i.e., $\overline{I'} = \overline{I}$) if and only if $e(I) = e(I')$. A similar result holds for tight closures and the Hilbert-Kunz multiplicities.

LEMMA 1.3 (cf. [HH2, Theorem 8.17]). *Let $I \subseteq I'$ be $\mathfrak{m}$-primary ideals in A.*

(1) *If $I' \subseteq I^*$, then $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(I')$.*
(2) *Assume further that A is equidimensional. Then the converse of* (1) *is also true.*

1.3. Minimal relative Hilbert-Kunz multiplicity. Our work is motivated by the following question.

QUESTION 1.4. What is the minimal value of the difference $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I')$ for $\mathfrak{m}$-primary ideals $I' \supseteq I$ with $l_A(I'/I) = 1$?

In order to represent the "difference", we define the following notion.

DEFINITION 1.5 (Relative Hilbert-Kunz multiplicity). Let L be an A-module, and let $N \subseteq M$ be finite A-submodules of L with $l_A(M/N) < \infty$. Then we set

$$e_{\mathrm{HK}}(N, M; L) := \liminf_{e\to\infty} \frac{l_A(M_L^{[p^e]}/N_L^{[p^e]})}{p^{de}}.$$

We call $e_{\mathrm{HK}}(N, M; L)$ the *relative Hilbert-Kunz multiplicity* with respect to $N \subseteq M$ of L. In particular, $e_{\mathrm{HK}}(I, I'; A) = e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I')$ for $\mathfrak{m}$-primary ideals $I \subseteq I'$ in A.

Using the notion of relative Hilbert-Kunz multiplicity, we introduce the following two notions.

DEFINITION 1.6 (Minimal relative Hilbert-Kunz multiplicity). Let z be a generator of the socle $\mathrm{Soc}(E_A) := \{x \in E_A \mid \mathfrak{m}x = 0\}$ of E_A. Then we put

$$m_{\mathrm{HK}}(A) := e_{\mathrm{HK}}(0, \mathrm{Soc}(E_A); E_A) = \liminf_{e\to\infty} \frac{l_A(A/\operatorname{ann}_A(z^{p^e}))}{p^{ed}},$$

where $z^{p^e} = \mathbb{F}_A^e(z) \in \mathbb{F}_A^e(E_A)$. We call $m_{\mathrm{HK}}(A)$ *the minimal relative Hilbert-Kunz multiplicity* of A. Also, we put

$$\widetilde{m}_{\mathrm{HK}}(A) := \inf\{e_{\mathrm{HK}}(I, I'; A) \mid I \subseteq I' \subseteq A \text{ such that } l_A(I'/I) = 1\}.$$

We call $\widetilde{m}_{\mathrm{HK}}(A)$ *the minimal relative Hilbert-Kunz multiplicity for cyclic modules* of A.

The following proposition justifies our definition of *minimal* relative Hilbert–Kunz multiplicity.

PROPOSITION 1.7. *$m_{\mathrm{HK}}(A)$ is the minimal number among all relative Hilbert-Kunz multiplicities of all A-modules. That is,*

$$m_{\mathrm{HK}}(A) = \inf\left\{ e_{\mathrm{HK}}(N, M; L) \;\middle|\; \begin{array}{l} L\text{: } A\text{-module} \\ N \subseteq M\text{: finite } A\text{-submodules of } L \\ \text{with } l_A(M/N) = 1. \end{array} \right\}.$$

In particular, $m_{\mathrm{HK}}(A) \le \widetilde{m}_{\mathrm{HK}}(A)$.

Proof. Since $E_A \cong E_{\hat{A}}$, $m_{\mathrm{HK}}(A) = m_{\mathrm{HK}}(\hat{A})$. Also, since $e_{\mathrm{HK}}(\widehat{N}, \widehat{M}; L \otimes_A \widehat{A}) = e_{\mathrm{HK}}(N, M; L)$, we may assume A is complete. Let L be an A-module and let $N \subseteq M$ be A-submodules of L with $l_A(M/N) = 1$.

Let z be a generator of the socle of E_A and take an element $x \in M \setminus N$ such that $M = N + Ax$ with $\mathfrak{m}x \subseteq N$. By Matlis duality, one can take a nonzero homomorphism $\phi \in \mathrm{Hom}_A(M, E_A)$ such that $\phi(N) = 0$ and $\phi(M) \neq 0$. Then we may assume $\phi(x) = z$, since $\phi(x)$ is a generator of $\mathrm{Soc}(E_A)$.

It suffices to show that $\mathrm{ann}_A(x^q + N^{[q]}) \subset \mathrm{ann}_A(z^q)$. But this is clear, since if $ax^q = 0$, then $az^q = a\phi(x^q) = \phi(ax^q) = 0$. □

Now let $(A, \mathfrak{m}, k)$ be a d-dimensional Cohen–Macaulay local ring of characteristic $p > 0$. Then the highest local cohomology $H^d_{\mathfrak{m}}(A)$ may be identified with $\varinjlim A/(a_1^n, \ldots, a_d^n)A$, where $a_1, a_2, \ldots, a_d$ is a system of parameters for A and the maps in the direct limit system are given by multiplication by $a = \prod_{i=1}^d a_i$. Any element $\eta \in H^d_{\mathfrak{m}}(A)$ can be represented as the equivalence class $[x + (a_1^n, \ldots, a_d^n)]$ for some $x \in A$ and some integer $n \ge 1$.

Considering the Frobenius action to $H^d_{\mathfrak{m}}(A)$, we have

$$\mathbb{F}^e_A(H^d_{\mathfrak{m}}(A)) \cong \varinjlim A/(a_1^{nq}, \ldots, a_d^{nq}) = H^d_{\mathfrak{m}}(A),$$

where $q = p^e$. Then $\eta^q = [x^q + (a_1^{nq}, \ldots, a_d^{nq})] \in H^d_{\mathfrak{m}}(A)$ for $\eta = [x + (a_1^n, \ldots, a_d^n)] \in H^d_{\mathfrak{m}}(A)$; see [Sm] for more details.

The following properties of m_{HK} follows from [WaY1, Theorem 1.5].

Proposition 1.8. *The following statements hold.*

(1) $0 \le m_{\mathrm{HK}}(A) \le \widetilde{m}_{\mathrm{HK}}(A) \le 1$.
(2) $\widetilde{m}_{\mathrm{HK}}(A) = 1$ (resp. $m_{\mathrm{HK}}(A) = 1$) *if and only if A is regular.*
(3) *If* $\widetilde{m}_{\mathrm{HK}}(A) > 0$*, then A is weakly F-regular.*
(4) *Suppose that A is F-finite. If* $m_{\mathrm{HK}}(A) > 0$*, then A is strongly F-regular.*

Proof. If A is *not* weakly F-regular, there exists an $\mathfrak{m}$-primary ideal I such that $I \neq I^*$. Taking an ideal I' with $I \subseteq I' \subseteq I^*$ and $l_A(I'/I) = 1$, we have $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(I')$ by Lemma 1.3(1). Hence $\widetilde{m}_{\mathrm{HK}}(A) = 0$. Also, if A is F-finite and not strongly F-regular, then $m_{\mathrm{HK}}(A) = 0$.

If A is regular, then $e_{\mathrm{HK}}(I) = l_A(A/I)$ for any $\mathfrak{m}$-primary ideal of A. Hence $m_{\mathrm{HK}}(A) = \widetilde{m}_{\mathrm{HK}}(A) = 1$. Conversely, if $\widetilde{m}_{\mathrm{HK}}(A) \ge 1$, then A is weakly F-regular and thus is Cohen–Macaulay (cf. [HH3]). Take a parameter ideal J of A. Then $e_{\mathrm{HK}}(J) = e(J) = l_A(A/J)$. By the assumption that $\widetilde{m}_{\mathrm{HK}}(A) \ge 1$, we get

$$e_{\mathrm{HK}}(\mathfrak{m}) \le e_{\mathrm{HK}}(J) - l_A(\mathfrak{m}/J) = l_A(A/J) - l_A(\mathfrak{m}/J) = 1.$$

Hence A is regular by [WaY1, Theorem 1.5]. □

In Section 3, we will give an affirmative answer to the following question in case of $\mathbb{Q}$–Gorenstein F-regular local rings.

QUESTION 1.9. Is the converse of Proposition 1.8(3) true?

REMARK 2. Aberbach and Leuschke [AL] proved that an F-finite local ring A is strongly F-regular if and only if its F-signature $s(A)$ (which is equal to $m_{\mathrm{HK}}(A)$ by Yao's result) is positive provided $s(A)$ exists.

The following question is related to the localization problem of F-regularity.

QUESTION 1.10. When does $\widetilde{m}_{\mathrm{HK}}(A) = m_{\mathrm{HK}}(A)$ hold?

We expect that this always holds. We will give a proof for Gorenstein local rings in the next section. See also [Ya] for a stronger result.

2. Gorenstein local rings

In this section, we prove that if $(A, \mathfrak{m})$ is a Gorenstein local ring, then $e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(J : \mathfrak{m})$ is independent of the choice of parameter ideal J of A. In fact, this invariant is equal to $m_{\mathrm{HK}}(A)$, defined in the previous section.

In the following, let $(A, \mathfrak{m}, k)$ be an excellent reduced local ring containing an infinite field of characteristic $p > 0$, unless specified.

THEOREM 2.1. *Suppose that A is Gorenstein. Then for any $\mathfrak{m}$-primary ideal J of A such that $\mathrm{pd}_A A/J < \infty$ and A/J is Gorenstein, we have*

$$e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(J : \mathfrak{m}) = m_{\mathrm{HK}}(A).$$

In particular, $\widetilde{m}_{\mathrm{HK}}(A) = m_{\mathrm{HK}}(A)$.

Proof. First, we consider the case of parameter ideals. Put $J = (a_1, \ldots, a_d)$. Since A is Gorenstein, $E_A \cong H^d_{\mathfrak{m}}(A)$. The generator z of $\mathrm{Soc}(E_A)$ can be written as $z = [b + J]$, where b is a generator of $\mathrm{Soc}(A/J)$. For any element $c \in A$ and for all $q = p^e$,

$$cz^q = cF^e_A([b + J]) = [cb^q + J^{[q]}] = 0 \in H^d_{\mathfrak{m}}(A)$$

if and only if there exists an integer $n \geq 1$ such that

$$cb^q \in (a_1^{nq}, \ldots, a_d^{nq}) : (a_1^{n-1} \cdots a_d^{n-1})^q = J^{[q]}.$$

It follows that $\mathrm{ann}_A z^q = J^{[q]} : b^q$. Hence we get

$$\begin{aligned} m_{\mathrm{HK}}(A) &= \lim_{e\to\infty} \frac{l_A(A/J^{[q]} : b^q)}{q^d} = \lim_{e\to\infty} \frac{l_A((J : \mathfrak{m})^{[q]}/J^{[q]})}{q^d} \\ &= e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(J : \mathfrak{m}), \end{aligned}$$

as required.

Next we consider the general case. Let J be an $\mathfrak{m}$-primary ideal such that $\mathrm{pd}_A A/J < \infty$ and A/J is Gorenstein. Take a parameter ideal $\mathfrak{q}$ which is contained in J. Then it is enough to show the following claim:

Claim. $e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(J : \mathfrak{m}) = e_{\mathrm{HK}}(\mathfrak{q}) - e_{\mathrm{HK}}(\mathfrak{q} : \mathfrak{m})$.

As $\mathfrak{q} \subseteq J$, there exists a natural surjective map $A/\mathfrak{q} \to A/J$. Also, since both $A/\mathfrak{q}$ and A/J are Artinian Gorenstein local rings, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to A = F'_d \to \cdots \to A^d & \longrightarrow & A & \longrightarrow & A/\mathfrak{q} \to 0 \\
\delta\downarrow \qquad\qquad\qquad \downarrow & & \downarrow & & \downarrow \mathrm{nat} \\
0 \to A = F_d \to \cdots \to A^n & \longrightarrow & A & \longrightarrow & A/J \to 0,
\end{array}$$

where the horizontal sequences are minimal free resolutions of $A/\mathfrak{q}$ and A/J, respectively. In particular, the map $F'_d \to F_d$ is given by the multiplication of an element (say δ). Then we have $J = \mathfrak{q} : \delta$. If we apply the Peskine–Szpiro functor to the above diagram, then we also get $J^{[q]} = \mathfrak{q}^{[q]} : \delta^q$ for all $q = p^e$.

Since $l_A(J : \mathfrak{m}/J) = 1$, there exists an element $a \in J : \mathfrak{m} \setminus J$ such that $J : \mathfrak{m} = J + aA$. Then one can easily see that $\mathfrak{m}\delta a \subseteq \mathfrak{q}$ and $\delta a \notin \mathfrak{q}$; thus $\mathfrak{q} : \mathfrak{m} = \mathfrak{q} + \delta a A$. Then, since $J^{[q]} : a^q = (\mathfrak{q}^{[q]} : \delta^q) : a^q = \mathfrak{q}^{[q]} : (\delta a)^q$, we get

$$\begin{aligned}
l_A((J : \mathfrak{m})^{[q]}/J^{[q]}) &= l_A(A/(J^{[q]} : a^q)) = l_A(A/(\mathfrak{q}^{[q]} : (\delta a)^q)) \\
&= l_A((\mathfrak{q} : \mathfrak{m})^{[q]}/\mathfrak{q}^{[q]})
\end{aligned}$$

for all $q = p^e$. The required assertion easily follows from this. □

By virtue of Theorem 2.1, we can prove that the converse of Proposition 1.8(3) is also true for Gorenstein local rings.

Corollary 2.2. *Let $e(A)$ denote the usual multiplicity of A. If A is weakly F-regular and Gorenstein, then $\widetilde{m}_{\mathrm{HK}}(A) = m_{\mathrm{HK}}(A) > 0$. If, in addition, $e(A) \geq 2$, then*

$$m_{\mathrm{HK}}(A) \leq \frac{e(A) - e_{\mathrm{HK}}(A)}{e(A) - 1}.$$

Proof. Suppose that A is weakly F-regular. Let J be any parameter ideal of A. Then, since $J : \mathfrak{m} \not\subseteq J = J^*$, we have $e_{\mathrm{HK}}(J) \neq e_{\mathrm{HK}}(J : \mathfrak{m})$ by Lemma 1.3(2). Hence we have $m_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(J : \mathfrak{m}) > 0$ by Theorem 2.1.

To see the last inequality, taking a minimal reduction J of $\mathfrak{m}$, we have

$$e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(\mathfrak{m}) \geq l_A(\mathfrak{m}/J) \cdot m_{\mathrm{HK}}(A).$$

This yields the required inequality, since $e_{\mathrm{HK}}(J) = e(J) = e(A)$. □

REMARK 3. In [HuL], Huneke and Leuschke independently proved a result similar to Corollary 2.2 with respect to F-signature. Also, Yao [Ya] extended this result to F-finite local rings A such that A_P is Gorenstein for every $P \in \operatorname{Spec} A \setminus \{\mathfrak{m}\}$.

EXAMPLE 2.3. Assume that A is a hypersurface local ring of multiplicity 2. Then we have $m_{\mathrm{HK}}(A) = 2 - e_{\mathrm{HK}}(A)$.

Proof. Let J be a minimal reduction of $\mathfrak{m}$. Since $l_A(A/J) = 2$ and $J : \mathfrak{m} = \mathfrak{m}$, we have $m_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(J : \mathfrak{m}) = e(J) - e_{\mathrm{HK}}(\mathfrak{m}) = 2 - e_{\mathrm{HK}}(A)$. □

Let A be a two-dimensional Gorenstein F-regular local ring which is not regular. Then $e(A) = 2$, since A has minimal multiplicity. Moreover, suppose that k is an algebraically closed field. Then it is known that the $\mathfrak{m}$-adic completion $\widehat{A}$ of A is isomorphic to the completion of the invariant subring by a finite subgroup $G \subseteq SL(2,k)$ which acts on a polynomial ring $k[x,y]$. Furthermore, we have $e_{\mathrm{HK}}(A) = 2 - 1/|G|$; see [WaY1, Theorem 5.1]. Hence $m_{\mathrm{HK}}(A) = 1/|G|$ by Example 2.3. This result will be generalized in Section 4.

By the above observation, we have an inequality $m_{\mathrm{HK}}(A) \leq \frac{1}{2}$ for hypersurface local rings with $\dim A = e(A) = 2$. We can extend this result to hypersurface local rings of higher dimension in the following form.

PROPOSITION 2.4. *Suppose that A is a hypersurface with $e(A) = \dim A = d \geq 1$. Then*

$$m_{\mathrm{HK}}(A) \leq \frac{1}{2^{d-1} \cdot (d-1)!}.$$

Proof. By Proposition 1.8(3) we may assume that A is a complete F-regular local domain. Let J be a minimal reduction of $\mathfrak{m}$. Take an element $x \in \mathfrak{m}$ such that $\mathfrak{m} = xA + J$. Then, since x^{d-1} is a generator of $\operatorname{Soc}(A/J)$, we have

$$m_{\mathrm{HK}}(A) = \lim_{q \to \infty} \frac{l_A(Ax^{(d-1)q} + J^{[q]}/J^{[q]})}{q^d}$$

by Theorem 2.1. For any $q = p^e$, we have the following claim.

CLAIM. $l_A(Ax^{(d-1)q} + J^{[q]}/J^{[q]}) \leq 2 \cdot l_A(A/\mathfrak{m}^{\lfloor \frac{q+1}{2} \rfloor})$.

To prove the claim, we put $B = A/J^{[q]}$, $y = x^{(d-1)q}$ and $\mathfrak{a} = \mathfrak{m}^{\lfloor \frac{q+1}{2} \rfloor}$. Then, since $y\mathfrak{a}^2 \subseteq x^{(d-1)q}\mathfrak{m}^q \subseteq \mathfrak{m}^{dq} \subseteq J^{[q]}$, we have $y\mathfrak{a}B \subseteq 0 : \mathfrak{a}B = K_{B/\mathfrak{a}B}$. By Matlis duality, we get

$$l_A(yB) \leq l_A(yB/y\mathfrak{a}B) + l_A(y\mathfrak{a}B) \leq 2 \cdot l_B(B/\mathfrak{a}B) \leq 2 \cdot l_A(A/\mathfrak{a}),$$

as required. Since $l_A(A/\mathfrak{m}^n) = \frac{e(A)}{d!}n^d + O(n^{d-1})$ for all large enough n, the assertion easily follows from the claim. □

Discussion 2.5. Let $(A,\mathfrak{m})$ be a three-dimensional F-regular hypersurface local ring. Then $e_{\mathrm{HK}}(A) \ge \frac{2}{3}e(A)$ by the following formula (see [BC], [BCP]):

$$e_{\mathrm{HK}}(A) \ge \frac{e(A)}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\theta}{\theta}\right)^{d+1} d\theta \;=\; \frac{e(A)}{2^d d!}\sum_{i=0}^{\lfloor \frac{d}{2}\rfloor}(-1)^i(d+1-2i)^d\binom{d+1}{i}.$$

In particular, if furthermore $e(A) = 3$, then $e_{\mathrm{HK}}(A) \ge \frac{2}{3}\cdot 3 = 2$. Thus $m_{\mathrm{HK}}(A) \le \frac{3-2}{3-1} = \frac{1}{2}$ by Corollary 2.2. On the other hand, Proposition 2.4 implies that $m_{\mathrm{HK}}(A) \le \frac{1}{8}$.

Question 2.6. Let d be an integer with $d \ge 2$, and let

$$A = k[[x_0, x_1, \ldots, x_d]]/(x_0^d + x_1^d + \cdots + x_d^d),$$

where k is a field of characteristic $p > 0$. Does $m_{\mathrm{HK}}(A) = 1/(2^{d-1}(d-1)!)$ hold if $p > d$?

3. Canonical covers

In the previous section, we have shown how to compute $m_{\mathrm{HK}}(A)$ in the case of Gorenstein local rings. In this section, we study the minimal Hilbert-Kunz multiplicity in the case of $\mathbb{Q}$–Gorenstein F-regular local rings using the canonical cover.

Let us recall the notion of canonical cover. Let A be a normal local ring and let I be a divisorial ideal (i.e., an ideal of pure height one) of A. Also, let $\mathrm{Cl}(A)$ denote the divisor class group of A. Suppose that $\mathrm{cl}(I)$ is a torsion element in $\mathrm{Cl}(A)$, that is, $I^{(r)} := \bigcap_{P\in \mathrm{Ass}_A(A/I)} I^r A_P \cap A$ is a principal ideal for some integer $r \ge 1$. Putting $r = \mathrm{ord}(\mathrm{cl}(I))$, one can write as $I^{(r)} = fA$ for some element $f \in A$. Then a $\mathbb{Z}_r$-graded A-algebra

$$B(I,r,f) := A \oplus I \oplus I^{(2)} \oplus \cdots \oplus I^{(r-1)} = \sum_{i=0}^{r-1} I^{(i)}t^i, \quad \text{where } t^r f = 1,$$

is called the r-*cyclic cover* of A with respect to I. Also, suppose that r is relatively prime to $p = \mathrm{char}(A) > 0$. Then $B(I,r,f)$ is a local ring with the unique maximal ideal $\mathfrak{n} := \mathfrak{m} \oplus I \oplus \cdots \oplus I^{(r-1)}$, and the natural inclusion $A \hookrightarrow B(I,r,f)$ is étale in codimension one; thus $B(I,r,f)$ is also normal.

We further assume that A admits a canonical module K_A. Note that one can regard K_A as an ideal of pure height one. The ring A is called $\mathbb{Q}$–*Gorenstein* if $\mathrm{cl}(K_A)$ is a torsion element in $\mathrm{Cl}(A)$. Put $r := \mathrm{ord}(\mathrm{cl}(K_A)) < \infty$. Then the r-cyclic cover with respect to K_A

$$B := A \oplus K_A \oplus K_A^{(2)} \oplus \cdots \oplus K_A^{(r-1)}$$

is called the *canonical cover* of A.

Using the canonical cover, we can reduce the $\mathbb{Q}$–Gorenstein case to the Gorenstein case. The main result of this section is the following.

THEOREM 3.1. *Let $(A, \mathfrak{m}, k)$ be a $\mathbb{Q}$–Gorenstein F-regular local ring of characteristic $p > 0$ and let $B = \oplus_{i=0}^{r-1} K_A^{(i)} t^i$ be the canonical cover of A, where r is the order of $\mathrm{cl}(K_A)$ in $\mathrm{Cl}(A)$ and $t^r f = 1$. Also, suppose $(r, p) = 1$. Then we have*

$$m_{\mathrm{HK}}(B) = r \cdot m_{\mathrm{HK}}(A).$$

The following corollary gives a partial answer to Question 1.9.

COROLLARY 3.2. *Let A be a $\mathbb{Q}$–Gorenstein F-regular local ring of characteristic $p > 0$ such that $(\mathrm{ord}(\mathrm{cl}(K_A)), p) = 1$. Then $m_{\mathrm{HK}}(A) > 0$.*

To prove Theorem 3.1, let us recall some properties of canonical covers.

LEMMA 3.3. *Let $(A, \mathfrak{m}, k)$ be a Cohen-Macaulay normal local ring, and suppose that A is $\mathbb{Q}$–Gorenstein. Let $B = \oplus_{i=0}^{r-1} K_A^{(i)} t^i$ be the canonical cover of A, where $K_A^{(r)} = fA$ and $t^r f = 1_A$. Then the following statements hold.*

1. *B is quasi-Gorenstein, that is, $B \cong K_B$ as B-modules. In particular, B is Gorenstein if it is Cohen-Macaulay.*
2. *If $(r, p) = 1$, then A is strongly F-regular if and only if so is B.*

If we further assume that B is Cohen-Macaulay, then we also have:

3. *The injective hull $E_B := E_B(B/\mathfrak{n})$ of $B/\mathfrak{n}$ is given as follows:*

$$E_B = \bigoplus_{i=0}^{r-1} H_{\mathfrak{m}}^d(K_A^{(i)}) t^i.$$

4. *$\mathrm{Soc}_B(E_B) = \mathrm{Hom}_B(B/\mathfrak{n}, E_B)$ is generated by zt, where z is a generator of the socle of $E_A \cong H_{\mathfrak{m}}^d(K_A)$.*

Proof. Assertion (1) follows from [TW, Sect.3] and assertion (2) follows from [Wa3, Theorem 2.7].

In the following, assume that B is Cohen-Macaulay. Then, since B is Gorenstein by (1) and $\mathfrak{m}B$ is $\mathfrak{n}$-primary, we have

$$E_B \cong H_{\mathfrak{n}}^d(B) \cong H_{\mathfrak{m}}^d(B) \cong \bigoplus_{i=0}^{r-1} H_{\mathfrak{m}}^d(K_A^{(i)}) t^i.$$

Thus we get assertion (3). To see (4), it is enough to show that $zt \in \mathrm{Soc}_B(E_B)$, since $\dim_k \mathrm{Soc}_B(E_B) = 1$. Namely, we must show that $az = 0$ in $H_{\mathfrak{m}}^d(K_A^{(i+1)})$ for all i with $1 \leq i \leq r-1$ and for all $a \in K_A^{(i)}$.

Fix an integer i with $1 \le i \le r-1$ and suppose that $0 \ne a \in K_A^{(i)}$. Applying the local cohomology functor to the short exact sequence

$$0 \to K_A \xrightarrow{a} K_A^{(i+1)} \to K_A^{(i+1)}/aK_A \to 0$$

implies that

$$0 = H_{\mathfrak{m}}^{d-1}(K_A^{(i+1)}) \to H_{\mathfrak{m}}^{d-1}(K_A^{(i+1)}/aK_A) \to H_{\mathfrak{m}}^{d}(K_A) \xrightarrow{a} H_{\mathfrak{m}}^{d}(K_A^{(i+1)}),$$

where the first vanishing follows from the fact that $K_A^{(i+1)}$ is a direct summand of a maximal Cohen-Macaulay A-module B. To get the lemma, it is enough to show the following claim:

Claim. $H_{\mathfrak{m}}^{d-1}(K_A^{(i+1)}/aK_A) \ne 0$.

Since A is Cohen-Macaulay, $aK_A \cong K_A$ is a maximal Cohen–Macaulay A-module, hence a divisorial ideal of A. If $K_A^{(i+1)}/aK_A = 0$, then $(i+1)\operatorname{div}(K_A) = \operatorname{div}(K_A) + \operatorname{div}(a)$, and thus $i \cdot \operatorname{cl}(K_A) = 0$, contradicting $r = \operatorname{ord}(\operatorname{cl}(K_A))$. Hence $K_A^{(i+1)}/aK_A \ne 0$ and $\dim K_A^{(i+1)}/aK_A = d-1$. We get the claim, as required. □

Proof of Theorem 3.1. We fix a system of parameters $x_1, \ldots, x_d$ of A. Since A is Cohen–Macaulay, we have $E_A = H_{\mathfrak{m}}^d(K_A) = \varinjlim K_A/\underline{x}^{[q]}K_A$. Also, one can regard the Frobenius map $\mathbb{F}_A^e$ in E_A as

$$F_A^e \colon E_A \to \mathbb{F}_A^e(E_A) \cong H_{\mathfrak{m}}^d(K_A^{(q)}) = \varinjlim K_A^{(q)}/\underline{x}^{[n]}K_A^{(q)}$$
$$\left([b + \underline{x}K_A] \mapsto [b^q + \underline{x}^{[q]}K_A^{(q)}]\right);$$

see [Wa3] for details. Thus we have

$$m_{\mathrm{HK}}(A) = \liminf_{q\to\infty} l_A\left(\frac{z^qA + \underline{x}^{[q]}K_A^{(q)}}{\underline{x}^{[q]}K_A^{(q)}}\right)\Big/ q^d. \tag{3.1}$$

On the other hand, since $zt \in K_At$ generates the socle of E_B by Lemma 3.3, we have

$$m_{\mathrm{HK}}(B) = \lim_{q\to\infty} l_A\left(\frac{z^qt^qB + \underline{x}^{[q]}B}{\underline{x}^{[q]}B}\right)\Big/ q^d \tag{3.2}$$

by Theorem 2.1. Also, as B is a $\mathbb{Z}/r\mathbb{Z}$-graded ring (in particular, $K_A^{(i+r)}t^{i+r} = K_A^{(i)}t^i$), (3.1) can be reformulated as follows:

$$m_{\mathrm{HK}}(B) = \sum_{i=0}^{r-1}\lim_{q\to\infty} l_A\left(\frac{z^qK_A^{(i)} + \underline{x}^{[q]}K_A^{(i+q)}}{\underline{x}^{[q]}K_A^{(i+q)}}\right)\Big/ q^d. \tag{3.3}$$

If necessary, we may assume that $q \equiv 1 \pmod r$. Taking a nonzero element $a_i \in K_A^{(i)}$ for each i with $0 \le i \le r-1$, we consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to K_q & \longrightarrow & \dfrac{z^q A + \underline{x}^{[q]} K_A^{(q)}}{\underline{x}^{[q]} K_A^{(q)}} & \xrightarrow{a_i} & \dfrac{z^q K_A^{(i)} + \underline{x}^{[q]} K_A^{(i+q)}}{\underline{x}^{[q]} K_A^{(i+q)}} & \longrightarrow & C_q \to 0 \\
\downarrow & & \downarrow \scriptstyle{inj.} & & \downarrow \scriptstyle{inj.} & & \downarrow \\
0 \to X_q & \longrightarrow & \dfrac{K_A^{(q)}}{\underline{x}^{[q]} K_A^{(q)}} & \xrightarrow{a_i} & \dfrac{K_A^{(i+q)}}{\underline{x}^{[q]} K_A^{(i+q)}} & \longrightarrow & Y_q \to 0.
\end{array}$$

In order to complete the proof of the theorem, it suffices to prove the following claim.

Claim. $\displaystyle\lim_{q\to\infty} \frac{l_A(K_q)}{q^d} = \lim_{q\to\infty} \frac{l_A(C_q)}{q^d} = 0.$

First, note that if N is a finitely generated A-module with $\dim N \le d-1$, then $l_A(N/\underline{x}^{[q]}N)/q^d = 0$. By the definition of Y_q, we have

$$Y_q = K_A^{(i+q)}/(a_i K_A^{(q)} + \underline{x}^{[q]} K_A^{(i+q)}) \cong \left(K_A^{(i+1)}/a_i K_A\right) \otimes_A A/\underline{x}^{[q]}.$$

Since $\dim K_A^{(i+1)}/a_i K_A \le d-1$, we get $\displaystyle\lim_{q\to\infty} l_A(Y_q)/q^d = 0$. On the other hand, as $q \equiv 1 \pmod r$, we have

$$\lim_{q\to\infty} \frac{l_A(K_A^{(q)}/\underline{x}^{[q]} K_A^{(q)})}{q^d} = e_{\mathrm{HK}}(\underline{x}) \cdot \mathrm{rank}_A\, K_A = e_{\mathrm{HK}}(\underline{x}),$$

$$\lim_{q\to\infty} \frac{l_A(K_A^{(i+q)}/\underline{x}^{[q]} K_A^{(i+q)})}{q^d} = e_{\mathrm{HK}}(\underline{x}) \cdot \mathrm{rank}_A\, K_A^{(i+1)} = e_{\mathrm{HK}}(\underline{x}).$$

That is, $\displaystyle\lim_{q\to\infty} l_A(X_q)/q^d = \lim_{q\to\infty} l_A(Y_q)/q^d = 0$ and thus $\displaystyle\lim_{q\to\infty} l_A(K_q)/q^d = 0$.

On the other hand,

$$\begin{aligned}
C_q &= \frac{z^q K_A^{(i)} + \underline{x}^{[q]} K_A^{(i+q)}}{a_i z^q A + \underline{x}^{[q]} K_A^{(i+q)}} \cong \frac{z^q K_A^{(i)}}{a_i z^q A + z^q K_A^{(i)} \cap \underline{x}^{[q]} K_A^{(i+q)}} \\
&= \frac{z^q K_A^{(i)}}{a_i z^q A + z^q [K_A^{(i)} \cap (\underline{x}^{[q]} K_A^{(i+q)} : z^q)]} \\
&\cong \frac{K_A^{(i)}}{a_i A + [K_A^{(i)} \cap (\underline{x}^{[q]} K_A^{(i+q)} : z^q)]}.
\end{aligned}$$

Since $\mathfrak{m}^{[q]} K_A^{(i)} \subseteq K_A^{(i)} \cap (\underline{x}^{[q]} K_A^{(i+q)} : z^q)$ by the choice of $z \in K_A$, we get

$$l_A(C_q) \le l_A(K_A^{(i)}/a_i A + \mathfrak{m}^{[q]} K_A^{(i)}) = l_A(K_A^{(i)}/a_i A \otimes_A A/\mathfrak{m}^{[q]}).$$

By a similar argument as above we obtain $\displaystyle\lim_{e\to\infty} l(C_q)/q^d = 0$, as required. □

Question 3.4. Let A be a weakly F-regular local ring, and let I be a divisorial ideal of A such that $\mathrm{cl}(I)$ has a finite order (say r). If $B = A \oplus It \oplus I^{(2)}t^2 \oplus \cdots \oplus I^{(r-1)}t^{r-1}$, the r-cyclic cover, does $m_{\mathrm{HK}}(B) = r \cdot m_{\mathrm{HK}}(A)$ (resp. $\widetilde{m}_{\mathrm{HK}}(B) = r \cdot \widetilde{m}_{\mathrm{HK}}(A)$) hold ?

4. Quotient singularities

In this section, as an application of Theorem 3.1, we study the minimal Hilbert-Kunz multiplicities for quotient singularities (i.e., the invariant subrings by a finite group; see below for the precise definition). In general, quotient singularities are not necessarily Gorenstein, but they are $\mathbb{Q}$–Gorenstein normal domains. Thus, using the canonical cover trick, we can reduce our problem to the case of Gorenstein rings.

Let k be a field and V a k-vector space of finite dimension (say $d = \dim_k V$). Assume that a finite subgroup G of $GL(V) \cong GL(d, k)$ acts linearly on $S := \mathrm{Sym}_k(V) \cong k[x_1, \ldots, x_d]$, a polynomial ring with d variables over k. Then

$$S^G := \{f \in S \,:\, g(f) = f \quad \text{for all } g \in G\}$$

is said to be the *invariant subring* of S by G.

In this section, we consider only the case of positive characteristic (say $p = \mathrm{char}(k)$), and assume that the order $|G|$ is non-zero in k, that is, $|G|$ is not divisible by p. Then, using the Reynolds operator

$$\rho \colon S \to S^G \quad \left(a \mapsto \frac{1}{|G|} \sum_{g \in G} g(a)\right),$$

we can show that S^G is a direct summand of S. Put $\mathfrak{n} = (x_1, \ldots, x_d)S$ and $\mathfrak{m} = \mathfrak{n} \cap S^G$. Then the ring $A = (S^G)_{\mathfrak{m}}$ is said to be a *quotient singularity* (by a finite group G). A quotient singularity is a $\mathbb{Q}$–Gorenstein strongly F-regular domain, but not always Gorenstein; see, e.g., [Wa1], [Wa2] for details.

In [WaY1], we gave a formula for Hilbert-Kunz multiplicity $e_{\mathrm{HK}}(A)$ of quotient singularities as follows.

Theorem 4.1 (cf. [WaY1, Theorem 2.7], [BCP]). *Under the same notation as above, we have*

$$e_{\mathrm{HK}}(I) = \frac{1}{|G|}\, l_A(S_{\mathfrak{n}}/IS_{\mathfrak{n}}),$$

for every $\mathfrak{m}$-primary ideal I in A. In particular, $e_{\mathrm{HK}}(A) = \frac{1}{|G|}\, \mu_A(S_{\mathfrak{n}})$, where $\mu_A(M)$ denotes the number of minimal system of generators of a finite A-module M.

The main purpose of this section is to prove the following theorem.

THEOREM 4.2. *Let $A = (S^G)_{\mathfrak{m}}$ be a quotient singularity by a finite group G as described above. Also, assume that G contains no pseudo-reflections. Then we have*

$$m_{\mathrm{HK}}(A) = \frac{1}{|G|}.$$

Proof. First, suppose that $G \subseteq SL(d,k)$. Then S^G is Gorenstein by [Wa1, Theorem 1a]. Since G acts linearly on S, S^G is a graded subring of S. Thus one can take a homogeneous system of parameters $a_1, \ldots, a_d$ of S^G with the same degree m. Also, we may assume that m is a multiple of $|G|$. Put $J = (a_1, \ldots, a_d)S^G$. Then, since S/JS is a homogeneous Artinian Gorenstein ring having the same Hilbert function as that of $S/(x_1^m, \ldots, x_d^m)S$, there exists an element $z \in S_{d(m-1)}$ which generates $\mathrm{Soc}(S/JS)$. Then we have $z \in S^G$. This follows from the proof of [Wa1, Theorem 1a], but since it is an essential point in the proof, we sketch the argument here.

To see that $z \in S^G$, it is enough to show that $z \in S^{\langle g \rangle}$ for any element $g \in G$. The property $z \in S^{\langle g \rangle}$ does not change if we consider $S \otimes_k \overline{k}$ instead of S, where $\overline{k}$ is the algebraic closure of k. Therefore we may assume $k = \overline{k}$ and further that g is diagonal. Then $x_1 \cdots x_d \in S^{\langle g \rangle}$ and $x_i^m \in S^{\langle g \rangle}$, since $\det(g) = 1$ and m is a multiple of $|G|$. If we put $(\underline{x})^{[m]} = (x_1^m, \cdots, x_d^m)$, then

$$\dim_k [S^{\langle g \rangle}/JS^{\langle g \rangle}]_{d(m-1)} = \dim_k [S^{\langle g \rangle}/(\underline{x})^{[m]} S^{\langle g \rangle}]_{d(m-1)} \geq 1.$$

On the other hand, since $JS^{\langle g \rangle} = JS \cap S^{\langle g \rangle}$, we have

$$\dim_k [S^{\langle g \rangle}/JS^{\langle g \rangle}]_{d(m-1)} \leq \dim_k [S/JS]_{d(m-1)} = 1.$$

It follows that $z \in S^G$, as required.

Now let J, z be as above. Then $JA : \mathfrak{m}A = (J,z)A$ and $JS : \mathfrak{n} = (J,z)S$. Hence

$$\begin{aligned} e_{\mathrm{HK}}(JA) - e_{\mathrm{HK}}(JA : \mathfrak{m}A) &= \frac{1}{|G|} l_A(S_{\mathfrak{n}}/JS_{\mathfrak{n}}) - \frac{1}{|G|} l_A(S_{\mathfrak{n}}/(J : \mathfrak{m})S_{\mathfrak{n}}) \\ &= \frac{1}{|G|} l_{S_{\mathfrak{n}}}(JS_{\mathfrak{n}} : \mathfrak{n}/JS_{\mathfrak{n}}) = \frac{1}{|G|}. \end{aligned}$$

The required assertion follows from Theorem 2.1.

Next, we consider the general case. If we put $H = G \cap SL(n,k)$, then S^H is Gorenstein by [Wa2, Theorem 1]. Further, since H is a normal subgroup of G and G/H is a finite subgroup of $k^\times$, G/H is a cyclic group. Say $G/H = \langle \sigma H \rangle$ and $r = |G/H|$. Also, $S^G = (S^H)^{\langle \sigma \rangle}$. Then $B = (S^H)_{\mathfrak{n} \cap S^H}$ is a cyclic r-cover of $A = (S^G)_{\mathfrak{m}}$. In fact, it is known that B is isomorphic to the canonical cover of A:

$$B \cong A \oplus K_A t \oplus K_A^{(2)} t^2 \oplus \cdots \oplus K_A^{(r-1)} t^{r-1},$$

where $K_A^{(r)} = fA$, $t^r f = 1$; see [TW] for details.

Since $m_{\mathrm{HK}}(B) = 1/|H|$, by Theorem 3.1, we get

$$m_{\mathrm{HK}}(A) = \frac{1}{r} m_{\mathrm{HK}}(B) = \frac{1}{(G:H)|H|} = \frac{1}{|G|},$$

as required. □

CONJECTURE 4.3. Under the same notation as in Theorem 4.2, $\widetilde{m}_{\mathrm{HK}}(A) = 1/|G|$.

5. Toric rings and Segre products

We first give a general formula for $m_{\mathrm{HK}}(A)$ in the case of a normal toric ring A. For simplicity, we denote the minimal relative Hilbert-Kunz multiplicity of the local ring at the unique graded maximal ideal by $m_{\mathrm{HK}}(A)$. To formulate our result, let us fix some notation.

Let M, $N \cong \mathbb{Z}^d$ be dual lattices, and denote the duality pairing of $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ with $N_{\mathbb{Z}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ by $\langle \quad , \quad \rangle \colon M_{\mathbb{R}} \otimes N_{\mathbb{R}} \to \mathbb{R}$. Let σ be a strongly convex rational polyhedral cone, and set $\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle m, n \rangle \geq 0$ for all $n \in \sigma\}$. Let $A = k[\sigma^\vee \cap M]$ be a normal toric ring, and let $n_1, \ldots, n_s$ be primitive generators of σ. Then $A = k[x^m \mid \langle m, n_i \rangle \geq 0 \quad \text{for all } i]$.

THEOREM 5.1. *Let k be a field of characteristic $p > 0$, and let $A = k[\sigma^\vee \cap M]$ be a normal toric ring. Under the above notation, we have*

$$m_{\mathrm{HK}}(A) = \mathrm{vol}\{m \in M_{\mathbb{R}} \mid 0 \leq \langle m, n_i \rangle \leq 1 \quad \textit{for all } i\},$$

where $\mathrm{vol}(W)$ *denotes the relative volume of an integral polytope $W \in M_{\mathbb{R}}$* (see [St, pp. 239]).

Proof. By [HaY, Section 4], we have

$$E_A = H^d_{\mathfrak{m}}(K_A) \cong \bigoplus_{\langle m, n_i \rangle \leq 0 \, (\forall i)} kx^m,$$

where the socle is generated by $z = 1$ and

$$E_A \otimes {}^e\!A \cong H^d_{\mathfrak{m}}(K_A^{(q)}) \cong \bigoplus_{\langle m, n_i \rangle \leq q-1 \ (\forall i)} kx^m.$$

Since the Frobenius action is given by $F^e \colon E_A \to F^e_A(E_A), x^m \mapsto x^{mq}$, the annihilator of $z^q = 1$ is given by the direct sum

$$\bigoplus_{0 \leq \langle m, n_i \rangle \leq q-1, \, m \neq 0} kx^m,$$

whose length is $\sharp\{m \in M \mid 0 \leq \langle m, n_i \rangle \leq q-1 \, (\forall i), m \neq 0\}$. We obtain the desired result by dividing by q^d and letting q tend to ∞. □

REMARK 4. In [Wa4], the first-named author gave a formula for Hilbert-Kunz multiplicities of normal toric rings.

EXAMPLE 5.2. Let k be a field and $A_n = k[x^{-n}T, x^{-n+1}T, \ldots, T, xT, yT, xyT]$, where x, y, T are variables and n is a non-negative integer. Then the generators of σ and $\sigma^\vee$ are given, respectively, by

$$\sigma = \langle (0,1,0), (-1,0,1), (0,-1,1), (1,-n,n) \rangle,$$
$$\sigma^\vee = \langle (-n,0,1), (1,0,1), (0,1,1), (1,1,1) \rangle.$$

Since the volume of the region given by

$$\{(x,y,z) \in \mathbb{R}^3 \mid 0 \le y \le 1, x \le z \le x+1, y \le z \le y+1, ny \le x+nz \le ny+1\}$$

is $5/(6(n+1))$, we have $m_{\mathrm{HK}}(A_n) = 5/(6(n+1))$.

Next, we will calculate $m_{\mathrm{HK}}(A)$ for a "Segre Product" of two polynomial rings. In the remainder of this section, let k be a perfect field of characteristic $p > 0$, and let $R = k[x_1, \ldots, x_r]$ (resp. $S = k[y_1, \ldots, y_s]$) be a polynomial ring with r variables (resp. s variables) over k. We regard these rings as homogeneous k-algebras with $\deg(x_i) = \deg(y_j) = 1$ as usual. We define the graded subring $A = R\#S$ of $R \otimes_k S$ by putting $A_n := R_n \otimes S_n$ for all integers $n \ge 0$. Then $A = R\#S$ is said to be the *Segre product* of R and S. In fact, the ring A is the coordinate ring of the Segre embedding $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \hookrightarrow \mathbb{P}^{rs-1}$.

Since the Segre product A is a direct summand of $R \otimes_k S$ (which is isomorphic to a polynomial ring with $r+s$ variables), it is a strongly F-regular domain. Further, it is known that $\dim A = r+s-1$ and $e(A) = \binom{r+s-2}{r-1}$; see [GW, Chapter 4] for more details.

Before giving a formula for $m_{\mathrm{HK}}(A)$ of Segre products, we recall related results. In [BCP], Buchweitz, Chen and Purdue have given the Hilbert-Kunz multiplicity $e_{\mathrm{HK}}(A)$ of A. Also, Eto and the second-named author [EtY] simplified their result in terms of "Stirling numbers of the second kind" as follows.

THEOREM 5.3 (cf. [BCP, 2.2.3], [EtY, Theorem 3.3], [Et]). *Suppose that $2 \le r \le s$ and put $d = r+s-1$. Let $A = k[x_1, \ldots, x_r]\#k[y_1, \ldots, y_s]$. Then*

$$e_{\mathrm{HK}}(A) = \frac{s!}{d!}\, S(d,s) - \frac{1}{d!} \sum_{k=1}^{r-1} \sum_{j=1}^{r-k} \binom{r}{k+j} \binom{s}{j} (-1)^{r+k} k^d,$$

where $S(n,k)$ denotes the Stirling number of the second kind; see below.

Stirling numbers of the second kind also play an important role in the study of the minimal Hilbert-Kunz multiplicity of the Segre product, so we recall their definition.

DEFINITION 5.4 ([St, Chapter 1, §1.4]). We denote by $S(n,k)$ the number of partitions of the set $[n] := \{1, \ldots, n\}$ into k blocks. The number $S(n,k)$ is called the *Stirling number of the second kind*.

The following properties are well-known; see [St].

FACT 5.5. *If we denote by $S(n,k)$ the Stirling number of the second kind, then*

$$\sum_{n\geq k} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k.$$

In particular,

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n,$$
$$S(n,2) = 2^{n-1}-1,$$
$$S(n,n-1) = \binom{n}{2}.$$

EXAMPLE 5.6. Let $A = R\#S = k[x_1,\, x_2]\#k[y_1,\dots,y_s]$, which is isomorphic to the Rees algebra $S[\mathfrak{n}t]$ over S. Then

$$e_{\mathrm{HK}}(A) = s\left(\frac{1}{2}+\frac{1}{(s+1)!}\right).$$

In the following, we will give a formula for the minimal Hilbert-Kunz multiplicity of the Segre product. Let A be the Segre product of R and S described as above, i.e., $A = R\#S = k[x_1,\,\dots,x_r]\#k[y_1,\,\dots,y_s]$, and suppose that $2\leq r\leq s$. Put $d = r+s-1 (= \dim A)$ and set

$$\mathfrak{m} = (x_1,\,\dots,x_r)R,\ \mathfrak{n} = (y_1,\dots,y_s)S,\quad \text{and } \mathfrak{M} = \mathfrak{m}\#\mathfrak{n} = \bigoplus_{n=1}^{\infty} R_n\otimes S_n.$$

Then the graded canonical module K_A of A is isomorphic to $K_R\#K_S$ by [GW, Theorem 4.3.1]. (In particular, A is Gorenstein if and only if $r=s$.) Thus, by virtue of [GW, Theorem 4.1.5], we get

$$E_A = H^d_{\mathfrak{M}}(K_A) = H^d_{\mathfrak{M}}(K_R\#K_S) = H^r_{\mathfrak{m}}(K_R)\#H^s_{\mathfrak{n}}(K_S) = E_R\#E_S.$$

Further, since E_R can be represented as a graded module $k[x_1^{-1},\,\dots,x_r^{-1}]$, which is called *the inverse system of Macaulay*, we have

$$E_A \cong k[x_1^{-1},\,\dots,x_r^{-1}]\#k[y_1^{-1},\,\dots,y_s^{-1}].$$

Then $z = 1\#1\in E_A$ generates the socle of E_A.
Using this, we obtain:

PROPOSITION 5.7. *Let $A = R\#S$ and $z = 1\#1$ be as above. Then:*

$$(5.1)\quad l_A(A/\operatorname{ann}_A(F_A^e(z)) = \#\left\{(a_1,\dots,a_r,b_1,\dots,b_s)\in\mathbb{Z}^{r+s}\;\middle|\;\begin{array}{l}0\le a_1,\,\dots,a_r\le q-1\\ 0\le b_1,\,\dots,b_s\le q-1\\ a_1+\cdots+a_r=b_1+\cdots+b_s\end{array}\right\}.$$

Proof. We use the same notation as in the above argument. Now we shall investigate the Frobenius action on z in E_A. First note that $\mathbb{F}_A^e(E_A)\cong\mathbb{F}_R^e(E_R)\#\mathbb{F}_S^e(E_S)$. Thus it is enough to investigate the Frobenius action of $z_1 = 1$ in E_R. Since $E_R = H_{\mathfrak{m}}^r(R)(-r)$, that is, $H_{\mathfrak{m}}^r(R)\cong(x_1\cdots x_r)^{-1}E_R$, the generator z_1 of $\operatorname{Soc}(E_R)$ corresponds to the element $w_1=(x_1\cdots x_r)^{-1}$ via this isomorphism. Then we have $F_R^e(w_1)=(x_1\cdots x_r)^{-q}$, since there exists an isomorphism

$$(x_1\cdots x_r)^{-1}k[x_1,\,\dots,x_r]\to H_{\mathfrak{m}}^r(R)=\varinjlim_n R/(x_1^n,\,\dots,x_r^n).$$
$$\left(x_1^{-a_1}\cdots x_r^{-a_r}\right)\mapsto[x_1^{a-a_1}\cdots x_r^{a-a_r}+(\underline{x}^a)],$$

where $a:=\max\{a_1,\,\dots,a_r\}$. If we identify $\mathbb{F}_R^e(E_R)$ with E_R, then

$$F_R^e(z_1)=(x_1\cdots x_r)\cdot F^e(w_1)=(x_1\cdots x_r)^{-(q-1)}.$$

Therefore

$$z^q=F_R^e(z_1)\#F_S^e(z_2)=(x_1\cdots x_r)^{-(q-1)}\#(y_1\cdots y_s)^{-(q-1)}\quad\text{in } E_A.$$

For any element $c=x_1^{a_1}\cdots x_r^{a_r}\#y_1^{b_1}\cdots y_s^{b_s}$ in R, we have

$$cF^e(z)\neq 0\quad\text{in } E_A\qquad\Longleftrightarrow\qquad\begin{cases}0\le a_1,\,\dots,a_r\le q-1,\\ 0\le b_1,\,\dots,b_s\le q-1,\\ a_1+\cdots+a_r=b_1+\cdots+b_s.\end{cases}$$

Thus we get the required assertion. □

We are now ready to state our main theorem in this section.

THEOREM 5.8. *Let $A = k[x_1,\,\dots,x_r]\#k[y_1,\,\dots,y_s]$, where $2\le r\le s$, and put $d = r+s-1$. Then*

$$m_{\mathrm{HK}}(A)=\frac{r!}{d!}\,S(d,r)+\frac{1}{d!}\sum_{k=1}^{r-1}\sum_{j=1}^{r-k}\binom{r}{k+j}\binom{s}{j}(-1)^{r+k}k^d,$$

where $S(n,k)$ denotes the Stirling number of the second kind; see below.

In particular,

$$e_{\mathrm{HK}}(A)+m_{\mathrm{HK}}(A)=\frac{r!\cdot S(d,r)+s!\cdot S(d,s)}{d!}.$$

The following two corollaries easily follow from Theorems 5.3 and 5.8.

COROLLARY 5.9. *Let $A = R\#S = k[x_1, x_2]\#k[y_1, \dots, y_s]$, which is isomorphic to the Rees algebra $S[\mathfrak{n}t]$ over S. Then*

$$m_{\mathrm{HK}}(A) = \frac{2^{s+1} - s - 2}{(s+1)!}.$$

COROLLARY 5.10. *Under the same notation as in Theorem 5.8, assume further that A is Gorenstein, that is, $r = s$. Then*

$$e_{\mathrm{HK}}(A) + m_{\mathrm{HK}}(A) = \frac{2 \cdot r!}{(2r-1)!} S(2r-1, r).$$

Proof of Theorem 5.8. If we put $\alpha_{r,n} := l_R(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \binom{n+r-1}{r-1}$ and $\alpha_{r,n,q} := l_R(\mathfrak{m}^n/\mathfrak{m}^{n-q}\mathfrak{m}^{[q]} + \mathfrak{m}^{n+1})$, then

$$\alpha_{r,n,q} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \alpha_{r,n-iq}.$$

In fact, $\alpha_{r,n,q}$ is the number of monomials of degree n which appear in the polynomial $\prod_{i=1}^{r}(1 + x_i + x_i^2 + \cdots + x_i^{q=1})$. Also, we have

$$\begin{aligned} e_{\mathrm{HK}}(A) = \lim_{q\to\infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n}\alpha_{s,n,q} \\ + \lim_{q\to\infty} \frac{1}{q^d} \sum_{n=0}^{s(q-1)} \alpha_{r,n,q}\alpha_{s,n} - \lim_{q\to\infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q}\alpha_{s,n,q}. \end{aligned}$$

By virtue of Proposition 5.7, we get

$$m_{\mathrm{HK}}(A) = \lim_{q\to\infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q}\alpha_{s,n,q}.$$

Hence the required assertion follows from the following lemma. □

LEMMA 5.11 (cf. [EtY, Lemmas 3.8 and 3.9]). *Under the same notation as above, we have*

$$\lim_{q\to\infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q}\alpha_{s,n} = \frac{r!}{d!} S(d, r),$$

$$\lim_{q\to\infty} \frac{1}{q^d} \sum_{n=0}^{r(q-1)} \alpha_{r,n,q}\alpha_{s,n,q} = \frac{r!}{d!} S(d, r) + \frac{1}{d!} \sum_{0<j<i\le r} \binom{r}{i}\binom{s}{j} (-1)^{r-i+j} (i-j)^d.$$

References

[AL] I. M. Aberbach and G. J. Leuschke, *The F-signature and strong F-regularity*, Math. Res. Lett. **10** (2003), 51–56. MR **2004b:**13003

[AM] I. M. Aberbach and B. MacCrimmon, *Some results on test elements*, Proc. Edinburgh Math. Soc. (2) **42** (1999), 541–549. MR **2000i:**13005

[BC] R. O. Buchweitz and Q. Chen, *Hilbert-Kunz functions of cubic curves and surfaces*, J. Algebra **197** (1997), 246–267. MR **98i:**13025

[BCP] R. O. Buchweitz, Q. Chen, and K. Pardue, *Hilbert-Kunz functions*, preprint, Algebraic Geometry e-print series, Feb. 4, 1997.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR **95h:**13020

[Co] A. Conca, *Hilbert-Kunz function of monomial ideals and binomial hypersurfaces*, Manuscripta Math. **90** (1996), 287–300. MR **97g:**13006

[Du] S. P. Dutta, *Frobenius and multiplicities*, J. Algebra **85** (1983), 424–448. MR **85f:**13022

[Et] K. Eto, *Hilbert-Kunz multiplicity of the Segre product of polynomial rings*, Rep. Res. Nippon Inst. Tech. **30** (2000), 317–327.

[EtY] K. Eto and K. Yoshida, *Notes on Hilbert-Kunz multiplicity of Rees algebras*, Comm. Algebra **31** (2003), 5943–5976. MR 2014910

[GW] S. Goto and K.-i. Watanabe, *On graded rings. I*, J. Math. Soc. Japan **30** (1978), 179–213. MR **81m:**13021

[HaM] C. Han and P. Monsky, *Some surprising Hilbert-Kunz functions*, Math. Z. **214** (1993), 119–135. MR **94f:**13008

[HaY] N. Hara and K. Yoshida, *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), 3143–3174. MR 1974679

[HH1] M. Hochster and C. Huneke, *Tight closure and strong F-regularity*, Mém. Soc. Math. France (N.S.) (1989), 119–133. MR **91i:**13025

[HH2] ———, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116. MR **91g:**13010

[HH3] ———, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62. MR **95d:**13007

[Hu] C. Huneke, *Tight closure and its applications*, CBMS Regional Conference Series in Mathematics, vol. 88, American Mathematical Society, Providence, R.I., 1996. MR **96m:**13001

[HuL] C. Huneke and G. J. Leuschke, *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. **324** (2002), 391–404. MR **2003j:**13011

[Ku1] E. Kunz, *Characterizations of regular local rings for characteristic p*, Amer. J. Math. **91** (1969), 772–784. MR 40#5609

[Ku2] ———, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), 999–1013. MR 55#5612

[Mc] B. MacCrimmon, *Weak F-regularity is strong F-regularity for rings with isolated non–$\mathbb{Q}$–Gorenstein points*, Trans. Amer. Math. Soc., to appear.

[Mo] P. Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983), 43–49. MR **84k:**13012

[PS] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. IHES **42** (1973), 47–119. MR 51#10330

[Se] G. Seibert, *The Hilbert-Kunz function of rings of finite Cohen-Macaulay type*, Arch. Math. (Basel) **69** (1997), 286–296. MR **98h:**13022

[Sm] K. E. Smith, *Tight closure of parameter ideals*, Invent. Math. **115** (1994), 41–60. MR **94k:**13006

[St] R. P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997. MR **98a:**05001

[TW] M. Tomari and K.-i. Watanabe, *Normal Z_r-graded rings and normal cyclic covers*, Manuscripta Math. **76** (1992), 325–340. MR **93j:**13002

[Wa1] K.-i. Watanabe, *Certain invariant subrings are Gorenstein. I*, Osaka J. Math. **11** (1974), 1–8. MR 50#7124

[Wa2] ———, *Certain invariant subrings are Gorenstein. II*, Osaka J. Math. **11** (1974), 379–388. MR 50#7124

[Wa3] ———, *F-regular and F-pure normal graded rings*, J. Pure Appl. Algebra **71** (1991), 341–350. MR **92g:**13003

[Wa4] ———, *Hilbert-Kunz multiplicity of Toric rings*, Proc. Inst. Nat. Sci. (Nihon Univ.) **35** (2000), 173–177.

[WaY1] K.-i. Watanabe and K. Yoshida, *Hilbert-Kunz multiplicity and an inequality between multiplicity and colength*, J. Algebra **230** (2000), 295–317. MR **2001h:**13032

[WaY2] ———, *Hilbert-Kunz multiplicity of two-dimensional local rings*, Nagoya Math. J. **162** (2001), 87–110. MR **2002e:**13040

[WaY3] ———, *Hilbert-Kunz multiplicity, McKay correspondence and good ideals in two-dimensional rational singularities*, Manuscripta Math. **104** (2001), 275–294. MR **2002d:**13024

[Wi] L. J. Williams, *Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism*, J. Algebra **172** (1995), 721–743. MR **96f:**13003

[Ya] Y. Yao, *Observations on the F-signature of local rings of characteristic p*, preliminary version, available at `http://www.math.ukans.edu/~yyao/eprint/f-sig.pdf`.

Kei-ichi Watanabe, Department of Mathematics, College of Humanities and Sciences, Nihon University, Setagaya-ku, Tokyo 156–0045, Japan
E-mail address: `watanabe@math.chs.nihon-u.ac.jp`

Ken-ichi Yoshida, Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464–8602, Japan
E-mail address: `yoshida@math.nagoya-u.ac.jp`