Illinois Journal of Mathematics
Volume 46, Number 1, Spring 2002, Pages 269–286
S 0019-2082

LOG-RESOLUTIONS, DERIVATIONS, AND EVOLUTIONS

REINHOLD HÜBL

ABSTRACT. Mazur's question on the evolutionary stability of reduced local algebras in characteristic zero and a problem of Huneke about derivations are related to the vanishing of cohomology groups.

In this note we examine two questions which—at a first glance—seem to be totally unrelated. The first question goes back to Huneke, and in its boldest form it might be formulated as follows:

Let k be a (perfect) field of positive characteristic $p > 0$ and let $(R, \mathfrak{m})$ be a regular local ring containing k and such that $R/\mathfrak{m}$ is finite over k. Let $C(R/k)$ be the subring of derivationally constant elements of R/k, i.e.,

$$C(R/k) = \{a \in R : \delta(a) = 0 \text{ for all } \delta \in \mathrm{Der}_k(R)\},$$

and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. If $x \in R$ with $\delta(x) \in I$ for all $\delta \in \mathrm{Der}_k(R)$, then is it true that there exists $c \in C(R/k)$ such that $x - c$ is in $\overline{I}$, the integral closure of I?

In characteristic 0 this question has a positive answer (see [Hü1]), but in the above situation only partial results are known (see [CH]). The question (in a weaker form—see [CH] for a precise formulation) has its origins in the theory of tight closure and was posed by Huneke to clarify the relations between rational singularities and singularities of F-rational type, a problem that meanwhile has been settled by Mehta and Srinivas [MS] and by Hara [Ha]. It later reappeared in the work of Huneke and Smith [HS] on the tight closure approach to Kodaira vanishing, and it can be used to analyze the depths of certain Rees-algebras (Huneke, private communications).

The second question has its origins in number theory and was posed by Mazur [Ma] in connection with Galois deformations. It asks whether any reduced local algebra $(R, \mathfrak{m})$ that is essentially of finite type over a field k of characteristic 0 is evolutionarily stable.

Received April 11, 2001; received in final form September 28, 2001.
2000 *Mathematics Subject Classification.* Primary 14F10. Secondary 13N05, 13A30, 11S25.
Partially supported by a Heisenberg-Stipendium of the DFG.

In this note we examine these questions from a birational point of view, using techniques and ideas developed by Lipman and by Ein and Lazarsfeld in connection with adjoint ideals and multiplier ideals; see [Li], [La], [Ei].

1. Log-resolution of ideals

Let $(R, \mathfrak{m})$ be a regular local ring, let $I \subseteq R$ be an ideal and set $Y = \operatorname{Spec}(R)$. A log-resolution of (R, I) is a projective birational map $\pi\colon X \to Y$ with exceptional divisor E such that

(1) $I\mathcal{O}_X = \mathcal{O}_X(-F)$ for some effective divisor F on X,
(2) $(E+F)_{\mathrm{red}}$ is a strictly normal crossing divisor.

If $(R, \mathfrak{m})$ is essentially of finite type over a field k of characteristic 0, then a log-resolution of (R, I) always exists; see [Hi].

Now let $(R, \mathfrak{m})$ be essentially of finite type over a perfect field k of characteristic $\operatorname{char}(k) = p > 0$, and let $I \subseteq R$ be an ideal. We say that (R, I) arises by reduction mod p, if there exists a regular local algebra $(S, \mathfrak{M})$, essentially of finite type over a field of characteristic 0, and an ideal $J \subseteq S$ such that $(R, \mathfrak{m})$ and I arise from $(S, \mathfrak{M})$ and J as described in [MS, §2] or [CH, §3]. By Hironaka's result, the pair (S, J) has a log-resolution, and by excluding a few primes if necessary, this log-resolution also reduces mod p. This is the situation that is of principal interest for us. Therefore, when saying that (R, I) arises by reduction mod p, we tacitly assume that it comes with a log-resolution, and, in particular, that the whole situation lifts to $W_2(k)$, the Witt-vectors of length 2 over k. Furthermore we always will assume that $d := \dim(R) < p$.

Recall that a $\mathbb{Q}$-divisor on a regular scheme X is an element of $\operatorname{Div}(X) \otimes \mathbb{Q}$, i.e., a formal linear combination of reduced and irreducible subschemes of codimension 1 with rational coefficients. For a $\mathbb{Q}$-divisor $D = \sum a_i D_i$ with pairwise distinct D_i we set

$\lceil D \rceil = \sum \lceil a_i \rceil D_i$, the round-up of D,
$\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$, the round-off of D,

where $\lceil a \rceil$ (resp. $\lfloor a \rfloor$) denotes the smallest integer greater or equal to (resp. the largest integer smaller or equal to) a.

Assume now that $(R, \mathfrak{m})$ is essentially of finite type over k, $\operatorname{char}(k) = p > \dim(R)$, and that $\pi\colon X \to Y$ is a log-resolution of (R, I). Furthermore let $E = \sum e_i E_i$ be the exceptional divisor of π. We first extend a result of Hara [Ha].

1.1 Proposition. *Let D be a $\mathbb{Q}$-divisor on X such that $D - \lfloor D \rfloor$ is supported on E, and such that for some $n > 0$ the sheaf $\mathcal{O}_X(-p^n D)$ is generated by global sections. Then*

$$H^j\left(X, \Omega_X^i(\log E)\left(-E - \lceil D \rceil\right)\right) = 0$$

for all i, j *with* $i + j > d = \dim(X)$. *In particular,*

$$H^j(X, \omega_X(-\lceil D \rceil)) = 0 \qquad \text{for all } j > 0,$$

where ω_X *denotes the canonical sheaf on* X.

Proof. By [DI] we have an isomorphism in the derived category

$$\varphi\colon \bigoplus_{i=0}^{d} \Omega^i(\log E)[-i] \longrightarrow F_*\left(\Omega_X^\bullet(\log E)\right),$$

where F denotes the Frobenius of X (note here that k is perfect). Using [Ha, (3.4) and (3.5)] we conclude that φ induces an isomorphism

$$\psi\colon \bigoplus_{i=0}^{d} \Omega_X^i(\log E)\left(-E - \lceil D \rceil\right)[-i] \longrightarrow F_*\left(\Omega_X^\bullet(\log E)\right)\left(-E - \lceil pD \rceil\right)$$

in the derived category. Applying this several times and making use of the Hodge-to-de Rham spectral sequence, we see that it suffices to prove that

$$H^j\left(X, \Omega_X^i(\log E)\left(-E - \lceil p^n D \rceil\right)\right) = 0$$

for some $n > 0$ and all i, j with $i + j > d$. Thus we may replace D by $\lceil p^n D \rceil$ and assume that D is a divisor with integral coefficients, and that $\mathcal{O}_X(-D)$ is generated by global sections. Assume now that F is an ample divisor for π, supported on the exceptional locus

$$F = \sum_{i=1}^{n} f_i E_i, \quad f_i < 0,$$

and choose $n > 0$ such that $\lfloor -(1/p^n)F \rfloor = 0$. Then it suffices to show that

$$H^j\left(X, \Omega_X^i(\log E)\left(-E - \left\lfloor -\frac{1}{p^n}F + D \right\rfloor\right)\right) = 0$$

for all $i + j > d$. As $\mathcal{O}_X(-D)$ is generated by global sections, $(1/p^n)F - D$ is an ample $\mathbb{Q}$-divisor, and the claim follows from [Ha, (3.8)]. □

1.2 Corollary. *Let* $(R, \mathfrak{m})$ *be a regular local algebra, essentially of finite type over a field* k *of characteristic* 0, *let* $I \subseteq R$ *be an ideal and let* $\pi\colon X \to Y$ *be a log-resolution of* (R, I). *If* D *is a* $\mathbb{Q}$*-divisor on* X *such that* $D - \lfloor D \rfloor$ *is supported on the exceptional divisor of* π *and all sufficiently high multiples of* $-D$ *are generated by global sections, then*

$$H^j\left(X, \Omega_X^i(\log E)\left(-E - \lceil D \rceil\right)\right) = 0$$

for all $i + j > d$.

Proof. This follows from 1.1 by reduction mod p. □

As a consequence of these vanishing results we obtain a positive answer to Lipman's vanishing conjecture [Li, (2.2)] in some cases:

1.3 Corollary. *In the situation of* 1.1 *or* 1.2 *let $J \subseteq R$ be an ideal of R such that $J\mathcal{O}_X$ is an invertible ideal. Then*

$$H^l(X, J\omega_X) = 0 \qquad \textit{for all } l > 0.$$

1.4 Remark. (i) If $\operatorname{char}(k) = 0$ then 1.3 was proved (by different methods, but in even greater generality) by Cutkosky (see [Li, Appendix]).

(ii) The vanishing result 1.3 is very useful in the study of results of Briançon–Skoda type; see [Li, (1.6), §2].

Log-resolutions play a crucial role in the study of multiplier ideals.

Definition. Given an ideal $\mathfrak{a} \subseteq R$ in a regular ring R, a log-resolution $\pi\colon X \to Y$ of $(R, \mathfrak{a})$ and a non-negative rational number c, the corresponding multiplier ideal is defined to be

$$\mathcal{I}(c \cdot \mathfrak{a}) = \pi_* \omega_{X/Y} \left(- \lfloor c \cdot F \rfloor\right),$$

where $\omega_{X/Y}$ denotes the relative canonical sheaf of X/Y (which we may take to be ω_X if R is local).

1.5 Remark. In a somewhat different context (only for local rings, but with no restrictions on the characteristic) multiplier ideals have already been studied by Lipman [Li], who called them adjoint ideals and wrote $\widetilde{\mathfrak{a}}$ for $\mathcal{I}(1 \cdot \mathfrak{a})$. Recently they have been examined and used very successfully by Ein, Lazarsfeld, and others (see [Ei], [La], [DEL], [ELS]). They have their origin in the (analytic) work of Nadel, Demailly, Siu, and others.

2. Evolutions, condition (NN) and log-resolutions

Suppose now that k is a field of characteristic 0 and that $(T, \mathfrak{m})/k$ is a local algebra, essentially of finite type, with $[T/\mathfrak{m} : k] < \infty$. An evolution of T/k is another local algebra $(S, \mathfrak{n})/k$, also essentially of finite type, together with a surjective map $\varepsilon\colon S \to T$ of local k-algebras, which induces an isomorphism

$$\varepsilon^*\colon \Omega^1_{S/k} \otimes T \to \Omega^1_{T/k}$$

on the level of differential forms. The algebra T/k is called evolutionarily stable, if $\varepsilon = \mathrm{id}_T$ is the only evolution of T/k. Mazur [Ma] asked, whether (in the present situation) every reduced local algebra T/k is evolutionarily stable. The corresponding question in positive characteristic has a negative answer (see [EM]), but for characteristic 0 no counterexamples are known.

The relation to birational geometry is given by the following theorem which summarizes some of the main results of [Hü2] and [HR] (see, in particular, [Hü2, §1 and §3] and [HR, Thm. 2 and Cor. 14]).

2.1 Theorem. *Let k be a field with* $\operatorname{char}(k) = 0$*, and let $(R, \mathfrak{m})/k$ be a smooth local algebra with $[R/\mathfrak{m} : k] < \infty$. Furthermore let $I \subseteq R$ be an ideal and let $T = R/I$. Then the following conditions are equivalent:*

(1) *If* $f \in I$ *with* $f^n \in I^{n+1}$ *for some* $n \in \mathbb{N}$, *then* $f \in \mathfrak{m}I$.
(2) *If* v *is a (normalized, discrete, rank one) valuation of* $K = Q(R)$, *nonnegative on* R *and positive on* I *(but not necessarily on* $\mathfrak{m}$*), and if* $f \in I$ *with* $v(f) \geq v(I) + 1$, *then* $f \in \mathfrak{m}I$.
(3) *If* v *is a Rees-valuation of* I, *and if* $f \in I$ *with* $v(f) \geq v(I) + 1$, *then* $f \in \mathfrak{m}I$.

If I *is a radical ideal, then* (1)–(3) *are also equivalent to the following conditions:*

(4) *If* $\pi\colon X \to Y$ *is a log–resolution of* (R, I) *with*
$$I\mathcal{O}_X = \mathcal{O}_X\left(\sum_{i=1}^{n} -a_i E_i\right)$$
for some $a_i > 0$, *and if in addition* $\mathfrak{m}\mathcal{O}_X$ *is invertible and*
$$\mathfrak{m}\mathcal{O}_X = \mathcal{O}_X\left(\sum_{i=\delta+1}^{n} -b_i E_i\right)$$
for some $b_i > 0$, *then*
$$H^0\left(X, \mathcal{O}_X\left(\sum_{i=1}^{\delta} -(a_i+1)E_i + \sum_{i=\delta+1}^{n} -(a_i+b_i)E_i\right)\right) \subseteq \mathfrak{m}I.$$
(5) *The algebra* T/k *is evolutionarily stable.*

We say that the pair (R, I) satisfies (NN) if condition (1) holds. Not all pairs (R, I) satisfy (NN), even if R is regular; whether $(R, \sqrt{I})$ satisfies (NN) is an open problem; see [Hü2, Conj. 1.3]. No counterexamples to this conjecture are known, even if we generalize it to arbitrary regular local rings. Here we concentrate on condition (4) of the above theorem. Clearly (3) implies (4), so this condition is (presumably) weaker than (NN), and it is also weaker than the condition that $\mathfrak{m}I$ be integrally closed. To obtain a positive answer to Mazur's question, one might try to identify large classes of pairs (R, I) satisfying 2.1 (4) and hope that these classes contain all radical ideals. In this direction we have the following result.

2.2 Corollary. *If* $I \subseteq R$ *is an* $\mathfrak{m}$*-primary ideal, then* (R, I) *satisfies* 2.1 (4) *if and only if* $\mathfrak{m}I$ *is integrally closed.*

Proof. In this situation we have $\delta = 0$ in (4) of 2.1. Furthermore the group $H^0\left(X, \mathcal{O}_X\left(\sum_{i=1}^{n} -(a_i+b_i)E_i\right)\right)$ is exactly the integral closure of $\mathfrak{m}I$. □

2.3 Corollary. *Suppose* I *is integrally closed with* $I \subseteq \mathfrak{m}^2$ *and* $\dim(R/I) = 1$. *If, for a general element* $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ *that is not a zero divisor of* R/I, *the pair* $(R/(x), (I, x)/(x))$ *satisfies* 2.1 (4), *then* (R, I) *satisfies* 2.1 (4).

Proof. By 2.2 the ideal $\mathfrak{m}/(x) \cdot (I, x)/(x)$ is integrally closed in $R/(x)$, and thus $(\mathfrak{m}I, x) \subseteq R$ is integrally closed (as can be seen easily by completing R and using x as a power series variable). But then $\mathfrak{m}I$ is integrally closed, by [HuR, (6.2)]. Hence (R, I) satisfies 2.1 (4). $\square$

Let $\pi\colon X \to Y = \operatorname{Spec}(R)$ be a log resolution of (R, I) such that $\mathfrak{m}\mathcal{O}_X$ is invertible as well. Write

$$I\mathcal{O}_X = \mathcal{O}_X(-F), \quad \mathfrak{m}\mathcal{O}_X = \mathcal{O}_X(-G)$$

with

$$F = \sum_{i=1}^{n} a_i E_i, \quad G = \sum_{j=\delta+1}^{n} b_j E_j$$

(with all $a_i, b_j > 0$), and set

$$H = \sum_{i=1}^{\delta} E_i.$$

2.4 Remark.

(i) (R, I) satisfies 2.1 (4) if and only if $H^0(X, \mathcal{O}_X(-F - G - H)) \subseteq \mathfrak{m}I$.
(ii) $\mathfrak{m}I$ is integrally closed if and only if $H^0(X, \mathcal{O}_X(-F - G)) \subseteq \mathfrak{m}I$.

Let K_X be a canonical divisor of X. As R is local, we may assume that $K_X = \sum c_i E_i$ is supported on the exceptional fibre of π with $c_i > 0$ for all i. We note that the integers c_i are uniquely determined by this.

2.5 Proposition. *Let* $\operatorname{ht}(I) = g$ *and* $\dim(R) = d$*, and assume that* I *is equidimensional without embedded components. If*

$$(g-1)F + (d-1)G - H \leq K_X, \tag{1}$$

then (R, I) *satisfies* 2.1 (4)*, and if*

$$(g-1)F + (d-1)G \leq K_X, \tag{2}$$

then $\mathfrak{m}I$ *is integrally closed.*

Proof. Suppose (1) is satisfied. Then

$$\mathcal{O}_X(-F - G - H) = \omega_X(-K_X - F - G - H) \subseteq \omega_X(-gF - dG)$$

and therefore

$$\begin{aligned} H^0(X, \mathcal{O}_X(-F - G - H)) &\subseteq H^0(X, \omega_X(-gF - dG)) \\ &= \mathcal{I}(I^g \mathfrak{m}^d) \\ &\subseteq \mathcal{I}(I^g) \cdot \mathcal{I}(\mathfrak{m}^d) \end{aligned}$$

by the subadditivity of multiplier ideals ([DEL]; see also [La, (5.2)]). By [Li, (1.3.2)(c)] we have

$$\mathcal{I}(\mathfrak{m}^d) = \mathfrak{m},$$

and therefore it remains to show that $\mathcal{I}(I^g) \subseteq I$. As I has no embedded components, it suffices to show that

$$\mathcal{I}(I^g)R_{\mathfrak{p}} \subseteq IR_{\mathfrak{p}}$$

for all $\mathfrak{p}$ that are minimal over I. This follows easily from [Li, (1.3.1) and (1.6)] (see also [ELS]). $\square$

2.6 Remark. The requirements of 2.5 are very strong, and in fact they will not be satisfied in general (see 2.7 below). They are used to prove the inclusion

$$H^0\left(X, \mathcal{O}_X(-F-G-H)\right) \subseteq H^0\left(X, \omega_X(-gF)\right) \cdot H^0\left(X, \omega_X(-dG)\right).$$

If $I = \mathfrak{p}$ is a prime ideal (so that 2.1 (4) is, in fact, equivalent to (NN)), it would be sufficient to have the weaker subadditivity property

$$\text{(3)} \quad H^0\left(X, \mathcal{O}_X(-F-G-H)\right) \subseteq H^0\left(X, \mathcal{O}_X(-H)\right) \cdot H^0\left(X, \omega_X(-dG)\right).$$

In fact, by [Li, (1.3.2)],

$$H^0\left(X, \omega_X(-dG)\right) = \mathfrak{m},$$

and furthermore

$$\begin{aligned} H^0\left(X, \mathcal{O}_X(-H)\right) &\subseteq H^0\left(X, \mathcal{O}_X(-H)\right) R_{\mathfrak{p}} \cap R \\ &\subseteq \mathfrak{p}R_{\mathfrak{p}} \cap R \\ &= \mathfrak{p}. \end{aligned}$$

Note that (3) is actually equivalent to $(R, \mathfrak{p})$ satisfying (NN). Indeed, we have

$$\mathfrak{p}\mathcal{O}_X \subseteq \mathcal{O}_X(-H),$$

and therefore we get

$$\mathfrak{p} = \overline{\mathfrak{p}} = H^0(X, \mathfrak{p}\mathcal{O}_X) \subseteq H^0\left(X, \mathcal{O}_X(-H)\right).$$

Hence, by 2.4 (i), (3) is equivalent to $(R, \mathfrak{p})$ satisfying (NN). (If I is a radical ideal, we may argue similarly.)

If $I = \mathfrak{p}$ is a prime ideal, certain parts of the condition in 2.5 will always be satisfied:

2.7 Proposition. *Write*

$$F = \sum_{i=1}^{n} a_i E_i, \quad G = \sum_{i=\delta+1}^{n} b_i E_i, \quad K_X = \sum c_i E_i,$$

with positive integers a_i, b_i, c_i. Assume that $I = \mathfrak{p}$ is a prime ideal. Then we have:

(i) $c_i \geq (d-1)b_i$ *for* $i \in \{\delta+1, \dots, n\}$.
(ii) *Let* $E_1, \dots, E_\rho$ *be those exceptional components of* π *with*

$$\pi(E_i) = \mathfrak{V}(\mathfrak{p}),$$

the set of primes containing $\mathfrak{p}$. *Then*

$$c_i \geq (g-1)a_i \quad \textit{for } i \in \{1, \dots, \rho\}.$$

Proof. (i) By the universal property of blowing-up, the map $\pi\colon X \to Y$ factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

where $g\colon Z \to Y$ denotes the blow-up of $\mathfrak{m}$. If T is the reduced exceptional divisor of g, then $K_Z \cong (d-1)T$, i.e., $\omega_Z^{-1} = \mathfrak{m}^{d-1}\mathcal{O}_Z$, and furthermore

$$K_X \cong f^*K_Z + \sum d_i E_i$$

with nonnegative d_i as Z is regular. Since $(f^*\omega_Z)^{-1} = \mathfrak{m}^{d-1}\mathcal{O}_X$, the claim follows by comparing coefficients of the appropriate exceptional components.

(ii) If $Y_\mathfrak{p} = \operatorname{Spec}(R_\mathfrak{p})$, $X_\mathfrak{p} = X \times_Y Y_\mathfrak{p}$, then the induced morphism $\pi_\mathfrak{p}\colon X_\mathfrak{p} \to Y_\mathfrak{p}$ is a log-resolution of the pair $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$, and we may argue as in (i). $\square$

2.8 Remark. If, in the situation of 2.7, $\operatorname{ht}(\mathfrak{p}) = d-1$, then 2.7 implies, in particular, that 2.5 (1) is satisfied if and only if 2.5 (2) is satisfied. Thus the example [Hü2, (4.4)] shows that there are regular local rings R and prime ideals $\mathfrak{p} \subseteq R$ such that 2.5 (1) is not satisfied for any log-resolution $\pi\colon X \to Y$ of $(R, \mathfrak{p})$ of the type considered in this section.

2.9 Question. Find all pairs (R, I) which satisfy 2.5 (1) (or 2.5 (2)).

2.10 Question. Is it possible to prove the inclusion

$$H^0\left(X, \mathcal{O}_X(-F-G-H)\right) \subseteq H^0\left(X, \mathcal{O}_X(-H)\right) \cdot H^0\left(X, \omega_X(-dG)\right)$$

by using the techniques developed by Ein and Lazarsfeld (and others) to study multiplier ideals and to prove their subadditivity?

Some results that seem to be related to the questions raised in this section have been obtained by Ein, Lazarsfeld and Smith [ELS]. They showed that, if $I = \mathfrak{p}$ is prime and if $\pi(E_1) = \mathfrak{V}(\mathfrak{p})$, the set of primes containing $\mathfrak{p}$, then

$$\pi_*(-l(c_1+1)E_1) \subseteq \mathfrak{p}^l \qquad \text{for all } l \in \mathbb{N}.$$

By analyzing the techniques of [DEL] and [La], we obtain the following: Suppose we are in the above situation. Let $q_i\colon X \times X \to X$ and $p_i\colon Y \times Y \to Y$

$(i = 1, 2)$ be the projections. Furthermore let $\pi_{(2)} \colon X \times X \to Y \times Y$ be the morphism induced by π and denote by $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X\times X}$ the ideal of the diagonal. Then condition (NN) can be related to the vanishing of certain cohomology groups. We state here a result along these lines, although so far we have not found any non-trivial examples where it applies:

2.11 Proposition. *Assume that $I \subseteq R$ is a prime ideal (or a radical ideal that is equidimensional of height $g < d = \dim(R)$). Suppose that there exist divisors A, B on X with*

(1) $H = \sum_{i=1}^{\delta} E_i \leq A \leq F + H$,
(2) $H^* = \sum_{i=\delta+1}^{n} E_i \leq B \leq G$,

such that

$$R^1\pi_{(2)*}\left(\mathcal{I}_\Delta \mathcal{O}_{X\times X}(-q_1^*A - q_2^*B)\right) = 0.$$

Then (R, I) satisfies (NN).

Proof. The vanishing of the cohomology group implies that the canonical map

$$\pi_{(2)*}\mathcal{O}_{X\times X}(q_1^*A - q_2^*B) \longrightarrow \pi_*\mathcal{O}_X(-A - B)$$

is surjective, and by the Künneth formula we have

$$\pi_{(2)*}\mathcal{O}_{X\times X}(-q_1^*A - q_2^*B) = p_1^*\left(\pi_*\mathcal{O}_X(-A)\right) \otimes p_2^*\left(\pi_*\mathcal{O}_X(-B)\right),$$

implying that

$$\begin{aligned}\pi_*\mathcal{O}_X(-A - B) &\subseteq \pi_*\mathcal{O}_X(-A) \cdot \pi_*\mathcal{O}_X(-B)\\ &\subseteq \pi_*\mathcal{O}_X(-H) \cdot \pi_*\mathcal{O}_X(-H^*)\\ &= I \cdot \mathfrak{m}.\end{aligned}$$

On the other hand, the assumptions clearly imply

$$\pi_*\mathcal{O}_X(-F - G - H) \subseteq \pi_*\mathcal{O}_X(-A - B),$$

and so the claim follows. □

2.12 Remark. Some of the results of [ELS] have been obtained (and strengthened) by Hochster and Huneke [HH], using tight closure techniques. It is not clear to the author whether tight closure and Frobenius may also be used in the study of evolutions.

3. Derivations and log-resolution

Throughout this section we assume that $(S, \mathfrak{n})$ is a regular local ring, essentially of finite type over a field K of characteristic 0 with $[S/\mathfrak{n} : K] < \infty$. We also assume that $J \subseteq S$ is an $\mathfrak{n}$-primary ideal and that $\Pi \colon \mathfrak{X} \to \mathfrak{Y} = \operatorname{Spec}(S)$ is a log-resolution of (S, J) with the following properties:

(1) $\Pi \colon \Pi^{-1}(\mathfrak{Y} \setminus \{\mathfrak{n}\}) \longrightarrow \mathfrak{Y} \setminus \{\mathfrak{n}\}$ is an isomorphism.

(2) The reduced exceptional locus

$$E = \Pi^{-1}(\{\mathfrak{n}\})_{\text{red}} = \sum_{i=1}^{n} E_i$$

is a strictly normal crossing divisor.

(3) $J\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}(-F)$ and $\mathfrak{m}\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}(-G)$, where

$$F = \sum_{i=1}^{n} a_i E_i \quad \text{and} \quad G = \sum_{i=1}^{n} b_i E_i$$

with positive coefficients a_i, b_i.

Furthermore, let k be a perfect field of characteristic $p > 0$, let $(R, \mathfrak{m})/k$ be a regular local algebra, essentially of finite type, and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Assume that (R, I) arises from (S, J) by reduction mod p and that (R, I) has a log-resolution

$$\pi\colon X \longrightarrow Y = \operatorname{Spec}(R)$$

arising from Π by reduction mod p and satisfying (1)–(3) accordingly. In particular, we have $I\mathcal{O}_X = \mathcal{O}_X(-F)$.

In this section we are interested in the following question.

Question. Suppose $r \in R$ is an element with $\delta(r) \in I$ for all $\delta \in \operatorname{Der}_k(R)$. Does there exist an $a \in C(R/k)$ such that $r - a$ is in $\overline{I}$, the integral closure of I?

Note that, in the present situation, $C(R/k) = R^p$, so the question amounts to asking whether the condition $\delta(r) \in I$ for all $\delta \in \operatorname{Der}_k(R)$ forces r to be in the integral closure of I up to a pth power in R.

Since, by [LS], $\overline{\mathfrak{a}^{n+d-1}} \subseteq \mathfrak{a}^n$ for each ideal $\mathfrak{a} \subseteq R$, a positive answer to this question would provide a positive answer to Huneke's original problem (see the introduction of [CH]).

3.1 Proposition. *If $H^1\left(X, \mathcal{O}_X\left(-\lceil (1/p)F \rceil\right)\right) = 0$, then the above question has a positive answer for R and I.*

Proof. (See also [CH, §3].) Let $f \in R$ with $\delta(f) \in I$ for all $\delta \in \operatorname{Der}_k(R)$. As R/k is smooth, this is equivalent to $d_{R/k}(f) \in I\Omega^1_{R/k}$. Furthermore, let $x \in X$ with $\pi(x) = \mathfrak{m}$. Then clearly $d_{\mathcal{O}_{X,x}/k}(f) \in I\Omega^1_{\mathcal{O}_{X,x}/k}$ by the functoriality of differential forms, and thus $\delta(f) \in I\mathcal{O}_{X,x}$ for all $\delta \in \operatorname{Der}_k(\mathcal{O}_{X,x})$. As $I\mathcal{O}_{X,x}$ defines a normal crossing divisor, we may apply [CH, (2.11)], and we conclude that for each $x \in X$ there exists a $c_x \in \mathcal{O}_{X,x}$ with

$$f - c_x^p \in I\mathcal{O}_{X,x}.$$

(Note that for $x \notin \pi^{-1}(\{\mathfrak{m}\})$ this is trivial.) Thus we can find a finite open cover $U_1 = \operatorname{Spec}(S_1), \ldots, U_t = \operatorname{Spec}(S_t)$ of X and $c_i \in S_i$ with

$$h_i := f - c_i^p \in IS_i.$$

From this we conclude

$$c_i^p - c_j^p \in \Gamma\left(U_i \cap U_j, \mathcal{O}_X(-F)\right),$$

which implies

$$c_i - c_j \in \Gamma\left(U_i \cap U_j, \mathcal{O}_X\left(-\left\lceil \frac{1}{p}F \right\rceil\right)\right).$$

Thus by our assumption there exist $\lambda_i \in \Gamma\left(U_i, \mathcal{O}_X\left(-\lceil (1/p)F \rceil\right)\right)$ with

$$\lambda_i - \lambda_j = c_i - c_j \quad \text{on } U_i \cap U_j,$$

so that $c_i - \lambda_i = c_j - \lambda_j$ on $U_i \cap U_j$. Hence there exists an $a \in R = \Gamma(X, \mathcal{O}_X)$ with

$$a \mid U_i = c_i - \lambda_i \quad \text{for } i \in \{1, \ldots, t\},$$

implying

$$f - a^p \mid U_i = h_i - \lambda_i^p \in \Gamma\left(U_i, \mathcal{O}_X(-F)\right) \quad \text{for all } i \in \{1, \ldots, t\}.$$

From this we conclude that

$$f - a^p \in \Gamma\left(X, \mathcal{O}_X(-F)\right) = \overline{I}.$$

□

3.2 Corollary. *In the above situation suppose that all high powers of ω_X^{-1} are generated by global sections. Then the question has a positive answer.*

Proof. In the present situation we have

$$H^1\left(X, \mathcal{O}_X\left(-\left\lceil \frac{1}{p}F \right\rceil\right)\right) = H^1\left(X, \omega_X\left(-K_X - \left\lceil \frac{1}{p}F \right\rceil\right)\right).$$

To see that this object vanishes we argue as in the proof of 1.1. Thus it suffices to prove that

$$H^i(X, \Omega_X^j\left(\log(E)\left(-E - p^n K_X - p^{n-1}F\right)\right) = 0$$

for all i, j with $i + j > d = \dim(R)$ and for some $n > 0$. This, however, follows from the proof of 1.1. □

3.3 Remark. X is obtained from $Y = \operatorname{Spec}(R)$ by a sequence of blow-ups of regular centers. If X arises from Y by blowing up only once, then ω_X^{-1} will be very ample, and the assumption of 3.2 is satisfied. However, in general ω_X^{-1} will not be ample, not even in the case when $\dim(Y) = 2$.

The next result states that the question has a positive answer "up to some discrepancy", which, however, depends on p.

3.4 THEOREM. *In the situation of* 3.1 *there exists a power* $J = I^l$ *of the ideal* I (*with* l *depending on the log-resolution of* (R, I) *and on the characteristic* $p = \operatorname{char}(k)$) *with the following property:*

For $f \in R$ *and* $n \in \mathbb{N}$ *with* $\delta(f) \in J^{n+1}$ *for all* $\delta \in \operatorname{Der}_k(R)$ *there exists an element* $a \in R$ *with*

$$f - a^p \in \overline{J^n}.$$

3.5 REMARK. (i) If we can make l independent of p, then this would answer Huneke's original question in the affirmative.

(ii) A similar result was already obtained in [CH, §2]. The result of 3.4, however, also provides an additional linear bound in p and thus a "weak" answer to Huneke's original question; see 3.7 below.

Proof. Let $\pi\colon X \to Y$ be the log-resolution of (R, I) with $I\mathcal{O}_X = \mathcal{O}_X(-F)$ for some effective

$$F = \sum_{i=1}^{m} a_i E_i$$

and write

$$K_X = \sum_{i=1}^{m} b_i E_i.$$

Note that $a_i, b_i > 0$ for all i since $\pi(E_i) = \{\mathfrak{m}\}$ for all i. Thus we may assume, after replacing I by a sufficiently high power $J = I^l$, that $p \cdot b_i \leq a_i$.

Let $f \in R$ with $\delta(f) \in J^{n+1}$ for some $n \in \mathbb{N}$ and all $\delta \in \operatorname{Der}_k(R)$. Arguing as in the proof of 3.1, we obtain an open affine cover $\{U_i = \operatorname{Spec}(S_i)\}$ of X and $c_i \in S_i$ such that

$$h_i = f - c_i^p \in J^{n+1} S_i,$$

i.e., such that

$$\begin{aligned} c_i - c_j &\in \Gamma\left(U_i \cap U_j, \mathcal{O}_X\left(-\left\lceil \frac{n+1}{p} F \right\rceil\right)\right) \\ &\subseteq \Gamma\left(U_i \cap U_j, \omega_X\left(-\left\lceil \frac{n+1}{p} F \right\rceil\right)\right), \end{aligned}$$

as K_X is effective. Since $\mathcal{O}_X(-F)$ and all its powers are generated by global sections, we conclude from 1.1 that

$$H^1\left(X, \omega_X\left(-\left\lceil \frac{n+1}{p} F \right\rceil\right)\right) = 0.$$

Thus there exist $\lambda_i \in \Gamma\left(U_i, \omega_X\left(-\left\lceil \frac{n+1}{p} F \right\rceil\right)\right)$ such that

$$\lambda_i - \lambda_j = c_i - c_j,$$

and as in the proof of 3.1 we obtain an element $a \in R$ with

$$a \mid U_i = c_i - \lambda_i.$$

This implies that

$$\begin{aligned} f - a^p \mid U_i = h - \lambda_i^p \mid U_i &\in \Gamma(U_i, \mathcal{O}_X(pK_X - (n+1)F) \\ &\subseteq \Gamma\left(U_i, \mathcal{O}_X(-nF)\right), \end{aligned}$$

from which we conclude

$$f - a^p \in \Gamma\left(X, \mathcal{O}_X(-nF)\right) = \overline{J^n}. \qquad \square$$

Question. In the above situation is it true that

$$H^1\left(X, \omega_X\left(-K_X - \left\lceil \frac{n+1}{p} F \right\rceil\right)\right) = 0$$

for all positive integers $n \in \mathbb{N}$, or at least for all sufficiently large integers n?

3.6 Remark. For the use in connection with tight closure techniques it often would be sufficient to answer the above question for all sufficiently large n. In fact, it would be sufficient to know that there exists an N_0 (which may depend on R and I) with the following property: If $n > N_0$ and if $r \in R$ with $\delta(r) \in I^n$ for all $\delta \in \mathrm{Der}_k(R)$, then $r - a \in \overline{I^n}$ for a suitable $a \in C(R/k)$.

3.7 Corollary. *Let $(S, \mathfrak{M})$ be a regular local ring of dimension d, essentially of finite type over a field k of characteristic 0, and such that $S/\mathfrak{M}$ is finite over k, and let $J \subseteq S$ be an $\mathfrak{M}$-primary ideal. Then there exists a nonnegative integer c with the following property:*

For almost all primes $p > 0$, if (R, I) arises from (S, J) by reduction mod p and if $f \in R$ and $n \in \mathbb{N}$, $n \geq cp + d$, with $\delta(f) \in I^n$ for all $\delta \in \mathrm{Der}_{\mathbb{F}_p}(R)$, then there exists an element $a \in R$ with

$$f - a^p \in \overline{I^{n-cp}} \subseteq I^{n-cp-d+1}.$$

Proof. Using a log-resolution arising from characteristic 0 and with the notations introduced in the proof of 3.4 we can choose an integer c such that $b_i \leq (c/2) \cdot a_i$ for all i. Then the proof of 3.4 goes through if in each reduction we take for J the $(c \cdot p)$th power of I. From this and the Briançon–Skoda theorem [LS] the claim follows. $\square$

By 3.4 and 3.7 the above question reduces to a question on the vanishing of cohomology groups. We can ask even more generally:

Question. Let $\pi \colon X \to Y = \mathrm{Spec}(R)$ be a log-resolution of $I \subseteq R$, and let $E = \sum_{i=1}^m E_i$ be the reduced exceptional divisor. Does there exist a number $c \in \mathbb{N}$ (depending on Y, X and $\mathrm{char}(k)$) such that

$$H^1\left(X, \omega_X\left(-\sum_{i=1}^m a_i E_i\right)\right) = 0$$

for all $a_i \geq c$, $i = 1, \ldots, m$, or equivalently

$$H^1\left(X, \mathcal{O}_X\left(-\sum_{i=1}^{m} a_i E_i\right)\right) = 0$$

for all $a_i \geq c$, $i = 1, \ldots, m$?

3.8 Remark. In [HR] it was shown that Mazur's question may also be related to the vanishing of certain cohomology groups. This raises the following question:

Question. Is it possible to determine all integers a_i for which

$$H^1\left(X, \omega_X\left(-\sum_{i=1}^{m} a_i E_i\right)\right) = 0,$$

or at least large classes of integers a_i for which this is true?

3.9 Remark. Suppose that $\pi(E) = \{\mathfrak{m}\}$ and let $U = Y \setminus \{\mathfrak{m}\}$. As

$$\pi\colon \pi^{-1}(U) \longrightarrow U$$

is an isomorphism, $H^i(X, \mathcal{M})$ is an Artinian module for each $i > 0$ and each coherent $\mathcal{O}_X$-module $\mathcal{M}$. Thus [EV, §11] can be modified to prove that for all positive integers $n \in \mathbb{N}$

$$(*) \qquad \bigoplus_{a+b=l} \lg_R H^a\left(X, \Omega^b_X(\log(E)) \otimes \mathcal{O}_X(-F)\right) \leq \bigoplus_{a+b=l} \lg_R H^a\left(X, \Omega^b_X\left(\log(E)\right) \otimes \mathcal{O}_X(-p^n F)\right).$$

Thus the above question has a positive answer for all those

$$F = \sum_{i=1}^{m} a_i E_i$$

such that $-F$ is ample, in which case the right-hand side of $(*)$ vanishes for $n \gg 0$. In general, however, we cannot hope for the vanishing of the right-hand side of $(*)$, even in the case $d = 3$, as the following example shows:

There exist pairs (R, I) with $\mathfrak{m}$-primary ideals I such that for each log-resolution $\pi\colon X \to Y$ and for any $c \in \mathbb{N}$ there exists a divisor

$$F = \sum_{i=1}^{m} \alpha_i E_i$$

with $\alpha_i \geq c$ for all $i \in \{1, \ldots, m\}$ and there exist a, b with $a + b = l \geq d + 1$ such that

$$H^a\left(X, \Omega^b_X(\log E) \otimes \mathcal{O}_X(-F)\right) \neq 0.$$

In fact, by [Hu] there exist regular local rings $(R, \mathfrak{m})$ (of dimension $d \geq 3$ and containing the rationals) and $\mathfrak{m}$-primary ideals $I \subseteq R$ such that I is integrally closed, but $\mathfrak{m}I$ is not integrally closed. We restrict ourselves to the case $d = 3$ and leave the general situation to the reader.

Let $\pi\colon X \to Y$ be a log-resolution of $\mathfrak{m}$ and I and write

$$\mathfrak{m}\mathcal{O}_X = \mathcal{O}_X\left(-\sum b_i E_i\right), \quad I\mathcal{O}_X = \mathcal{O}_X\left(-\sum a_i E_i\right),$$

with positive integers a_i, b_i. A minor adaption of [HR, Cor. 17] to the present situation shows that either

$$H^1\left(X, \mathcal{O}_X\left(\sum -(a_i - b_i)E_i\right)\right) \neq 0 \tag{1}$$

or

$$H^2\left(X, \mathcal{O}_X\left(\sum -(a_i - 2b_i)E_i\right)\right) \neq 0. \tag{2}$$

Write $K_X = \sum c_i E_i$. In case (1) we conclude from $(*)$ that for all $n \geq 0$ either

$$H^1\left(X, \omega_X\left(\sum -p^n(c_i + a_i - b_i)E_i\right)\right) \neq 0$$

or

$$H^2\left(X, \Omega_X^2(\log(E)) \otimes \mathcal{O}_X\left(\sum -p^n(c_i + a_i - b_i)E_i\right)\right) \neq 0.$$

Arguing as in 2.7 (i) we conclude that $c_i \geq 2b_i$. Thus the coefficients grow arbitrarily large. In case (2) we proceed similarly. This, however, does not imply that the question raised above has a negative answer.

In 3.1 we require the vanishing of $H^1\left(X, \mathcal{O}_X\left(-\lceil (1/p)F \rceil\right)\right)$, where $I\mathcal{O}_X = \mathcal{O}_X(-F)$. The following result shows that the first cohomology group of the inverse of this line bundle vanishes, provided $d \geq 3$.

3.10 Proposition. *In the situation of* 3.1 *let $r \in \mathbb{Q}_+$ be a positive rational number. Then*

$$H^j\left(X, \mathcal{O}_X\left(\lceil rF \rceil\right)\right) = 0 \quad \textit{for } j \in \{1, \dots, d-2\}.$$

Proof. Let $Z = \pi^{-1}(\{\mathfrak{m}\})$ and let $U = \pi^{-1}(Y \setminus \{\mathfrak{m}\})$. Then we have a long exact sequence

$$\dots \to H_Z^j\left(X, \mathcal{O}_X\left(\lceil rF \rceil\right)\right) \to H^j\left(X, \mathcal{O}_X\left(\lceil rF \rceil\right)\right) \to H^j\left(U, \mathcal{O}_X\left(\lceil rF \rceil\right) \mid U\right) \to \dots$$

As $\mathcal{O}_X\left(\lceil rF \rceil\right) \mid U = \mathcal{O}_U$ and $U \cong Y \setminus \{\mathfrak{m}\}$, we have

$$H^j(U, \mathcal{O}_X\left(\lceil rF \rceil\right) \mid U) = H^j(U, \mathcal{O}_U) = H_{\mathfrak{m}}^{j+1}(R) = 0$$

for $0 < j \leq d-2$ since R is Cohen–Macaulay. Denoting by $\widehat{M}$ the completion of an R-module M we have, by formal duality,

$$\operatorname{Hom}_k(H_Z^j\left(X, \mathcal{O}_X\left(\lceil rF \rceil\right)\right), k) \cong H^{d-j}\left(X, \omega_X\left(-\lceil rF \rceil\right)\right)^{\widehat{}} = 0$$

for $j < d$ (by 1.1), and the claim follows. □

3.11 Remark. If $r \geq 0$, then in general we do not have

$$H^{d-j}\left(X, \omega_X\left(\lceil rF \rceil\right)\right)^{\widehat{}} = 0$$

for all $j < d$, and thus we cannot apply the above argument to $-\lceil (1/p)F \rceil$. However, setting, for $r \geq 0$,

$$\overline{I^r} := H^0\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right)$$

(the "integral closure of I^r") and assuming that $d \geq 2$, the following conditions are equivalent:

(1) $H^1\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) = 0$,
(2) $H^1_Z\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) = R/\overline{I^r}$.

Proof. We have a long exact sequence

$$\begin{aligned} \cdots \to H^0\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) \to H^0(U, \mathcal{O}_U) \to H^1_Z\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) \\ \to H^1\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) \to H^1(U, \mathcal{O}_U) \to \cdots, \end{aligned}$$

where we have used the fact that $\mathcal{O}_X\left(-\lceil rF \rceil\right) \mid U = \mathcal{O}_U$.

If $d \geq 3$, then $H^1(U, \mathcal{O}_U) = H^2_{\mathfrak{m}}(R) = 0$ and $H^0(U, \mathcal{O}_U) = R$, implying the claim in this case.

If $d = 2$, we still have $H^0(U, \mathcal{O}_U) = R$. Furthermore we have the commutative diagram

$$\begin{tikzcd} H^1(U, \mathcal{O}_X\left(-\lceil rF \rceil\right) \mid U) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & H^2_Z\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) \arrow[d, "\gamma"] \\ H^1(U, \mathcal{O}_U) = H^2_{\mathfrak{m}}(R) \arrow[r, "\delta"] & H^2_Z(\mathcal{O}_X) \end{tikzcd}$$

where all maps are the canonical ones. Since $(R, \mathfrak{m})$ is regular, it is pseudo-rational by [LT, §4], and therefore δ is injective. As β is an isomorphism, we conclude that α is injective. This shows that (1) and (2) are equivalent in the two-dimensional case as well. □

3.12 Remark. (i) The integral closure of generalized powers of ideals has also been considered by McAdam, Ratliff and Sally [MRS]. In the above situation, however, little seems to be known about local cohomology with supports in the exceptional fibre Z and its relations to the integral closure of these powers in general.

(ii) If $r \in \mathbb{Q}_+$ is such that all coefficients of rF are less than or equal than 1, then

$$H^1\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) = 0$$

by [CH, 3.4], and therefore we have

$$H^1_Z\left(X, \mathcal{O}_X\left(-\lceil rF \rceil\right)\right) = R/\overline{I^r}$$

in this case as well.

References

[CH] S. D. Cutkosky and R. Hübl, *Vanishing of differentials along ideals and non–archimedean approximation*, Math. Ann. **317** (2000), 295–308.

[DEL] J. P. Demailly, L. Ein, and R. Lazarsfeld, *A subadditivity property for multiplier ideals*, Michigan Math. J. **48** (2000), 137–156.

[DI] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270.

[Ei] L. Ein, *Multiplier ideals, vanishing theorems and applications*, Algebraic geometry (Santa Cruz 1995), Proc. Symp. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997; pp. 203–219.

[ELS] L. Ein, R. Lazarsfeld, and K. Smith, *Uniform bounds and symbolic powers on smooth varieties*, Invent. Math. **144** (2001), 141–152.

[EM] D. Eisenbud and B. Mazur, *Evolutions, symbolic squares, and fitting ideals*, J. Reine Angew. Math. **488** (1997), 189–201.

[EV] H. Esnault and E. Viehweg, *Lectures on vanishing theorems*, DMV–Seminar, vol. 20, Birkhäuser, 1992.

[Ha] N. Hara, *A characterization of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. **120** (1998), 981–996.

[Hi] H. Hironaka, *Resolution of singularities over an algebraic variety over a field of characteristic* 0. *I, II*, Ann. of Math. (2) **79** (1964), 109–203, 205–326.

[HH] M. Hochster and C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. **147** (2002), 349–369.

[Hü1] R. Hübl, *Derivations and the integral closure of ideals*, Proc. Amer. Math. Soc. **127** (1999), 3503–3511.

[Hü2] ———, *Evolutions and valuations associated to an ideal*, J. Reine Angew. Math. **517** (1999), 81–101.

[HR] R. Hübl and A. Rechenauer, *Valuations of ideals, evolutions and the vanishing of cohomology groups*, Geometric and combinatorial aspects of commutative algebra (Messina, 1999), Lecture Notes in Pure and Appl. Math., vol. 217, Marcel Dekker, New York, 2001, pp. 245–258.

[Hu] C. Huneke, *The primary components and integral closures of ideals in 3-dimensional regular local rings*, Math. Ann. **275** (1986), 617–635.

[HuR] C. Huneke and J. Ribbe, *Symbolic squares in regular local rings*, Math. Z. **229** (1998), 31–44.

[HS] C. Huneke and K. Smith, *Tight closure and the Kodaira Vanishing Theorem*, J. Reine Angew. Math. **484** (1997), 127–152.

[La] R. Lazarsfeld, *Multiplier ideals for algebraic geometers*, Preprint.

[Li] J. Lipman, *Adjoints of ideals in regular local rings*, Math. Res. Letters **1** (1994), 739–755.

[LS] J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon–Skoda*, Michigan Math. J. **28** (1981), 199–232.

[LT] J. Lipman and B. Teissier, *Pseudo–rational local rings and a theorem of Briançon–Skoda*, Michigan Math. J. **28** (1981), 97–116.

[Ma] B. Mazur, *Deformations of Galois representations and Hecke algebras*, Lecture notes, Harvard, 1994.

[MRS] S. McAdam, L. J. Ratliff, and J. D. Sally, *Integrally closed projectively equivalent ideals*, Commutative algebra (Berkeley, CA, 1987), Springer-Verlag, New York, 1989, pp. 407–416.

[MS] V. B. Mehta and V. Srinivas, *A characterization of rational singularities*, Asian J. Math. **1** (1997), 249–271.

NWF I–Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: Reinhold.Huebl@Mathematik.Uni-Regensburg.de