Illinois Journal of Mathematics
Volume 46, Number 1, Spring 2002, Pages 111–128
S 0019-2082

APPROXIMATION THEOREMS IN O-MINIMAL STRUCTURES

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ABSTRACT. We prove that, in any o-minimal structure, definable C^r mappings can always be approximated by definable C^{r+1} mappings. As an application we obtain definable triviality for pairs of definable proper submersions.

Introduction

In this note we study the following interesting problem in Differential Topology: the approximation of C^r mappings by C^s mappings, where $r < s < \infty$. We study this problem in an *o-minimal context*, that is, we work on C^r mappings definable in an o-minimal structure expanding a real closed field. Classical arguments for this problem (for example, convolutions; see [9]) cannot be used here, as the integration of definable functions does not necessarily produce definable functions.

Using this approximation result, we can extend to the o-minimal context the triviality results for submersions proved by Coste and Shiota [4] and by the author [7] in the semialgebraic case.

Let us recall the basic notions involved. A *structure expanding a real closed field* R is a collection $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$, where each $\mathcal{S}_n$ is a Boolean subalgebra of subsets of the affine space R^n that contains all algebraic sets of R^n and such that $A \times B \in \mathcal{S}_{m+n}$ if $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, and $\pi(A) \in \mathcal{S}_n$ if $\pi : R^{n+1} \to R^n$ is the projection on the first n coordinates and $A \in \mathcal{S}_{n+1}$. The elements of $\mathcal{S}_n$ are called the *definable* subsets of R^n. The structure $\mathcal{S}$ is said to be *o-minimal* if the elements of $\mathcal{S}_1$ are precisely the finite unions of points and intervals. Of course, the first model for an o-minimal geometry is semialgebraic geometry. Nice references on the subject are [3] and [5]. In the following, we shall always work in an o-minimal structure expanding a real closed field R.

It is easy to translate to an arbitrary real closed field R the usual notions of differentiability over $\mathbb{R}$. Basic results on differentiability for semialgebraic

Received February 2, 2001; received in final form December 7, 2001.
2000 *Mathematics Subject Classification.* Primary 03C64. Secondary 57R12.
Partially supported by DGICYT, PB98-0756-C02-01.

functions defined over R can be found in [1]. Van den Dries' work [5] gives results of this kind for definable functions. We use the following notation. If a C^k function $f : U \to R$, where $U \subset R^n$ is an open definable subset, is definable in an o-minimal structure $\mathcal{S}$, we say that f is (of class) $\mathcal{D}^k_{\mathcal{S}}$ (or simply $\mathcal{D}^k$ when there is no ambiguity about $\mathcal{S}$). In other words, $\mathcal{D}^k$ means "definable of class C^k". In [10], the notion of a $\mathcal{D}^r$ manifold is introduced. We can construct the usual objects attached to a $\mathcal{D}^r$ manifold, such as tangent and normal bundles (see [3] and [10]).

We establish our results for $r < \infty$ because the $\mathcal{D}^\infty$ category is not well behaved (see [13]).

Our approximation theorem replaces the approximation theorem for the Nash case established by Shiota [11] and allows us to obtain the equivalence of the $\mathcal{D}^r$ and $\mathcal{D}^{r+1}$ categories. This can be applied, for instance, to construct tubular neighbourhoods without loss in the order of differentiability.

We remark here that [12, Sec. II.6] contains results of this type, although the proofs given there are quite different and difficult to follow, even in the case of the reals.

In Section 1, we state and prove the approximation theorem for $\mathcal{D}^r$ mappings. We prove the results in several steps, obtaining at each step a partial approximation result. One of these steps is a result that is of great interest in itself: it asserts the existence of $\mathcal{D}^r$ tubular neighbourhoods for $\mathcal{D}^r$ manifolds. We apply these results to prove a result (Theorem 1.10) about the smoothing of definable corners. In Section 2 we explain how the triviality of pairs of proper definable submersions follows, once our approximation theorem is available. To prove the triviality results, we introduce the concepts of the definable spectrum and an elementary extension.

This paper is based on our Ph.D. thesis [6]. We wish to thank our advisors Michel Coste and Jesús M. Ruiz for their help and support.

1. Approximation theorems in o-minimal structures

In this section we prove the following approximation theorem for definable mappings in an o-minimal structure:

THEOREM 1.1. *Given two $\mathcal{D}^k$ manifolds $X \subset R^n$ and $Y \subset R^m$, each $\mathcal{D}^{k-1}$ mapping $f : X \to Y$ admits a $\mathcal{D}^k$ approximation $\widetilde{f} : X \to Y$.*

(This result corresponds to [6, Th. 4.7.1].)

To make the statement of the theorem precise, we must first define a topology in the spaces of $\mathcal{D}^r$ mappings, for $r < \infty$. Once this is done, we will prove several partial results, which will then lead us to the proof of the main theorem.

Let $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ be an o-minimal structure expanding the real closed field R. Let X and Y be $\mathcal{D}^r$ submanifolds of R^n and R^m, respectively. Fix $k \le r$.

We denote by $\mathcal{D}^k_{\mathcal{S}}(X,Y)$ the set of $\mathcal{D}^k$ mappings $X \to Y$. We write $\mathcal{D}^k(X,Y)$ when there is no confusion about $\mathcal{S}$, and $\mathcal{D}^k(X)$ when $Y = R$.

To define a topology on $\mathcal{D}^k(X,Y)$, consider first the case $Y = R$. We use the following notation. If V is a $\mathcal{D}^{k-1}$ vector field on X and $f \in \mathcal{D}^k(X)$, we write Vf for the *derivative of f along V*, that is, $Vf(x) = Df(x)(V(x))$ for each $x \in X$. We take a finite family $\{V_1, \dots, V_p\}$ of $\mathcal{D}^{k-1}$ vector fields on X that span the tangent space to X at each point of X, that is,

$$\langle V_1(x), \dots, V_p(x)\rangle = T_x X \text{ for each } x \in X.$$

For each definable continuous positive function ε on X let

$$U_\varepsilon = \left\{ g \in \mathcal{D}^k(X) : |V_{i_1} \cdots V_{i_j} g| < \varepsilon \text{ for } 1 \le i_1, \dots, i_k \le p, j \le k \right\}.$$

Then the sets $\{h + U_\varepsilon\}_\varepsilon$ form a neighbourhood basis of $\mathcal{D}^k(X)$ at h, which defines a topology on $\mathcal{D}^k(X)$.

The topology on $\mathcal{D}^k(X, R^m) = \mathcal{D}^k(X,R) \times \cdots \times \mathcal{D}^k(X,R)$ is simply the product topology. For a manifold $Y \subset R^m$, $\mathcal{D}^k(X,Y)$ is a subset of $\mathcal{D}^k(X,R^m)$, and we can restrict to $\mathcal{D}^k(X,Y)$ the topology of $\mathcal{D}^k(X,R^m)$.

We can always choose a finite set $\{V_1, \dots, V_p\}$ of vector fields on X that verifies the above condition (specifically, we can take the projections on the tangent spaces $\mathrm{T}_x X$ of the vector fields $\partial/\partial x_i$, where $x_1, \dots, x_n$ are the coordinates in R^n), and it is easy to check that the topology does not depend on the choice of $\{V_1, \dots, V_p\}$.

We will call this topology the $\mathcal{D}^k$ *topology.* It is a definable version of the strong Whitney topology.

Proposition 1.2. *Given a $\mathcal{D}^r$ submanifold $X \subset R^n$ and a* closed *$\mathcal{D}^r$ submanifold $Y \subset X$, the* restriction mapping

$$\mathrm{res} : \mathcal{D}^r(X) \to \mathcal{D}^r(Y) : f \mapsto f_{|Y}$$

is continuous for the $\mathcal{D}^k$ topology.

Proof. (See [6, Prop. 4.1.2].) Consider a set $\{V_1, \dots, V_q\}$ of $\mathcal{D}^{r-1}$ vector fields on X such that $\{V_1(x), \dots, V_q(x)\}$ generates $\mathrm{T}_x X$ at each $x \in X$. Reordering the V_i's if necessary, we can assume that there exists $p \le q$ such that $\{V_1(x), \dots, V_p(x)\}$ generates $\mathrm{T}_x Y$ for each $x \in Y$.

By this construction, if we take a neighbourhood U_ε of 0 as above, we can construct a definable function $\overline{\varepsilon} : X \to R$ such that $0 \in U_{\overline{\varepsilon}} \subset \mathrm{res}^{-1}(U_\varepsilon)$. To construct $\overline{\varepsilon}$, let T be an open definable tubular neighbourhood of Y in X, and $\pi : T \to Y$ a $\mathcal{D}^{r-1}$ retraction (see Remark 1.4 below). Take a $\mathcal{D}^r$ partition of unity $\{\theta, 1-\theta\}$ of X subordinated to the covering $\{T, X \setminus Y\}$, and set $\overline{\varepsilon} = \theta(\varepsilon \circ \pi) + (1-\theta)$. □

If Y is not closed, we obtain the continuity of

$$\mathrm{res} : \mathcal{D}^r(U) \to \mathcal{D}^r(Y)$$

in an open definable neighbourhood U of Y in X (we can take $U = X \setminus \mathrm{Bd}(Y)$).

A main property of this topology is that *the* $\mathcal{D}^k$ *diffeomorphisms form an open subset of* $\mathcal{D}^k(X, Y)$. (To see this, it is enough to repeat the argument given in [11, Lemma II.1.7] for the semialgebraic case.) Moreover, *the* $\mathcal{D}^k$ *embeddings form an open subset.*

Another easy, but useful result is the following:

PROPOSITION 1.3. *Let* $X \subset R^m$, $Y \subset R^n$ *and* $Z \subset R^p$ *be* $\mathcal{D}^k$ *manifolds. Let* $h : Y \to Z$ *be a* $\mathcal{D}^k$ *mapping. Then the mapping*

$$\begin{aligned} h_* : \mathcal{D}^k(X, Y) &\to \mathcal{D}^k(X, Z) \\ f &\mapsto h_*(f) = h \circ f \end{aligned}$$

is continuous (for the $\mathcal{D}^k$ *topology).*

Proof. See [6, Prop. 4.1.3]. □

For our approximation results, we will need the following construction.

REMARK 1.4. Let M be a $\mathcal{D}^r$ submanifold of R^n. The usual normal bundle is a $\mathcal{D}^{r-1}$ manifold, and there exists a $\mathcal{D}^{r-1}$ diffeomorphism from a definable open neighbourhood of the zero section $M \times \{0\}$ onto a definable open neighbourhood Ω of M in R^n. This neighbourhood Ω is called a $\mathcal{D}^{r-1}$ *tubular neighbourhood* of M, and as usual we have a $\mathcal{D}^{r-1}$ retraction $\pi : \Omega \to M$ and a $\mathcal{D}^{r-1}$ "square of the distance function" $\rho : \Omega \to R$. (See [3, Th. 6.11] or [12, Lemma II.5.1] for details. In the latter reference, the result is established in a somewhat different setting.)

Having defined the topology and our auxiliary construction, we now turn to approximations. We begin by proving an approximation theorem under some additional restrictions. We fix the following notation. We write $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n) \in R^{m+n}$. Given $a = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$, we set $|a| = a_1 + \cdots + a_m$, $|b| = b_1 + \cdots + b_n$ and

$$\frac{\partial^{|a|+|b|} f}{\partial x^a \partial y^b} = \frac{\partial^{|a|+|b|} f}{\partial x^{a_1} \cdots \partial x^{a_m} \partial y^{b_1} \cdots \partial y^{b_n}}.$$

THEOREM 1.5. *Let* $f : R^{m+n} \to R$ *be a* $\mathcal{D}^r$ *function. Assume that* f *is* $\mathcal{D}^{r+1}$ *off* $\{0\} \times R^n$ *and that the mappings*

$$R^n \to R : y \mapsto \frac{\partial^{|a|} f}{\partial x^a}(0, y)$$

are $\mathcal{D}^{r+1}$ *for each* $a \in \mathbb{N}^m$, $0 \leq |a| \leq r$. *Let* $\delta : R^{m+n} \to R$ *be a positive continuous definable function. Then there exists a* $\mathcal{D}^{r+1}$ *function* $\widetilde{f} : R^{m+n} \to R$ *such that*

$$\left| \frac{\partial^{|a|+|b|}}{\partial x^a \partial y^b}(f - \widetilde{f}) \right| < \delta$$

for $0 \leq |a| + |b| \leq r$.

Proof. By Taylor's formula, we have

$$f(x,y) = f(0,y) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(0,y)x_i + \cdots + \sum_{|a|=r-1} \frac{1}{a!}\frac{\partial^{r-1} f}{\partial x^a}(0,y)x^a$$

$$+ \sum_{|a|=r} \frac{1}{a!}\frac{\partial^r f}{\partial x^a}(\xi,y)x^a$$

for a suitable ξ in the segment $[0,x]$. We define

$$\theta(x,y) = f(0,y) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(0,y)x_i + \cdots + \sum_{|a|=r} \frac{1}{a!}\frac{\partial^r f}{\partial x^a}(0,y)x^a$$

and

$$Q(x,z,y) = \sum_{|a|=r} \frac{1}{a!}\left(\frac{\partial^r f}{\partial x^a}(z,y) - \frac{\partial^r f}{\partial x^a}(0,y)\right)x^a,$$

so that for each x and y there exists $\xi \in [0,x]$ such that

$$f(x,y) = \theta(x,y) + Q(x,\xi,y).$$

Note that Q is a polynomial in x whose coefficients are the continuous functions

$$\Phi_a(z,y) = \frac{1}{a!}\left(\frac{\partial^r f}{\partial x^a}(z,y) - \frac{\partial^r f}{\partial x^a}(0,y)\right).$$

Consider the set

$$A = \{(x,y,\xi) \in R^m \times R^n \times R^m : f(x,y) = \theta(x,y) + Q(x,\xi,y),\ \xi \in [0,x]\}.$$

This set is definable, and for every $x \in R^m$ and $y \in R^n$ there exists ξ such that (x,y,ξ) belongs to A. Hence, by Definable Choice [5, Prop. 6.1.2] there exists a definable function $\zeta : R^m \times R^n \to R^m$ such that for each $x \in R^m$ and $y \in R^n$, $(x,y,\zeta(x,y)) \in A$. Moreover, as $\xi \in [0,x]$, we see that $\zeta(0,y) = 0$ and $\zeta(x,y) \to 0$ as $(x,y) \to (0,y_0)$. Hence

$$f(x,y) = \theta(x,y) + \sum_{|a|=r} \frac{1}{a!}\chi_a(x,y)x^a, \quad \chi_a(x,y) = \Phi_a(\zeta(x,y),y),$$

where the χ_a's are definable functions such that $\chi_a(0,y) = 0$ and $\chi_a(x,y) \to 0$ as $(x,y) \to (0,y_0)$. We set

$$Q(x,y) = \sum_{|a|=r} \frac{1}{a!}\chi_a(x,y)x^a.$$

Let $\epsilon_1 : R \to R$ be a *sufficiently small* positive $\mathcal{D}^{r+1}$ function and define $\epsilon : R^n \to R$ by $\epsilon(y) = \epsilon_1(|y|^2)$. We can choose ϵ_1 such that the derivatives

$$\frac{\partial^{|a|}\epsilon}{\partial y^a} = \sum_{q \le [a/2]} \mathcal{K}_{a,q} \epsilon_1^{(|a|-|q|)}(|y|^2) y^{a-2q},$$

where the $\mathcal{K}_{a,q}$'s are constants, and $[a/2]$ is the integer part of $a/2$, are bounded by constants. Consider the set

$$U_\epsilon = \left\{(x,y) \in R^{m+n} : |x| < \epsilon(y)\right\}.$$

We define

$$\lambda(x,y) = \mu\left(\frac{|x|^2}{\epsilon^2(y)}\right),$$

where $\mu : R \to R$ is a $\mathcal{D}^{r+1}$ function such that $\mu \equiv 1$ in a neighbourhood of 0 and $\mu \equiv 0$ outside the interval $[-1,1]$. Finally, we set

$$\widetilde{f}(x,y) = \theta(x,y) + (1 - \lambda(x,y))Q(x,y).$$

The function $\widetilde{f}$ is $\mathcal{D}^{r+1}$ off $\{0\} \times R^n$ because all functions used in its definition are $\mathcal{D}^{r+1}$. Moreover, in a neighbourhood of $\{0\} \times R^n$, $\widetilde{f}$ is equal to θ, and hence is $\mathcal{D}^{r+1}$. Thus, $\widetilde{f}$ is $\mathcal{D}^{r+1}$.

We next show that $\widetilde{f}$ is a good approximation. As $\widetilde{f} = f$ off U_ϵ, it is enough to consider U_ϵ. We first observe that $|f(x,y) - \widetilde{f}(x,y)| = |\lambda(x,y)||Q(x,y)| \le |Q(x,y)|$. But since $(x,y) \in U_\epsilon$ and $Q(0,y) = 0$, we can make $|Q(x,y)|$ arbitrary small by taking ϵ small enough.

We next consider the derivatives. Using the above notation, we have

$$\frac{\partial^{|a|+|b|}(f - \widetilde{f})}{\partial x^a \partial y^b} = \sum_{p \le a} \sum_{q \le b} \mathcal{A}_{p,q} \frac{\partial^{|p|+|q|}\lambda}{\partial x^p \partial y^q} \frac{\partial^{|a-p|+|b-q|}Q}{\partial x^{a-p} \partial y^{b-q}},$$

where $p = (p_1, \ldots, p_m)$, $q = (q_1, \ldots, q_n)$ and the $\mathcal{A}_{p,q}$'s are constants. By induction one proves easily that

$$\frac{\partial^{|a|}\lambda}{\partial x^a} = \left(\frac{1}{\epsilon(y)}\right)^{|a|} \sum_{q \le [a/2]} \mathcal{C}_{a,q}\, \mu^{(|a|-|q|)}\left(\frac{|x|^2}{\epsilon^2(y)}\right) x^{a-2q} \left(\frac{1}{\epsilon(y)}\right)^{|a|-2|q|},$$

where the $\mathcal{C}_{a,q}$'s are constants.

Hence, to study $\frac{\partial^{|a|+|b|}\lambda}{\partial x^a \partial y^b}$ we must analyze the derivatives with respect to y of expressions of the form

$$\mu^{(\gamma)}\left(\frac{|x|^2}{\epsilon^2(y)}\right)\left(\frac{1}{\epsilon(y)}\right)^{2\gamma}.$$

Again by induction, one sees that each term

$$\frac{\partial^{|k|}}{\partial y^k}\left(\mu^{(\gamma)}\left(\frac{|x|^2}{\epsilon^2(y)}\right)\right)$$

is a sum of terms of the form

$$\mu^{(\gamma+|s|)}\left(\frac{|x|^2}{\epsilon^2(y)}\right)|x|^{2|s|}\left(\frac{1}{\epsilon(y)}\right)^{2|s|+|\alpha|+|\beta|}\Phi_{\alpha,\beta,s},$$

where $\alpha_i+\beta_i\le k_i$, $1\le s_i\le \alpha_i$, and $\Phi_{\alpha,\beta,s}$ is a polynomial in ϵ and its derivatives. On the other hand,

$$\frac{\partial^{|k|}}{\partial y^k}\left(\frac{1}{\epsilon(y)^a}\right)=\frac{1}{\epsilon(y)^{a+k}}\Lambda_k,$$

where Λ_k is a polynomial in ϵ and its derivatives.

The derivatives of Q are

$$\frac{\partial^{|a+b|}Q}{\partial x^a\partial y^b}=\sum_{|q|=r-(|a|+|b|)}\delta_q(x,y)x^q,$$

where δ_q is a definable function such that $\delta(0,y)=0$ and $\delta(x,y)\to 0$ when $(x,y)\to(0,y_0)$.

Combining these calculations, we see that each term

$$\frac{\partial^{|p+q|}\lambda}{\partial x^p\partial y^q}\frac{\partial^{|a-p|+|b-q|}Q}{\partial x^{a-p}\partial y^{b-q}}$$

is a sum of terms of the form

$$\mu^{(|p|-|\nu|+|s|)}\left(\frac{|x|^2}{\epsilon^2(y)}\right)|x|^{2|s|}x^{p-2\nu}\left(\frac{1}{\epsilon(y)}\right)^{2|s|+|\alpha|+|\beta|}\cdot$$

$$\cdot\left(\frac{1}{\epsilon(y)}\right)^{2|p|-2|\nu|+|q|-|k|}\Xi_{k,\alpha,\beta,s}(y)\delta_t(x,y)x^t,$$

where $\nu_i\le[p_i/2]$, $k_i\le q_i$, $\alpha_i+\beta_i\le k_i$, $1\le s_i\le\alpha_i$, $|t|=r-(|a-p|+|b-q|)$, and $\Xi_{k,\alpha,\beta,s}$ is a polynomial in ϵ and its derivatives.

To bound the above expression, it is enough to bound

$$|x|^{2|s|}x^{p-2\nu}\left(\frac{1}{\epsilon(y)}\right)^{2|s|+|\alpha|+|\beta|}\left(\frac{1}{\epsilon(y)}\right)^{2|p|-2|\nu|+|q|-|k|}x^t.$$

But since we assumed that $|x|<\epsilon(y)$, this expression is bounded by

$$\epsilon(y)^{2|s|}\epsilon(y)^{|p|-2|\nu|}\frac{1}{\epsilon(y)^{2|s|+|\alpha|+|\beta|}}\frac{1}{\epsilon(y)^{2|p|-2|\nu|+|q|-|k|}}\epsilon(y)^{|t|}$$

$$=\frac{\epsilon(y)^{|p|}\epsilon(y)^{r-|a|+|p|-|b|+|q|}}{\epsilon(y)^{|\alpha|+|\beta|}\epsilon(y)^{2|p|+|q|-|k|}}=\epsilon(y)^{r-(|a|+|b|)}\epsilon(y)^{|k|-(|\alpha|+|\beta|)},$$

and it is easily seen that the last expression is bounded. □

We now prove an approximation theorem for arbitrary functions on R^n, without any restrictions.

THEOREM 1.6. *Let $f : R^n \to R$ be a $\mathcal{D}^r$ function. Let $\delta : R^n \to R$ be a positive continuous definable function. There exists a $\mathcal{D}^{r+1}$ approximation $\widetilde{f} : R^n \to R$ of f such that*

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}(f - \widetilde{f})\right| < \delta$$

for $0 \leq |\alpha| \leq r$.

Proof. We first choose a finite stratification $R^n = \bigcup M_i$ such that, for each i, M_i is a $\mathcal{D}^p$ cell, M_i is $\mathcal{D}^p$ diffeomorphic to R^{d_i} (where $d_i = \dim M_i$), and $f_{|M_i} : M_i \to R$ is a $\mathcal{D}^p$ function, for an integer p that is large enough so that the conditions stated below hold (see [5, Ch. 7.3]).

For each i, M_i has a $\mathcal{D}^{p-1}$ tubular neighbourhood (see [3, Th. 6.11]), that is, a definable open neighbourhood T_i of M_i in R^n, a $\mathcal{D}^{p-1}$ submersive retraction $\tau_i : T_i \to M_i$ and a $\mathcal{D}^{p-1}$ "square of distance to M_i" function $\rho_i : T_i \to R$. We can also assume that, for each i, there is a $\mathcal{D}^{p-1}$ diffeomorphism $\phi_i : T_i \to R^n$ such that $\phi_i(M_i) = \{0\} \times R^{d_i}$. (Note that the tubular neighbourhood is trivial because M_i is $\mathcal{D}^p$ diffeomorphic to an affine space.)

We define

$$M^{\leq k} = \bigcup_{\operatorname{codim}(M_i) \leq k} M_i.$$

As $\{M_i\}$ is a stratification, for each $k = 0, \ldots, n$, the union of strata of dimension $< n-k$ is a closed subset of R^n, and hence its complement $M^{\leq k}$ is an *open* subset of R^n. In this situation, we will prove by induction on k that there exists a $\mathcal{D}^{r+1}$ function $\widetilde{f}^{\leq k} : M^{\leq k} \to R$ such that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}(f - \widetilde{f}^{\leq k})(x)\right| \leq \eta_k(x)$$

for each $x \in M^{\leq k}$, where $\eta_k : R^n \to R$ is a nonnegative continuous definable function such that $\eta_k < \delta$ and $\eta_k \equiv 0$ on $R^n \backslash M^{\leq k}$.

In the case $k = 0$, $M^{\leq 0}$ is a union of open definable subsets of R^n, so we can just take $\widetilde{f}^{\leq 0} = f$ and $\eta_0 = 0$. Assume $k > 0$. By induction, there is a $\mathcal{D}^{r+1}$ function $\widetilde{f}^{<k} : M^{<k} \to R$ such that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}(f - \widetilde{f}^{<k})\right| \leq \eta_{k-1}$$

for a certain nonnegative continuous definable function $\eta_{k-1} : R^n \to R$ satisfying $\eta_{k-1} < \delta$ and $\eta_{k-1} \equiv 0$ on $R^n \backslash M^{<k}$. Let us consider one stratum $N = M_i$ such that $\operatorname{codim} N = k$. Let $T = T_i$ be a tubular neighbourhood of N, $\tau = \tau_i$ and $\rho = \rho_i$. For T small enough, N is the unique stratum of codimension $\geq k$ that intersects T.

The functions $\widetilde{f}^{<k} : T \cap M^{<k} \to R$ and $f : N \to R$ are $\mathcal{D}^{r+1}$. Since η_{k-1} is continuous and $\eta_{k-1} \equiv 0$ over N, we see that $\widetilde{f}^{<k}$ and f define a $\mathcal{D}^r$ function $f^{\leq k} : T \to R$.

Consider the closed subset $F = R^n \backslash M^{\leq k}$. By [3, Th. 6.17] there is a nonnegative continuous definable function $\nu : R^n \to R$ such that $\nu^{-1}(0) = F$. We then define $\eta_k = \max\{\eta_{k-1}, \frac{\nu}{\nu+1}\delta\}$, which is a nonnegative continuous definable function such that $\eta_k < \delta$ and $\eta_k \equiv 0$ on $R^n \backslash M^{\leq k}$.

Let $\phi = \phi_i : T \to R^n$ be a $\mathcal{D}^{p-1}$ diffeomorphism such that $\phi(N) = \{0\} \times R^d$, where $d = d_i$. Consider the function $g = f^{\leq k} \circ \phi^{-1} : R^n \to R$. If p is large enough, then g is $\mathcal{D}^{\min\{r,p-1\}} = \mathcal{D}^r$, $g_{|\{0\}\times R^d}$ is $\mathcal{D}^{\min\{r+1,p-1\}} = \mathcal{D}^{r+1}$ and g is $\mathcal{D}^{\min\{r+1,p-1\}} = \mathcal{D}^{r+1}$ off $\{0\} \times R^d$. We can also assume that the functions $y \mapsto \frac{\partial^a g_i}{\partial y^a}(0, y) : R^d \to R$ are $\mathcal{D}^{r+1}$. Thus, by Theorem 1.5, we can approximate g by a $\mathcal{D}^{r+1}$ function $\widetilde{g} : R^n \to R$ such that

$$\left| \frac{\partial^{|\alpha|}}{\partial y^\alpha}(g - \widetilde{g}) \right| < \delta_1$$

for $|\alpha| \leq r$ and a certain definable function δ_1 to be determined later.

Let us denote by x the coordinates in T and by y the coordinates in R^n. Consider now the $\mathcal{D}^{r+1}$ function $h = \widetilde{g} \circ \phi : T \to R$. We have

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}(f - h) = \frac{\partial^{|\alpha|}}{\partial x^\alpha}((g - \widetilde{g}) \circ \phi),$$

and this can be expressed as a finite sum of terms

$$\frac{\partial^{|\beta|}(g - \widetilde{g})}{\partial y^\beta} \left[\frac{\partial^{|\gamma_1|}\phi_1}{\partial x^{\gamma_1}} \right]^{k_1} \cdots \left[\frac{\partial^{|\gamma_n|}\phi_n}{\partial x^{\gamma_n}} \right]^{k_n},$$

where $|\beta| \leq |\alpha|$ and $\sum k_i|\gamma_i| \leq |\alpha|$. Since δ_1 can be taken arbitrarily small, the above expressions can also be made arbitrarily small. Hence h is a good $\mathcal{D}^{r+1}$ approximation of f in T, that is, we can assume that

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha}(f - h) \right| < \eta_k$$

over T.

Finally, we want to "glue" h with the function $\widetilde{f}^{<k}$ defined over the strata of codim $< k$ in order to obtain a global approximation. By the construction of h, we can assume that $h = \widetilde{f}^{<k}$ outside a small neighbourhood of N. Hence it is enough to define the $\mathcal{D}^{r+1}$ approximation $\widetilde{f}^{\leq k}$ of f over $M^{\leq k}$ by

$$\widetilde{f}^{\leq k}(x) = \begin{cases} h(x) & \text{if } x \in T, \\ \widetilde{f}^{<k}(x) & \text{if } x \in M^{<k} \backslash T, \end{cases}$$

to complete the proof. □

We remark that the above argument works also for functions $f : U \to R$, where $U \subset R^n$ is an open definable subset.

Remark 1.7. If $f : R^n \to R$ is a $\mathcal{D}^r$ function, F is a closed definable subset of R^n and $U \subset R^n$ is a definable open neighbourhood of F such that $f_{|R^n \setminus F}$ is $\mathcal{D}^{r+1}$, then we can take the $\mathcal{D}^{r+1}$ approximation $\widetilde{f}$ above such that $\widetilde{f} = f$ on $R^n \setminus U$. To see this, it is enough to take a stratification compatible with F.

Our next step is to prove the approximation theorem for arbitrary functions.

Theorem 1.8. *Let $X \subset R^n$ be a $\mathcal{D}^r$ submanifold. Let $f : X \to R$ be a $\mathcal{D}^{r-1}$ function. Then there exists a $\mathcal{D}^r$ function $\widetilde{f} : X \to R$ which is an approximation of f in the $\mathcal{D}^{r-1}$ topology.*

Proof. We consider a definable tubular neighbourhood U of X in R^n and a $\mathcal{D}^{r-1}$ retraction $\tau : U \to X$. Then $g = f \circ \tau : U \to R$ is a $\mathcal{D}^{r-1}$ extension of f to U. By the above theorem, there exists a $\mathcal{D}^r$ approximation $\widetilde{g} : U \to R$ of g. By the continuity of the restriction (note that X is closed in U), we conclude that $\widetilde{f} = \widetilde{g}_{|X}$ is a $\mathcal{D}^r$ approximation of f. □

To prove Theorem 1.1, we need a result that is interesting in itself. In [3] a construction of $\mathcal{D}^{r-1}$ tubular neighbourhoods for $\mathcal{D}^r$ manifolds was given. Using the above results, we now prove the existence of $\mathcal{D}^r$ tubular neighbourhoods.

Theorem 1.9. *Let X be a $\mathcal{D}^r$ submanifold of R^n. Then there exists a $\mathcal{D}^r$ tubular neighbourhood of X.*

Proof. If $\dim X = k$, we have the mapping

$$\psi : X \to \mathbb{G}_{n,n-k} : x \mapsto \mathrm{N}_x X$$

that sends each point x in X to the normal space to X at x. This mapping is of class $\mathcal{D}^{r-1}$. The Grassmannian $\mathbb{G}_{n,n-k}$ is a $\mathcal{D}^\infty$ submanifold of R^{n^2}. Hence there exists an open definable neighbourhood U of $\mathbb{G}_{n,n-k}$ in R^{n^2} and a $\mathcal{D}^r$ retraction $\tau : U \to \mathbb{G}_{n,n-k}$. The mapping ψ can be regarded as a mapping $X \to R^{n^2}$. Thus we can approximate it by a $\mathcal{D}^r$ mapping $\widetilde{\psi} : X \to R^{n^2}$. If this approximation is close enough, then we have $\widetilde{\psi}(X) \subset U$. Hence the map $\tau \circ \widetilde{\psi} : X \to \mathbb{G}_{n,n-k}$ is a $\mathcal{D}^r$ approximation of ψ.

Finally, if we take $\widetilde{\psi}$ close enough so that the relation

$$\tau \circ \widetilde{\psi}(x) + \mathrm{T}_x X = R^n \text{ for each } x \in X$$

holds, we can repeat the proof of [3, Th. 6.11], replacing $\mathrm{N}X$ by the subset

$$\left\{(x, v) \in X \times R^n : v \in \tau \circ \widetilde{\psi}(x)\right\},$$

and thus obtain a bent tubular neighbourhood such that the mappings π and ρ are of class $\mathcal{D}^r$. $\square$

Proof of Theorem 1.1. By the above result, we can assume that there is a $\mathcal{D}^r$ retraction $\tau : U \to Y$, where U is a definable neighbourhood of Y in R^n. Let $h : X \to R^n$ be a $\mathcal{D}^r$ approximation of f, regarded as a mapping $X \to R^n$. If this approximation is close enough, we have $h(X) \subset U$, and thus, by Proposition 1.3, $\widetilde{f} = \tau \circ h : X \to Y$ is a $\mathcal{D}^r$ approximation of f. $\square$

Next, we use Theorem 1.1 to *smooth* corners in $\mathcal{D}^r$ manifolds ($r > 0$).

THEOREM 1.10. *Consider two positive integers $r < s$. Let $M \subset R^n$ be a $\mathcal{D}^r$ submanifold that contains a closed subset S such that $M \backslash S$ is a $\mathcal{D}^s$ submanifold. Given an open definable neighbourhood U of S in R^n, there exists a $\mathcal{D}^s$ submanifold $N \subset R^n$ and a $\mathcal{D}^r$ diffeomorphism $h : N \to M$ which is a $\mathcal{D}^r$ approximation of the inclusion $N \hookrightarrow R^n$ and satisfies $h_{|N\backslash U} = \mathrm{Id}$ (that is, $N \backslash U = M \backslash U$).*

Proof. As M is a $\mathcal{D}^r$ submanifold of R^n, we can cover M by finitely many open definable subsets $M_1, \ldots, M_k$ such that for each i, $i = 1 \ldots k$, a suitable projection $\pi_{i_1,\ldots,i_d} : R^n \to R^d : (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_d})$ induces a $\mathcal{D}^r$ diffeomorphism $M_i \to V_i$ onto a definable open subset V_i of R^d, that is, there exists a $\mathcal{D}^r$ mapping $\phi_i : V_i \subset R^d \to R^{n-d}$ such that

$$M_i = \mathrm{graph}(\phi_i).$$

The proof of this result (see [1, Cor. 9.3.10]) shows that for each point $x \notin S$ there exists a neighbourhood V^x of x in M such that the projection $\pi_{i_1,\ldots,i_d|V^x}$ induces a $\mathcal{D}^s$ diffeomorphism onto its image. Hence ϕ_i is $\mathcal{D}^s$ in $V_i \backslash \pi_{i_1,\ldots,i_d}(S)$.

We now prove the theorem by induction on k. For $k = 1$ (omitting the indices in this case) we have the $\mathcal{D}^r$ mapping $\phi : V \subset R^d \to R^{n-d}$ such that $M = \mathrm{graph}(\phi)$. Consider the projection $\pi : R^n \to R^d$. If $F = \pi(S)$, the restriction $\phi_{|V\backslash F}$ is $\mathcal{D}^s$. By Remark 1.7, we can find a $\mathcal{D}^s$ approximation of ϕ, $\widetilde{\phi} : V \to R^{n-d}$, such that $\widetilde{\phi} = \phi$ on $V \backslash \pi(U)$. Hence, $N = \mathrm{graph}(\widetilde{\phi})$ is a $\mathcal{D}^s$ submanifold of R^n, and $h : N \to M : (x, y) \mapsto (x, \phi(x))$ is a $\mathcal{D}^r$ diffeomorphism approximating the inclusion map $N \hookrightarrow R^n$ and satisfying $h_{|N\backslash U} = \mathrm{Id}$.

If $k > 1$, we consider an open definable covering of M, $M = \bigcup_{i=1}^k N_i$, such that, for each i, $N_i \subset \overline{N_i} \subset M_i$. Let us assume that $n = d + 1$. (The general case is similar.) For each i let us consider a $\mathcal{D}^r$ nonnegative function $\delta_i : R^d \to R$ such that $\delta_i \equiv 0$ on $R^d \backslash \pi_{i_1,\ldots,i_d}(\overline{N_i})$. Let us take M_1, and let $\widetilde{\phi_1}$ be a $\mathcal{D}^s$ δ_1-approximation of ϕ_1 on $\pi_{i_1,\ldots,i_d}(N_i)$. The graph of $\widetilde{\phi_1}$ automatically glues with $M_1 \backslash N_1$ to give the graph of a $\mathcal{D}^r$ function. Let M_1^* be this new graph. Then $M_1^* \cup M_2 \cup \cdots \cup M_k$ is a $\mathcal{D}^r$ manifold that satisfies the conditions in the statement of the theorem. We can repeat the argument for N_2, smoothing the corners in $N_2 \backslash \overline{N_1}$. Continuing in this manner, we obtain for each i a $\mathcal{D}^s$ manifold N satisfying the conditions of the theorem. We observe that, for each i, we smooth the corners in ∂N_i, because $\partial N_i \subset \bigcup_{j \neq i} N_j$. □

From the above smoothing result, we deduce the following theorem.

THEOREM 1.11. *For $r > 0$ any $\mathcal{D}^r$ submanifold of R^n is $\mathcal{D}^r$ diffeomorphic to a $\mathcal{D}^{r+1}$ submanifold. Two $\mathcal{D}^r$-diffeomorphic $\mathcal{D}^{r+1}$ submanifolds of R^n are $\mathcal{D}^{r+1}$ diffeomorphic.*

Proof. As above, we can stratify our manifold in a finite union of strata, each of which is a $\mathcal{D}^{r+1}$ submanifold. Let F be the union of the strata of codimension > 0, which is a definable closed subset. The result now follows readily from Theorem 1.10.

For the second assertion, we use the approximation theorem. Given a $\mathcal{D}^r$ diffeomorphism between $\mathcal{D}^{r+1}$ submanifolds, we can approximate it by a $\mathcal{D}^{r+1}$ mapping. If this approximation is close enough, is also a diffeomorphism. □

2. An application: triviality of definable families

A definable subset A of $R^p \times R^n$ is called a *definable family of subsets of R^n parametrized by R^p*. For each $t \in R^p$, the fiber of the family A at the point t is the definable subset $A_t = \{x \in R^n : (t, x) \in A\}$.

We can consider a more general situation. Let $N \subset R^n$ and $P \subset R^p$ be two $\mathcal{D}^r$ manifolds and $g : N \to P$ a $\mathcal{D}^r$ mapping. If we define

$$X^g = \{(x, y) \in R^p \times R^n : x = g(y)\} \subset R^p \times R^n,$$

then we can regard g as a family of definable sets, where X^g_t is just $g^{-1}(t)$.

Let X and Y be two definable subsets of $R^p \times R^n$ and $R^p \times R^m$, respectively. We consider both subsets as definable families parametrized by R^p. A *definable family of mappings* from X to Y is a definable mapping $f : X \to Y$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{proj}"'] & & Y \arrow[dl, "\mathrm{proj}"] \\ & R^p & \end{tikzcd}$$

Here the mappings proj are the projections on the first p coordinates. We obtain a family of mappings $f_t : X_t \to Y_t$, $t \in R^p$, defined by $f(t, x) = (t, f_t(x))$.

Equivalently, given two mappings $f : X \to Y$ and $g : Y \to \mathbb{R}^p$, we can regard the pair (f, g) as a family of mappings

$$\{f_t : X_t \to Y_t\}_{t \in R^p}$$

by taking $X_t = (g \circ f)^{-1}(t)$, $Y_t = g^{-1}(t)$, and $f_t = f_{|X_t}$.

We are interested in the triviality of such families. By the above observation, we can express this triviality of families of definable objects in terms of the triviality of definable mappings and pairs of definable mappings.

Given two $\mathcal{D}^r$ manifolds N and P and a $\mathcal{D}^r$ mapping $g : N \to P$, we say that g is $\mathcal{D}^r$ *trivial* if there exist a point $p \in P$ and a $\mathcal{D}^r$ diffeomorphism $\gamma = (\gamma_0, g) : N \to g^{-1}(p) \times P$. This means that the associated family X^g is trivial, that is, each fiber $g^{-1}(y)$, $y \in P$, is $\mathcal{D}^r$ diffeomorphic to $g^{-1}(p)$.

Let M be a $\mathcal{D}^r$ manifold and $f : M \to N$ another $\mathcal{D}^r$ mapping. We say that (f, g) is $\mathcal{D}^r$ *trivial* if there exist p and γ as before and a $\mathcal{D}^r$ diffeomorphism $\theta = (\theta_0, g \circ f) : M \to (g \circ f)^{-1}(p) \times P$ such that $f \circ \theta_0 = \gamma_0 \circ f$, in other words, if the following diagram is commutative:

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & P \\ \Big\downarrow \theta & & \Big\downarrow \gamma & & \Big\downarrow \mathrm{Id} \\ (g \circ f)^{-1}(p) \times P & \xrightarrow{f \times \mathrm{Id}} & g^{-1}(p) \times P & \xrightarrow{\pi} & P. \end{array}$$

We will prove two triviality results for a certain type of $\mathcal{D}^r$ mappings. Recall that a $\mathcal{D}^r$ mapping $f : M \to N$ is a *submersion at a point* $x \in M$ if the differential $df_x : \mathrm{T}_x M \to \mathrm{T}_{f(x)} N$ is surjective. We say that f is a *submersion* if it is a submersion at each point $x \in M$. Let $X \subset R^m$ and $Y \subset R^n$ be two definable subsets and $f : X \to Y$ a continuous definable mapping. We say that f is *definably proper* if the inverse image of a closed and bounded definable set is again closed and bounded; basic results on definably proper mappings can be found in [5, Ch. 6.4].

With these definitions we can now state our triviality results.

Theorem 2.1. *Let $p : M \to R^l$ be a surjective proper $\mathcal{D}^r$ submersion. Then p is $\mathcal{D}^r$ trivial.*

Theorem 2.2. *Let M and N be $\mathcal{D}^r$ manifolds and let $f : M \to N$ and $g : N \to R^l$ be proper surjective $\mathcal{D}^r$ submersions. Then (f, g) is $\mathcal{D}^r$ trivial.*

The latter result corresponds to [6, Th. 6.3.1].

We will only sketch the proofs of these results, which in the semialgebraic category have been proved in [4] and [7]. To prove the results in the present setting, we need to develop o-minimal versions of the results and constructions that were used for the semialgebraic case, and our approximation result enables us to do this. For detailed proofs we refer the reader to [6].

One main construction used in the semialgebraic case is the real spectrum (see [1, Ch. 7]). A construction suitable for our o-minimal context was given by M. Coste [2] and is called the *definable spectrum*. We embed R^p in a compactification $\widetilde{R^p}$ whose points α are ultrafilters of definable sets. Given a definable subset $A \subset R^p$, we consider the set $\widetilde{A} = \{\alpha \in \widetilde{R^p} : \alpha \ni A\} \subset \widetilde{R^p}$. If

$\alpha \in \widetilde{R^p}$, we define $k(\alpha)$ to be the set of germs of definable functions $f : A \to R$ for $A \in \alpha$; that is, if $A, B \in \alpha$ and $f : A \to R$, $g : B \to R$ are definable functions, we say that f and g are equivalent is there exists a definable subset $C \in \alpha$ such that $C \subset A \cap B$ and $f_{|C} = g_{|C}$. Then $k(\alpha)$ is the set of equivalence classes for the above equivalence relation, and we denote by $f(\alpha) \in k(\alpha)$ the equivalence class of $f : A \to R$. In fact, $k(\alpha)$ is a real closed field.

For a definable family of definable sets or mappings, parametrized by R^p, we extend the definition of the fiber of the family to points in $\widetilde{R^p}$. Consider a definable family $X \subset R^p \times R^n$ and $\alpha \in \widetilde{R^p}$. If $f = (f_1, \dots, f_n) : A \to R^n$ is a definable mapping and $A \in \alpha$, we denote by $f(\alpha)$ the point $(f_1(\alpha), \dots, f_n(\alpha)) \in k(\alpha)^n$. The *fiber* of X at α is the set X_α of those $f(\alpha) \in k(\alpha)^n$ such that there exists $A \in \alpha$ on which f is defined and $(t, f(t)) \in X$ for all $t \in A$. In other words, $f(\alpha)$ belongs to X_α if and only if there exists $A \in \alpha$ such that $f(t) \in X_t$ for all $t \in A$.

Given $\alpha \in \widetilde{R^p}$, let $\mathcal{S}_n(\alpha)$ be the set of all fibers X_α, for a definable subset X of $R^p \times R^n$. Then the collection $\mathcal{S}(\alpha) = (\mathcal{S}_n(\alpha))_{n \in \mathbb{N}}$ is an o-minimal structure expanding the real closed field $k(\alpha)$.

If $X \subset R^p \times R^n$ and $Y \subset R^p \times R^m$ are definable families of definable subsets and $f : X \to Y$ is a definable family of mappings, we define, for $\alpha \in \widetilde{R^p}$, the *fiber* of f at α, $f_\alpha : X_\alpha \to Y_\alpha$, by considering the graph. This mapping is definable in $\mathcal{S}(\alpha)$.

The motivation behind the use of the real spectrum is the following: Given a definable family of definable sets or mappings, parametrized by R^p, and a point $\alpha \in \widetilde{R^p}$, the fiber of the family at α verifies a certain property (expressed by first-order formulas) if and only if there exists a definable subset $S \subset R^p$, $S \in \alpha$, such that all the fibers at points $t \in S$ verify the same property. Furthermore, by studying the properties of generic fibers we obtain global properties, not just fiberwise properties (see Proposition 2.4 below).

If $A \in R^n$ is a definable set, A can be described in the form $A = \{x \in R^n : \Phi(x)\}$, where Φ is a *first-order formula* of the language of the o-minimal structure. For example, if $X \subset R^p \times R^n$ is a definable family, we can write $X = \{(t, x) \in R^p \times R^n : \Phi(t, x)\}$, for a certain first-order formula Φ. If $\alpha \in \widetilde{R^p}$, then $X_\alpha = \{x \in k(\alpha)^n : \Phi_\alpha(x)\}$, where Φ_α is the *generic fiber formula* of the family of formulas $\Phi_t = \Phi(t, \cdot)$.

For a function $f : U \to R$ on an open definable subset $U \subset R^n$, the fact that f is $\mathcal{D}^r$ can be expressed by first-order formulas. In general, all concepts involving definable differentiability, up a finite order, can be expressed by first-order formulas. This allows us to transfer properties from fibers of a given family to the generic fiber at a point of the definable spectrum. The following result makes this precise.

PROPOSITION 2.3. *Let $B \subset R^p$ and $X \subset B \times R^n$ be definable subsets such that, for each $b \in B$, X_b is a $\mathcal{D}^r$ submanifold of R^n. Then, for any $\alpha \in \widetilde{B}$, X_α is a $\mathcal{D}^r$ submanifold of $k(\alpha)^n$.*

Proof. See [6, Prop. 5.1.1]. □

It is more interesting to transfer properties from the generic fibers to the original family. The following proposition is an example of a result of this type.

PROPOSITION 2.4. *Let $B \subset R^p$ and $X \subset B \times R^n$ be definable subsets. Let $\alpha \in \widetilde{B}$ be such that X_α is a $\mathcal{D}^r$ submanifold of $k(\alpha)^n$. Then there exists a $\mathcal{D}^r$ submanifold $M \subset R^p$, with $M \subset B$, such that $M \in \alpha$, $X \cap (M \times R^n)$ is a $\mathcal{D}^r$ submanifold of $M \times R^n$, and the projection $\pi : X \cap (M \times R^n) \to M$ is a submersion.*

Proof. See [6, Prop. 5.1.6]. □

Another important construction is the following. In [4], to prove the semialgebraic version of Theorem 2.1, the authors construct "Nash models" of Nash manifolds over smaller real closed ground fields: Given a Nash manifold $N \subset R^n$ and given a real closed subfield $R' \subset R$, we can construct a Nash manifold N' defined over R' whose extension N'_R to R is Nash diffeomorphic to N. In [7], this construction is extended to mappings: Given a proper Nash submersion $f : M \to N$ defined over R, and a subfield R' as before, we can construct over R' a proper Nash submersion $f' : M' \to N'$ whose extension f'_R to R is "diffeomorphic" to f. In order to translate this construction to the o-minimal situation, we need the following definition.

Let $(R', \mathcal{S}')$ and $(R, \mathcal{S})$ be two o-minimal structures expanding real closed fields such that $R' \subset R$. We say that $(R, \mathcal{S})$ is an *elementary extension* of $(R', \mathcal{S}')$, and we write $(R', \mathcal{S}') \prec (R, \mathcal{S})$, if, for each n, there exists an *extension mapping* $\mathcal{S}'_n \to \mathcal{S}_n : A' \mapsto A'_R$ with the following properties:

(1) $A'_R \cap R'^n = A'$ for each $A' \in \mathcal{S}'_n$.
(2) $A' \mapsto A'_R$ commutes with boolean operations.
(3) $(A' \times B') = A'_R \times B'_R$ for each $A' \in \mathcal{S}'_n$, $B' \in \mathcal{S}'_m$.
(4) If $\pi' : R'^{n+1} \to R'^n$ and $\pi : R^{n+1} \to R^n$ are the projections on the first n coordinates, and $A' \in \mathcal{S}'_{n+1}$, then $\pi(A'_R) = \pi'(A')_R$.
(5) If A' is semialgebraic, then A'_R is the usual extension of semialgebraic sets (see [1]).
(6) For each $B \in \mathcal{S}_m$, there exist $n \in \mathbb{N}$, $A' \in \mathcal{S}'_{m+n}$ and $a \in R^n$ such that $B = \{x \in R^m : (x, a) \in A'_R\}$.

We remark that, if $A' \in \mathcal{S}'_n$ is empty, its projection to $R'^0 = R^0$ (singleton) is empty; hence, by (4), ${A'}_R$ is also empty.

For the o-minimal setting we have the following theorem, which is patterned after [4] and [7]; see [6, Th. 5.3.2] and [6, Prop. 6.2.2]. The proof in the new setting requires our approximation theorem.

THEOREM 2.5. *Let $(R', \mathcal{S}') \prec (R, \mathcal{S})$ be an elementary extension.*

(1) *Let $X \subset R^n$ be a $\mathcal{D}^r_{\mathcal{S}}$ submanifold. Then there exists a $\mathcal{D}^r_{\mathcal{S}'}$ submanifold Y such that Y_R is $\mathcal{D}^r_{\mathcal{S}}$ diffeomorphic to X.*
(2) *Let $f : M \to N$ be a proper $\mathcal{D}^r_{\mathcal{S}}$ submersion between the $\mathcal{D}^r_{\mathcal{S}}$ submanifolds M and N of R^n. Then there exist two $\mathcal{D}^r_{\mathcal{S}'}$ submanifolds M' and N' of $R'^{n'}$, a proper $\mathcal{D}^r_{\mathcal{S}'}$ submersion $f' : M' \to N'$ and $\mathcal{D}^r_{\mathcal{S}}$ diffeomorphisms $\alpha : N'_R \to N$ and $\beta : M'_R \to M$ such that $f \circ \beta = \alpha \circ f'_R$.*

We observe that, given an o-minimal structure $(R, \mathcal{S})$ and a point $\alpha \in \widetilde{R^p}$, the o-minimal structure $(k(\alpha), \mathcal{S}(\alpha))$ is an elementary extension of $(R, \mathcal{S})$.

Using these results, we can prove the following o-minimal "Hardt type" Theorem (see [8]).

THEOREM 2.6. *Let $B \subset R^p$, $X \subset B \times R^n$ and $Y \subset B \times R^m$ be definable sets, and $f : X \to Y$ a definable family of definable mappings. Assume that $f_t : X_t \to Y_t$ is a proper $\mathcal{D}^r$ submersion between $\mathcal{D}^r$ manifolds, for each $t \in B$. Then we can stratify B in a disjoint union of $\mathcal{D}^r$ manifolds, say $B = S_1 \sqcup \cdots \sqcup S_r$, such that f has a $\mathcal{D}^r$ trivialization over each S_i of the form*

$$\begin{array}{ccc} S_i \times X_{t_i} & \xrightarrow{\sim} & X \cap (S_i \times R^n) \\ \downarrow{\scriptstyle \mathrm{Id} \times f_{t_i}} & & \downarrow{\scriptstyle f_{|X \cap (S_i \times R^n)}} \\ S_i \times Y_{t_i} & \xrightarrow{\sim} & Y \cap (S_i \times R^m), \end{array}$$

for certain elements $t_i \in S_i$.

Proof. Let us take a point α in the real spectrum $\widetilde{B}$ associated to B. The fiber $f_\alpha : X_\alpha \to Y_\alpha$ is a proper $\mathcal{D}^r$ submersion over $k(\alpha)$ (that is, a $\mathcal{D}^r_{\mathcal{S}_\alpha}$ mapping). By Proposition 2.5 we obtain a $\mathcal{D}^r$ submersion $f' : X' \to Y'$ between the $\mathcal{D}^r$ manifolds X' and Y', defined over R, and $\mathcal{D}^r$ diffeomorphisms γ and σ such that the following diagram is commutative:

$$\begin{array}{ccc} X'_{k(\alpha)} & \xrightarrow{\gamma} & X_\alpha \\ \downarrow{\scriptstyle f'_{k(\alpha)}} & & \downarrow{\scriptstyle f_\alpha} \\ Y'_{k(\alpha)} & \xrightarrow{\sigma} & Y_\alpha \end{array}$$

But, as $f'_{k(\alpha)}$ is the fiber $(\mathrm{Id} \times f')_\alpha$ of the *constant family* $\mathrm{Id} \times f' : B \times X' \to B \times Y'$, there exist a $\mathcal{D}^r$ manifold $S \subset R$, $S \in \alpha$, and $\mathcal{D}^r$ diffeomorphisms ζ and η, compatible with the projections, such that the following diagram is commutative:

$$\begin{array}{ccc} S \times X' & \xrightarrow{\zeta} & X \cap (S \times R^n) \\ \Big\downarrow{\scriptstyle \mathrm{Id} \times f'} & & \Big\downarrow{\scriptstyle f_{|X \cap (S \times R^n)}} \\ S \times Y' & \xrightarrow{\eta} & Y \cap (S \times R^m) \end{array}$$

(We call $g = \mathrm{Id} \times f' : B \times X' \to B \times Y'$ a constant family because the fiber g_t is equal to f' for each $t \in B$.) By the compactness of $\widetilde{B}$, sub-stratifying if necessary, we can assume that there exists a finite $\mathcal{D}^r$ stratification $\{S_i\}$ of B such that a diagram of the above form holds for each S_i. □

Using the above constructions, we now prove the triviality results Theorems 2.1 and 2.2. To prove the first result we follow the argument in [4] for the semi-algebraic case. Consider the case $l = 1$ and let $p : M \subset R^n \to R$ be a surjective proper $\mathcal{D}^r$ submersion. We consider the definable family of $\mathcal{D}^r$ manifolds M^p, where for each $t \in R$, $M^p_t = p^{-1}(t)$. Let us take $\alpha \in \widetilde{R}$ and consider the generic fiber M^p_α, that is a $\mathcal{D}^r$ manifold over $k(\alpha)$. We can find a *model* for this manifold, that is, there exists a $\mathcal{D}^r$ manifold M' over R whose extension to $k(\alpha)$, $M'_{k(\alpha)}$, is $\mathcal{D}^r$ diffeomorphic to M^p_α.

The manifold $M'_{k(\alpha)}$ can be regarded as the generic fiber at α of the trivial family $R \times M'$. Since the families M^p and $R \times M'$ are $\mathcal{D}^r$ diffeomorphic at $\alpha \in \widetilde{R}$, there exists a definable subset $I \subset R$, with $I \in \alpha$, such that both families are diffeomorphic on I. But this just means that M^p (that is, p) is $\mathcal{D}^r$ trivial on I. By the compactness of $\widetilde{R}$, there exist definable subsets $I_1, \ldots, I_k \subset R$ such that $R = \bigcup_i I_i$ and p is trivial on each I_i. We can assume that the I_i's are open intervals or singletons. If $I_i = \{a_i\}$, we trivialize p on an open interval containing a_i (see [4, Th. 2.4]), using a $\mathcal{D}^r$ tubular neighbourhood. Next, we can assume that there exist open intervals $I_1, \ldots, I_k \subset R$ such that $R = \bigcup_i I_i$ and p is trivial on each I_i. Finally, we glue the local trivializations together in order to obtain a global trivialization of p.

If $l > 1$, we replace again M by $M^p = \{(x, y) \in R^l \times R^n : x = p(y)\} \subset R^l \times R^n$ and p by the projection $R^l \times R^n \to R^l$. Hence we can regard $p : M \to R^l = R \times R^{l-1}$ as a family of proper submersions parametrized by R. For $\alpha \in \widetilde{R}$, we consider the mapping $p_\alpha : M_\alpha \to k(\alpha)^{l-1}$. This mapping is a $\mathcal{D}^r$ proper submersion, and hence, by induction hypothesis, it is $\mathcal{D}^r$ trivial. By an argument similar to the one given above, this implies that p is trivial on a definable subset of the form $I \times R^{l-1}$, where I is an interval or a singleton such that $I \in \alpha$. By compactness, we can assume that there exist definable subsets $I_1, \ldots, I_k$ with the above property such that $R = \bigcup_i I_i$. We can again assume that each subset I_i is an open interval. By a rather technical argument, these local trivializations on open definable subsets of the form $I_i \times R^{l-1}$ can be glued together to give a global trivialization..

We now sketch the proof of Theorem 2.2. By Theorem 2.1, we can assume that $M = R^l \times X$ (resp. $N = R^l \times Y$), where X (resp. Y) is a $\mathcal{D}^r$ manifold, and $f : R^l \times X \to R^l \times Y$ is a family of $\mathcal{D}^r$ mappings. (We can forget about g.) Assume that $l = 1$ and take $\alpha \in \widetilde{R}$. Then the generic fiber $f_\alpha : X_{k(\alpha)} \to Y_{k(\alpha)}$ is a proper submersion. By Theorem 2.5, there exists a $\mathcal{D}^r$ proper submersion $f' : X \to Y$ whose extension to $k(\alpha)$, $f'_{k(\alpha)} : X_{k(\alpha)} \to Y_{k(\alpha)}$, is "$\mathcal{D}^r$ diffeomorphic" to f_α (in the sense of Theorem 2.5). But $f'_{k(\alpha)}$ can be regarded as the generic fiber at α of the trivial family $(\mathrm{Id}, f) : R \times X \to R \times Y$. This implies that there exists a definable subset $I \subset R$ such that $I \in \alpha$ and both families are $\mathcal{D}^r$ diffeomorphic on I. But this just means that f is $\mathcal{D}^r$ trivial on I. Again, using the compactness of the real spectrum and a differential topology argument (using $\mathcal{D}^r$ tubular neighbourhoods), we obtain that f is $\mathcal{D}^r$ trivial on certain open intervals $I_1, \dots, I_k \subset R$ such that $R = \bigcup_i I_i$. Finally, using $\mathcal{D}^r$ partitions of unity (see [6, Th. 3.4.2]) and our approximation theorem, we obtain a global trivialization by glueing together the local trivializations. The case when $l > 1$ is similar to the case $l > 1$ in the proof of Theorem 2.1. (We again refer to [6] for the details.)

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