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THE AUTOMORPHISM GROUP OF DOMAINS WITH BOUNDARY POINTS OF INFINITE TYPE

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ABSTRACT. Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded domain. We prove that if $\partial\Omega$ contains a (small) smooth curve of points of infinity type, then the automorphism group $\mathrm{Aut}(\Omega)$ is compact. This result implies the Greene-Krantz conjecture for a special class of domains. The proof makes no use of scaling techniques.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$, $n > 1$, and denote by $\mathrm{Aut}(\Omega)$ its holomorphic automorphisms group. It is known that, in the compact-open topology, $\mathrm{Aut}(\Omega)$ is a Lie group and is non-compact if and only if there exists a point $P \in \Omega$ and a sequence $\{F_n\} \subset \mathrm{Aut}(\Omega)$ such that

$$\lim_{n\to\infty} F_n(P) = Q \in \partial\Omega. \tag{1.1}$$

Several important results have been proved for bounded domains with non-compact automorphism group (for an extensive review see, e.g., [IK]). In particular, for a domain $\Omega \subset \mathbb{C}^2$ with a boundary $\partial\Omega$, which satisfies some additional regularity conditions, the following fact holds ([Br], [BP]): if $\mathrm{Aut}(\Omega)$ is non-compact and if the limit point $Q \in \partial\Omega$ in (1.1) is of finite type, then Ω is biholomorphic to a domain of the form

$$\Omega \cong \{\mathrm{Re}\, w + q(z, \overline{z}) < 0\}, \tag{1.2}$$

where q is a homogeneous polynomial in z and $\overline{z}$.

For a result for the unbounded case see also [Ef].

On the other hand, the following conjecture was formulated by R. Greene and S. Krantz [GK].

GREENE-KRANTZ CONJECTURE. *If the automorphism group* $\mathrm{Aut}(\Omega)$ *of a smoothly bounded domain* $\Omega \subset \mathbb{C}^n$ *is non-compact, then any point* $Q \in \partial\Omega$*, which is limit of a sequence* $F_n(P)$ *as in* (1.1)*, is of finite type.*

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In several papers, this conjecture has been proved under additional assumptions on the domain Ω or on the boundary $\partial\Omega$ (see, e.g., [GK], [Ka], [KK]). All these results provide strong evidence that the Greene-Krantz conjecture is true in full generality, but, at the moment, a complete proof is still lacking. In this paper we make a new contribution in the area of the Greene-Krantz conjecture. Namely, we prove the following result.

THEOREM 1.1. *Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain, with C^∞ boundary, and satisfying the following two conditions:*

(a) *There exists a point $P \in \partial\Omega$ and a system of complex coordinates (z, w) defined on a connected neighborhood U of P so that $(z(P), w(P)) = (0, 0)$ and*

$$\Omega \cap U = \{\ Q \in U\ :\ \operatorname{Re} w(Q) + \phi(z(Q), \overline{z(Q)}) < 0\ \}, \tag{1.5}$$

where ϕ is smooth, subharmonic and strictly positive at all points different from the origin, where it vanishes at any order, i.e.,

$$\lim_{z\to 0} \frac{\phi(z)}{|z|^N} = 0,\ \forall N \geq 0. \tag{1.6}$$

(b) *Any point $P \in \partial\Omega \setminus \overline{U}$ is a boundary point of finite type.*

Then $\operatorname{Aut}(\Omega)$ *is compact.*

We remark that in Theorem 1.1 any point $Q \in U \cap \partial\Omega$ with coordinate $z(Q) = 0$ is a point of infinite type (see Definition 2.3 below). Therefore, our result can be considered as a proof of the Greene-Krantz conjecture for the case in which the set of points of infinite type constitute a special smooth curve contained in $\partial\Omega$.

We point out that similar results have been proved by H. Kang [Ka] and by K.-T. Kim and S. Krantz [KK].

H. Kang proved the compactness of $\operatorname{Aut}(\Omega)$ for a bounded domain $\Omega \subset \mathbb{C}^2$ which is *globally* defined by

$$\{(z, w) :\ |w|^2 + \phi(z) < 0\}, \tag{1.7}$$

where ϕ satisfies the hypothesis (a) of Theorem 1.1. Kang's proof depends strongly on the fact that, by the existence of a global defining function of the form (1.7), the automorphism group $\operatorname{Aut}(\Omega)$ contains the compact, 1-parameter group of automorphisms $\Gamma = \{\ (z, w) \to (z, e^{it}w),\ t \in \mathbb{R}\ \}$. Our results may be considered as a generalization of Kang's theorem, which makes no hypothesis on $\operatorname{Aut}(\Omega)$.

K.-T. Kim and S. Krantz proved that for a domain $\Omega \subset \mathbb{C}^2$, for which $\operatorname{Aut}(\Omega)$ is non-compact, it may never occur that, on a neighborhood of a limit point Q which satisfies (1.1), the domain Ω is of the form

$$\{\ (z, w) \in \mathbb{C}^2\ : \operatorname{Re} w + \phi(|z|^2) = 0\ \},$$

where ϕ is a $\mathcal{C}^\infty$ function, with $\phi(x) > 0$, $\phi''(x) > 0$ for any $x > 0$, and such that the map

$$\Phi : (0, \epsilon) \subset \mathbb{R} \to \mathbb{R}, \qquad \Phi(x) \stackrel{\text{def}}{=} -\frac{1}{\log \phi(x)},$$

extends to a C^∞ function over a neighborhood of 0, vanishing up to finite order at the origin. In contrast to the result of Kim and Krantz, our Theorem 1.1 makes no assumption on the function ϕ, aside from the positivity and the vanishing at any order at the origin. On the other hand, in our proof, we need the hypothesis (b), which is a global property of $\partial\Omega$, while the hypotheses of Kim and Krantz are of purely local nature. This is mainly due to the fact that our proof makes no use of scaling techniques, while Kim and Krantz's paper is heavily based on them. For this reason, we believe that a suitable combination of the ingredients of our paper with those of the paper by Kim and Krantz can produce an important step towards a general solution of the Greene-Krantz conjecture.

Before concluding this section, we remark that hypothesis (b) in Theorem 1.1 is needed to prove the boundary regularity of the automorphisms in $\mathrm{Aut}(\Omega)$ near P. Once such a boundary regularity of the automorphisms is established, all the other arguments are of purely local nature. On the other hand, the boundary regularity of the automorphisms is basically obtained by local subelliptic estimates for the $\overline{\partial}$-Neumann problem on a neighborhood of the point P of weak pseudoconvexity. Such estimates are consequences of hypothesis (b), by means of the results in [Bo] and [Ca]. So, in order to remove (or to replace) hypothesis (b), it is necessary to find an alternative proof of the quoted local subelliptic estimates, which is an interesting question in itself. The interested reader is referred to [Bel], where local subelliptic estimates similar to those needed here are obtained in neighborhoods of points of strong pseudoconvexity.

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2. Preliminaries and basic results

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $\{F_j\} \subset \mathrm{Aut}(\Omega)$ a sequence of automorphisms of Ω. Observe that, by Montel's Theorem, there exists a subsequence $\{F_{j_i}\}$ such that both sequences $\{F_{j_i}\}$ and $\{F_{j_i}^{-1}\}$ converge uniformly on compact subsets of Ω to a pair of holomorphic maps $F, G : \Omega \to \mathbb{C}^n$.

Moreover, by the results in [BL], if a bounded domain $\Omega \subset \mathbb{C}^n$ is smooth and satisfies the so-called *Condition R*, then any automorphism $F \in \mathrm{Aut}(\Omega)$

extends smoothly up to the boundary. (For the definition of *Condition R*, we refer to [BL].)

These observations are completed by the following theorem by S. Bell, which represents a major tool for the proof of our result.

THEOREM 2.1. [Be] *Let $\Omega \subset \mathbb{C}^n$ be a bounded, smooth domain with $\mathbb{C}^\infty$ boundary and which satisfies Condition R. Let also $\{F_j\} \subset \mathrm{Aut}(\Omega)$ be a sequence such that both $\{F_j\}$ and $\{F_j^{-1}\}$ converge uniformly on compact subsets of Ω to two holomorphic maps F, $G : \Omega \to \mathbb{C}^n$, respectively.*

Then only one of the following two cases occurs:

1. *$F \in \mathrm{Aut}(\Omega)$ and the smooth extensions to $\overline{\Omega}$ of the automorphisms F_j converge uniformly on compact subsets of $\overline{\Omega}$.*
2. *There exist two points $P_1, P_2 \in \partial\Omega$ such that the smooth extensions to $\overline{\Omega}$ of the maps F_j converge uniformly on compact subsets of $\overline{\Omega}\setminus\{P_1\}$ to the constant map $F(z_1, \dots, z_n) \equiv P_2$; in this case, the determinants $\det(JF_j)$ of the holomorphic Jacobians converge to 0 in the $\mathcal{C}^\infty$ sup norm on $\overline{\Omega} \setminus \{P_1\}$.*

This theorem has the following direct corollary.

COROLLARY 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, smooth domain with $\mathbb{C}^\infty$ boundary and which satisfies Condition R. Then $\mathrm{Aut}(\Omega)$ is compact if and only if, for any sequence $\{F_j = (f_j^1, \dots, f_j^n)\} \subset \mathrm{Aut}(\Omega)$, there exists at least one index $1 \le k \le n$ such that the sequence of holomorphic maps $\{f_m^k\}$ admits a subsequence that converges uniformly in $\overline{\Omega}$.*

Proof. If $\mathrm{Aut}(\Omega)$ is a compact group, any sequence $\{F_j\} \in \mathrm{Aut}(\Omega)$ admits a subsequence $\{F_{j_i}\}$ converging to an element $F \in \mathrm{Aut}(\Omega)$. In addition we can assume that $F_{j_i}^{-1}$ too is a converging sequence.

By the previous remarks and Theorem 2.1, the sequence $\{F_{j_i}\}$ and any of the sequences $\{f_{j_n}^k\}$, $1 \le k \le n$, converge uniformly on compact subsets of $\overline{\Omega}$.

Conversely, suppose that $\mathrm{Aut}(\Omega)$ is non-compact and let $\{F_j = (f_j^1, \dots, f_j^n)\}$ $\subset \mathrm{Aut}(\Omega)$ be a sequence with no converging subsequence. In particular, for any subsequence of F_j, there exists a subsequence which satisfies (2) of Theorem 2.1. On the other hand, if there exists a value $1 \le k \le n$ and a subsequence $\{f_{j_i}^k\} \subset \{f_j^k\}$, which converges uniformly on compact subsets of $\overline{\Omega}$, by considering a suitable subsequence, we may assume that the corresponding subsequence $\{F_{j_i}\} \subset \{F_j\}$ converges to the constant map $F(z_1, \dots, z_n) \equiv P$ and that

$$\lim_n \sup_{z \in \overline{\Omega}} |f_{j_i}^k(z) - z_k(P)| = 0. \tag{2.1}$$

On the other hand, since each F_{j_i} is an automorphism, we have that for any n

$$\sup_{z\in\overline{\Omega}}|f^k_{j_i}(z)-P| = \sup_{\overline{z}\in\Omega}|z_k - z_k(P)| = C > 0 \tag{2.2}$$

for some positive constant $C = C(\Omega)$. It is clear that (2.1) and (2.2) are in contradiction. □

Let us now consider smoothly bounded domains in $\mathbb{C}^2$. First, we recall the definition of boundary points of finite (infinite) type for smooth domains in $\mathbb{C}^2$.

Definition 2.3. Let $\Omega \subset \mathbb{C}^2$ be a smooth domain with C^∞ boundary. We say that a point $P \in \partial\Omega$ is a *point of type* k if there exists at least one smooth holomorphic curve which has a contact with $\partial\Omega$ of order k at P, and there is no smooth holomorphic curve with contact with $\partial\Omega$ of order greater than k at the same point.

We say that P is a *point of infinite type* if, for any $k \in \mathbb{N}$, there exists at least one smooth holomorphic curve which has a contact with $\partial\Omega$ of order greater or equal to k at P.

From now on, we will assume that $\Omega \subset \mathbb{C}^2$ is a bounded domain which satisfies the hypothesis of Theorem 1.1. There is no loss of generality if we assume that $P = (0,0)$ and that $\Omega \cap U$ coincides with the set

$$\Omega \cap U = \{\ (z,w) \in U\ :\ 2\,\mathrm{Re}\, w + \phi(z,\overline{z}) < 0\ \}, \tag{2.3}$$

where:

(i) $\phi : U \to \mathbb{R}$ depends only on z and $\bar{z}$ and it is subharmonic in z in a neighborhood of the origin;

(ii) $\phi(z,\bar{z})$ vanishes at infinite order at 0, i.e., for any $N \geq 0$

$$\lim_{z\to 0}\frac{\phi(z)}{|z|^N} = 0\ ;$$

(iii) $\phi(z,\bar{z}) \geq 0$ on U and $\phi(z,\bar{z}) = 0$ if and only if $z = 0$;

(iv) any $P \in \partial\Omega \setminus \overline{U}$ is a point of finite type.

We will also denote by $P_\infty(\partial\Omega)$ the subset of $\partial\Omega$ given by all points of infinite type of $\partial\Omega$. From the previous remarks, it follows immediately that

$$\overline{U} \supset P_\infty(\partial\Omega) \supset \{(0,i\epsilon) \in \overline{U}\}.$$

Moreover the following notation

$$I = \{\epsilon\ : (0,i\epsilon) \in P_\infty(\partial\Omega)\}$$

will be used.

In particular, all points of $\partial\Omega$, except a $1_{\mathbb{R}}$-dimensional set, are of finite type. By a result of H. Boas ([Bo]; see also [Si]), this implies that *any domain* $\Omega \subset \mathbb{C}^2$*, which satisfies* (i)–(iv)*, also satisfies Condition R.*

For shortness of notation, given $F = (f^1, f^2) \in \mathrm{Aut}(\Omega)$ will indicate the components f^1, f^2 of the composition $F \circ \xi^{-1}$ of F with the previously system of coordinates $\xi : U \to \mathbb{C}^2$ present in the statement in the theorem. So, all previous remarks and, in particular, Theorem 2.1 and Corollary 2.2, apply to our bounded domain $\Omega \subset \mathbb{C}$.

Now, since any point of finite type is mapped onto a point of finite type, the smooth extension to the boundary of any $F = (f^1, f^2) \in \mathrm{Aut}(\Omega)$ maps $P_\infty(\partial\Omega)$ bijectively onto itself. In particular, for any $F = (f^1, f^2) \in \mathrm{Aut}(\Omega)$ we have that

$$\mathrm{Re}\, f^2(0, i\epsilon) = 0, \qquad \text{for any } \epsilon \in I. \tag{2.4}$$

This observation leads to the following useful lemma.

Lemma 2.4. *Let $F = (f^1, f^2) \in \mathrm{Aut}(\Omega)$ and let $V \subset \mathbb{C}$ be the set*

$$V = \{\ w \in \mathbb{C}\ :\ (0, w) \in U \cap \Omega\ \}.$$

Then the holomorphic function

$$\tilde{f}^2 : V \subset \mathbb{C} \to \mathbb{C}, \qquad \tilde{f}^2(w) = f^2(0, w),$$

extends holomorphically to an open set $\hat{V} \subset \mathbb{C}$, which properly contains V and $\{i\epsilon,\, \epsilon \in \overset{\circ}{I}\}$. Moreover, the extension $\tilde{f}^2 : \hat{V} \to \mathbb{C}$ satisfies the inequality

$$\sup_{w \in \hat{V}} |\tilde{f}^2(w)| \leq \sup_{(z,w) \in \Omega \cap U} |f^2(z, w)| \tag{2.5}$$

Proof. By construction, the domain of definition $V \subset \mathbb{C}$ of the holomorphic function $\tilde{f}^2$ contains the segment $i\epsilon$, $\epsilon \in \overset{\circ}{I}$, in the boundary and, by (2.4), the real part $\mathrm{Re}\, \tilde{f}^2$ admits a smooth extension at these points and it vanishes there. Therefore, $\tilde{f}^2$ admits the required holomorphic extension by Schwartz's Reflection Principle. The inequality (2.5) is a direct consequence of the construction by reflection of the extended map $\tilde{f}^2 : \hat{V} \to \mathbb{C}$. $\square$

A classical argument based on the Hopf Lemma shows that if $\rho(z, w)$ is a defining function of Ω on a neighborhood $\mathcal{U}_P$ of a point P, then for any $F = (f^1, f^2) \in \mathrm{Aut}(\Omega)$, $(\rho \circ F)(z, w)$ is also a defining function on a neighborhood $\mathcal{U}_{F^{-1}(P)}$ of $F^{-1}(P)$. In particular, there exists a smooth function $h(z, w)$ which is strictly positive on $\mathcal{U}_{F(P)}$ and such that, for any $(z, w) \in \mathcal{U}_{F(P)}$,

$$(\rho \circ F)(z, w) = h(z, w) \cdot \rho(z, w).$$

Using this observation and the fact that $F(P_\infty(\partial\Omega)) = P_\infty(\partial\Omega)$, we get the existence of a smooth function $k(z, w)$, which is defined on a neighborhood $\mathcal{U} \subseteq U$ of $P_\infty(\partial\Omega) \cap U$, strictly positive, and such that

$$2\,\mathrm{Re}\, w + \phi(z) = k(z, w)[2\,\mathrm{Re}\, f^2(z, w) + \phi(f^1(z, w))] \tag{2.6}$$

at all points $(z, w) \in \mathcal{U}$. This fact leads to the second technical lemma, which is needed for the proof of our main theorem.

LEMMA 2.5. *Let $F = (f^1, f^2) \in \mathrm{Aut}(\Omega)$ and $\mathcal{U} \subset U$ be a neighborhood of $P_\infty(\partial\Omega) \cap U$ where* (2.6) *holds. Then, for any $(z, w) \in \mathcal{U}$,*
(a) $f^1(0, w) \equiv 0$;
(b) $f^2(z, w) = f^2(w)$.

Proof. (a) Let $k(z, w) > 0$ be the smooth positive function which appears in (2.6). Then, for any $\epsilon \in \overset{\circ}{I}$, we have that

$$0 = k(0, i\epsilon)\phi(f^1(0, i\epsilon)), \tag{2.7}$$

and hence that $\phi(f^1(0, i\epsilon)) = 0$. Since the function $\tilde{f}^1(w) \overset{\text{def}}{=} f^1(0, w)$ is holomorphic on any w which is sufficiently close to 0 and satisfies $\mathrm{Re}\, w < 0$, from property (iv) and (2.7) we conclude that $f^1(0, w) \equiv 0$.

(b) We claim that for any $N \geq 1$ and any $(0, i\epsilon) \in P_\infty(\partial\Omega)$

$$\frac{\partial^N}{\partial z^N} \Big(2\,\mathrm{Re}\, f^2(z, w) + \phi(f^1(z, w), \overline{f^1(z, w)}) \Big)\Big|_{(0, i\epsilon)} = 0. \tag{2.8}$$

In fact, for any $(0, i\epsilon) \in P_\infty(\partial\Omega)$ we have that $F(0, i\epsilon) \in P_\infty(\partial\Omega) \subset \partial\Omega$ and hence

$$0 = 2\,\mathrm{Re}\, f^2(0, i\epsilon) + \phi(f^1(0, i\epsilon), \overline{f^1(0, i\epsilon)}).$$

From (2.6) it follows that

$$0 = k(0, i\epsilon) \cdot \frac{\partial}{\partial z} \Big(2\,\mathrm{Re}\, f^2(z, w) + \phi(f^1(z, w), \overline{f^1(z, w)}) \Big)\Big|_{(0, i\epsilon)},$$

which implies (2.8) for $N = 1$. Taking the N-th derivative with respect to the variable z of (2.6) and using an inductive argument, it follows that (2.8) holds also for any $N > 1$.

From (a), (2.8) and the property (iii) we get that for any $N \geq 1$ and any $(0, i\epsilon) \in P_\infty(\partial\Omega)$

$$\frac{\partial^N f^2}{\partial z^N}(0, i\epsilon) = 0. \tag{2.9}$$

Using the same arguments as for (a), we see that (2.9) implies (b). $\square$

From Lemmas 2.4, 2.5, Theorem 2.1 and Corollary 2.2, we finally obtain the following crucial fact.

LEMMA 2.6. *Assume that* $\mathrm{Aut}(\Omega)$ *is not compact. Let $\{F_j = (f_j^1, f_j^2)\} \subset$* $\mathrm{Aut}(\Omega)$ *be a sequence so that the sequences $\{F_j\}$ and $\{F_j^{-1}\}$ both converge uniformly on compact subsets of on Ω and which satisfy* (2) *of Theorem 2.1. Then the point P_1 of Theorem 2.1* (2) *cannot be of the form $(0, i\epsilon)$, $\epsilon \in \overset{\circ}{I}$.*

Proof. Suppose not and assume, e.g., that P_1 is the point $P_1 = (0,0)$. (The other cases can be treated in the very same way.) We claim that the hypotheses imply that the sequence of holomorphic maps $\{f_j^2\}$, given by the second components of the maps F_j, converges uniformly on $\overline{\Omega}$. This will give a contradiction with Corollary 2.2 and will conclude the proof.

By the uniform convergence of the sequence $\{F_j = (f_j^1, f_j^2)\}$ outside P_1, the claim is proved if we show the uniform convergence of the sequence $\{f_j^2\}$ on a neighborhood of $(0,0)$. By Lemmas 2.4 and 2.5 (b), we know that $f_j^2(z,w) = f_j^2(w)$ and that, for $r > 0$ sufficiently small, all maps $f_j^2(w)$ are holomorphic on $\overline{\Delta}_r(0) = \{\, |w| < r \,\}$ and

$$\sup_{\overline{\Delta}_r(0)} |f_n^2| = \sup_{\overline{\Delta}_r(0)\cap\overline{\Omega}\cap U} |f_j^2|.$$

This implies the uniform convergence because of the boundedness of the maps $f_j^2|_{\overline{\Omega}}$. □

3. Proof of Theorem 1.1

As in the previous section, we will always denote by $\Omega \subset \mathbb{C}^2$ a bounded domain which satisfies the hypothesis of Theorem 1.1 and, in particular, satisfies the conditions (i)–(iv) listed after (2.3).

Let us assume that $\mathrm{Aut}(\Omega)$ is not compact. Consider a sequence $\{F_j\} \in \mathrm{Aut}(\Omega)$ which admits no converging subsequence. By the remarks of Section 2, there is no loss of generality if we assume that $\{F_j\}$ and $\{F_j^{-1}\}$ converge on compact subsets of Ω and that case (2) of Theorem 2.1 holds.

We want to show that this leads to a contradiction. The contradiction will arise by showing that the absolute value of the holomorphic jacobian determinant of $\{F_j\}$ is at any $P = (0, i\epsilon)$, $\epsilon \in \overset{\circ}{I}$, and for every j larger than a constant $c(\epsilon)$.

From Lemma 2.5 we have that for any n

$$F_j = (zg_j(z,w), h_j(w)), \qquad F_j^{-1} = (z\tilde{g}_j(z,w), \tilde{h}_j(w)), \tag{3.1}$$

for some holomorphic functions g_j, $\tilde{g}_j$, h_j and $\tilde{h}_j$. Furthermore, by (2.6), for any j there exist two $\mathcal{C}^\infty$, strictly positive, real functions $k_j(z,w)$ and $\tilde{k}_j(z,w)$ defined on a neighborhood $\mathcal{U}$ of $P_\infty(\partial\Omega)$ such that

$$2\,\mathrm{Re}\,w + \phi(z) = k_j(z,w)[2\,\mathrm{Re}\,h_j(w) + \phi(zg_j(z,w))], \tag{3.2}$$

$$2\,\mathrm{Re}\,w + \phi(z) = \tilde{k}_j(z,w)[2\,\mathrm{Re}\,\tilde{h}_j(w) + \phi(z\tilde{g}_j(z,w))]. \tag{3.3}$$

Claim 3.1. *For any $\epsilon \in \overset{\circ}{I}$ and any j, we have $|g_j(0, i\epsilon)| \geq 1$.*

Proof. First we note that $g_j(0, i\epsilon) \neq 0$ for $\epsilon_o \in \overset{\circ}{I}$ (F_j is an automorphism).

From (2.4) and (3.2) we have $\operatorname{Re} h_j(i\epsilon) = 0$ and

$$\frac{\phi(z)}{\phi(zg_j(z,i\epsilon))} = k_j(z,i\epsilon) \tag{3.4}$$

Now, assume that there exists $\epsilon_o \in \overset{\circ}{I}$ and a natural number n_o such that $0 < |g_{n_o}(0,i\epsilon_o)| < 1$. By continuity, this implies that for any $|z| < \delta$ with $\delta > 0$ sufficiently small,

$$0 < |g_{n_o}(z,i\epsilon_o)| < 1. \tag{3.5}$$

Let us denote by C_o the supremum $C_o \overset{\text{def}}{=} \sup_{|z|<\delta} k_{n_o}(z,i\epsilon_o)$. From (3.4) we have that, for any integer $M > 0$ and any $|z| < \delta$,

$$\frac{\phi(z)}{|z|^M} \leq C_o|g_{n_o}(z,i\epsilon_o)|^M \frac{\phi(zg_{n_o}(z,i\epsilon_o))}{|z|^M|g_{n_o}(z,i\epsilon_o)|^M}.$$

From (3.5) it follows that for any M sufficiently large,

$$\frac{\phi(z)}{|z|^M} < \frac{\phi(zg_{n_0}(z,i\epsilon_o))}{|z|^M|g_{n_0}(z,i\epsilon_o)|^M}. \tag{3.6}$$

On the other hand, (3.5) implies also that $|zg_{n_0}(z,i\epsilon_o)| < |z| < \delta$. Hence (3.6) holds also if we replace everywhere z by $zg_{n_o}(z,i\epsilon_o)$. In this way we obtain the inequality

$$\frac{\phi(zg_{n_o}(z,i\epsilon_o))}{|z|^M|g_{n_0}(z,i\epsilon_o)|^M} < \frac{\phi(zg_{n_o}(z,i\epsilon_o)g_{n_o}(zg_{n_o}(z,i\epsilon_o),i\epsilon_o))}{|z|^M|g_{n_o}(z,i\epsilon_o)|^M|g_{n_o}(zg_{n_o}(z,i\epsilon_o),i\epsilon_o)|^M}, \tag{3.7}$$

which holds for any $|z| < \delta$. Combining (3.6) with (3.7) we get

$$\frac{\phi(z)}{|z|^M} < \frac{\phi(zg_{n_o}(z,i\epsilon_o)g_{n_o}(zg_{n_o}(z,i\epsilon_o),i\epsilon_o)}{|z|^M|g_{n_o}(z,i\epsilon_o)|^M|g_{n_o}(zg_{n_o}(z,i\epsilon_o),i\epsilon_o)|^M} = \frac{\phi(z\psi_1(z,i\epsilon_o))}{|z\psi_1(z,i\epsilon_o)|^M}, \tag{3.8}$$

where we denoted by $\psi_1(z,i\epsilon_o)$ the value

$$\psi_1(z,i\epsilon_o) = g_{n_o}(z,i\epsilon_o)g_{n_o}(zg_{n_o}(z,i\epsilon_o),i\epsilon_o)$$

Iterating the above argument N times, we obtain the inequality

$$\frac{\phi(z)}{|z|^M} < \frac{\phi(z\psi_N(z,i\epsilon_o))}{|z\psi_N(z,i\epsilon_o)|^M}, \tag{3.9}$$

where $\psi_N(z,i\epsilon_o)$ is defined recursively for any $N \geq 1$ by

$$\psi_N(z,i\epsilon_o) = g_{j_o}(z,i\epsilon_o)\psi_{N-1}(zg_{j_o}(z,i\epsilon_o),i\epsilon_o).$$

For a fixed value of z, the sequence $\{z\psi_N(z,i\epsilon_o)\}$ tends to 0 when N goes to infinity, while the left hand side of (3.9) is a positive real number. This gives a clear contradiction with condition (ii) in Section 2 and concludes the proof of the claim. $\square$

Claim 3.2. *For any $\epsilon \in \overset{\circ}{I}$ there exists a positive constant $C(\epsilon) > 0$ such that $|\frac{\partial h_j}{\partial w}(0,i\epsilon)| > C(\epsilon)$ for any j.*

Proof. Inserting the automorphism $F_j^{-1}(z,w)$ into (3.2) and using (3.3), we have for any (z,w) in a suitable neighborhood of the origin

$$\begin{aligned} k_j(F_j^{-1}(z,w))[2\operatorname{Re} w + \phi(z)] &= 2\operatorname{Re}\tilde{h}_j(w) + \phi(z\tilde{g}_j(z,w)) \\ &= \frac{1}{\tilde{k}_j(z,w)}[2\operatorname{Re} w + \phi(z)]. \end{aligned}$$

From this it follows that

$$k_j(z,w) = \frac{1}{\tilde{k}_j(F_j(z,w))} = \frac{1}{\tilde{k}_j(zg_j(z,w), h_j(w))}$$

and, in particular, that for any $\epsilon \in \overset{\circ}{I}$

$$k_j(0,i\epsilon)\tilde{k}_j(0,h_j(i\epsilon)) = 1. \tag{3.10}$$

For a fixed value $\epsilon \in \overset{\circ}{I}$, identity (3.10) implies that either the sequence $\{k_j(0,i\epsilon_o)\}$ or the sequence $\{\tilde{k}_j(0,\tilde{h}_j(i\epsilon_o)\}$ is bounded by some constant $C > 0$. Assume that $k_j(0,i\epsilon_o) < C$ for all n. Then, by taking the derivative of both sides of (3.2) with respect to w at the point $(0,i\epsilon_o)$, we get that

$$1 = \left| k_j(0,i\epsilon_o)\frac{\partial h_j(i\epsilon_o)}{\partial w}\right| < C\left|\frac{\partial h_j(i\epsilon_o)}{\partial w}\right|,$$

and this implies the claim with $C(\epsilon) = 1/C$.

On the other hand, if we assume that $\tilde{k}_j(0,\tilde{h}_j(i\epsilon_o)) < C$ for all j, then by replacing $i\epsilon_o$ by $\tilde{h}_j(i\epsilon_o)$ at all places, the same argument proves again the claim with $C(\epsilon) = 1/C$. $\square$

From Claims 3.1 and 3.2 and (3.1) we have that for any j and any $\epsilon \in \overset{\circ}{I}$,

$$\begin{aligned} |\det(JF_j)(0,i\epsilon)| &= \left|\det\begin{bmatrix} g_j(0,i\epsilon) & 0 \\ 0 & \frac{\partial h_j}{\partial w}(0,i\epsilon)\end{bmatrix}\right| \\ &= \left| g_j(0,i\epsilon)\cdot\frac{\partial h_j}{\partial w}(0,i\epsilon)\right| > C(\epsilon) > 0. \end{aligned} \tag{3.11}$$

On the other hand, by Theorem 2.1 (2) and Lemma 2.6, we have that, for any $\epsilon \in \overset{\circ}{I}$, the value of the determinant $\det(JF_j)(0,i\epsilon)$ tends to 0 when j goes to infinity. This fact clearly contradicts (3.11) and it concludes the proof.

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