EXTENSION OF UNITARY OPERATORS

BY

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Let $(X, \mathfrak{M}, \mu)$ be a measure space; the space $(X, \mathfrak{M}, \mu)$ will ordinarily be written X. A *measure-preserving transformation* on X is a function T whose domain is X, and whose range is a subset of X, such that if E is a measurable set of X then $T^{-1}E$ is measurable and $\mu(T^{-1}E) = \mu(E)$. Let T be a measure-preserving transformation on X. If there exists a measure-preserving transformation S on X such that $ST(x) = TS(x) = x$ a.e. then T is *invertible.*

Let T be an invertible measure-preserving transformation on X. It is well known that if $f \in \mathfrak{L}^2(X)$ then $fT \in \mathfrak{L}^2(X)$ and that the correspondence $f \to fT$ is a unitary operator on $\mathfrak{L}^2(X)$. This is called the unitary operator *induced* by T.

The following theorem is due to Kakutani [5].

Theorem. *Let U be a unitary operator on a separable Hilbert space $\mathfrak{H}$. Then there exists a measure space $(X, \mathfrak{M}, \mu)$ isomorphic to the space of the interval* $[0, 1)$ *with Lebesgue measure, a measure-preserving transformation T on X, and a subspace $\mathfrak{K}$ of $\mathfrak{L}^2(X)$ invariant under the unitary operator U_T induced by T, such that the restriction of U_T to $\mathfrak{K}$ is unitarily equivalent to U.*

Remark. Kakutani's proof involves a non-trivial argument based on properties of Gauss functions. Some of the details were not published in [5]. The proof presented here has a more elementary approach, although it does not develop the facts as completely as Kakutani's does.

Proof. The proof is based on a form of the spectral theorem for unitary operators. If $(Y, \mathfrak{N}, \nu)$ is a measure space and φ is a complex-valued measurable function on Y such that $|\varphi(y)| = 1$ for every point y of Y, then for every function f in $\mathfrak{L}^2(Y)$ the product $\varphi f \in \mathfrak{L}^2(Y)$, and the mapping

$$V : f \to \varphi f$$

is a unitary operator on $\mathfrak{L}^2(Y)$. By a certain form of the spectral theorem [2, pp. 911–912], [3] there exist a space Y and a function φ such that the operator V is unitarily equivalent to U. We therefore replace U and $\mathfrak{H}$ by V and $\mathfrak{L}^2(Y)$. Moreover, in case U has no proper values we may take Y to be normal in the sense of [4]; in case the proper vectors of U span $\mathfrak{H}$ then Y may be taken to be a countable set and we may assume that $\nu(Y) = 1$ and if $y \in Y$ then $\{y\} \in \mathfrak{M}$ and $\nu\{y\}) > 0$; and if neither of these holds then we may take Y to be the normalized union of two such spaces as occur in the first two cases.

Let $(C, \mathfrak{C}, \gamma)$ be the normalized measure space of the unit circle $\{z : |z| = 1\}$

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in the complex plane—with the usual structure obtained from that of the measure space of $[0, 1)$ with Lebesgue measure by means of the one-to-one correspondence $z = e^{2\pi i x}$ between C and $[0, 1)$.

By Fubini's theorem, it is easily seen that if for every function f in $\mathfrak{L}^2(Y)$ a function f^* on $Y \times C$ is defined by $f^*(y, z) = f(y) \cdot z$, then the association $f \to f^*$ is an isometric isomorphism from $\mathfrak{L}^2(Y)$ onto a subspace of $\mathfrak{L}^2(Y \times C)$. Let this subspace be $\mathcal{K}$.

If $(y, z) \in Y \times C$ write $T(y, z) = (y, \varphi(y)z)$. Then it is straightforward (using Fubini's theorem)—see, e.g., [1]—that T is an invertible measure-preserving transformation on $Y \times C$. We note that if $f \in \mathfrak{L}^2(Y)$ and $(y, z) \in Y \times C$ then

$$(Vf)^*(y, z) = \varphi(y)f(y)z = f^*(y, \varphi(y)z) = f^*T(y, z) = [U_T(f^*)](y, z).$$

It follows that $\mathcal{K}$ is invariant under U_T and that the restriction of U_T to $\mathcal{K}$ is unitarily equivalent to V. From the nature of the space Y and the normality of C it is easily checked that $Y \times C$ is normal. By a well-known characterization theorem [4, p. 339] the space $Y \times C$ is isomorphic to the space of $[0, 1)$ with Lebesgue measure. End of proof.

References

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