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DECOMPOSITIONS THAT DESTROY SIMPLE CONNECTIVITY

BY

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We shall be concerned with a monotone decomposition of R^3 with only one nondegenerate decomposition element X. We use g to denote the decomposition map and $g(R^3)$ the decomposition space. Also, D denotes a disk. To determine if $g(R^3)$ is simply connected we shall be concerned with whether maps of Bd D into $g(R^3)$ can be extended to D.

At the Summer Institute on Set Theoretic Topology at Wisconsin in 1955 I gave a talk entitled "What topology is here to stay" in which I envisioned decompositions of R^3 as a very viable area for research. I mentioned R.L. Moore's monotone decomposition theorem [3] for S^2 which states that if G is a nondegenerate upper semicontinuous decomposition of S^2 each of whose elements is a continuum that does not separate S^2, then the decomposition space is S^2. I pointed out that the theorem was false if one replaced S^2 by S^3 and gave as an example the decomposition whose only nondegenerate element is a circle. The earlier version of the Summary of Lectures and Seminars [1] reported on page 26 that the reason I gave that the decomposition space differed from S^3 was that it *is not simply connected*. The second printing of [1] made the correction by replacing the *is not simply connected* part of the statement by *does not remain simply connected* on the removal of some point. It was also claimed there and in [2] that the decomposition space of S^3 (or R^3) whose only nondegenerate element is a solenoid is not simply connected. When I was assembling copies of my publications it was called to my attention that a proof of this claim had not been published. It is the purpose of this paper to fill that gap. Other claims were made in [2] about the simple connectivity of other monotone decompositions (perhaps with many nondegenerate elements) of R^3, but we shall not treat them in this paper.

Richard Skora read an early draft of this paper and made valuable suggestions for improving some proofs.

1. X is a standard solenoid

In this case X is the intersection of smooth unknotted tori $T_1, T_2, \ldots$ where T_{i+1} winds around T_i smoothly more than once, the meridional cross sections

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of T_{i+1} are round planar disks that lie in those of T_i, and the diameters of the meridional cross sections of the T_i's converge to 0 as i increases without limit. Sometimes the restriction of "more than once" is omitted and a circle is permitted to be a solenoid—but we shall not do that in this paper.

THEOREM 1. *If X is a standardly embedded solenoid, $g(R^3)$ is neither simply connected nor locally simply connected.*

Proof. If we seek a map f of Bd D into $g(R^3)$ that cannot be extended to D, we should seek one such that $g(X) \in f(\text{Bd } D)$ because if $g(X) \notin f(\text{Bd } D)$ there is an extension of $g^{-1}f$ on Bd D to take D into R^3. This extension followed by g would extend f to map D into $g(R^3)$.

Let pq be an arc in a meridional cross section of T_1 that intersects X only in its end points where these end points belong to different arc components of X. We show that $g(R^3)$ is not simply connected by showing that a homeomorphism f of Bd D onto $g(pq)$ cannot be extended to map D into $g(R^3)$. Since for each open subset U of $g(R^3)$ containing $g(X)$ there is a pq with $g(pq)$ in U, this will also show that $g(R^3)$ is not locally simply connected. See Figure 1.

Assume f is a homeomorphism of Bd D onto $g(pq)$ and that f can be extended to a map F of D into $g(R^3)$. It would be nice if $F^{-1}(g(X))$ were 0-dimensional, so we adjust F to get a new map F_2 where $F_2^{-1}(g(X))$ is 0-dimensional. First, let F_1 be a map of D into $g(R^3)$ such that $F_1 = F$ on the component of $D - F^{-1}(g(X))$ intersecting Bd D and F_1 takes the rest of D to $g(X)$. Next we let k be a map of D onto itself that is the identity on Bd D and whose point inverses are the components of $F_1^{-1}(g(X))$ and points of $D - F_1^{-1}(g(X))$. Moore's decomposition theorem [3] mentioned earlier is used to get k. Then $F_2 = F_1k^{-1}$. For simplicity we suppose $F = F_2$.

Let a_0b_0 be a spanning arc of D such that $F(a_0b_0)$ misses $g(X)$ and the subdisk D_0 of D bounded by the union of a_0b_0 and the subarc of Bd D from a_0 to b_0 through $f^{-1}(g(X))$ lies in $F^{-1}(g(\text{Int } T_1))$. Since $g^{-1}(F(a_0))$ and $g^{-1}(F(b_0))$ lie in the same meridional cross section of T_1, we can speak of the number of times that $g^{-1}F(a_0b_0)$ winds around T_1.

For some large r let a_rb_r be a spanning subarc of D_0 such that a_r lies on BdD between a_0 and $f^{-1}g(X)$, b_r lies on Bd D between b_0 and $f^{-1}g(X)$, $F(a_rb_r)$ misses $g(X)$, and $F(a_rb_r)$ lies in $g(T_r)$. Let D_r be the subdisk of D_0 bounded by union of a_0b_0, a_rb_r and two subarcs of Bd D. See Figure 1.

Since p and q belong to different arc components of X, for large r, $g^{-1}F(a_rb_r)$ winds around T_1 many times—even more than $g^{-1}F(a_0b_0)$ does. Let $y(r)$ be the number of times that $g^{-1}(F(\text{Bd } D_r))$ winds around T_1. Suppose r is so large that $y(r) > 0$.

Let $z(s)$ be the number of times that T_s winds around T_1. We suppose s is so large that $z(s) > y(r)$ and $g^{-1}F(\text{Bd } D_r)$ misses Bd T_s. We suppose that on D_r near $F^{-1}(g(\text{Bd } T_s))$, F has enough general position so that $D \cap F^{-1}g(\text{Bd } T_s)$ is the union of a finite number of mutually disjoint simply closed

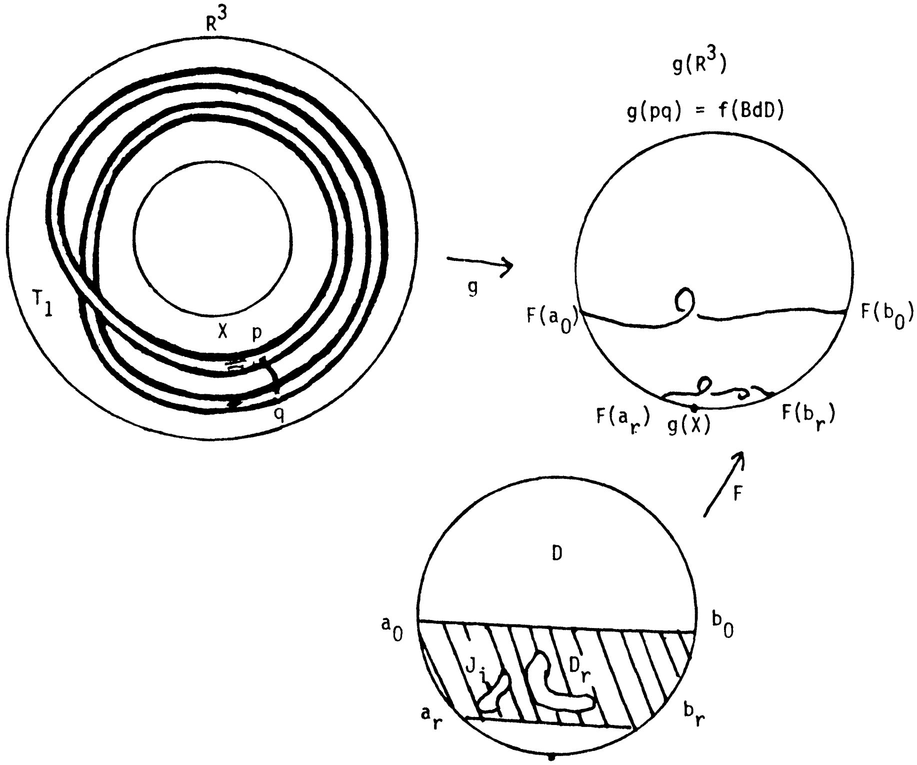

FIG. 1

curves $J_1, J_2, \ldots, J_n$. Since each $g^{-1}F(J_i)$ lies on Bd T_s, it winds around T_1 some integral multiple of $z(s)$.

Let E_r be the finitely holed D_r obtained by deleting from D_r the interiors of the subdisks of D_r bounded by the J_i's. We now come to the contradiction caused by the assumption tha f on Bd D could be extended to F on D. The boundary of each of the holes of E_r winds around T_1 some integral multiple of $z(s)$, but $y(r)$ is not an integral multiple of $z(s)$.

2. X is an embedded solenoid

The complement of an embedded solenoid may be quite different from the complement of a standardly embedded solenoid. We no longer can speak of tori about the embedded solenoid. However, we still find that the decomposition space is not simply connected.

THEOREM 2. *If X is an embedded solenoid, $g(R^3)$ is neither simply connected nor locally simply connected.*

Proof. We use X' to denote the standardly embedded solenoid of Theorem 1, $p'q'$ the arc called pq there, and g' the decomposition map called g there. Let β be a homeomorphism of X onto X' and pq an arc from $\beta^{-1}(p')$ to $\beta^{-1}(q')$ in R^3 that intersects X only at p and q. This is possible since the dimension of X is 1. Extend β to a map of R^3 onto itself that takes pq homeomorphically to $p'q'$. For convenience, call the extension β. It is a map rather than a homeomorphism. We finish the proof of Theorem 2 by showing that $g(pq)$ cannot be shrunk to a point in $g(R^3)$.

Let f be a homeomorphism of Bd D onto $g(pq)$. Assume that f can be extended to map F taking D into $g(R^3)$. This leads to the contradiction that a homeomorphism of Bd D onto $g'(p'q')$ can be extended to a map $g'\beta g^{-1}F$ of D into $g'(R^3)$. See Figure 2. The proof of Theorem 1 showed that no homeomorphism of Bd D onto $g'(p'q')$ could be extended to map D into $g'(R^3)$.

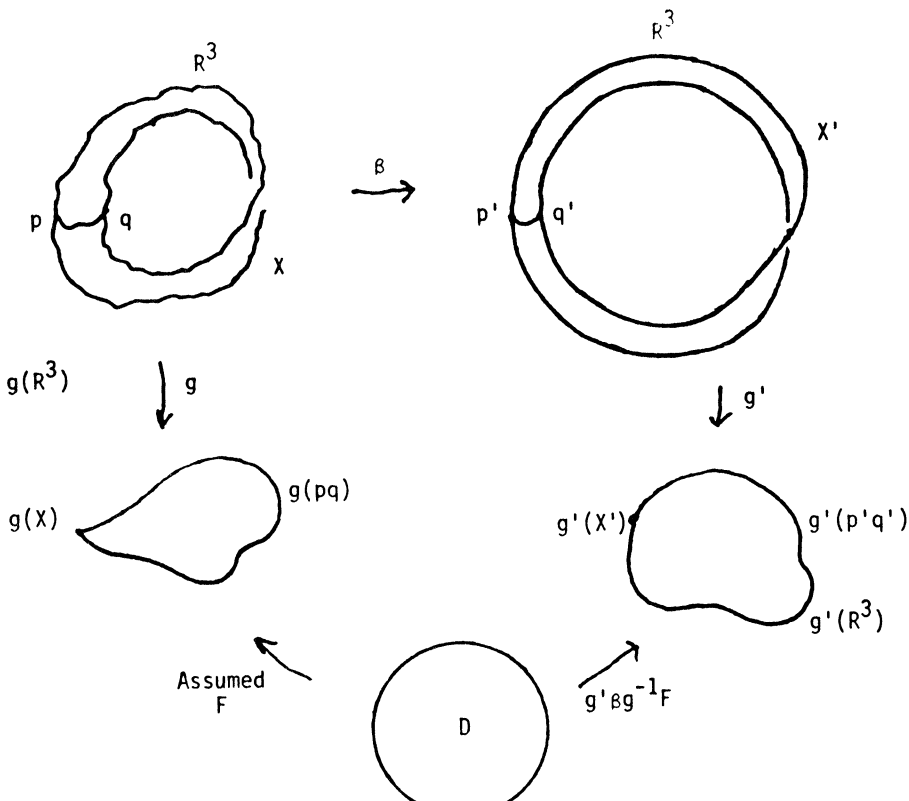

FIG. 2

Since pq could be picked close to X, this also shows that $g(X)$ is not locally simply connected.

3. X is unlike-a-solenoid

We say that a disk D can be converted to a disk with finitely many holes E if there is a finite collection of mutually disjoint disks in Int D and E is obtained from D by removing the interiors of these subdisks. These interiors are called holes in D and E is called a *finitely holed* D. We call D a finitely holed D even if there are no holes and $D = E$.

Let N_i be the $1/i$-neighborhood of X in R^3—that is, the set of points of R^3 whose distance from X is less than $1/i$.

If $f(\text{Bd } D) \subset N_i - X$, we say that *$f$ can be pulled in $N_i - X$ arbitrarily close to X* if for arbitrary large s, f can be extended to take a finitely holed D into $N_i - X$ so that the boundary of each hole is sent into N_s. Note that s is picked before the extension. If a different s had been chosen, we might have needed a different extension. (If f can be extended to map D into N_i, we could have picked an extension independent of s. If f can be extended to map D into $N_i - X$, then technically the definition says that f can be pulled arbitrarily close to X even though $f(D)$ misses X. This is because we call D a finitely holed D.) We say that *X is unlike-a-solenoid* if for each N_i there is an $N_{r(i)}$ such that each map f of Bd D into $N_{r(i)} - X$ can be pulled in $N_i - X$ arbitrarily close to X.

THEOREM 3. *If X is unlike-a-solenoid, then $g(R^3)$ is simply connected and locally simply connected.*

Proof. We first show that $g(R^3)$ is locally simply connected at $g(X)$. We show that if U is a neighborhood in $g(R^3)$ of $g(X)$, there is a neighborhood V of $g(X)$ such that each map f of Bd D into V can be extended to take D into U. We use f as a map of Bd D into V and $g^{-1}f$ to send Bd D into R^3.

To show that $g(R^3)$ is locally simply connected, without loss of generality we pick U to be $g(N_j)$ and V to be $g(N_{r(j)})$ where $r(j)$ is an integer such that any map of Bd D into $N_{r(j)} - X$ can be pulled in $N_j - X$ arbitrarily close to X on a finitely holed D.

We consider the sequence $n_1, n_2, \ldots$ where $n_1 = j$, and $n_{i+1} = r(n_i)$. Although the r's are defined for maps of Bd D into $R^3 - X$, we realize that in the case of the f of Bd D into $V = g(N_{r(j)})$ we wish to extend it to a map from D into $U = g(N_j)$ and this $f(\text{Bd } D)$ may contain $g(X)$.

It may be that $f^{-1}(g(X))$ has several components. We wish to avoid this. With that purpose in mind we suppose D is a round planar disk and let C be the convex hull of $f^{-1}(g(X))$ and partially extend f to send C to $g(X)$. For convenience we call the extension f. Now f is defined except on a collection

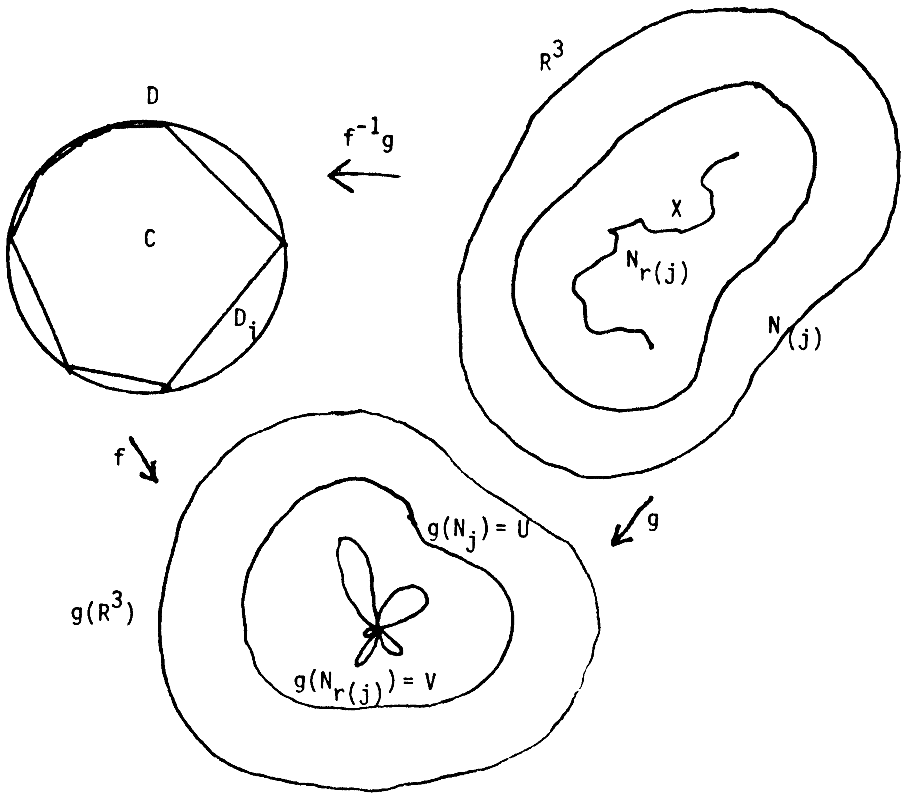

FIG. 3

(possibly infinite) of open disks. We call these disks D_i's. For each of these disks, f sends the straight subarc of its boundary to $g(X)$ and the open curved part into $g(R^3 - X)$. We have simplified the situation so that $f^{-1}g(X)$ intersects each Bd D_i in a connected set. See Figure 3.

Let D_i be one of the subdisks on whose interior f has not been defined. Each Bd $D_i \subset g(N_{n_2})$ and for all but a finite number of these D_i's $f(\text{Bd } D_i) \subset g(N_{n_3})$. For this finite number where $f(\text{Bd } D_i) \not\subset g(N_{n_3})$, we extend f to a part of their interiors.

Let ab be a spanning arc of D_i that cuts D_i into two subdisks D_i', D_i'' where $f(\text{Bd } D_i' - ab)$ misses $g(X)$ and the closure of $f(\text{Bd } D_i'' - ab)$ lies in $g(N_3)$. Extend f to take ab into $g(N_{n_3} - X)$. Extend f further on a finitely holed D_i' to take the finitely holed D_i' into $g(N_{n_1} - X)$ where the boundary of the holes go into $g(N_{n_3})$. To get the extension we use the hypothesis that X is unlike-a-solenoid.

We now find that f is defined except on open disks whose boundaries are sent by f into $g(N_{n_3})$. In fact on all but a finite number of these disks, f sends their boundaries into $g(N_{n_4})$. We extend f into a part of the interiors of this finite collection so that now f is defined except on open disks whose boundaries are sent into $g(N_{n_4})$. Using the hypothesis that X is unlike-a-solenoid we pick the extension on the new part to take this new part into $g(N_{n_2})$.

The extension is extended a countable number of times and finally we define the extension to take the remaining part of D to $g(X)$. We have now shown that $g(R^3)$ is locally simply connected. That it is simply connected follows from the following theorem.

THEOREM 4. *If $g(R^3)$ is locally simply connected, it is simply connected.*

Proof. Let f be a map of Bd D into $g(R^3)$. We show that f is simply connected by showing that f can be extended to map D into $g(R^3)$.

It follows from the local simple connectivity of $g(R^3)$ that there is a neighborhood U of $g(X)$ such that each map of Bd D into U can be extended to map D into $g(R^3)$. Suppose D is a round disk and $A_1, A_2, \ldots, A_n$ are the components of Bd $D - f^{-1}g(X)$ that are not sent into U by f. Let B_i be an open arc on A_i such that each $f(A_i - B_i) \subset U$. Let C_i be a straight arc in D joining the two components of $A_i - B_i$.

Extend f to C_i so that the extension takes C_i into $U - g(X)$. Call the extension f. On the subdisk D_i of D bounded by C_i and a part of A_i, extend $g^{-1}f$ to take each D_i into R^3 and follow this extension by g to extend f to take D_i into $g(R^3)$. The boundary of the remaining part of D is sent by f into U so the local connectivity of $g(R^3)$ shows that f can be extended to the rest of D. This shows that $g(R^3)$ is simply connected.

4. X is solenoid-like

We say that X is solenoid-like if there is a neighborhood N of X in R^3 such that for any neighborhood N' of X there is a map f of Bd D into $N' - X$ which cannot be pulled in $N - X$ on a finitely holed D arbitrarily close to X. One might note if X is solenoid-like, then it is untrue that X is unlike-a-solenoid.

THEOREM 5. *If X is solenoid-like, $g(R^3)$ is not locally simply connected.*

Proof. We show that for some neighborhood $U = g(N)$ of $g(X)$, and each smaller neighborhood $V = g(N')$ of $g(X)$ there is a map of Bd D into V that cannot be extended to map D into U. Here we use N, N', f as in definition of solenoid-like and use gf for the map of Bd D into $V - g(X)$ that cannot be extended to map D into U. If gf could be extended by F to send D into U, $g^{-1}F$ would show that f can be pulled in $N - X$ on a finitely holed D arbitrarily close to X.

Question Recall that X is a continuum in R^3 and $g(R^3)$ is the decomposition space whose only nondegenerate point inverse is X. Is $g(R^3)$ locally simply connected if it is simply connected? If $g(R^3)$ is not locally simply connected, could it be simply connected?

5. Necessary and sufficient conditions

Theorems 3 and 5 provide a necessary and sufficient condition that $g(R^3)$ is not locally simply connected. However the condition is dependent on the embedding of X and does not say that if X' is homeomorphic to X then $g(R^3) = R^3/X$ is locally simply connected if and only if R^3/X' is.

THEOREM 6. *A necessary and sufficient condition that $g(R^3)$ not be locally simply connected is that X be solenoid-like.*

Proof. The sufficiency is provided by Theorem 5 and the necessity by Theorem 3.

THEOREM 7. *If X and X' are homeomorphic continua in R^3 and* $\dim X = 1$, *then $g(R^3) = R^3/X$ fails to be locally simply connected if and only if $g'(R^3) = R^3/X'$ does.*

Proof. The proof of Theorem 7 is modelled after that of Theorem 2.

Suppose $g'(R^3)$ is not locally simply connected. Then it is solenoid-like and there are a neighborhood N' of X' and a sequence of mutually disjoint simple closed curves $J_1', J_2', \dots$ in $N' - X'$ such that J_1' lies in the $1/i$ neighborhood of X' and a map f_i' of Bd D onto J_i' that cannot be pulled in $N' - X'$ arbitrarily close to X' on a finitely-holed D.

Let β be a homeomorphism of X onto X'. We now pick a sequence of simple closed curves $J_1, J_2, \dots$ in $R^3 - X$ so that β can be extended to a homeomorphism taking $X \cup J_1 \cup J_2 \cup \dots$ onto $X' \cup J_1' \cup J_2' \cup \dots$. We assume $J_1, J_2, \dots, J_{i-1}$ have been found with β extended to them and describe J_i and β on it.

Express J_i' as the union of arcs $a_1'a_2', a_2'a_3', \dots, a_n'a_{n+1}'$ each of diameter less than $1/i$. Let b_j' be a point of X' in the $1/i$ neighborhood of a_j'. Note that the distance between two adjacent b_j''s is less than $3/i$. Let a_j be a point of

$$R^3 - (X \cup J_1 \cup \cdots \cup J_{i-1})$$

in the $1/i$ neighborhood of $\beta^{-1}(a_j')$. Note that distance between two adjacent a_j's is less than $\varepsilon + 2/i$ where ε is a positive number such that the image under β^{-1} of any $3/i$-subset of X' has diameter less than ε. Hence adjacent a_j's are close if i is large. We suppose the a_j's are distinct and let J_i be a simple closed curve in

$$R^3 - (X \cup J_1 \cup J_2 \cup \cdots \cup J_{i-1})$$

which is the union of arcs $a_1a_2, a_2a_3, \ldots, a_na_{n+1}$ where a_ja_{j+1} lies in the $1/i$ neighborhood of the straight line interval from a_j to a_{j+1}. It is in showing that there are such a_ja_{j+1}'s in $R^3 - (X \cup J_1 \cup J_2 \cup \cdots \cup J_{i-1})$ that we use the fact that dim $X = 1$. The homeomorphism β is extended to J_i so that $\beta(a_j) = a_j'$.

Suppose all J_i's are defined. Call the extended homeomorphism β. Now extend β to a map taking R^3 onto R^3 and call this extension β also. This final extension need not be a homeomorphism. Let f_i be a homeomorphism of Bd D onto J_i such that $\beta f_i = f_i'$.

Assume that $g(R^3) = R^3/X$ is locally simply connected. We prove the theorem by showing that this assumption is false. Local simple connectivity of $g(R^3)$ implies that for a large r, gf_r on Bd D can be extended to a map F_r taking D into $g\beta^{-1}(N')$. Let C be the set of points of D that are carried by $\beta g^{-1}F_r$ onto X' or points of X'. Change D to E, a finitely holed D in $D - C$, so that the boundaries of the holes of E are very close to C. Although $\beta g^{-1}F_r$ need not be a map on D, it is one on E. Also $\beta g^{-1}F_r = f_r'$ on Bd D. Hence $\beta g^{-1}F_r$ on E shows that f_r' on Bd D can be pulled in $N' - X'$ arbitrarily close to X on a finitely holed D. This contradicts the selection of J_r' and f_r'.

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