ON H-SPACES THAT ARE CW-COMPLEXES I

BY
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A topological space that has a continuous multiplication with a homotopy unit is called an H-space. In [3] A. H. Copeland, Jr. describes a binary operation on the set of homotopy classes $[A; X]$ of basepoint-fixed maps of a topological space A into an H-space X and proves that an exact sequence of homomorphisms results from a sequence $B, * \to A, * \to A, B$ of inclusion maps when (A, B) is a CW-pair. I. M. James proves in [6] that if A is a CW-complex and X is an H-space, then $[A; X]$ is a loop. It follows that when the H-space X is itself a CW-complex, it shares many of the properties of a loop. In this paper we examine such spaces from the point of view of the theory of loops.

By means of exact sequences of sets of homotopy classes we show that the set $HS(X)$ of homotopy classes of H-structure maps on a given H-space X is in one-to-one correspondence with the set of homotopy classes of maps

$$f : X^2, X \vee X \to X, e.$$

Corresponding to the loop theoretic notions power associativity and diassociativity, we define the concepts homotopy power associativity and homotopy diassociativity for H-spaces, and we show that if a CW-complex X is an H-space, then X is homotopy power associative when cat $X \leqq 3$, and X is homotopy diassociative when cat $X \leqq 2$. We also obtain results on homotopy commutativity and homotopy associativity. In the sequel to this paper results obtained here will be applied to the study of H-spaces in terms of their Postnikov systems.

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1. Preliminary considerations and conventions

Recall that a *loop* is a set M together with a binary operation, written multiplicatively, having the following two properties:

(i) there is a two-sided identity element 1,

(ii) for every two elements $a, b \in M$ the equations

$$a \cdot x = b, \qquad y \cdot a = b,$$

admit unique solutions in M.

(The standard reference for the theory of loops is [2].)

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Let $f : M \to N$ be a homomorphism of loops. The subset $fM \subset N$ is clearly a subloop; fM is called the *image* of f, denoted $\operatorname{Im} f$. The homomorphism f is called an *epimorphism* if $fM = N$, and f is called a *monomorphism* if f is a one-to-one function. The subset $f^{-1}1 \subset M$ is clearly a subloop of M; $f^{-1}1$ is called the *kernel* of f and is denoted by $\operatorname{Ker} f$. A subloop $K \subset M$ is said to be a *normal* subloop if K is the kernel of a homomorphism.

The following "generalized Lagrange theorem" for loops is well known. (See [2, Chap. IV, Sec. 1].)

THEOREM 1.1. *Let $f : M \to N$ be a loop epimorphism. Then, for each element $x \in N$, the set $f^{-1}x$ is in one-to-one correspondence with* $\operatorname{Ker} f$.

It follows from Theorem 1.1 that a homomorphism is a monomorphism if and only if its kernel consists of the identity element alone. Thus we can deal with exact sequences of loops in the same way as with exact sequences of groups. Factor loops and direct products of loops are defined in the same way as for groups. (It is well known that the "five-lemma" holds for arbitrary groups. One can show, in fact, that the "five-lemma" also holds for loops. Likewise, a number of other standard group-theoretic devices used by algebraic topologists hold for loops.)

Let M be a loop. Then every element $m \in M$ possesses both a left inverse m^L and a right inverse m^R. If $m^L = m^R$ for each $m \in M$, we denote this element by m^{-1}, and, in this case, the loop M is said to be *inversive*. Thus a commutative loop is inversive. An associative loop is, in fact, a group.

The topological spaces we treat will be assumed to be provided with a basepoint, usually denoted by $*$; all spaces will be assumed to be Hausdorff spaces. We denote the closed unit interval $[0, 1]$, taken in the usual topology, by I and choose $0 \in I$ as basepoint. If $f : A \to X$ is a map of topological spaces, it will be understood that $f(*) = *$. Only those subspaces of a topological space that are closed and contain the basepoint will be considered. If (A, B) is a topological pair, the expression $f : A, B \to X$ will mean $f : A, B \to X, *$; similarly $h : X \to A, B$ will mean $h : X, * \to A, B$. Likewise, if $g : A \to X$ is a map (preserving basepoints), then $f \simeq g$ will be understood to mean that f is homotopic to g (rel $*$). The cartesian product space $A \times B$ has $(*, *)$ as its basepoint. If $f : A \to X$ and $g : B \to Y$ are maps, then $f \times g : A \times B \to X \times Y$ is the map defined by the formula

$$(f \times g)(a, b) = (f(a), g(b)).$$

The n-fold cartesian product space $X \times \cdots \times X$ will be denoted by X^n.

If A is a CW-complex [9], then the basepoint will always be taken as a subcomplex of A, and only those subcomplexes of A containing the basepoint will be considered. If A and B are CW-complexes, then $A \times B$ is the cartesian product space with the product cellular decomposition. We shall, moreover, always assume that $A \times B$ is a CW-complex whenever A and B are CW-complexes. (The reader may, if he wishes, assume that all CW-complexes are countable.) This convention is justified by the fact that most of the results we obtain for CW-complexes hold more generally for spaces in the category $\mathcal{W}$ of spaces having the homotopy type of a CW-complex together with the fact that if $A \in \mathcal{W}$ and $B \in \mathcal{W}$, then $A \times B \in \mathcal{W}$. (See [7, Proposition 3].) The precise statements of the more general results are left to the reader.

Let X be a topological space with basepoint e. The subspace $(X \times e) \cup (e \times X)$ of X^2 is denoted $X \vee X$; the *folding map* $\nabla : X \vee X \to X$ is defined by

$$\nabla(x, e) = x = \nabla(e, x).$$

Let $i : X \vee X \to X^2$ be the inclusion map. If there is a map $\mu : X^2 \to X$ such that $\mu i \simeq \nabla$, then μ is called an *H-structure map*, and we say that (X, μ) is an *H-space*; the basepoint $e \in X$ is called the *identity element*. Whenever X is an H-space, we shall always choose its identity element as basepoint. If $\mu i = \nabla$, then μ is called a *strong* structure map, and (X, μ) is said to be a *strong H-space*. Let (X, μ) and (Y, ν) be H-spaces. A map $\pi : X \to Y$ is said to be an *H-map* if the maps

$$\pi\mu, \; \nu(\pi \times \pi) : X^2 \to Y$$

are homotopic. The *transposition map* $T : X^2 \to X^2$ is defined by $T(x, y) = (y, x)$. An H-space (X, μ) is said to be *homotopy commutative* if $\mu \simeq \mu T$. Let $1 : X \to X$ be the identity map. Then (X, μ) is said to be *homotopy associative* if $\mu(1 \times \mu) \simeq \mu(\mu \times 1)$. Let $0 : X \to X$ be the constant map onto e. Then (X, μ) is said to be *homotopy inversive* if there is a map $\eta : X \to X$ such that $\mu(1 \times \eta) \simeq \mu(\eta \times 1) \simeq 0$. The map η is called a *homotopy inverse* for μ.

Let X be a topological space, and let (A, B) be a topological pair. We denote the set of homotopy classes (rel B) of maps $f : A, B \to X$ by $[A, B; X]$, and we denote the homotopy class of each such map f by $[f]$. When $B = *$, we abbreviate $[A, B; X]$ by writing $[A; X]$. If (C, D) is also a topological pair, then every map $\phi : A, B \to C, D$ induces a function

$$\phi^* : [C, D; X] \to [A, B; X]$$

defined for each element $[f] \in [C, D; X]$ by $\phi^*[f] = [f\phi]$. We observe that if C is the space obtained from (A, B) by identifying B to a single point $*$, and if $\phi : A, B \to C$ is the identification map, then the induced function $\phi^* : [C; X] \to [A, B; X]$ is a one-to-one correspondence.

Dually, if Y is a topological space, then every map $\psi : X \to Y$ induces a function $\psi_* : [A, B; X] \to [A, B; Y]$ defined for each element $[f] \in [A, B; X]$ by $\psi_*[f] = [\psi f]$. A map $\psi : X \to Y$ is called a *weak homotopy equivalence* if the induced function $\psi_* : \pi_n(X) \to \pi_n(Y)$ is a one-to-one correspondence in each dimension $n = 0, 1, \cdots$. If $\psi : X \to Y$ is a weak homotopy equivalence, and if A is a CW-complex, then the induced function

$$\psi_* : [A; X] \to [A; Y]$$

is a one-to-one correspondence. (See [6, Lemma 3.4].) One can easily prove that if B is a subcomplex of A, then $\psi_* : [A, B; X] \to [A, B; Y]$ is also a one-to-one correspondence whenever ψ is a weak homotopy equivalence.

Let A be a topological space. The *diagonal map* $\triangle : A \to A^2$ is defined by the formula $\triangle(a) = (a, a)$. Let (X, μ) be an H-space. The structure map μ induces a binary operation on the set $[A; X]$ as follows. For any two maps $f, g : A \to X$ the product $[f] \cdot [g] \in [A; X]$ is defined by

$$[f] \cdot [g] = [\mu(f \times g)\triangle].$$

Whenever we wish to emphasize that the binary operation on $[A; X]$ is induced by the H-structure map μ, we shall do so by writing $[A; X]_\mu$. Let $0 : A \to X$ denote the constant map onto the basepoint of X. Then $[0] \in [A; X]_\mu$ is a two-sided identity element. If μ is homotopy commutative, then $[A; X]_\mu$ is commutative; if μ is homotopy associative, then $[A; X]_\mu$ is associative; and if μ is homotopy inversive, then $[A; X]_\mu$ is inversive. In this connection, Copeland has proved the following result. (See [3, Theorems 3.2B, 3.2C, and 3.2D].)

THEOREM 1.2. *Let (X, μ) be an H-space.*

(a) *μ is homotopy inversive if and only if $[X; X]_\mu$ is inversive.*

(b) *μ is homotopy commutative if and only if $[X^2; X]_\mu$ is commutative.*

(c) *μ is homotopy associative if and only if $[X^3; X]_\mu$ is associative.*

Let A and B be topological spaces, and let (X, μ) and (Y, ν) be H-spaces. If $\phi : A \to B$ is a map, then the induced function $\phi^* : [B; X]_\mu \to [A; X]_\mu$ is a homomorphism. (See [3, Theorem 3.2A].) If $\pi : X \to Y$ is an H-map, then $\pi_* : [A; X]_\mu \to [A; Y]_\nu$ is a homomorphism. (See [3, Theorem 3.4A].)

A result attributed to W. Hurewicz asserts that if (X, μ) is an H-space, then there are a strong H-space (Y, ν) and a weak homotopy equivalence $\pi : Y \to X$ such that π is an H-map. It follows that if A is a CW-complex, then $\pi_* : [A; Y]_\nu \to [A; X]_\mu$ is an isomorphism. Hence, in studying properties of the sets $[A; X]_\mu$, one can assume that (X, μ) is a strong H-space without loss of generality.

The following theorem, which is fundamental for our investigations here, was proved by I. M. James. (See [6, Theorem 1.1].)

THEOREM 1.3. *Let A be a CW-complex, and let (X, μ) be an H-space. Then $[A; X]_\mu$ is a loop.*

It follows easily from Theorem 1.3 that if A is a CW-complex and $B \subset A$ is a subcomplex, then $[A, B; X]_\mu$ is a loop. In fact, if C is the space obtained from (A, B) by identifying B to a point $*$, and if $\phi : A, B \to C$ is the identification map, then $\phi^* : [C; X]_\mu \to [A, B; X]_\mu$ is a loop isomorphism.

Let R, S, T be topological spaces or topological pairs, and let $\phi : R \to S$ and $\psi : S \to T$ be maps. For any topological space X the sequence

$$[T; X] \xrightarrow{\psi^*} [S; X] \xrightarrow{\phi^*} [R; X]$$

is said to be *exact* whenever the image of ψ^* coincides with the set of elements of $[S; X]$ mapped by ϕ^* onto $[0]$. Copeland has shown that if A is a CW-complex and B is a subcomplex of A, and if $i : B \to A$ and $j : A \to A, B$ are the inclusion maps, then for each topological space X the sequence

$$\cdots \to [A, B; X] \xrightarrow{j^*} [A; X] \xrightarrow{i^*} [B; X]$$

is exact. (See [3, Theorem 3.3C].) It follows from Theorem 1.3 that if (X, μ) is an H-space, then the sequence

$$\cdots \to [A, B; X]_\mu \xrightarrow{j^*} [A; X]_\mu \xrightarrow{i^*} [B; X]_\mu$$

is an exact sequence of loops and homomorphisms; that is, $\operatorname{Im} j^* = \operatorname{Ker} i^*$.

The concept of retractile subcomplex of a CW-complex was introduced by I. M. James in [6]; "retractile" generalizes the notion of "retract". We give an equivalent definition which was suggested by J. D. Stasheff.

Let A be a CW-complex. A subcomplex B is said to be *retractile* in A provided that, for every topological space X, whenever $f : A \to X$ is a map such that $fB = *$ and $f \simeq 0$, then $f \simeq 0$ (rel B).

A basic property of retractile subcomplexes is described in the following proposition.

PROPOSITION 1.4. *Let X be an H-space, and let B be a retractile subcomplex of a CW-complex A. If $f, g : A \to X$ are homotopic maps that agree on B, then $f \simeq g$ (rel B).*

This is Corollary 4.4 of [6].

COROLLARY 1.5. *Let (X, μ) be an H-space, let B be a retractile subcomplex of a CW-complex A, and let $j : A \to A, B$ be the inclusion map. Then the induced homomorphism $j^* : [A, B; X]_\mu \to [A; X]_\mu$ is a monomorphism.*

For suppose $f, g : A, B \to X$ are maps such that $fj \simeq gj$. Then, since fj and gj coincide on B with the constant map, it follows that $fj \simeq gj$ (rel B). Therefore, $f \simeq g$; so the homomorphism $j^* : [A, B; X]_\mu \to [A; X]_\mu$ is a one-to-one function.

COROLLARY 1.6. *Let (X, μ) be an H-space, let B be a retractile subcomplex of a CW-complex A, and let C be the complex obtained by identifying B to a point. If $\phi : A \to C$ is the identification map, then the induced homomorphism*

$$\phi^* : [C; X]_\mu \to [A; X]_\mu$$

is an isomorphism.

Let (X, μ) be a strong H-space with identity element e, and let $T : X^2 \to X^2$ denote the transposition map. Then μT is also a strong H-structure map; hence μ coincides with μT on $X \vee X$. We say that a strong H-space (X, μ) is *strongly homotopy commutative* provided that $\mu \simeq \mu T$ (rel $X \vee X$). We denote by $T^3(X)$ the subspace of X^3 defined as follows:

$$T^3(X) = (e \times X^2) \cup (X \times e \times X) \cup (X^2 \times e).$$

If (X, μ) is a strong H-space, then the maps $\mu(1 \times \mu), \mu(\mu \times 1) : X^3 \to X$ coincide on $T^3(X)$. We say that a strong H-space (X, μ) is *strongly homotopy associative* provided that $\mu(1 \times \mu) \simeq \mu(\mu \times 1)$ (rel $T^3(X)$).

Let X be a CW-complex and suppose that (X, ν) is an H-space. Then it follows from the homotopy extension theorem [9] that there is a strong structure map μ for X such that $\mu \simeq \nu : X^2 \to X$; moreover, the identity map $1 : X \to X$ is an H-map. Thus, in studying homotopy properties of H-spaces that are CW-complexes, it suffices to restrict attention to strong H-spaces. Likewise, in studying the homotopy properties of homotopy commutative or homotopy associative H-spaces that are CW-complexes, one can also restrict attention to the strong cases without loss of generality; specifically, we have the following result.

PROPOSITION 1.7. *Let X be a CW-complex, and suppose that (X, μ) is a strong H-space.*

(a) *If (X, μ) is homotopy commutative, then (X, μ) is strongly homotopy commutative.*

(b) *If (X, μ) is homotopy associative, then (X, μ) is strongly homotopy associative.*

For $X \vee X$ is a retractile subcomplex of X^2, and $T^3(X)$ is a retractile subcomplex of X^3. (See [6, p. 165].) Therefore if $\mu \simeq \mu T : X^2 \to X$, then by Proposition 1.4 we have

$$\mu \simeq \mu T \quad (\text{rel } X \vee X).$$

Likewise, if $\mu(1 \times \mu) \simeq \mu(\mu \times 1)$, then $\mu(1 \times \mu) \simeq \mu(\mu \times 1)$ (rel $T^3(X)$).

2. The homotopy classes of H-structure maps

Let X be a topological space with basepoint e, let $\nabla : X \vee X \to X$ be the folding map, and let $k : X \vee X \to X^2$ be the inclusion. Let A and B be topological spaces, and let $i : A \vee B \to A \times B$ be the inclusion.

THEOREM 2.1. *X is an H-space with identity element e if and only if the function $i^* : [A \times B; X] \to [A \vee B; X]$ sends $[A \times B; X]$ onto $[A \vee B; X]$ for all spaces A and B.*

For suppose that (X, μ) is an H-space. Let $f : A \vee B \to X$ be any map. Then the function $g : A \vee B \to X \vee X$ defined by the formulas

$$g(a, *) = (f(a, *), e), \qquad g(*, b) = (e, f(*, b)),$$

is clearly well defined and continuous; moreover $f = \nabla g$. Define the map $F : A \times B \to X$ by the formula

$$F(a, b) = \mu(f(a, *), f(*, b)).$$

Since $\mu k \simeq \nabla$, we have

$$Fi = \mu k g \simeq \nabla g = f.$$

Therefore $i^*[F] = [f]$, and so i^* is an epimorphism.

Conversely, suppose that for all spaces A and B the function i^* sends $[A \times B; X]$ onto $[A \vee B; X]$. Then set $A = B = X$; we have $i = k$. It follows that there is a map $\mu : X^2 \to X$ such that $\mu k \simeq \nabla$. Then (X, μ) is an H-space, and the theorem is proved.

Let A and B be CW-complexes, and let

$$i : A \vee B \to A \times B \quad \text{and} \quad j : A \times B \to A \times B, A \vee B$$

be the inclusion maps.

COROLLARY 2.2. *If (X, μ) is an H-space, then*

$$[0] \to [A \times B, A \vee B; X]_\mu \xrightarrow{j^*} [A \times B; X]_\mu \xrightarrow{i^*} [A \vee B; X]_\mu \to [0]$$

is an exact sequence of loops.

The subcomplex $A \vee B$ is retractile in $A \times B$. (See [6, p. 165].) Hence exactness at $[A \times B, A \vee B; X]$ follows from Corollary 1.5. Exactness at $[A \times B; X]$ has already been discussed in Section 1. Exactness at $[A \vee B; X]$ follows from Theorem 2.1. Thus the corollary is proved.

Let X be a topological space, and let e be the basepoint. Following Copeland [3] we use the symbol $HS(X)$ to denote the set of homotopy classes of H-structure maps on X that have e as identity element. The following theorem generalizes a result of Copeland. (See Theorem 5.5A of [3].)

THEOREM 2.3. *Let X be a CW-complex. If X is an H-space, then $HS(X)$ is in one-to-one correspondence with $[X^2, X \vee X; X]$.*

Let $\nabla : X \vee X \to X$ be the folding map, and let $i : X \vee X \to X^2$ and $j : X^2 \to X^2, X \vee X$ be the inclusion maps. Then $HS(X) = (i^*)^{-1}[\nabla]$. By Theorem 1.1, $(i^*)^{-1}[\nabla]$ is in one-to-one correspondence with Ker i^*, and by Corollary 2.2, Ker i^* is in one-to-one correspondence with $[X^2, X \vee X; X]$.

3. H-spaces, category, and groups

Let A be a topological space. The *Lusternik-Schnirelmann category* of A is at most n, written cat $A \leqq n$, if A is the union of n open subsets, each contractible in A. (The standard reference for Lusternik-Schnirelmann category is [4].)

Let A be a connected CW-complex. The subcomplex $T^n(A) \subset A^n$ is defined to be the subspace consisting of all points $(a_1, \cdots, a_n)$ such that $a_i = *$ for some i, $1 \leqq i \leqq n$. Let $\triangle : A \to A^n$ be the diagonal map,

$$\triangle(a) = (a, a, \cdots, a),$$

and let $i : T^n(A) \to A^n$ be the inclusion map. G. W. Whitehead shows in [8] that cat $A \leqq n$ if and only if there is a map $\psi : A \to T^n(A)$ such that $i\psi \simeq \triangle$. We call the map ψ a *category-n structure map.*

THEOREM 3.1. *Let (X, μ) be an H-space, and let A be a connected CW-complex. If* cat $A \leqq 3$, *then $[A; X]_\mu$ is a group.*

It follows from remarks made in Section 1 that we may assume, without loss of generality, that (X, μ) is a strong H-space. By Theorem 1.3 the set $[A; X]_\mu$ is a loop; hence it suffices to show that $[A; X]_\mu$ is associative.

Let $\psi : A \to T^3(A)$ be a category-3 structure map, let $i : T^3(A) \to A^3$ be the inclusion map, and let $\triangle : A \to A^3$ be the diagonal map. Then $i\psi \simeq \triangle$. Let $j : T^3(X) \to X^3$ be the inclusion map. Then, since μ is a strong H-structure map, we have $\mu(1 \times \mu)j = \mu(\mu \times 1)j$. Let $f, g, h : A \to X$ be any three maps, and let us denote by $T^3(f, g, h) : T^3(A) \to T^3(X)$ the map induced by $f \times g \times h : A^3 \to X^3$. Consider the element $[f]\cdot([g]\cdot[h]) \in [A; X]_\mu$. We have

$$\begin{aligned}[f]\cdot([g]\cdot[h]) &= [\mu(1 \times \mu)(f \times g \times h)\triangle] \\ &= [\mu(1 \times \mu)(f \times g \times h)i\psi] \\ &= [\mu(1 \times \mu)j(T^3(f, g, h))\psi] \\ &= [\mu(\mu \times 1)j(T^3(f, g, h))\psi] \\ &= ([f]\cdot[g])\cdot[h].\end{aligned}$$

Therefore $[A; X]_\mu$ is associative, and the theorem is proved.

COROLLARY 3.2. *If X is a connected CW-complex,* cat $X \leqq 3$, *and (X, μ) is an H-space, then $[X; X]_\mu$ is a group.*

We note that if X satisfies the hypothesis of Corollary 3.2, then (X, μ) is an inversive H-space. For the group $[X; X]_\mu$ is inversive, and so, by Theorem 1.2(a), μ is inversive. Thus, for example, if X is an $(n-1)$-connected CW-complex of dimension less than $3n$, then every H-structure map on X is homotopy inversive; for cat $X \leqq 3$ by Proposition 2.5 of [1].

THEOREM 3.3. *Let (X, μ) be an H-space, and let A be a connected CW-complex. If* cat $A \leqq 2$, *then $[A; X]_\mu$ is an abelian group.*

The set $[A; X]_\mu$ is a group by Theorem 3.1. That $[A; X]_\mu$ is commutative follows from Corollary 4D of [3]. A direct proof of commutativity can also be constructed along the lines of our proof of Theorem 3.1.

4. Homotopy power associativity and homotopy diassociativity

Let P be one of the following properties of loops:

(a) inversivity,

(b) commutativity,

(c) associativity,

and let P' denote the corresponding homotopy property of H-spaces. Theorem 1.2 states that an H-space (X, μ) has property P' if and only if $[X^n; X]_\mu$ has property P for the appropriate value of n. In this section we extend the list of homotopy properties P' by defining homotopy power associativity and homotopy diassociativity. Results complementing Theorem 1.2 with respect to these properties are obtained in Theorem 4.1 and in Theorem 4.4.

Recall that a loop M is said to be *power associative* provided that the subloop $\{a\}$ generated by any element $a \in M$ is a cyclic group. Equivalently, M is power associative provided that M is inversive and the equations

$$a \cdot (a \cdot a) = (a \cdot a) \cdot a, \qquad a^{-1} \cdot (a \cdot a) = (a \cdot a) \cdot a^{-1} = a$$

hold for every element $a \in M$. We define next an analogous concept for H-spaces.

Let (X, μ) be a homotopy inversive H-space, and let $\eta : X \to X$ be a homotopy inverse map. We abbreviate $\mu(x, y)$ to $x \cdot y$ and write x^{-1} for $\eta(x)$. Let $1 : X \to X$ be the identity map, and let $h_1, k_1, h_2, k_2 : X \to X$ be the maps defined as follows:

$$h_1(x) = x \cdot (x \cdot x), \qquad k_1(x) = (x \cdot x) \cdot x,$$

$$h_2(x) = x^{-1} \cdot (x \cdot x), \qquad k_2(x) = (x \cdot x) \cdot x^{-1}.$$

We say that an inversive H-space (X, μ) is *homotopy power associative* provided that $h_1 \simeq k_1$ and $h_2 \simeq k_2 \simeq 1$.

THEOREM 4.1. *Let X be a CW-complex, and let (X, μ) be an H-space. Then (X, μ) is homotopy power associative if and only if $[X; X]_\mu$ is a power associative loop.*

Suppose that (X, μ) is homotopy power associative. If $f : X \to X$ is any map, then the element $[f] \cdot ([f] \cdot [f]) \in [X; X]_\mu$ is represented by the map $h_1 f : X \to X$, and the element $([f] \cdot [f]) \cdot [f]$ is represented by the map $k_1 f$. Since $h_1 \simeq k_1$, it follows that

$$[f] \cdot ([f] \cdot [f]) = ([f] \cdot [f]) \cdot [f].$$

In a similar manner one can show that the relation $h_2 \simeq k_2 \simeq 1$ leads to the equation

$$[f]^{-1} \cdot ([f] \cdot [f]) = ([f] \cdot [f]) \cdot [f]^{-1} = [f].$$

Therefore $[X; X]_\mu$ is a power associative loop.

Conversely, suppose that $[X; X]_\mu$ is a power associative loop. Then

$[X; X]_\mu$ is inversive, and so, by Theorem 1.2 (a), (X, μ) is a homotopy inversive H-space. The map $h_1 : X \to X$ represents the element $[1]\cdot([1]\cdot[1]) \in [X; X]_\mu$, and the map k_1 represents $([1]\cdot[1])\cdot[1]$. But $[X; X]_\mu$ is power associative; hence $[1]\cdot([1]\cdot[1]) = ([1]\cdot[1])\cdot[1]$. Therefore $h_1 \simeq k_1$. The proof that $h_2 \simeq k_2 \simeq 1$ is similar. Thus the theorem is proved.

COROLLARY 4.2. *If (X, μ) is a homotopy power associative H-space, then $[A; X]_\mu$ is a power associative loop for every CW-complex A.*

The proof is straightforward and will be omitted.

COROLLARY 4.3. *Let X be a connected CW-complex, and suppose that* $\operatorname{cat} X \leqq 3$. *Then every H-structure map on X is homotopy power associative.*

For if μ is any H-structure map on X, then, by Corollary 3.2, $[X; X]_\mu$ is a group. Therefore, by Theorem 4.1, (X, μ) is a homotopy power associative H-space, and the proof of the corollary is completed.

Let M be a loop. Recall that M is said to be *diassociative* provided that the subloop $\{a, b\}$ generated by any two elements $a, b \in M$ is a group. Equivalently, M is diassociative provided that M is inversive and the following equations are satisfied for any two elements $a, b \in M$:

$$a\cdot(a\cdot b) = (a\cdot a)\cdot b, \qquad a\cdot(b\cdot a) = (a\cdot b)\cdot a, \qquad a\cdot(b\cdot b) = (a\cdot b)\cdot b,$$

$$a^{-1}\cdot(a\cdot b) = b, \qquad a = (a\cdot b)\cdot b^{-1}, \qquad a^{-1}\cdot(b\cdot a) = (a^{-1}\cdot b)\cdot a.$$

We now define an analogous concept for H-spaces.

Let (X, μ) be an H-space with a homotopy inverse map $\eta : X \to X$. For any $x \in X$ denote $\eta(x)$ by x^{-1}, and for any two points $x, y \in X$ denote $\mu(x, y)$ by $x\cdot y$. Define maps $h_i, k_i : X^2 \to X$, $i = 1, \cdots, 6$, by the equations

$$\begin{aligned}
h_1(x, y) &= x\cdot(x\cdot y), & k_1(x, y) &= (x\cdot x)\cdot y,\\
h_2(x, y) &= x\cdot(y\cdot x), & k_2(x, y) &= (x\cdot y)\cdot x,\\
h_3(x, y) &= x\cdot(y\cdot y), & k_3(x, y) &= (x\cdot y)\cdot y,\\
h_4(x, y) &= x^{-1}\cdot(x\cdot y), & k_4(x, y) &= y,\\
h_5(x, y) &= x^{-1}\cdot(y\cdot x), & k_5(x, y) &= (x^{-1}\cdot y)\cdot x,\\
h_6(x, y) &= x, & k_6(x, y) &= (x\cdot y)\cdot y^{-1}.
\end{aligned}$$

We say that an inversive H-space (X, μ) is *homotopy diassociative* provided that $h_i \simeq k_i$, $i = 1, \cdots, 6$.

The following theorem, together with the corollaries, generalizes a result of I. M. James. (See [6, Theorem 10.2].)

THEOREM 4.4. *Let X be a CW-complex, and suppose that (X, μ) is an H-space. Then (X, μ) is homotopy diassociative if and only if $[X^2; X]_\mu$ is a diassociative loop.*

The proof is similar to our proof of Theorem 4.1 and will be omitted.

COROLLARY 4.5. *If (X, μ) is a homotopy diassociative H-space, then $[A; X]_\mu$ is a diassociative loop for every CW-complex A.*

COROLLARY 4.6. *Let X be a connected CW-complex, and suppose* cat $X \leqq 2$. *Then every H-structure map on X is homotopy diassociative.*

Suppose that (X, μ) is an H-space. Then it follows from Theorem 3.1 that $[X^2; X]_\mu$ is a group. For, by Theorem 9 of [4], cat $X^2 \leqq 3$. Therefore, by Theorem 4.4, (X, μ) is homotopy diassociative; hence the corollary is proved.

We remark that the notion of "strongly homotopy diassociative" can be defined for strong H-spaces in the obvious manner, and a result similar to Proposition 1.7 holds with respect to this property.

We conclude with an application of Corollary 4.6 to the seven-dimensional sphere S^7. This space is, in fact, a diassociative topological loop if the multiplication is taken to be the one induced by the multiplication of Cayley numbers. There are, however, 120 distinct homotopy classes of H-structure maps on S^7, none of which is homotopy associative [5], yet each of which is homotopy diassociative.

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