A NOTE ON HOMOTOPY IN HOMEOMORPHISM SPACES¹

BY

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1. INTRODUCTION

In some papers soon to be published or under preparation ([7] and [8]) concerning homotopy properties of the space of homeomorphisms on a 2-manifold, the following situation occurs. There is a mapping f of S^{k} into the space H of homeomorphisms of the compact metric space K, usually a 2-manifold. A certain subset, L, of K, usually a point or simple closed curve, onto itself. has the property that if $f_L(x)$ is the restriction to L of f(x), then there is a homotopy θ_t such that for each x in S^k , $\theta_0(x) = f_L(x)$, $\theta_1(x)$ is the identity map of L onto itself, and $\theta_t(x)$ is a homeomorphism of L into K. It is then desirable to extend each $\theta_t(x)$ to $\theta_t^*(x)$ to obtain the homotopy θ_t^* in H such that for each x in S^k , $\theta_0^*(x) = f(x)$, $\theta_1^*(x)$ is the identity map of K onto itself, and $\theta_t^*(x)$ is a homeomorphism of K onto itself. Such an extension can sometimes be obtained by looking at $S^k \times K \times I$ and using various theorems about regular mappings. In Section 3 of this note is proved a general elementary theorem that permits these homotopy extensions to be constructed. In fact H is proved, under certain conditions, to be a fiber space (in the sense of Hu, [9, p. 63]). Section 2 contains the necessary theorem about regular mappings and some related theorems, useful in [7] and [8], and Section 4 contains an application of the theorem of Section 3 to the space of homeomorphisms of S^1 This application and variations of it are useful in [7] and into an annulus. [8].

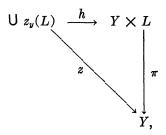
2. Regular mappings

A mapping z of a metric space X onto a metric space Y is completely regular if for each positive number ε and point y of Y, there is a positive number δ such that $d(y, y') < \delta$, $y, y' \in Y$, implies that there is a homeomorphism of $z^{-1}(y)$ onto $z^{-1}(y')$ that moves no point as much as ε (i.e. an ε -homeomorphism). In [2]–[6] were proved theorems giving conditions that imply that (X, f, Y) is a locally trivial fiber space or a direct product. In this section are given other such conditions, easily deduced from the earlier ones.

Suppose that K is a compact metric space, L is a closed subset of K and z is a completely regular mapping of a complete metric space X onto a finite (covering) dimensional space Y such that for each point y of Y, there is a homeomorphism z_y of K onto $z^{-1}(y)$. Suppose, further, that there is a homeomorphism h of $\bigcup z_y(L)$ onto $Y \times L$ such that the diagram

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where π is the projection map, is commutative and that z is completely regular in the strong sense that there is an ε -homeomorphism g of $z^{-1}(y)$ onto $z^{-1}(y')$ such that if $x \in z_y(L)$, then h(x) and hg(x) have the same L-coordinate in $Y \times L$. Finally, denote by H(K, L) the space of homeomorphisms of K onto itself leaving L pointwise fixed, this space being topologized by the uniform convergence metric.

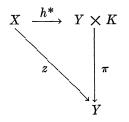
Note 1. A metric space is LC^k if for each point y and each positive number ε , there is a positive number δ such that every mapping of an *n*-sphere, S^n , $n \leq k$, into the δ -neighborhood of y is homotopic to 0 in the ε -neighborhood of y.

Note 2. The *n*-sphere S^n is considered as the boundary of the *n*-ball R^{n+1} .

Note 3. A mapping f of X into Y is homotopy *n*-regular if for each point x of $f^{-1}(y)$ and positive number ε , there is a positive number δ such that every mapping of the k-sphere, $k \leq n$, into $S(x, \delta) \cap f^{-1}(y')$, $y' \in Y$, is homotopic to 0 in $S(x, \varepsilon) \cap f^{-1}(y')$, where $S(x, \varepsilon)$ is the (open) ε -neighborhood of x. In [2]-[6] it was proved that under most circumstances, if K and L are manifolds of low dimension and f is homotopy *n*-regular, then f is completely regular.

Note 4. If M and N are topological spaces with subsets M_1 and N_1 respectively, the notations $M_1 \times N_1$ and (M_1, N_1) will be used to denote the subspace of $M \times N$ consisting of those points (x, y) such that $x \in M_1$ and $y \in N_1$.

THEOREM 2.1. If H(K, L) is locally connected and Y is a cell, then h may be extended to a homeomorphism h^* of X onto $Y \times K$ such that the diagram



is commutative.

Proof. This is the lemma of [6].

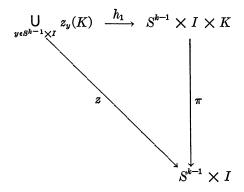
COROLLARY. This theorem remains true if h^* is already defined on $Y' \times 0$ (where $Y = Y' \times I$ and Y' is a cell of dimension one less than that of Y).

THEOREM 2.2. If H(K, L) is LC^k and $Y = S^k \times I$, then h may be extended to the required h^* .

Proof. It follows from Lemma 5.6 of [5] and its proof that the space of homeomorphisms of $S^k \times K$ onto itself leaving each (x, K) invariant and $S^k \times L$ pointwise fixed is locally connected. The proof of Theorem 5 of [2] or of the lemma of [6] now applies.

COROLLARY. This remains true if h^* is already defined on $z^{-1}(x, I)$ for some point x and/or it is already defined on $(S^k, 0)$.

THEOREM 2.3. If $Y = \mathbb{R}^k \times I$, H(K, L) is LC^k , and h extends to a homeomorphism h_1 of $\bigcup z_y(K)$, $y \in S^{k-1} \times I$, onto $S^{k-1} \times I \times K$ such that the diagram



is commutative, then h_1 may be extended to the required h^* .

Proof. It also follows from the proof of Lemma 5.6 of [5] that the space of homeomorphisms of $\mathbb{R}^k \times K$ onto itself leaving each (x, K) invariant and $(\mathbb{R}^k \times L) \cup (S^{k-1} \times K)$ pointwise fixed is locally connected, so the theorem follows as above.

COROLLARY. This remains true if h^* is already defined on $z^{-1}(x, I)$ for some point x and/or it is already defined on $(R^k, 0)$.

THEOREM 2.4. If H(K, L) is LC^k and its homotopy groups vanish in dimensions $\leq k$ and Y has covering dimension $\leq k + 1$, then h^* exists as above. If h^* is already defined on the inverse under z of a closed subset of Y, then the theorem still holds.

Proof. This is a consequence of Theorem 2 of [2]. The second sentence above does not follow directly from the statement of that theorem, but the selection theorem of Michael [12] used in its proof and stated in [2] as Theorem M makes the truth of that sentence evident.

3. Homotopy in homeomorphism spaces

In this section, let H denote the space of homeomorphisms of K onto itself and H_L the space of homeomorphisms of L into K that are extendable to elements of H. Suppose that these homeomorphism spaces have the properties that (1) H(K, L) is locally connected and (2) for each $\varepsilon > 0$ and $f \in H_L$, there exists a $\delta > 0$ such that if g is an element of H_L and $d(f, g) < \delta$, then if f^* is an element of H and $f = f^* | L$, then there is an element g^* of H such that $d(f^*, g^*) < \varepsilon$ and $g^* | L = g$. If K is a 2-manifold and L is a simple closed curve or point or if K is a 3-manifold and L is a tamely imbedded 2-manifold, then these conditions are satisfied ([3], [4] and [5]), so Theorem 3.1 below is applicable to the study of the space of homeomorphisms on a 2 or 3-manifold.

Let p denote the mapping of H into H_L such that $p(f^*) = f^* | L$. Then for each element f of H_L , $p^{-1}(f)$ is homeomorphic to H(K, L).

THEOREM 3.1. The space H is a fiber space over the base space H_L with fiber H(K, L) and projection map p.

Proof. Let ϕ be a mapping of \mathbb{R}^k into H and θ a mapping of $\mathbb{R}^k \times I$ into H_L such that $\theta(x, 0) = p\phi(x)$ for each x in \mathbb{R}^k . Consider $\mathbb{R}^k \times I \times K$ as the X of Section 2 and the projection map π of this on $\mathbb{R}^k \times I$ as z. Let $z_{y,t}$ be a homeomorphism of K into $\pi^{-1}(y, t) = (y, t, K)$ taking the point x of L into $(y, t, \theta(y, t)(x))$. Such a homeomorphism exists as a consequence of the definition of H_L . Let h be the mapping of $\bigcup z_{y,t}(L)$ into $\mathbb{R}^k \times I \times L$ carrying $(y, t, \theta(y, t)(x))$ onto (y, t, x). Let h_1 be the homeomorphism of $\mathbb{R}^k \times 0 \times K$ onto itself taking $(y, 0, \phi(y)(x))$ onto (y, 0, x). Conditions (1) and (2) above guarantee that the requirements in the hypotheses of Theorem 2.1 and its corollary are met, so h and h_1 extend to a homeomorphism h^* of $\mathbb{R}^k \times I \times K$ onto itself leaving each (y, t, K) invariant.

For each x in K and y in \mathbb{R}^k , let $\theta^*(y, t)(x) = x^1$, where $h^*(y, t, x^1) = (y, t, x)$. Then, since $h^*(y, 0, \phi(y)(x)) = (y, 0, x)$, $\theta^*(y, 0) = \phi(y)$. If $x \in L$, $h^*(y, t, \theta(y, t)(x)) = (y, t, x)$; hence $\theta^*(y, t)(x) = \theta(y, t)(x)$ —i.e. $p\theta^*(y, t) = \theta(y, t)$. Therefore the covering homotopy property holds for simplices and the theorem is proved.

COROLLARY 1. Let f^* be a mapping of S^k into H and θ_t a homotopy of pf^* in H_L such that $\theta_0(x) = pf^*(x)$ for each x in S^k . Then there is a homotopy θ_t^* of f^* in H such that $p\theta_t^*(x) = \theta_t(x)$ for each x in S^k .

COROLLARY 2. Let f^* be a mapping of \mathbb{R}^{k+1} into H, θ_t a homotopy of pf^* in H_L such that $\theta_0(x) = pf^*(x)$ for each x in \mathbb{R}^{k+1} and ϕ_t^* a homotopy of $f^* | S^k$ in H such that for each x in S^k , $\phi_0^*(x) = f^*(x)$ and $p\phi_t^*(x) = \theta_t(x)$ for each t. Then ϕ_t^* can be extended to a homotopy θ_t^* of f^* in H such that for each t and each x in \mathbb{R}^{k+1} , $\theta_0^*(x) = f^*(x)$ and $p\theta_t^*(x) = \theta_t(x)$ for each t. Then ϕ_t^{k+1} , $\theta_0^*(x) = f^*(x)$ and $p\theta_t^*(x) = \theta_t(x)$ and for x in S^k , $\theta_t^*(x) = \phi_t^*(x)$.

Proof of Corollaries 1 and 2. These are just alternative forms of the require-

ment that H be a fiber space with base H_L and projection p. (See Hu, p. 63.)

COROLLARY 3. If M is a closed subset of L and all mappings in H and H_L have the further properties that they leave M pointwise fixed (conditions (1) and (2) remain satisfied), then H is a fiber space with base space H_L , projection p, and fiber H(K, L).

Proof. The proof follows that for Theorem 3.1.

4. Application to annuli

The above theorems can be used to prove that if K is a 2-sphere, then (1) if L consists of only one or two points, $\pi_1(H(K, L)) = Z$ and $\pi_i H(K, L) = 0$ for i > 1 and (2) if L is finite and contains more than two points, then $\pi_i H(K, L) = 0$ for each *i*. These statements were proved by McCarty in [11]. In [6], it is proved that if K is a disc with holes and L is its boundary, then $\pi_i(H(K, L)) = 0$ for each *i*. The techniques of [6] can be used to prove without difficulty that is K has *n* holes, then H has the same homotopy groups as the space of homeomorphisms on S^2 leaving *n* points fixed. The following theorems, whose proof uses these facts, is useful in [7] and variations of it are useful in [8], where surfaces more complicated than the annulus must be considered.

THEOREM 4.1. Let K be an annulus, L a simple closed curve in the interior of K that separates the boundary curves of K, and x a point of L. Then $\pi_i(H_L) = 0$ for i > 1, $\pi_1(H_L) = Z$, and the identity component of the space of homeomorphisms of L into int K that leave x fixed is homotopically trivial.

Proof. It follows from [3, Theorem 2] that H(K, L) is homotopically trivial. In particular, it is connected $(\pi_0(H(K, L)) = 0)$. The spaces H, H_L and H(K, L) satisfy the requirements for Theorem 3.1, as noted earlier. Hence H is a fiber space over H_L with fiber H(K, L). Since $\pi_i(H(K, L)) = 0$ for $i \ge 0$, the exactness of the homotopy sequence of this fibering (see Hu, p. 152) yields $\pi_i(H) = \pi_i(H_L)$ for $i \ge 1$. But $\pi_i(H) = 0$ for i > 1 and $\pi_1(H) = Z$, as noted above, so the first two conclusions of the theorem follow.

Let $H_{L,x}$ denote the space of homeomorphisms of L into int K that leave x fixed. Then it follows from Corollary 3 to Theorem 3.1 that H(K, x) is a fiber space with base space $H_{L,x}$ and fiber H(K, L). Since $\pi_i(H(K, L)) = 0$ for $i \ge 0$, $\pi_i(H(K, x)) = \pi_i(H_{L,x})$ for $i \ge 1$. But H(K, x) has the homotopy groups of the space of homeomorphisms on a disc with three holes. Therefore $\pi_i(H_{L,x}) = 0$ for $i \ge 1$.

References

- 1. J. W. ALEXANDER, On the deformation of an n-cell, Proc. Nat. Acad. Sci., vol. 9 (1923), pp. 406-407.
- ELDON DYER AND M.-E. HAMSTROM, Completely regular mappings, Fund. Math., vol. 45 (1958), pp. 103-118.

- 3. M.-E. HAMSTROM AND ELDON DYER, Regular mappings and the space of homeomorphisms on a 2-manifold, Duke Math. J., vol. 25 (1958), pp. 521-532.
- 4. M.-E. HAMSTROM, Regular mappings whose inverses are 3-cells, Amer. J. Math., vol. 82 (1960), pp. 393-429.
- 5. ———, Regular mappings and the space of homeomorphisms on a 3-manifold, Mem. Amer. Math. Soc., number 40 (1961).
- 6. , Some global properties of the space of homeomorphisms on a disc with holes, Duke Math. J., vol. 29 (1962), pp. 657-662.
- 7. ——, The space of homeomorphisms on a torus, Illinois J. Math., vol. 9 (1965), pp. 59-65.
- 8. , Homotopy in the space of homeomorphisms on a 2-manifold, preliminary report, Notices Amer. Math. Soc., vol. 11 (1964), p. 333.
- 9. S. T. Hu, Homotopy theory, New York, Academic Press, 1959.
- 10. H. KNESER, Die deformationssätze der einfach zussammenhängenden Flächen, Math. Zeitschrift, vol. 25 (1926), pp. 362–372.
- 11. G. S. McCARTY, JR., *Homeotopy groups*, Trans. Amer. Math. Soc., vol. 106 (1963), pp. 293-304.
- E. A. MICHAEL, Continuous selections, II, Ann. of Math. (2), vol. 64 (1956), pp. 562– 580.

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